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A Simple Approach to the Study of Global Asymptotic Stability of Some Modified Continuous-Time Epidemiological Models for Distributed Denial of Service attacks

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Abstract. In this work, we first revisit two recognized continuous-time epidemiological models for distributed denial of service (DDoS) attacks on targeted sources in computer networks, which are described by systems of nonlinear ordinary differential equations (ODEs) with complex dynamics. These models were formulated and analyzed in existing literature but the global asymptotic stability (GAS) of disease-free equilibrium (DFE) points has not been established.

Our main objective is to perform a rigorous mathematical analysis for the complete GAS of the two mathematical models under consideration. We use a simple approach, which is based on utilizing the cascade structure of the ODE systems, to study the GAS problem. More clearly, by taking advantage of the cascade structure, the GAS analysis of the original nonlinear systems is reduced to the GAS analysis of simple linear systems. After that, the GAS analysis of the reduced linear systems is completed in a straightforward manner. As an important consequence, the GAS is confirmed not only for the DEE points but also for possible disease-endemic equilibrium (DEE) points.

The theoretical findings improve the results presented in the benchmark works. Furthermore, the present approach can be applied to a broad range of mathematical models arising in real-world applications, with a specific focus on DDoS attacks. To show advantages of the proposed approach, we consider some other mathematical models of DDoS attacks constructed previously. It is proved that the used approach is not only simple but also useful in investigating the GAS of the mathematical models being considered.

Finally, the theoretical insights are illustrated by a set of illustrative numerical experiments, in which the validity of the theoretical findings is supported.

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1. Introduction

It is well-known that cyber attacks have always been a serious and permanent threat to the safety of computer systems, services, equipment and data both of organizations and individuals. For this reason, many efforts have been made to find solutions and strategies against cyber attacks. Inspired by classical and standard epidemiological models (see, for instance, [2, 5, 12, 13, 14, 25]), a great number of mathematical models for studying the propagation of malware, computer viruses, worms, and distributed denial of service (DDoS) attacks, etc. have been constructed and analyzed [6, 7, 8, 9,

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10, 21, 26, 27, 28, 29, 30, 31, 32, 35, 36, 40, 41, 42, 43, 44, 45, 46]. The study of these models is very useful in discovering characteristics and mechanisms of cyber attacks. Consequently, strategies and measures to prevent cyber attacks can be suggested.

In this work, we revisit two modified epidemiological models for DDoS attacks on targeted sources in computer networks, which were proposed and analyzed in [10] and [31]. These models are represented by systems of nonlinear ordinary differential equations (ODEs), which describe the interaction between targeted nodes and attacking nodes. In each epidemiological model, the entire population of nodes is divided into two sub-populations, namely attacking and targeted populations. However, the number of compartments in each sub-population in each model is different. On the other hand, although they are built on basic principles of mathematical epidemiology, technical assumptions corresponding to each model are also different. For the sake of convenience, we will briefly recall the mathematical formulations under consideration in Subsection 2.2 (see (5) and (9)).

It was proved in [10, 31] that each ODE model being considered always possesses a disease-free equilibrium (DFE) point for all values of the parameters, whereas, a unique disease-endemic equilibrium (DEE) point exists if the basic reproduction number of the attacking population is greater than 1. An important common feature of both works [10] and [31] is that only the global asymptotic stability (GAS) of the DEE points was established based on the geometric approach proposed by Li and Muldowney [23], whereas only the local asymptotic stability of the DFE points was confirmed. However, previous studies on global dynamics of epidemiological models suggest a DFE point is often globally asymptotically stable if the basic reproduction number is less than or equal 1 (see, for instance, [2, 5, 25]). Moreover, it is important to remark that the GAS of DFE points of epidemiological models is very important since they correspond to the case of epidemics being extinguished. In particular, the GAS of the DFE points of the two ODE models being considered can suggest strategies and measures to prevent cyber attacks (see Remark 3.2).

Motivated by the above reason, our objective is to establish the GAS of the DFE points of the models (5) and (9). To achieve this objective, we use a simple approach, which is based on using the cascade structure of the ODE models and appropriate Lyapunov functions, to investigate the GAS of the DFE points. More clearly, we first analyze the GAS of the systems corresponding to the attacking populations by suitable Lyapunov functions. After that, by taking advantage of the cascade structure, the GAS analysis of the original nonlinear systems is reduced to the GAS analysis of linear systems of ODEs with constant coefficients. Consequently, the GAS not only of the DFE points but also of DEE points is easily obtained, which improves the results constructed in the benchmark works [10, 31]. On the other hand, the present approach is simpler than the geometric approach used in [10, 31] and can be applied to a broad range of mathematical models arising in real-world applications, with a particular focus on DDoS attacks. In Section 5, we show that the used approach is not only simple but also useful in examining the GAS of some other mathematical models of DDoS attacks, which were formulated in [1, 21, 26, 36].

It should be emphasized that the cascade structure has been utilized in [11] to study the GAS of some malware and computer virus propagation models described by ODEs. Although the Lyapunov stability theory [22, 24] has been one of the most successful approaches to the GAS problem of dynamical systems (see, e.g., [16, 17, 18, 19, 20, 37, 38, 39]), the construction of Lyapunov functions for the ODE models (5) and (9) is not a trivial task. However, thanks to the present approach, the GAS analysis is done in simple manner, in which Lyapunov function candidates for the sub-populations of the models (5) and (9) can be determined easily.

The plan of this work is as follows:
Some preliminaries, auxiliary results and the mathematical models under consideration are provided in Section 2. The analysis of GAS is presented in Sections 3. Numerical experiments supporting the theoretical assertions are conducted in Section 4. In Section 5, the GAS analysis of some other mathematical models considering the effect of external attacking nodes is considered. The last section includes some discussions and conclusions.

2. Preliminaries and auxiliary results

This section provides some preliminaries and auxiliary results that will be used in the next sections.

2.1. Global asymptotic stability of continuous-time dynamical systems. Consider a general continuous-time autonomous dynamical system of the form

$$\frac{dy(t)}{dt} = f(y(t)), \quad t \geq 0, \qquad y(0) = y_0 \in \mathbb{R}^n, \tag{1}$$

where y is a vector function and the right-hand side function f is assumed to satisfy suitable conditions such that solutions of (1) are unique [15, 34]. Assume that $y^* = 0$ is an equilibrium point of (1), that is $f(y^*) = 0$. We now state the Lyapunov stability theorem for continuous-time dynamical systems of the form (1) [15] (see also [22, 24, 34]).

Theorem 2.1. *Let $y^* = 0$ be an equilibrium point for (1). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that*

$$\begin{aligned} &V(0) = 0 \quad \text{and} \quad V(y) > 0, \quad \forall y \neq 0,\\ &\|y\| \to \infty \Longrightarrow V(y) \to \infty,\\ &\dot{V}(y) < 0, \quad \forall y \neq 0, \end{aligned}$$

then $y^* = 0$ is globally asymptotically stable.

Before ending this subsection, we recall from [33] a result for the GAS of two cascade connected nonlinear systems. Consider a triangular system

$$\dot{y_1} = f_1(y_1), \tag{2}$$

$$\dot{y_2} = f_2(y_1, y_2), \tag{3}$$

where $y_1 \in \mathbb{R}^{m_1}$ and $y, f_2 \in \mathbb{R}^{m_2}$. The variable of the space $\mathbb{R}^{m_3} = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ is denoted by y_3. Assume that $f_1(0) = f_2(0, 0) = 0$. Then, $y_2 = 0$ is an equilibrium point of

$$\dot{y_2} = f_2(0, y_2). \tag{4}$$

Theorem 2.2. *If (4) and (2) are both globally asymptotically stable, and every orbit of (2)-(3) are bounded for $t > 0$, then (2)-(3) is globally asymptotically stable.*

2.2. The mathematical models of DDoS attacks and their dynamical qualitative properties. In this subsection, we recall the mathematical models of DDoS attacks given in [10] and [31].

In both models, the entire populations of nodes are divided into two sub-populations, namely, targeted and attacking populations. In this first model [10], the targeted population is partitioned into three compartments, which are
(1) susceptible compartment: S_t;
(2) infected compartment: I_t;
(3) recovered compartment: R_t.
Meanwhile, the attacking population is partitioned only into two sub-classes, namely
(1) susceptible compartment: S;
(2) infected compartment: I.
Based on principles of mathematical epidemiology and a series of technical hypotheses, the following ODE model was given to describe the interaction between the compartments of the attacking population and targeted population [10]:

$$\begin{aligned}\frac{dS_t}{dt} &= -\beta S_t I + \xi_t R_t,\\ \frac{dI_t}{dt} &= \beta S_t I - \gamma I_t,\\ \frac{dR_t}{dt} &= \gamma I_t - \xi_t R_t,\\ \frac{dS}{dt} &= \mu - \beta SI - \mu S + \xi I,\\ \frac{dI}{dt} &= \beta SI - (\xi + \mu) I,\end{aligned} \tag{5}$$

where S_t, I_t and R_t stand for the number of susceptible, infected and recovered nodes in the targeted population, respectively, and S and I represent the number of susceptible and infected nodes in the attacking population, respectively. Since $dS_t/dt + dI_t/dt + dR_t/dt = dS/dt + dI/dt = 0$, the total populations are constant. Without loss of generality, we can assume that $S_t + I_t + R_t = S + I = 1$ for $t \geq 0$. Then, (5) admits the following set as a positively invariant set

$$\Omega_1 = \{(S_t, I_t, R_t, S, I) \in \mathbb{R}^5_+ \, | S_t + I_t + R_t = S + I = 1\}. \tag{6}$$

In the model (5):
(1) μ represents the rate of addition of new vulnerable systems and their removal from the network;
(2) β is the infectivity contact rate; γ is the recovery rate for the targeted systems;
(3) ξ_t and ξ are the rates at which the recovered targeted nodes and the disinfected attacking hosts again become susceptible to the attack.

For the model (5), the basic reproduction number for the attacking population was calculated as

$$\mathcal{R}^1_{0a} = \frac{\beta}{\xi + \mu}. \tag{7}$$

It was proved in [10] that:
(1) the model (5) always has a DFE point $E^0_1 = (S^0_t, I^0_t, R^0_t, S^0, I^0) = (1, 0, 0, 1, 0)$ for all values of the parameters, whereas, a unique DEE point E^*_1 exists if and

only if $\mathcal{R}_{0a}^1 > 1$. Furthermore, if E_1^* exists, it is given by

$$
\begin{aligned}
S_t^* &= \frac{\xi_t}{\xi_t + \left(1 + \dfrac{\xi_t}{\gamma}\right)(\beta - \xi - \mu)}, \\
I_t^* &= \frac{\xi_t}{\dfrac{\xi_t \gamma}{\beta - \xi - \mu} + (\gamma + \xi_t)}, \\
R_t^* &= 1 - S_t^* - I_t^*, \\
I^* &= \frac{\beta - \xi - \mu}{\beta}, \\
S^* &= 1 - I^*.
\end{aligned} \tag{8}
$$

(2) the DFE point is locally asymptotically stable if $\mathcal{R}_{0a}^1 < 1$ and is unstable if $\mathcal{R}_{0a}^1 > 1$;
(3) the DEE point is not only locally asymptotically stable but also globally asymptotically stable if it exists.

In the second model [31], the entire population of nodes is also divided into attacking and targeted populations but the targeted population is divided into four compartments:
(1) susceptible compartment: S_t;
(2) infected compartment: I_t;
(3) quarantined compartment: Q_t;
(4) recovered compartment: R_t;

and the attacking population is also divided in two compartments: susceptible (S) and infected (I) compartments. Let us denote by S_t, I_t, Q_t and R_t the number of susceptible targeted, infectious targeted, quarantine targeted and recovered targeted nodes and by S, I the number of susceptible nodes and infectious attacking nodes, respectively. Based on some technical hypotheses (see (H1)-(H6) in [31]), the following dynamic model was given in [31]

$$
\begin{aligned}
\frac{dS_t}{dt} &= -\beta S_t I + \epsilon_t R_t, \\
\frac{dI_t}{dt} &= \beta S_t I - \gamma I_t, \\
\frac{dQ_t}{dt} &= \gamma I_t - \eta Q_t, \\
\frac{dR_t}{dt} &= \eta Q_t - \epsilon_t R_t, \\
\frac{dS}{dt} &= \mu - \beta S I - \mu S + \epsilon I, \\
\frac{dI}{dt} &= \beta S I - (\mu + \epsilon) I,
\end{aligned} \tag{9}
$$

where
(1) μ is the rate of newborn and natural death of nodes from the network;
(2) β is the infectivity contact rate;
(3) γ is the quarantine rate;
(4) η is the recovery rate;

(5) ϵ is the rate at which recovered targeted nodes are again susceptible;
(6) ϵ_t is the rate at which attacking nodes become susceptible.
Similarly to the model (5), we also assume that $S_t + I_t + Q_t + R_t = S + I = 1$ for $t \geq 0$. So, the model (9) admits the following set as a positively invariant set

$$\Omega_2 = \{(S_t, I_t, Q_t, R_t, S, I) \in \mathbb{R}^6_+ | S_t + I_t + Q_t + R_t = S + I = 1\}. \tag{10}$$

For the model (9), the basic reproduction number for the attacking population was calculated as

$$\mathcal{R}^2_{0a} = \frac{\beta}{\epsilon + \mu}. \tag{11}$$

Dynamics of (9) was established as follows [31]:
(1) the model (9) always has a DFE point E^0_2 for all values of the parameters; whereas, a unique DEE point E^*_2 exists if and only if $\mathcal{R}^2_{0a} > 1$;
(2) the DFE point is locally asymptotically stable if $\mathcal{R}^2_{0a} < 1$ and is unstable if $\mathcal{R}^2_{0a} > 1$;
(3) the DEE point is not only locally asymptotically stable but also globally asymptotically stable if it exists.
It is clear that only the local stability of the DEE points was confirmed. In the next section, we will establish the complete GAS of the models (5) and (9).

3. Global asymptotic stability analysis

In this section, we analyze the complete GAS of (5) and (9). First, we need the following simple result, which is useful in analyzing the GAS problem.

Lemma 3.1. *Consider a dynamical system described by a scalar ODE*

$$\frac{dy}{dt} = y(\lambda_1 - \lambda_2 y), \quad y(0) = y_0 \in \mathbb{R}, \tag{12}$$

where λ_1 and λ_2 are real numbers with $\lambda_2 > 0$. Then, we have
(1) If $\lambda_1 \leq 0$, then $y^0 = 0$ is a globally asymptotically stable equilibrium point of (12) with respect to the set $\mathbb{R}_+$.
(2) If $\lambda_1 > 0$, then $y^* = \lambda_1/\lambda_2$ is a globally asymptotically stable equilibrium point of (12) with respect to the set $\mathbb{R}_+ - \{0\}$.

Proof. First, it follows from (12) that

$$y(t) = y(0)e^{\int_0^t y(\tau)(\lambda_1 - \lambda_2 y(\tau))d\tau}.$$

Therefore, $y(t) \geq 0$ for $t > 0$ whenever $y(0) \geq 0$. Furthermore, if $y(0) > 0$, then $y(t) > 0$ for $t > 0$. So, (12) admits $\mathbb{R}_+$ and $\mathbb{R}_+ - \{0\}$ as positively invariant sets.

To prove the GAS of the trivial equilibrium point $y^0 = 0$, consider a Lyapunov function $V_1(y) = y$. Then, the derivative of V_1 along solutions of (12) satisfies

$$\frac{dV_1}{dt} = \frac{dy}{dt} = \lambda_1 y - \lambda_2 y^2 \leq -\lambda_2 y^2.$$

Hence, by Lyapunov stability theorem (Theorem 2.1), y^0 is globally asymptotically stable.

To show the GAS of the unique positive equilibrium point $y^* = \lambda_1/\lambda_2$, consider a Lyapunov function candidate (see [16, 17, 18, 19, 20, 37, 38, 39])

$$V_2(y) = y - y^* \ln\left(\frac{y}{y^*}\right) - y^*.$$

Then, we have

$$\frac{dV_2}{dt} = \frac{y - y^*}{y}\frac{dy}{dt} = \frac{y - y^*}{y}\lambda_2 y(y^* - y) = -\lambda_2(y - y^*)^2,$$

which implies the GAS of y^*. The proof is complete. □

3.1. Global stability analysis of (5). We now analyze the complete GAS of (5). Since Ω_1 given in (6) is a positively invariant set of (5), it is sufficient to consider the following reduced system

$$\begin{aligned}
\frac{dI_t}{dt} &= \beta I(1 - I_t - R_t) - \gamma I_t,\\
\frac{dR_t}{dt} &= \gamma I_t - \xi_t R_t,\\
\frac{dI}{dt} &= \beta(1 - I)I - (\xi + \mu)I,
\end{aligned} \tag{13}$$

on a feasible set given by

$$\Omega_1^* = \{(I_t, R_t, I) \in \mathbb{R}^3_+ \,|I_t + R_t \le 1, \quad I \le 1\}. \tag{14}$$

Now, the DEE and DFE points are reduced to $\widetilde{E}_1^0$ and $\widetilde{E}_1^*$, respectively, where

$$\widetilde{E}_1^0 = (I_t^0, R_t^0, I^0) = (0, 0, 0), \quad \widetilde{E}_1^* = (I_t^*, R_t^*, I^*). \tag{15}$$

Here, I_t^*, R_t^* and I^* are defined in (8). The following theorem is the main result of this subsection.

Theorem 3.2 (GAS analysis of the $S_tI_tR_tSI$ model (5))**.** *(i) The DFE point $\widetilde{E}_1^0$ of the reduced model (13) is globally asymptotically stable whenever $\mathcal{R}_{0a}^1 := \dfrac{\beta}{\xi + \mu} \le 1$.*

(ii) Suppose that $\mathcal{R}_{0a}^1 := \dfrac{\beta}{\xi + \mu} > 1$. Then, the DEE point $\widetilde{E}_1^*$ of the reduced model (13) is globally asymptotically stable if $I(0) > 0$.

Proof. **Proof of Part (i).** First, consider the last equation of (13), which can be written in the form

$$\frac{dI}{dt} = I\big[(\beta - \xi - \mu) - \beta I\big]. \tag{16}$$

Note that $(\beta - \xi - \mu) \le 0$. So, Lemma 3.1 indicates that the trivial equilibrium point $I^0 = 0$ of (16) is globally asymptotically stable. Now, by applying Theorem 2.2, we only need to study the GAS of the following reduced system of (13)

$$\begin{aligned}
\frac{dI_t}{dt} &= -\gamma I_t,\\
\frac{dR_t}{dt} &= \gamma I_t - \xi_t R_t.
\end{aligned} \tag{17}$$

Here we have substituted $I = 0$ into the first equation of (13). The system (17) has a unique equilibrium point $e^0 = (0, 0)$. We now show that e^0 is globally asymptotically stable. Indeed, consider a Lyapunov function candidate defined by

$$V_3(I_t, R_t) = \frac{1}{2}\tau I_t^2 + \frac{1}{2}R_t^2, \quad \tau > 0.$$

Then, the derivative of V_3 along solutions of (17) satisfies

$$\begin{aligned}\frac{dV_3}{dt} &= \tau I_t \frac{dI_t}{dt} + R_t \frac{dR_t}{dt} = -\tau\gamma I_t^2 + \gamma I_t R_t - \xi_t R_t^2 \\ &= -\xi_t \left(R_t^2 - 2\frac{\gamma}{2\xi_t} I_t R_t + \frac{\gamma^2}{4\xi_t^2} I_t^2\right) + \left(\frac{\gamma^2}{4\xi_t} - \tau\gamma\right) I_t^2 \\ &= \xi_t \left(R_t - \frac{\gamma}{2\xi_t} I_t\right)^2 + \left(\frac{\gamma^2}{4\xi_t} - \tau\gamma\right) I_t^2.\end{aligned}$$

So, if $\tau > \gamma/(4\xi_t)$, then V_3 satisfies the Lyapunov stability theory. Consequently, e^0 is a globally asymptotically stable equilibrium point of (17). Combining this with the GAS of $I^0 = 0$ of (16), we conclude that $\widetilde{E}_1^0$ of (13) is globally asymptotically stable.
Proof of Part (ii). Assume that $\beta > \xi + \mu$. Then, $I^* = (\beta - \xi - \mu)/\beta$ is a globally asymptotically stable equilibrium point of (16). Thanks to Theorem 2.2, it is sufficient to consider the following system obtained by substituting $I = I^*$ into the first two equations of (13)

$$\begin{aligned}\frac{dI_t}{dt} &= \beta I^*(1 - I_t - R_t) - \gamma I_t, \\ \frac{dR_t}{dt} &= \gamma I_t - \xi_t R_t.\end{aligned} \tag{18}$$

Note that $e^* = (I_t^*, R_t^*)$ is a unique equilibrium point of (18). The next step is to prove that e^* is globally asymptotically stable. Indeed, it is easy to see that (18) can be rewritten in the form

$$\begin{aligned}\frac{dI_t}{dt} &= -(\beta I^* + \gamma)(I_t - I_t^*) - \beta I^*(R_t - R_t^*), \\ \frac{dR_t}{dt} &= \gamma(I_t - I_t^*) - \xi_t(R_t - R_t^*).\end{aligned} \tag{19}$$

Consider a Lyapunov function candidate given by

$$V_4(I_t, R_t) = \frac{1}{2}\gamma(I_t - I_t^*)^2 + \beta I_* \frac{1}{2}(R_t - R_t)^*.$$

Then,

$$\frac{dV_4}{dt} = \gamma(I_t - I_t^*)\frac{dI_t}{dt} + \beta I^*(R_t - R_t^*)\frac{dR_t}{dt} = -\gamma(\beta I^* + \gamma)(I_t - I_t^*)^2 - \beta I^* \xi_t (R_t - R_t^*)^2,$$

which implies the GAS of e^*. This is the desired conclusion. This proof is complete. □

Remark 3.1. From Theorem 3.2, the GAS not only of the DEE point but also of the DFE point of the full model (5) is obtained. This improves the conclusions in [10].

3.2. Global stability analysis of (9). In this subsection, the complete GAS of (9) will be established. Since Ω_2 defined in (10) is a positively invariant set of (9), it is sufficient to consider the following reduced system

$$\begin{aligned}
\frac{dI_t}{dt} &= \beta I(1 - I_t - Q_t - R_t) - \gamma I_t,\\
\frac{dQ_t}{dt} &= \gamma I_t - \eta Q_t,\\
\frac{dR_t}{dt} &= \eta Q_t - \epsilon_t R_t,\\
\frac{dI}{dt} &= \beta(1 - I)I - (\mu + \epsilon)I,
\end{aligned} \tag{20}$$

on a feasible region defined by

$$\Omega_2^* = \{(I_t, Q_t, R_t, I) \in \mathbb{R}_+^4 \,|I_t + Q_t + R_t \le 1;\;\; I \le 1\}. \tag{21}$$

Now, the DEE point is reduced to $\widetilde{E}_2^0$ with

$$\widetilde{E}_2^0 = (I_t^0, Q_t^0, R_t^0, I^0) = (0, 0, 0, 0).$$

Note that, $E_2^* = (S_t^*, I_t^*, R_t^*, S^*, I^*)$ is given by

$$\begin{aligned}
I^* &= \frac{\beta - \mu - \epsilon}{\beta},\\
S^* &= 1 - I^*,\\
I_t^* &= \frac{\beta I^*}{\beta I^* + \beta I^* \dfrac{\gamma}{\eta} + \beta I^* \dfrac{\gamma}{\epsilon_t} + \gamma},\\
Q_t^* &= \frac{\gamma}{\eta} I_t^*,\\
R_t^* &= \frac{\gamma}{\epsilon_t} I_t^*,\\
S_t^* &= 1 - S_t^* - I_t^* - R_t^*.
\end{aligned} \tag{22}$$

So, the DEE point of (9) is simplified to $\widetilde{E}_2^* = (I_t^*, Q_t^*, R_t^*, I^*)$. The following theorem is the main result of this subsection.

Theorem 3.3 (GAS analysis of the $S_tI_tQ_tR_tSI$ model (9)). *(i) The DFE point $\widetilde{E}_2^0$ of the reduced model (20) is globally asymptotically stable whenever $\mathcal{R}_{0a}^2 = \dfrac{\beta}{\mu + \epsilon} \le 1$. (ii) Suppose that $\mathcal{R}_{0a}^2 = \dfrac{\beta}{\mu + \epsilon} > 1$. Then, the DEE point $\widetilde{E}_2^*$ of the reduced model (20) is globally asymptotically stable if $I(0) > 0$.*

Proof. **Proof of Part(i).** Consider the last equation of (20), which can be re-written in the form

$$\frac{dI}{dt} = I\Big[(\beta - \mu - \epsilon) - \beta I\Big]. \tag{23}$$

If follows from Lemma 3.1 that $I^0 = 0$ is globally asymptotically stable as $\beta - \mu - \epsilon \leq 0$. So, we only need to consider the following reduced system of (20)

$$\begin{aligned} \frac{dI_t}{dt} &= -\gamma I_t, \\ \frac{dQ_t}{dt} &= \gamma I_t - \eta Q_t, \\ \frac{dR_t}{dt} &= \eta Q_t - \epsilon_t R_t, \end{aligned} \tag{24}$$

which is obtained by substituting $I = 0$ into the first three equations of (20).

The system (24) has a unique equilibrium point $f^0 = (0, 0, 0)$. We will show that this equilibrium point is globally asymptotically stable. Note that (24) also has the cascade structure. From the first equation of (24) that

$$I_t(t) = I_t(0)e^{-\gamma t},$$

which implies that $I_t^0 = 0$ is globally asymptotically stable. So, the GAS analysis of (24) is reduced to the GAS analysis of

$$\begin{aligned} \frac{dQ_t}{dt} &= -\eta Q_t, \\ \frac{dR_t}{dt} &= \eta Q_t - \epsilon_t R_t, \end{aligned} \tag{25}$$

Similarly to the proof of Part (i) of Theorem 3.2, we obtain that $(0, 0)$ is a globally asymptotically stable equilibrium point of (25) by using a Lyapunov function

$$V(Q_t, R_t) = \frac{1}{2}\tau^* Q_t^2 + \frac{1}{2}R_t^2, \qquad \tau^* > \frac{\eta}{4\epsilon_t}.$$

Therefore, $f^0 = (0, 0, 0)$ of (24) is globally asymptotically stable. This is the desired conclusion. The proof of this part is complete.

Proof of part (ii). Since $\beta > \mu + \epsilon$, $I^* = (\beta - \mu - \epsilon)/\beta$ is a globally asymptotically stable equilibrium point of (23). So, it is sufficient to focus on the reduced system

$$\begin{aligned} \frac{dI_t}{dt} &= \beta I^*(1 - I_t - Q_t - R_t) - \gamma I_t, \\ \frac{dQ_t}{dt} &= \gamma I_t - \eta Q_t, \\ \frac{dR_t}{dt} &= \eta Q_t - \epsilon_t R_t. \end{aligned} \tag{26}$$

It is easy to verify that (26) possesses a unique positive equilibrium point $f^* = (I_t^*, Q_t^*, R_t^*)$, where I_t^*, Q_t^* and R_t^* is given in (22).

By using stability theory of systems of linear ODEs [4, 34], it is enough to show that all eigenvalues λ of the Jacobian matrix of (26) evaluated at f^* satisfy $Re(\lambda) < 0$. The Jacobian matrix of (26) at f^* is

$$J(f^*) = \begin{pmatrix} -(\beta I^* + \gamma) & -\beta I^* & -\beta I^* \\ \gamma & -\eta & 0 \\ 0 & \eta & -\epsilon_t \end{pmatrix}.$$

The characteristic polynomial is given by

$$P(x) = x^3 + a_1 x^2 + a_2 x + a_3,$$

where

$$
\begin{aligned}
a_1 &= \epsilon_t + \eta + \gamma + I^*\beta,\\
a_2 &= \eta(\gamma + I^*\beta) + \epsilon_t(\eta + \gamma + I^*\beta) + I^*\beta\gamma,\\
a_3 &= \epsilon_t[\eta(\gamma + I^*\beta) + I^*\beta\gamma] + I^*\beta\eta\gamma.
\end{aligned}
$$

It is clear that $a_1 > 0, a_2 > 0$ and $a_3 > 0$. On the other hand,

$$
\begin{aligned}
a_1a_2 - a_3 = {} & (\beta^2\epsilon_t + \beta^2\eta + \beta^2\gamma)(I^*)^2\\
& + (\beta\epsilon_t^2 + 2\beta\epsilon_t\eta + 2\beta\epsilon_t\gamma + \beta\eta^2 + 2\beta\eta\gamma + \beta\gamma^2)I^*\\
& + \epsilon_t^2\eta + \epsilon_t^2\gamma + \epsilon_t\eta^2 + 2\epsilon_t\eta\gamma + \epsilon_t\gamma^2 + \eta^2\gamma + \eta\gamma^2 > 0.
\end{aligned}
$$

From the Routh-Hurwitz criteria [2, Theorem 4.4 and Section 4.5], we conclude that all the roots x^* of $P(x)$ satisfy $Re(x^*) < 0$. This shows that f^* is globally asymptotically stable. This is the desired conclusion. The proof is complete. □

Remark 3.2. We deduce from the complete GAS of (5) and (9) that a condition for the suppression of cyber attacks is that the reproduction numbers of the attacking populations are less than or equal to 1.

4. Numerical simulation

In this section, numerical examples are reported to support the theoretical findings. Here, we use the classical four stage Runge-Kutta (RK4) method (see [3, 34]) with a small step size $h = 10^{-4}$ to numerically solve the models (5) and (9).

Example 4.1 (GAS of the model (5))**.** Consider (5) with the following sets of parameters given in Table 1.

TABLE 1. The parameters used in Example 4.1.

Set	β	γ	ξ_t	ξ	μ	$\mathcal{R}_{0a}^1$	GAS equilibrium point
1	0.2	0.02	0.1	0.15	0.1	0.8	(1, 0, 0, 1, 0)
2	0.25	0.01	0.1	0.2	0.3	0.5	(1, 0, 0, 1, 0)
3	0.4	0.05	0.2	0.1	0.15	1.6	(0.2105, 0.6316, 0.1579, 0.6250, 0.3750)
4	0.8	0.06	0.2	0.25	0.15	2	(0.1034, 0.6897, 0.2069, 0.5, 0.5)

Figures 1-4 depict phase spaces of the reduced system (13) over the time interval $[0, 1000]$. In each figure, each blue curve represents a phase space associated with a particular initial data, the red circle marks the position of the globally asymptotically stable equilibrium point and the green arrows show the evolutionary trajectory of the model.

It is clear that the GAS of the DFE and DEE points of the model (5) is confirmed. So, the theoretical assertions presented in Subsection 3.1 are supported.

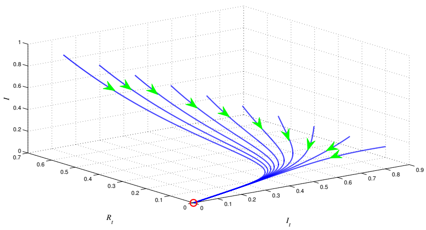


Figure 1. The phase spaces of the reduced system (13) for Set 1 of the parameters.

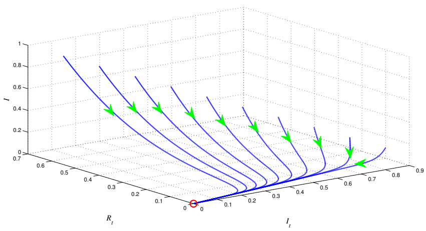


Figure 2. The phase spaces of the reduced system (13) for Set 2 of the parameters.

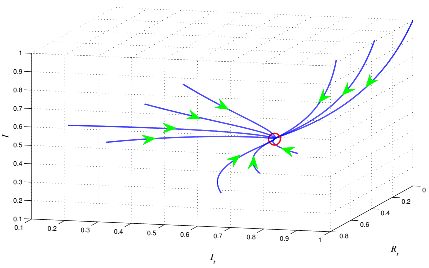


Figure 3. The phase spaces of the reduced system (13) for Set 3 of the parameters.

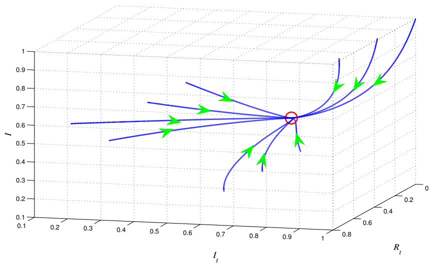


FIGURE 4. The phase spaces of the reduced system (13) for Set 4 of the parameters.

Example 4.2 (GAS of the model (9))**.** Consider (9) with the following two sets of parameters given in Table 2.

TABLE 2. The parameters used in Example 4.2.

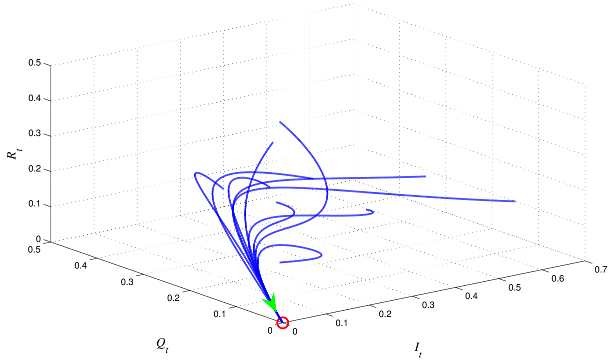
Case	β	γ	η	ϵ	ϵ_t	μ	$\mathcal{R}_{0a}^2$	GAS equilibrium point
1	0.4	0.15	0.15	0.25	0.3	0.25	0.8	(1, 0, 0, 0, 1, 0)
2	0.5	0.1	0.25	0.3	0.15	0.3	0.8333	(1, 0, 0, 0, 1, 0)
3	0.6	0.05	0.2	0.25	0.1	0.15	1.500	E^1
4	0.4	0.1	0.25	0.1	0.2	0.1	2.0000	E^2

In Table 2, E^1 and E^2 are given by: $E^1 = (0.1250, 0.5000, 0.1250, 0.2500, 0.6667, 0.3333)$ and
$E^2 = (0.2083, 0.4167, 0.1667, 0.2083, 0.5000, 0.5000)$.

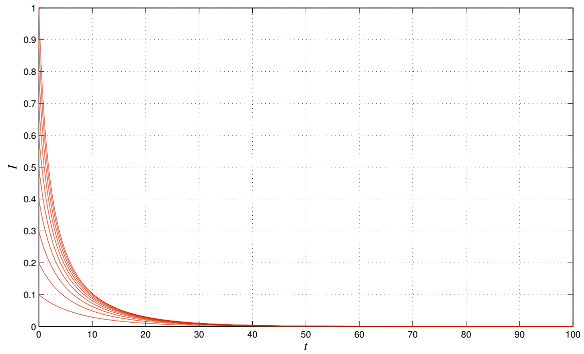
We now observe dynamics of the reduced system (20) on the time interval $[0, 100]$. The solutions are sketched in Figures 5-8. It is clear that the GAS of the DFE and DEE points of the model (9) is confirmed, which supports the theoretical assertions constructed in Subsection 3.2.

5. GAS analysis of some mathematical models considering the effect of external attacking nodes

In this section, to show advantages of the approach used in Section 3, we revisit some mathematical models of DDoS attacks, which consider the effect of external attacking nodes and were proposed recently in [1, 21, 26, 36]. Our objective is show that the approach is useful in studying the GAS of the mathematical models under consideration. Here, we will ignore arguments and detailed algebraic manipulations because they are performed similarly as in Section 3.



(A) The phase spaces for (I_t, Q_t, R_t).



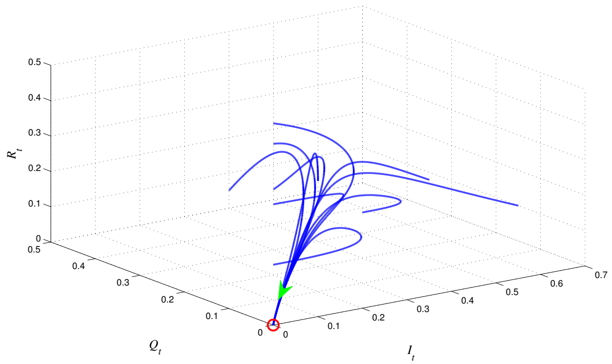
(B) The component I.

FIGURE 5. The solutions of the reduced system (20) for Set 1 of the parameters.

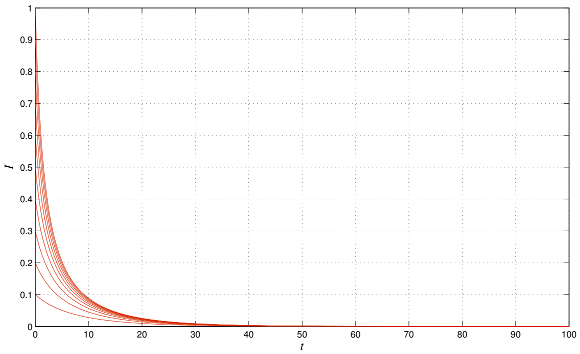
In [26], an ODE model integrating the effect of external nodes in the attacking population was proposed in the following form

$$
\begin{aligned}
\frac{dS_t}{dt} &= -\beta S_t I_a + \epsilon_t R_t, \\
\frac{dI_t}{dt} &= \beta S_t I_a - \gamma I_t, \\
\frac{dR_t}{dt} &= \gamma I_t - \epsilon R_t, \\
\frac{dS_a}{dt} &= -\beta S_a I_a - \mu S_a + \epsilon_a I_a + \sigma E_a - \alpha S_a, \\
\frac{dI_a}{dt} &= \beta S_a I_a - \mu_a I_a - \epsilon_a I_a - \alpha I_a, \\
\frac{dE_a}{dt} &= \alpha S_a + \alpha I_a - \sigma E_a + \mu - \mu E_a.
\end{aligned} \tag{27}
$$

In the model (27), the targeted population is classified to three classes:



(A) The phase spaces for (I_t, Q_t, R_t).



(B) The component I.

FIGURE 6. The solutions of the reduced system (20) for Set 2 of the parameters.

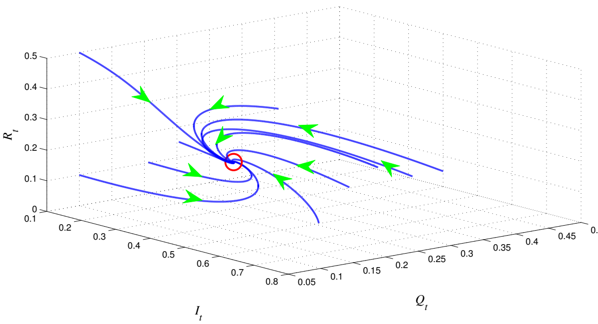
- susceptible targeted nodes: S_t;
- infectious targeted nodes: I_t;
- recovered targeted nodes: R_t;

and the attacking population is classified to three classes:

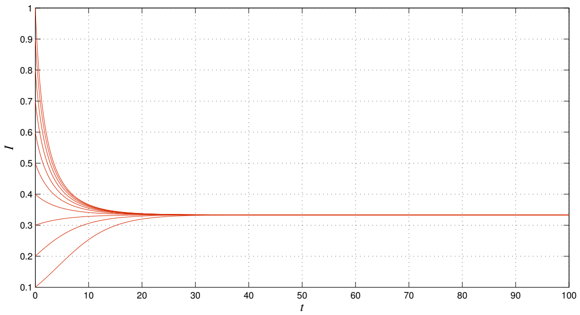
- susceptible attacking nodes: S_a;
- infectious attacking nodes: I_a;
- external attacking nodes: E_a;

Also, all the parameters are assumed to be positive because of their implications in epidemiology. The description and explanation of the model are given in [26].

In [36], another ODE system modeling attacks of computer viruses on targeted networks was proposed, in which the effect of external attacking nodes is considered.



(A) The phase spaces for (I_t, Q_t, R_t).



(B) The component I.

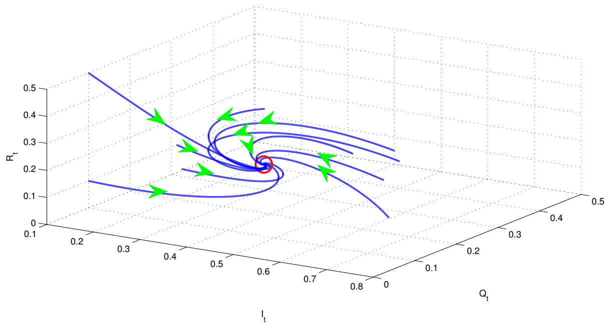
FIGURE 7. The solutions of the reduced system (20) for Set 3 of the parameters.

This system has the form

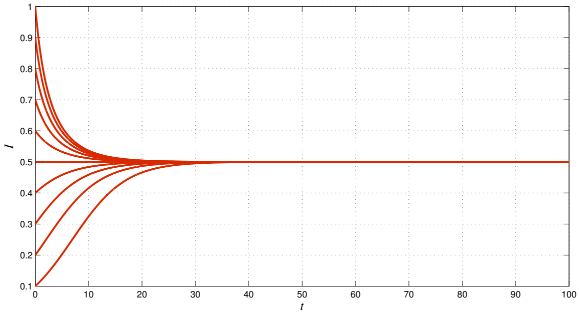
$$
\begin{aligned}
\frac{dS_t}{dt} &= -\beta_1 S_t I_a + \alpha R_t,\\
\frac{dI_t}{dt} &= \beta_1 S_t I_a - \gamma I_t,\\
\frac{dR_t}{dt} &= \gamma I_t - \alpha R_t,\\
\frac{dS_a}{dt} &= -\beta_2 S_a I_a + k_1 E_a - \eta S_a - \mu S_a,\\
\frac{dI_a}{dt} &= \beta_2 S_a I_a + k_2 E_a - \eta I_a - \mu I_a,\\
\frac{dE_a}{dt} &= \eta S_a + \eta I_a - k_1 E_a - k_2 E_a - \mu E_a + \xi,
\end{aligned} \tag{28}
$$

The description and explanation of the model are given in [36] (see Table 1 in [36]).

For the models (27) and (28), the basic reproduction numbers for the targeted and attacking populations were determined. For each model, it was shown that a DFE point always exists for all values of the parameters, whereas a unique DEE point



(A) The phase spaces for (I_t, Q_t, R_t).



(B) The component I.

FIGURE 8. The solutions of the reduced system (20) for Set 4 of the parameters.

exists if and only if the basic reproduction number of the attacking population is greater than 1. Moreover, the DFE point is locally asymptotically stable if the basic reproduction number is less than 1 and is unstable otherwise. Meanwhile, based on the geometric approach [23], the DEE point was proved to be globally asymptotically stable.

By applying the approach used in Section 3, we can establish the complete GAS of (27) and (28), which improves the results in [26, 36].

In [21], Kumari et al. introduced an ODE model for virus dynamics of distributed attacks on targeted networks. The GAS analysis of this model presented in [21] was quite complex. However, thanks to the present approach, the GAS problem can be solved easily.

The present approach can be applied to more complex models of DDoS attacks, for example, to a mathematical model considering two levels of security (low-security and high-security) in the targeted population formulated in [1]. The complete GAS of this model can be easily accomplished by using the proposed approach.

6. Concluding remarks and discussions

As the first and also the main conclusion of this study, we have established the complete GAS of two modified epidemiological models for DDoS attacks, which are described by systems of nonlinear ordinary differential equations (ODEs) with complex dynamics. By a simple approach, which is based on utilizing the cascade structure of the ODE models and suitable Lyapunov function candidates, the GAS problem has been resolved easily. By taking advantage of the cascade structure, the GAS analysis of the original nonlinear systems is reduced to the GAS analysis of simple linear systems. After that, the GAS analysis of the reduced systems is completed in a straightforward manner. As an important consequence, the GAS has been confirmed not only for the DEE points but also for the DEE points. Therefore, the obtained results improve the ones constructed in [10, 31]. Also, the complete GAS can suggest strategies and measures to prevent attacks (Remark 3.2).

The present approach is simple and can be applied to a broad range of mathematical models arising in real-world applications, with a specific focus on DDoS attacks. In Section 5, we have shown that the approach is not only simple but also useful in analyzing the GAS of some mathematical models of DDoS attacks under the effect of external attacking nodes.

In the near future, we are going to extend the approach and obtained results to the study of mathematical models of DDoS attacks. In particular, the following issues will be of particular interest:

- Proposing extended versions of the ODE models under consideration and examining their dynamics and applications.
- Constructing discrete-time models for DDoS attacks with applications in computational modeling and predicting real-life cyber attack scenarios.

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