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On the extension of Longstaff and Evans-Keef-Okunev models related to the pricing of zero-coupon bonds

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Abstract. We obtain the term structure of interest rate of a zero-coupon bond and we extend two classical models - the model of Longstaff and the model of Evans-Keef-Okunev, using the McShane stochastic calculus.

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1. Finance is one of the fastest developing areas in the modern banking and corporate word. This, together with the sophistication of modern financial products, provides a rapidly growing impetus for new mathematical models and modern mathematical methods.

When an evolution of a financial asset is affected by exterior disturbances, its time-development can offen be dicribe by a system of ordinary differential equations, provided that the disturbances are smooth functions. But, for round reasons financial analysts want to apply the theory when the noises belong to a larger class, including for example white noise. A unified theory was give by E.J.McShane ([7],[8]) who introduced so called belated integrals and stochastic differential systems which enjoying the following three properties: inclusiveness, consistency and stability. McShane's calculus had proved to very valuable in modeling and it finding applications in physics, engineering and economics.

In McShane's Calculus, the standard equation

$$X^i(t,\omega) = X^i(0,\omega) + \int_0^t f^i(s, X(s,\omega))ds + \tag{1}$$

$$+\sum_{j=1}^{r} \int_0^t g_j^i(s, X(s,\omega))dz_j(s,\omega) + \sum_{j,k=1}^{r} \int_0^t h_{j,k}^i(s, X(s,\omega))dz_j(s,\omega)dz_k(s,\omega)$$

in the adequate hypotheses can be replaced by what he calls a **canonical extension** (canonical form or canonical system) of equation (1):

$$X^i(t,\omega) = X^i(0,\omega) + \int_0^t f^i(s, X(s,\omega))ds + \tag{2}$$

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$$\sum_{j=1}^{r}\int_{0}^{t} g_j^i(s,X(s,\omega))dz_j(s,\omega)+\frac{1}{2}\sum_{j,k=1}^{r}\int_{0}^{t} g_{j,k}^i(s,X(s,\omega))dz_j(s,\omega)dz_k(s,\omega)$$

in which

$$g_{j,k}^i(t,x,\omega)=\sum_{m=1}^{n}[\partial g_j^i(t,x,\omega)/\partial x^m]g_k^m(t,x,\omega)$$

$i=1,2,...,n;\ j,k=1,2,...,r;\ t\in[0,a];\ x\in I\!R^n.$

The solution of the stochastic differential equation in canonical form has a stability under modification of the z_j that is not possessed by the equations with other choices for $g_{j,k}^i$ (including the traditional choice zero). We can be sure that any other extension that gives different solutions certainly lacks stability.

Moreover, the solutions of the canonical extension do not depend on the coordinate system in which we choose to expres them, the property that is not in general possessed by other equations. Also, it must be mentioned an especial suitability for retaining adequate agrement with experiment when the noises are idealized to "white noise" (see [10, pp 228-234], [11], [19], [20]).

In conclusion, McShane's Calculus had proved to be very valuable in modeling and in finding application in finances under canonical form.

2. Let T^* be a fixed horizon date for all market activities. A *bond* is a contract, paid for up-front, that yields a known amount on a known date in the future, the *maturity* date, $T\le T^*$. The bond may also pay a known cash divident (the *coupon*) at fixed times during the life of the contract. If there is no coupon the bond is known as *zero-coupon bond.*

In the financial analysis sense, by a zero-coupon bond (or a *discount bond*) of a maturiy T we mean a financial security paying to its holder one unit of cash at a prespecified date T in the future. This mean that, by convention, the bond's principal (known also as *face value* or *nominal value*) is one dollar.

We assume throughout that bonds are *default-free*, that is, the possibility of default by the bond's issuer is excluded. The price of a zero-coupon bond of maturity T at any instant $t\le T$ will be denote by $B(t,T)$; it is thus obvious that $B(T,T)=1$ for any maturity date $T\le T^*$. Since there are no other payments to the holder, in practice a discount bond sells for less than the principal before maturity - that is, at a discount (hence the name). This is because one could no incetive to invest in a discount bond costing more than its face value.

We assume that, for any fixed maturity $T\le T^*$, the bond price $B(t,T)$ follows a strictly positive and adapted process on a filtered probability space $(\Omega,\mathcal{F},I\!P)$. The problem of valuting a bond can be illustrated in the question " How much should I pay now to get a guaranteed \$1 in 10 years' time ?"

Let us consider a zero-coupon bond with a fixed maturity date $T\le T^*$. The simple *rate of return* from holding the bond over the time interval $[t,T]$ equals

$$\frac{1-B(t,T)}{B(t,T)}=\frac{1}{B(t,T)}-1.$$

The equivalent rate of return, with continuous compounding, is commonly referred, ro as a *yield-to-maturity* on a bond, i.e. an adapted process $Y(t,T)$ defined by the formula

$$Y(t,T) = -\frac{1}{T-t}\ln B(t,T), \;\; \forall\, t \in [0,T].$$

The *term structure of interest rate*, known also as the *yield curve*, is the function that related the yield $Y(t,T)$ to maturity T. It is obvious that, for arbitrary fixed maturity date T, there is a one-to-one correspondence between the bond price process $B(t,T)$ and its yield-to-maturity process $Y(t,T)$ and we have the formula

$$B(t,T) = e^{-Y(t,T)(T-t)}, \;\forall\, t \in [0,T].$$

Suppose we hold one bond. The change in the value of that bond is a time-step dt (from t to $t+dt$) is $\frac{dB}{dt}dt$. Arbitrage consideration (see [13], [22]) lead as to equate this with the return from a bank deposit receiving interest at a rate $r(t)$. Thus we conclude that

$$\frac{dB}{dt} = r(t)$$

and the solution of this ordinary differential equation is

$$B(t) = e^{\int_t^T r(\tau)d\tau} B(T)$$

or equivalent, the function B_t solves the differential equation $dB_t = r_t B_t dt$, with the conventional initial condition $B_0 = 1$.

3. In view of our uncertainty about the future course of interest rate, it is natural to model it as a random variable. Most traditional stochastic interest rate models are based on the exogenous specification of a *short-term rate of interest*. We write r_t to denote the *instantaneous interest rate* (also referred to as a *short-term interest rate*, or *spot interest rate* for borrowing or leading prevailing at time t over the infinitesimal time interval $[t, t+dt]$. In a stochastic setup, the short-term interest rate is modelled as an adapted process, say r, defined on a filtered probability space $(\Omega, \mathcal{F}, I\!\!P)$ for some $T^* > 0$. We assume throughout that r is a stochastic process with almost all sample path integrable on $[0,T]$ with respect to the Lebesgue measure.

We suppose that the interest rate r is gouverned by a stochastic differential equation by the McShane type of the following form:

$$dr_t = \mu(t,r)dt + \sigma(t,r)dz_t + \rho(t,r)(dz_t)^2.$$

Pricing of a bond is technically harder than pricing an option, since there is no underlying asset with which to hedge (see [13], [22]). In this situation only alternative is to hedge with bonds of different maturity dates. For this reason we setup a portofolio containing two bonds with different maturities, T_1 and T_2. The bond with maturity T_1 has the price P_1 and the bond with maturity T_2 has price P_2. We denote the value of this portofolio with V. Thus we have that

$$V = x_1 P_1 + x_2 P_2 \tag{3}$$

with the condition

$$x_1 + x_2 = 1\;. \tag{4}$$

We suppose that

$$\frac{dP_i(t,r,T_i)}{P_i(t,r,T_i)} = \mu_p(t,r)dt + \sigma_p(t,r)dz_t + \rho(t,r)(dz_t)^2, \;\; i = 1,2, \tag{5}$$

where

$$\mu_p(t,r) = \frac{1}{P(t,r)}(P_t + \mu(t,r)P_r) \tag{6}$$

$$\sigma_p = \frac{\sigma(t,r)}{P(t,r)}P_r \tag{7}$$

$$\rho_p(t,r) = \frac{1}{P(t,r)}[\rho(t,r)P_r + \frac{\sigma^2(t,r)}{2}P_{rr}] \tag{8}$$

where $P_t = \frac{\partial P}{\partial t}$, $P_r = \frac{\partial P}{\partial r}$ and $P_{rr} = \frac{\partial^2 P}{\partial r^2}$.

We assume that the portofolio is without risk (risk-free) i.e.

$$\frac{dV}{V} = rdt\ . \tag{9}$$

Thus, we obtain

$$\begin{aligned}\frac{dV}{V} = [x_1\mu_1(t,r) + x_2\mu_2(t,r)]dt + [x_1\sigma_1(t,r) + x_2\sigma_2(t,r)]dz_t + \\ +[x_1\rho_1(t,r) + x_2\rho_2(t,r)](dz_t)^2\end{aligned} \tag{10}$$

and (using the no arbitrage condition, see [13]) we obtain that

$$\begin{cases} x_1\mu_1(t,r) + x_2\mu_2(t,r) = r & (a) \\ x_1\sigma_1(t,r) + x_2\sigma_2(t,r) = 0 & (b) \\ x_1\rho_1(t,r) + x_2\rho_2(t,r) = 0 & (c) \end{cases} \tag{11}$$

From (12).(a) and (12).(b) results that

$$x_2 = \frac{r\sigma_2(t,r)}{\mu_1(t,r)\sigma_2(t,r) - \mu_2(t,r)\sigma_1(t,r)},\ x_2 = \frac{-r\sigma_1(t,r)}{\mu_1(t,r)\sigma_2(t,r) - \mu_2(t,r)\sigma_1(t,r)} \tag{12}$$

and from the relation (5) results that

$$\frac{\mu_1(t,r) - r}{\sigma_1(t,r)} = \frac{\mu_2(t,r) - r}{\sigma_2(t,r)} \stackrel{not}{=} \lambda(t,r), \tag{13}$$

λ is called, in financial literature as the *risk premium* or the *market price for risk.*

In a similarly way, from (12).(b) and (12).(c) we obtain that

$$\frac{\rho_1(t,r)}{\sigma_1(t,r)} = \frac{\rho_2(t,r)}{\sigma_2(t,r)} \stackrel{not}{=} \eta(t,r), \tag{14}$$

η can be interpreted as a *supplementary (bonus) risk premium* .

$$\mu_p(t,r) = r + \lambda(t,r)\sigma_p(t,r) \tag{15}$$

and from (15)

$$\rho_p(t,r) = \eta(t,r)\sigma_p(t,r)\ . \tag{16}$$

These with (7), (8) and (9) yield the system

$$\begin{cases} P_t + (\mu(t,r) - \lambda(t,r)\sigma(t,r))P_r = rP \\ (\rho(t,r) - \eta(t,r)\sigma(t,r))P_r + \frac{1}{2}\sigma^2(t,r)P_{rr} = 0 \end{cases} \tag{17}$$

and this implies that the equation of structure term for pricing on bond is

$$P_t + (\mu + \rho - \sigma(\lambda + \eta))P_r + \frac{1}{2}\sigma^2 P_{rr} = rP \tag{18}$$

with the final condition

$$P(T,r) = 1\ . \tag{19}$$

In almost situation $\lambda(t,r)$ if fixed on the market and analogous $\eta(t,r)$ and hence they are determinsitic functions.

4. In this paragraph we present two classical models pricing of a zero-coupon bond.

Example 1. *We start with the model developement by F. A. Longstaff ([9]) for the classical Itô case and we assume that the dynamic of the interest rate is gouverned by the following stochastic differential equation by McShane type*

$$dr_t = -a\sqrt{r}dt + \sigma\sqrt{r}dz_t + l(dz_t)^2 \ . \tag{20}$$

which is in the canonical form for $l = \frac{\sigma^2}{4}$. *We take* $\mu(t,r) = -a\sqrt{r}$, $\sigma(t,r) = \sigma\sqrt{r}$ *and* $\rho(t,r) = l = \frac{\sigma^2}{4}$ *in the equation of structure term* (19) *and we have*

$$P_t + [-a\sqrt{r} - \sigma\sqrt{r}(\lambda+\eta) + \frac{\sigma^2}{4}]P_r + \frac{\sigma^2}{2}rP_{rr} = rP, \quad P(T,r) = 1. \tag{21}$$

We take

$$\sqrt{r} = x, \quad \tau = T - t$$

and we have that

$$P_\tau = (\frac{\alpha}{2}P_x + \frac{l}{2x}P_x - \frac{\sigma^2}{8x}P_x) + \frac{\sigma^2}{8}P_{xx} - Px^2 - Pl.$$

We search a solution by the following form

$$P(\tau,x) = e^{A(\tau)x^2 + B(\tau)x + C(\tau)}, \tag{22}$$

then we obtain

$$\begin{cases} A(\tau) = \frac{\sqrt{2}\tanh\left[\frac{-\sqrt{2}\tau\sigma^2 - 2\ \sqrt{2}}{2\ \sigma}\right]}{\sigma} \\ B(\tau) = -\cosh\left[\frac{\tau\sigma}{\sqrt{2}}\right]^{-\frac{2(a+\sigma(\lambda+\eta))}{\sigma^2}} + \frac{A(\tau)}{a+\sigma(\lambda+\eta)}\frac{\sigma^2}{2} \\ C(\tau) = \int_0^\tau \Big(\frac{\sigma^2}{4}A(s) + \frac{a+\sigma(\lambda+\eta)}{4}B^2(s) - \frac{a+\sigma(\lambda+\eta)}{2}B(s)\Big)ds \end{cases} \tag{23}$$

(the explicite analytical expression for $C(\tau)$ *was obtained but it is in the respect to Gamma and Hypergeometric functions and was omitted to write here).*

Remark 1. *The solution of the equation* (21) *it is*

$$r_t = [2\sqrt{r_0} + at + \sigma z_t]^2 \ . \tag{24}$$

We observe that the coefficient functions of equation (21) *satisfy a non-lipschitz condition as in* [5] *and thus the uniqueness of the solution of this equation it is assured.*

Example 2. *We consider the model of L. T. Evans, S. P. Keef, J. Okunev (see* [8]*) for Itô case for the evolution of short-term interest rate and we assume an adequate equation in the McShane sense as the following form*

$$dr_t = ae^{-kt}\sqrt{r}dt + \sigma e^{-kt}\sqrt{r}dz_t + le^{-2kt}(dz_t)^2 \tag{25}$$

with $l = \frac{\sigma^2}{4}$ *in the canonical form.*

In this case, the equation of structure term (19) *become*

$$P_t + (-ae^{-kt}\sqrt{r} + \frac{\sigma^2}{4}e^{-2kt} - \sigma e^{-2kt}(\lambda+\eta)\sqrt{r})P_r + \frac{1}{2}\sigma^2 e^{-2kt}P_{rr} = rP \ . \tag{26}$$

In a similar manner as in the above example we search an explicite formula for the price of a zero-coupon bond by the form (23).

The functions $A(\tau)$, $B(\tau)$ *and* $C(\tau)$ *result from the following differential system:*

$$\begin{cases} A'(\tau) = \frac{\sigma^2}{2} e^{-2k(T-\tau)} A^2(\tau) - 1 \\ B'(\tau) = a - \sigma(\lambda+\eta) e^{-k(T-\tau)} A(\tau) + e^{-2k(T-\tau)} \frac{\sigma^2}{2} A(\tau) B(\tau) \\ C'(\tau) = e^{-k(T-\tau)} \frac{a-\sigma(\lambda-\eta)}{2} B(\tau) + e^{-2k(T-\tau)} \frac{\sigma^2}{8} (2A(\tau) + B^2(\tau)) \end{cases}$$

with initial conditions $A(0) = B(0) = C(0) = 0$.

We note that, in this case, the expressions for $A(\tau)$, $B(\tau)$ *and* $C(\tau)$ *can be obtained but they are in respect to Bessel and Hypergeometric functions and they were not given here. Naturally, if we replace parameters of model (*a, σ, λ, η *and* k*) with their estimations we obtain a simplified forms for this functions.*

The solution of the equation (26) *it is*

$$r_t = \left[\sqrt{r_0} + \frac{a}{2} t + \frac{\sigma}{2} \int_0^t e^{-ks} dz_s \right]^2 . \tag{27}$$

and from Theorem 2 [5] *we have an unique solution.*

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