Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 52(1), 2025, Pages 101–132, DOI: 10.52846/ami.v52i1.1930
ISSN: 1223-6934

Theoretical and Numerical Studies for a Delayed Swelling Porous Thermoelastic Soils Model

Chaima Boulkheloua, Houssem Eddine Khochemane, and Lamine Bouzettouta

Abstract. In this paper, we consider a one-dimensional swelling problem in porous elastic soils with second sound and a constant internal delay, where the heat conduction is given by Cattaneo's law. We show that the system is well-posed using the semigroup approach. Then, based on the energy method as well as by constructing a suitable Lyapunov functional, we prove that the unique dissipation given only by the second sound is strong enough to provoke an exponential decay of the solution without any relationship between the system parameters. For the numerical study, we discretize the continuous problem by performing a temporal discretization using Euler scheme and the classical finite difference method for spatial discretization. To solve the discretized problem, we propose to introduce a fixed point algorithm and derive the condition for which the proposed algorithm converges. Finally, we present some numerical test to illustrate the theoretical results by taking different delay weights and show that the studied system is highly reactive to small delays.

2020 *Mathematics Subject Classification.* 35L70, 35B40, 93D20.
Key words and phrases. Swelling soils, semigroup theory, exponential stability, Lyapunov functional, numerical study, time delay.

1. Introduction

Swelling porous media have attracted many researchers and this is due to its prevalence in a lot of disparate fields including soil science, hydrology, forestry, geotechnical, chemical, mechanical engineering. Among the important researches that have been realized in this area is the study of the asymptotic behavior of the swelling soils that belongs to porous media theory in the case of fluid saturation. The swelling soils are caused by the chemical attraction of water where water molecules are incorporated in the clay structure in between the clay plates separating and destabilizing the mineral structure. Furthermore, the clay's particle has the properties is that it consists of lattice hydrated aluminum and magnesium silicate minerals which form a unit (particle). Thus the clay's particle is a mixture of clay platelets and absorbed water (vicinal water). For a brief descriptions concerning the details historical development/review related to the general theory of the mixtures, we refer the readers to Bedford and Drumheller [1] and Eringen [7].

The basic field equations for the theory of swelling of one-dimensional porous elastic soils are given by

$$\rho_u u_{tt} = T_x + P_1 + F_1,\ \rho_z z_{tt} = H_x - P_2 + F_2, \tag{1}$$

Received April 10, 2024. Accepted November 11, 2024.

where the constituents u and z represent the displacement of the fluid and elastic solid material. The parameters ρ_u and ρ_z are the densities of each constituent which are assumed to be strictly positive constants. T, H are the partial tensions, F_1, F_2 are the external forces, P_1, P_2 are internal body forces associated with the dependent variables u, z. Here we assume that the constitutive equations of partial tensions are given as in [12] by

$$\begin{pmatrix} T \\ H \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} u_x \\ z_x \end{pmatrix},$$

where $\mathcal{M}$ is a positive definite symmetric array, i.e.,

$$\alpha_1\alpha_3 > \alpha_2^2. \tag{2}$$

Many investigations have been realized regarding the theory of swelling porous elastic soils and among them, we cite the work of Quintanilla [24] when he considered the following problem

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} - \beta_1 T_x + \xi\left(z_t - u_t\right) - \mu_z z_{xxt} = 0, \text{ in } (0, L) \times (0, \infty), \\ \rho_u u_{tt} - \mu u_{xx} - a_2 z_{xx} - \beta_2 T_x - \xi\left(z_t - u_t\right) = 0, \text{ in } (0, L) \times (0, \infty), \\ cT_t - \beta_1 z_{xt} - \beta_2 u_{xt} - kT_{xx} = 0, \text{ in } (0, L) \times (0, \infty), \end{cases} \tag{3}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \ u_t(x,0) = u_1(x), \ z(x,0) = z_0(x), \\ z_t(x,0) &= z_1(x), \ T(x,0) = T_0(x), \ x \in (0, L), \end{aligned}$$

and homogeneous Dirichlet boundary conditions

$$u(x,t) = z(x,t) = T(x,t) = 0, \ x = 0, L, \ t \in (0,\infty).$$

Under the following condition on the constants

$$\beta_1 = \beta_2 = 0, \ a_2^2 < a_1\xi, \ a_3 > 0,$$

the author established an exponential stability result for the solution of (3) in the isothermal case ($\Delta T = 0$). Furthermore, in the nonisothermal case and $\beta_1, \beta_2 \neq 0$, he showed that the combination of the thermal effects with the elastic effects provokes exponential stability.

In [29], Wang and Guo considered a problem of swelling of one-dimensional porous elastic soils given by

$$\begin{cases} \rho_u u_{tt} = \alpha_1 u_{xx} + \alpha_2 z_{xx}, \text{ in } (0, L) \times (0, \infty), \\ \rho_z z_{tt} = \alpha_3 z_{xx} + \alpha_2 u_{xx} + \rho_z \gamma(x) z_t, \text{ in } (0, L) \times (0, \infty), \\ u(0,t) = u_x(L,t) = z(0,t) = z_x(L,t) = 0, \ t \in (0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in (0, L), \\ z(x,0) = z_0(x), \ z_t(x,0) = z_1(x), \ x \in (0, L), \end{cases} \tag{4}$$

where $\gamma(x)$ is an internal viscous damping function satisfying the condition

$$\int_0^L \gamma(x)\, dx > 0,$$

and they proved an exponential stability of the system by using the spectral method. We refer the reader to [2, 16] for some other interesting related results.

In [27], the authors considered the following system

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, \text{ in } (0,L)\times(0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \gamma(t) g(u_t) = 0, \text{ in } (0,L)\times(0,\infty), \\ u(0,t) = u_x(L,t) = z(0,t) = z_x(L,t) = 0, \ t\in(0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x\in(0,\ L), \\ z(x,0) = z_0(x), \ z_t(x,0) = z_1(x), \ x\in(0,\ L), \end{cases}$$

and under some properties of convex functions they showed that the dissipation given only by the nonlinear damping term $\gamma(t) g(u_t)$ is strong enough to provoke an exponential decay rate.

Recently, in [26], the authors considered the following swelling problem in porous elastic soils with fluid saturation, viscous damping and a time delay term

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + \xi_1 z_t + \xi_2 z_t(x,t-\tau) = 0, \text{ in } (0,L)\times(0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} = 0, \text{ in } (0,L)\times(0,\infty), \\ z(0,t) = z_x(L,t) = u(0,t) = u_x(L,t) = 0, \ t\in(0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x\in(0,\ L), \\ z(x,0) = z_0(x), \ z_t(x,0) = z_1(x), \ x\in(0,\ L). \end{cases}$$

Under the appropriate assumption on the weight of the delay term, they established an exponential decay of the solution.

Motivated by the above mentioned work, in this paper we consider the following problem

$$\begin{cases} \rho_z\, z_{tt} = \alpha_1\, z_{xx} + \alpha_2\, u_{xx}, & in\ (0,L)\times(0,\infty) \\ \rho_u\, u_{tt} = \alpha_3\, u_{xx} + \alpha_2\, z_{xx} + \beta\,\theta_x - \mu\, u_t(x,t-\tau), & in\ (0,L)\times(0,\infty) \\ c\,\theta_t = -q_x + \beta\, u_{tx}, & in\ (0,L)\times(0,\infty) \\ \tau_0\, q_t = -q - k\,\theta_x, & in\ (0,L)\times(0,\infty) \end{cases} \tag{5}$$

with the boundary conditions

$$z(0,t) = z_x(L,t) = u(0,t) = u_x(L,t) = \theta_x(0,t) = \theta(L,t) = q(0,t) = 0\ ,\ t>0 \tag{6}$$

and the initial data

$$\begin{cases} z(x,0) = z_0(x) \ , \ z_t(x,0) = z_1(x) \ , \ \theta(x,0) = \theta_0(x) \ , \ x\in(0,L) \\ u(x,0) = u_0(x) \ , \ u_t(x,0) = u_1(x) \ , \ q(x,0) = q_0(x) \ , \ x\in(0,L) \\ u_t(x,-t) = f_0(x,t)\ , \quad in\ (0,L)\times(0,\tau) \end{cases} \tag{7}$$

where $\theta = \theta(x,t)$ represents the temperature difference, $q = q(x,t)$ is the heat flux and the coefficients $\alpha_1, \alpha_2, \alpha_3, c, \beta, \tau_0, k$ are positive constants represent the constitutive parameters defining the coupling among the different components of the materials, μ is a real number, $\tau > 0$ represents the time delay. The initial data $z_0, z_1, u_0, u_1, \theta_0, q_0$ belongs to a suitable functional space and f_0 is a history function. From physical point of view, it is well known that the model using the classic Fourier's law ($\tau_0 = 0$) leads to the physical paradox of infinite speed of heat propagation. To avoid this physical paradox, one of the theories that treats this problem is the advent of the second sound effects observed experimentally in materials at a very low temperature. The second sound effects emerge when heat is transported by a wave propagation process instead of the usual diffusion. This theory suggests to replace the classic Fourier's law by the Cattaneo's law ($\tau_0 > 0$). For more papers related to the second sound and its influence on the asymptotic behavior of solutions for different types of problems, we refer the reader to [3, 6, 9, 14, 19, 20, 18, 31].

Our aims in this paper are: First, we prove the existence and uniqueness of the solution of the system by using the semigroup arguments. Second, based on the multipliers method, we construct a suitable Lyapunov functional which allows us to estimate the energy of the system and we prove that the unique dissipation given by the second sound is strong enough to stabilize exponentially the system regardless of the wave speeds of the system or any other condition on coefficients in spite of the existence of the delay without the damping mechanism to control the undesirable delay effects that may be a source of instability of systems which are uniformly asymptotically stable in the absence of delay unless additional control terms have been used (see[4, 5, 11, 30]). Also, the introducing of delay may lead to ill-posedness, as shown in many works such as [5, 25] and the references therein. For the numerical part, we perform a temporal discretization based on the Euler scheme and the classical finite difference method for spatial discretization. In order to solve the discretized problem, we introduce a fixed point algorithm and we search for the necessary condition for which the proposed algorithm converges. Finally, we present the results of some numerical experiments to validate the theoretical result using MATLAB software and by taking different delay weights, we show that the considered system is very reactive to small delays which confirms that the time delay is not arbitrary in the considered problem.

The paper is organized as follows: In Section 2, we give the existence and uniqueness result of solutions of the problem (5) by using some results from the semigroup theory. In section 3, we establish the exponential stability result. In section 4, we give a numerical study with illustrative examples.

2. Well-posedness

In this section, we give the existence and uniqueness of solutions for the system (5) using semigroup theory.

First, we introduce as in [17], new dependent variable

$$\varphi(x,\rho,t) = u_t(x,t-\rho\tau) \qquad in\ (0,L)\times(0,1)\times(0,\infty). \tag{8}$$

A simple differentiation shows that φ satisfies

$$\tau\varphi_t(x,\rho,t) + \varphi_\rho(x,\rho,t) = 0 \qquad in\ (0,L)\times(0,1)\times(0,\infty). \tag{9}$$

Hence problem (5) takes the form:

$$\begin{cases} \rho_z z_{tt} = \alpha_1 z_{xx} + \alpha_2 u_{xx} & in\ (0,L)\times(0,\infty) \\ \rho_u u_{tt} = \alpha_3 u_{xx} + \alpha_2 z_{xx} + \beta\theta_x - \mu\varphi(x,1,t) & in\ (0,L)\times(0,\infty) \\ c\,\theta_t = -q_x + \beta u_{tx} & in\ (0,L)\times(0,\infty) \\ \tau_0 q_t = -q - k\,\theta_x & in\ (0,L)\times(0,\infty) \\ \tau\varphi_t = -\varphi_\rho & in\ (0,L)\times(0,1)\times(0,\infty) \end{cases} \tag{10}$$

with the boundary and the initial data

$$\begin{cases} z(0,t) = z_x(L,t) = u(0,t) = u_x(L,t) = \theta_x(0,t) = \theta(L,t) = q(0,t) = 0\ ,\ t>0 \\ z(x,0) = z_0(x)\ ,\ z_t(x,0) = z_1(x)\ ,\ \theta(x,0) = \theta_0(x)\ ,\ x\in(0,L) \\ u(x,0) = u_0(x)\ ,\ u_t(x,0) = u_1(x)\ ,\ q(x,0) = q_0(x)\ ,\ x\in(0,L) \\ \varphi(x,\rho,0) = f_0(x,\rho\tau) \qquad in\ (0,L)\times(0,\tau) \end{cases} \tag{11}$$

Second, we introduce the vector function $U = (z, v, u, \psi, \theta, q, \varphi)^T$, with $v = z_t$, and $\psi = u_t$.

We consider the following spaces:

$$\begin{aligned}
\tilde{H}^1(0,L) &= \left\{ f \in H^1(0,L) ;\ f(0) = 0 \right\}, \\
\tilde{H}^1_*(0,L) &= \left\{ f \in H^1(0,L) ;\ f(L) = 0 \right\}, \\
\tilde{H}^2(0,L) &= H^2(0,L) \cap \tilde{H}^1(0,L),
\end{aligned}$$

and

$$\mathcal{H} = \tilde{H}^1(0,L) \times L^2(0,L) \times \tilde{H}^1(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2((0,L) \times (0,1)).$$

Then $\mathcal{H}$, along with the inner product

$$\begin{aligned}
\left\langle U, \tilde{U} \right\rangle_{\mathcal{H}} &= \rho_z \int_0^L v\tilde{v}dx + \rho_u \int_0^L \psi\tilde{\psi}dx + \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1} \right) \int_0^L u_x \tilde{u}_x dx \\
&+ \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right) \left(\frac{\alpha_2}{\sqrt{\alpha_1}} \tilde{u}_x + \sqrt{\alpha_1} \tilde{z}_x \right) dx + c \int_0^L \theta\tilde{\theta}dx \\
&+ \frac{\tau_0}{k} \int_0^L q\tilde{q}dx + |\mu| \tau \int_0^L \int_0^1 \varphi(x,\rho,t) \tilde{\varphi}(x,\rho,t) \, d\rho \, dx, \qquad (12)
\end{aligned}$$

is a Hilbert space for any $U = (z, v, u, \psi, \theta, q, \varphi)^T \in \mathcal{H}$ and $\tilde{U} = (\tilde{z}, \tilde{v}, \tilde{u}, \tilde{\psi}, \tilde{\theta}, \tilde{q}, \tilde{\varphi})^T \in \mathcal{H}$.

The system (10) can be rewritten as follows:

$$\begin{cases} U_t + (\mathcal{A} + \mathcal{B})U = 0, \ t > 0, \\ U(x,0) = U_0(x) = (z_0, z_1, u_0, u_1, \theta_0, q_0, f_0)^T, \end{cases}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} -v \\ -\frac{\alpha_1}{\rho_z} z_{xx} - \frac{\alpha_2}{\rho_z} u_{xx} \\ -\psi \\ -\frac{\alpha_3}{\rho_u} u_{xx} - \frac{\alpha_2}{\rho_u} z_{xx} - \frac{\beta}{\rho_u}\theta_x + \frac{|\mu|}{\rho_u}\psi + \frac{\mu}{\rho_u}\varphi(x,1,t) \\ \frac{1}{c} q_x - \frac{\beta}{c}\psi_x \\ \frac{1}{\tau_0} q + \frac{k}{\tau_0}\theta_x \\ \frac{1}{\tau}\varphi_\rho \end{pmatrix}.$$

The domain of $\mathcal{A}$ is given by

$$\begin{aligned}
D(\mathcal{A}) = \Big\{ & U \in \mathcal{H} \mid z, u \in \tilde{H}^2(0,L) ;\ v, \psi, q \in \tilde{H}^1(0,L) ; \\
& \theta \in \tilde{H}^1_*(0,L) ; \varphi, \varphi_\rho \in L^2((0,L) \times (0,1)) \\
& z_x(L) = u_x(L) = \theta_x(0) = 0 \Big\},
\end{aligned}$$

and the operator $\mathcal{B} : D(\mathcal{B}) = \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{B}U = -\frac{|\mu|}{\rho_u} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \psi \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we have the following existence and uniqueness result

Theorem 2.1. *Let $U_0 \in \mathcal{H}$ and assume that (2) holds. Then, there exists a unique solution $U \in C(\mathbb{R}_+, \mathcal{H})$ for problem (10)-(11). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Proof. We use the semi-group approach. So we prove that $\mathcal{A}$ is a maximal monotone operator and that $\mathcal{B}$ is a Lipschitz continuous operator.

First, we prove that $\mathcal{A}$ is monotone. Let $U \in D(\mathcal{A})$, then we have

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = |\mu| \int_0^L \psi^2 dx + \mu \int_0^L \psi \varphi(x,1,t)\, dx + \frac{1}{k} \int_0^L q^2 dx + |\mu| \int_0^L \int_0^1 \varphi_\rho \varphi\, d\rho\, dx. \tag{13}$$

Using integration by parts and the fact that $\varphi(x,0,t) = \psi(x,t)$, the last term in the right-hand side of (13) gives

$$\int_0^L \int_0^1 \varphi_\rho \varphi\, d\rho\, dx = \frac{1}{2} \int_0^L \varphi^2(x,1,t)\, dx - \frac{1}{2} \int_0^L \psi^2 dx.$$

Also, using Young inequality we get

$$-\mu \int_0^L \psi \varphi(x,1,t)\, dx \leq \frac{|\mu|}{2} \int_0^L \psi^2 dx + \frac{|\mu|}{2} \int_0^L \varphi^2(x,1,t)\, dx.$$

Consequently, (13) yields

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}} \geq \frac{1}{k} \int_0^L q^2 dx \geq 0.$$

Therefore, the operator $\mathcal{A}$ is monotone. Next, we prove that the operator $\mathcal{I} + \mathcal{A}$ is surjective. For any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathcal{H}$, we prove that there exists a unique $U \in D(\mathcal{A})$ such that

$$(\mathcal{I} + \mathcal{A})\, U = F. \tag{14}$$

The problem (14) , leads to solve the following system

$$\begin{cases} z - v = f_1 \in \tilde{H}^1(0,L), \\ \rho_z\, v - \alpha_1\, z_{xx} - \alpha_2\, u_{xx} = \rho_z\, f_2 \in L^2(0,L), \\ u - \psi = f_3 \in \tilde{H}^1(0,L), \\ (\rho_u + |\mu|)\,\psi - \alpha_3\, u_{xx} - \alpha_2\, z_{xx} - \beta\,\theta_x + \mu\,\varphi(x,1,t) = \rho_u\, f_4 \in L^2(0,L), \\ c\,\theta + q_x - \beta\,\psi_x = c\, f_5 \in L^2(0,L), \\ (\tau_0 + 1)\, q + k\,\theta_x = \tau_0\, f_6 \in L^2(0,L), \\ \tau\,\varphi + \varphi_\rho = \tau\, f_7 \in L^2((0,L)\times(0,1)). \end{cases} \tag{15}$$

The last equation in (15) with $\varphi(x,0,t) = \psi(x,t)$ has a unique solution given by

$$\varphi(x,\rho,t) = \psi(x,t)\, e^{-\tau\rho} + \tau\, e^{-\tau\rho} \int_0^\rho e^{\tau\sigma} f_7(x,\sigma,t)\, d\sigma. \tag{16}$$

From the sixth equation in (15) , we define

$$\theta(x,t) = \frac{\tau_0 + 1}{k} \int_x^L q(y)\, dy - \frac{\tau_0}{k} \int_x^L f_6(y)\, dy. \tag{17}$$

Inserting $v = z - f_1$, $\psi = u - f_3$, (16) , (17) in $(15)_2$, $(15)_4$, $(15)_5$, we get

$$\begin{cases} -\alpha_1\, z_{xx} + \rho_z\, z - \alpha_2\, u_{xx} = g_1 \in L^2(0,L), \\ -\alpha_3\, u_{xx} + \lambda\, u - \alpha_2\, z_{xx} + \dfrac{\beta(\tau_0+1)}{k} q = g_2 \in L^2(0,L), \\ q_x + \dfrac{c(\tau_0+1)}{k} \displaystyle\int_x^L q(y)\, dy - \beta\, u_x = g_3 \in L^2(0,L), \end{cases} \tag{18}$$

where

$$g_1 = \rho_z\,(f_1 + f_2) \in L^2(0,L),$$

$$g_2 = \rho_u f_4 + \lambda\, f_3 + \frac{\beta\,\tau_0}{k} f_6 - \tau\,\mu\, e^{-\tau} \int_0^1 e^{\tau\sigma} f_7(x,\sigma,t)\, d\sigma \in L^2(0,L),$$

$$g_3 = c\, f_5 + \frac{c\,\tau_0}{k} \int_x^L f_6(y)\, dy - \beta\, f_{3_x} \in L^2(0,L),$$

$$\lambda = \rho_u + |\mu| + \mu\, e^{-\tau}.$$

To solve (18), we consider

$$B((z,u,q);(\tilde{z},\tilde{u},\tilde{q})) = \mathcal{G}(\tilde{z},\tilde{u},\tilde{q}), \tag{19}$$

where $B : \left[\tilde{H}^1(0,L) \times \tilde{H}^1(0,L) \times L^2(0,L)\right]^2 \to \mathbb{R}$ is the bilinear form defined by

$$
\begin{aligned}
B\left((z,u,q);(\tilde z,\tilde u,\tilde q)\right) = {} & \rho_z \int_0^L z\,\tilde z\,dx + \lambda \int_0^L u\,\tilde u\,dx + \frac{\tau_0+1}{k}\int_0^L q\,\tilde q\,dx + \alpha_1 \int_0^L z_x\,\tilde z_x\,dx \\
& + \alpha_3 \int_0^L u_x\,\tilde u_x\,dx + \alpha_2 \int_0^L u_x\,\tilde z_x\,dx + \alpha_2 \int_0^L z_x\,\tilde u_x\,dx \\
& + \frac{c(\tau_0+1)^2}{k^2} \int_0^L \left(\int_x^L q\,(y)\;dy\right)\left(\int_x^L \tilde q\,(y)\;dy\right) dx \\
& + \frac{\beta\,(\tau_0+1)}{k}\left(\int_0^L q\,\tilde u\,dx - \int_0^L u\,\tilde q\,dx\right),
\end{aligned}
$$

and $\mathcal{G} : \tilde H^1(0,L) \times \tilde H^1(0,L) \times L^2(0,L) \to \mathbb{R}$ is the linear form given by

$$
\mathcal{G}\left(\tilde z,\tilde u,\tilde q\right) = \int_0^L g_1\,\tilde z\,dx + \int_0^L g_2\,\tilde u\,dx + \frac{\tau_0+1}{k}\int_0^L g_3 \left(\int_x^L \tilde q\,(y)\;dy\right) dx.
$$

Let $\mathcal{V} = \tilde H^1(0,L) \times \tilde H^1(0,L) \times L^2(0,L)$ equipped with the norm

$$
\|(z,u,q)\|_V^2 = \|z\|_2^2 + \|u\|_2^2 + \|q\|_2^2 + \|u_x\|_2^2 + \left\| \frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1}\,z_x \right\|_2^2,
$$

then, we can easily prove that

$$
\begin{aligned}
\left|B\left((z,u,q);(z,u,q)\right)\right| = {} & \rho_z \int_0^L z^2 dx + \lambda \int_0^L u^2 dx + \frac{\tau_0+1}{k}\int_0^L q^2 dx \\
& + \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1}\,z_x\right)^2 dx \\
& + \frac{c(\tau_0+1)^2}{k^2}\int_0^L \left(\int_x^L q\,(y)\;dy\right)^2 dx \\
\geq {} & M_0 \left\|(z,u,q)\right\|_V^2 ,
\end{aligned}
$$

where $M_0 = min\left\{\rho_z\,,\,\lambda,\,\frac{\tau_0+1}{k}, 1, \alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right\}$. Thus, B is coercive. Moreover, we can easily see that B and $\mathcal{G}$ are bounded. Consequently, by Lax-Milgram Lemma, system (18) has a unique solution $(z,u,q) \in \mathcal{V}$ satisfying (19).
Substituting z and u in $(15)_1$ and $(15)_3$, respectively, we obtain

$$
v, \psi \in \tilde H^1(0,L)\,,
$$

then, inserting q in (17) and $(15)_6$ we get

$$
\theta \in \tilde H^1_*(0,L)\,.
$$

Similarly, inserting ψ in (16) and bearing in mind $(15)_7$, we obtain

$$\varphi, \varphi_\rho \in L^2\left((0,L)\times(0,1)\right).$$

Moreover, if we take $(\tilde{z},\tilde{u}) \equiv (0,0) \in \tilde{H}^1(0,L) \times \tilde{H}^1(0,L)$, then (19) reduces to

$$\int_0^L q\,\tilde{q}\,dx + \frac{c(\tau_0+1)}{k}\int_0^L \left(\int_x^L q(y)\,dy\right)\left(\int_x^L \tilde{q}(y)\,dy\right)dx - \beta\int_0^L u\,\tilde{q}\,dx$$
$$= \int_0^L g_3\left(\int_x^L \tilde{q}(y)\,dy\right)dx, \forall\tilde{q}\in L^2(0,L). \tag{20}$$

That is

$$-\int_0^L (q-\beta\,u)\left(\int_x^L \tilde{q}(y)\,dy\right)_x dx$$
$$= \int_0^L \left[-\frac{c(\tau_0+1)}{k}\left(\int_x^L q(y)\,dy\right)+g_3\right]\left(\int_x^L \tilde{q}(y)\,dy\right)dx, \forall\tilde{q}\in L^2(0,L), \tag{21}$$

which implies

$$q_x = \beta\,u_x - \frac{c(\tau_0+1)}{k}\left(\int_x^L q(y)\,dy\right) + g_3,\ \in L^2(0,L).$$

So,

$$q \in H^1(0,L),$$

and the other hand, we have (21). Thus

$$q(0)\int_0^L \tilde{q}(y)\,dy = 0,\ \ \forall\tilde{q}\in L^2(0,L).$$

Since $\tilde{q} \in L^2(0,L)$ is arbitrary. Then,

$$q(0) = 0.$$

Consequently

$$q \in \tilde{H}^1(0,L).$$

If we choose $(\tilde{u},\tilde{q}) \equiv (0,0) \in H^1(0,L) \times L^2(0,L)$ in (19) we have

$$\int_0^L (\alpha_1 z_x + \alpha_2 u_x)\,\tilde{z}_x dx = \int_0^L (g_1 - \rho_z z)\,\tilde{z}\,dx, \forall\tilde{z}\in\tilde{H}^1(0,L). \tag{22}$$

This last is also true for any function $\phi \in C^1(0,L)$, $\phi(0) = 0$ which is in $\tilde{H}^1(0,L)$, thus

$$-\alpha_1 z_{xx} - \alpha_2 u_{xx} = g_1 - \rho_z z,\ \in L^2(0,L).$$

Similarly, if we select $(\tilde{z}, \tilde{q}) \equiv (0,0) \in H^1(0,L) \times L^2(0,L)$ in (19), we find

$$\int_0^L (\alpha_3 u_x + \alpha_2 z_x) \tilde{u}_x dx = \int_0^L \left(g_2 - \lambda u - \frac{\beta(\tau_0+1)}{k} q \right) \tilde{u}\, dx, \forall \tilde{u} \in \tilde{H}^1(0,L). \quad (23)$$

This last is also true for any function $\phi \in C^1(0,L)$, $\phi(0) = 0$ which is in $\tilde{H}^1(0,L)$, thus

$$-\alpha_3 u_{xx} - \alpha_2 z_{xx} = g_2 - \lambda u - \frac{\beta(\tau_0+1)}{k} q, \in L^2(0,L).$$

Therefore,

$$u_{xx}, z_{xx} \in L^2(0,L).$$

So,

$$z, u \in \tilde{H}^2(0,L).$$

Finally, from $(15)_6$, we get

$$\theta_x(0) = 0,$$

and from (22), (23), we find

$$\begin{aligned} \tilde{z}(L)[\alpha_1 z_x(L) + \alpha_2 u_x(L)] &= 0, \\ \tilde{u}(L)[\alpha_3 u_x(L) + \alpha_2 z_x(L)] &= 0. \end{aligned}$$

Since $\tilde{z}, \tilde{u} \in \tilde{H}^1(0,L)$ are arbitrary. Then

$$z_x(L) = u_x(L) = 0.$$

Hence, there exists a unique $U \in D(\mathcal{A})$ such that (14) is satisfied. Consequently, the operator $\mathcal{A}$ is maximal. With this, we conclude that $\mathcal{A}$ is a maximal monotone operator. On the other hand, it is obvious that operator $\mathcal{B}$ is Lipschitz continuous. Consequently, $\mathcal{A} + \mathcal{B}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathcal{H}$. Therefore, the well-posedness result follows from the Lummer Phillips theorem (see [22]). □

3. Exponential decay

In this section, we state and prove technical lemmas needed for the proof of our stability result.

Lemma 3.1. *Let $(z, u, \theta, q, \varphi)$ be the solution of (10)-(11). Then, the energy functional $E(t)$, defined by*

$$\begin{aligned} E(t) = {} & \frac{\rho_z}{2} \int_0^L z_t^2 dx + \frac{\rho_u}{2} \int_0^L u_t^2 dx + \frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L u_x^2 dx \\ & + \frac{1}{2} \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 dx + \frac{c}{2} \int_0^L \theta^2 dx \\ & + \frac{\tau_0}{2k} \int_0^L q^2 dx + \frac{|\mu| \tau}{2} \int_0^L \int_0^1 \varphi^2(x, \rho, t)\, d\rho\, dx, \end{aligned} \quad (24)$$

satisfies

$$E'(t) \leq |\mu| \int_0^L u_t^2 dx - \frac{1}{k} \int_0^L q^2 dx. \tag{25}$$

Proof. Multiplying $(10)_1$, $(10)_2$, $(10)_3$, $(10)_4$ by z_t , u_t , θ , $\frac{1}{k}q$ respectively, integrating over $(0, L)$, and Multiplying$(10)_5$ by $|\mu|\,\varphi$, integrating over $(0, L) \times (0, 1)$ then,using integration by part and taking into account the boundary conditions and summing them up, we obtain

$$\begin{aligned} &\frac{d}{2dt} \int_0^L \left(\rho_u u_t^2 + \rho_z z_t^2 + c\theta^2 + \alpha_3 u_x^2 + \alpha_1 z_x^2 + 2\alpha_2 z_x u_x + \frac{\tau_0}{k} q^2 \right) dx \\ &+ \frac{d}{dt} \frac{|\mu|\,\tau}{2} \int_0^L \int_0^1 \varphi^2 (x, \rho, t)\, d\rho\, dx \\ &= \frac{|\mu|}{2} \int_0^L u_t^2 dx - \frac{1}{k} \int_0^L q^2 dx - \frac{|\mu|}{2} \int_0^L \varphi^2 (x, 1, t)\, dx - \mu \int_0^L u_t\, \varphi (x, 1, t)\, dx. \end{aligned} \tag{26}$$

Using the fact that

$$\alpha_3 u_x^2 + \alpha_1 z_x^2 + 2\alpha_2 z_x u_x = \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1} \right) \int_0^L u_x^2 dx + \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 dx. \tag{27}$$

Then, using Young's inequality on the last term in (26) we have

$$-\mu \int_0^L u_t\, \varphi (x, 1, t)\, dx \leq \frac{|\mu|}{2} \int_0^L u_t^2 dx + \frac{|\mu|}{2} \int_0^L \varphi^2 (x, 1, t)\, dx. \tag{28}$$

Inserting (27) and (28) in (26), we get (24) and (25). □

Lemma 3.2. *Let $(z, u, \theta, q, \varphi)$ be the solution of (10) - (11). Then, the functional*

$$I_1(t) = \rho_u \int_0^L u_t u dx - \frac{\alpha_2}{\alpha_1} \rho_z \int_0^L z_t u dx, \ t \geq 0,$$

satisfies for any $\varepsilon_1 > 0$,

$$\begin{aligned} I_1'(t) \leq &-\frac{1}{2} \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1} \right) \int_0^L u_x^2 dx + \left(\rho_u + \frac{\alpha_2^2 \rho_z^2}{4\varepsilon_1 \alpha_1^2} \right) \int_0^L u_t^2 dx + \varepsilon_1 \int_0^L z_t^2 dx \\ &+ C_0 \int_0^L \left(\theta^2 + \varphi^2 (x, 1, t) \right) dx, \ \forall t \geq 0. \end{aligned} \tag{29}$$

Proof. By differentiating $I_1(t)$, using $(10)_1$,$(10)_2$ and integrating by parts together with the boundary conditions, we obtain

$$I_1'(t) = -\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + \rho_u \int_0^L u_t^2 dx - \beta \int_0^L u_x \theta dx - \frac{\alpha_2}{\alpha_1}\rho_z \int_0^L u_t z_t dx - \mu \int_0^L u\,\varphi(x,1,t)\,dx. \tag{30}$$

Young's and Poincaré's inequalities lead to

$$-\mu \int_0^L u\,\varphi(x,1,t)\,dx \le \frac{1}{4}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + C_0 \int_0^L \varphi^2(x,1,t)\,dx, \tag{31}$$

$$-\beta \int_0^L u_x \theta dx \le \frac{1}{4}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x^2 dx + C_0 \int_0^L \theta^2 dx, \tag{32}$$

and

$$-\frac{\alpha_2}{\alpha_1}\rho_z \int_0^L u_t z_t dx \le \varepsilon_1 \int_0^L z_t^2 dx + \frac{\alpha_2^2\rho_z^2}{4\varepsilon_1\alpha_1^2}\int_0^L u_t^2 dx. \tag{33}$$

Substituting (31) , (32) and (33) in (30), we get (29). □

Lemma 3.3. *Let (z,u,θ,q,φ) be the solution of (10)-(11). Then, the functional*

$$I_2(t) = \rho_u \alpha_2 \int_0^L u_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right)dx - \frac{\alpha_2^2}{\alpha_1}\rho_z \int_0^L z_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right)dx,\ t \ge 0,$$

satisfies, for any $\varepsilon_2 > 0$,

$$I_2'(t) \le -\frac{\alpha_2^2\rho_z}{2\sqrt{\alpha_1}}\int_0^L z_t^2 dx + C_1 \int_0^L u_t^2 dx + C_{\varepsilon_2}\int_0^L \left(\theta^2 + u_x^2 + \varphi^2(x,1,t)\right)dx + \varepsilon_2 \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)^2 dx. \tag{34}$$

Proof. By differentiating $I_2(t)$, using $(10)_1$,$(10)_2$ and integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned} I_2'(t) = &-\frac{\alpha_2^2\rho_z}{\sqrt{\alpha_1}}\int_0^L z_t^2 dx + \alpha_2\left(\rho_u\sqrt{\alpha_1} - \frac{\alpha_2^2\rho_z}{\alpha_1\sqrt{\alpha_1}}\right)\int_0^L u_t z_t dx \\ &- \alpha_2\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\int_0^L u_x\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)dx + \frac{\alpha_2^2\rho_u}{\sqrt{\alpha_1}}\int_0^L u_t^2 dx \\ &- \alpha_2\beta\int_0^L \theta\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u_x + \sqrt{\alpha_1}z_x\right)dx - \mu\,\alpha_2\int_0^L \varphi(x,1,t)\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}\,z\right)dx. \end{aligned} \tag{35}$$

Using Young's and Poincaré's inequalities, we get

$$\alpha_2\left(\rho_u\sqrt{\alpha_1} - \frac{\alpha_2^2\rho_z}{\alpha_1\sqrt{\alpha_1}}\right)\int_0^L u_t z_t dx \le \frac{\alpha_2^2\rho_z}{2\sqrt{\alpha_1}}\int_0^L z_t^2 dx + C_1\int_0^L u_t^2 dx, \tag{36}$$

$$
\begin{aligned}
&-\mu\,\alpha_2 \int\limits_0^L \varphi\,(x,1,t) \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u + \sqrt{\alpha_1}\, z\right) dx \\
&\leq \frac{\varepsilon_2}{3} \int\limits_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1}\, z_x\right)^2 dx + C_{\varepsilon_2} \int\limits_0^L \varphi^2\,(x,1,t)\, dx,
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&-\alpha_2 \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L u_x \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right) dx \\
&\leq \frac{\varepsilon_2}{3} \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx + C_{\varepsilon_2} \int_0^L u_x^2 dx,
\end{aligned} \tag{38}
$$

and

$$
-\alpha_2\beta \int_0^L \theta \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right) dx \leq \frac{\varepsilon_2}{3} \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx + C_{\varepsilon_2} \int_0^L \theta^2 dx. \tag{39}
$$

Inserting (36)-(39) in (35), we obtain (34). □

Lemma 3.4. *Let $(z, u, \theta, q, \varphi)$ be the solution of (10) - (11). Then, the functional*

$$
I_3(t) = \rho_u \int_0^L u u_t dx + \rho_z \int_0^L z z_t dx,\ t \geq 0,
$$

satisfies

$$
\begin{aligned}
I_3'(t) \leq &-\int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx - \frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L u_x^2 dx \\
&+ \rho_z \int_0^L z_t^2 dx + \rho_u \int_0^L u_t^2 dx + C_2 \int_0^L \left(\theta^2 + \varphi^2\,(x,1,t)\right) dx.
\end{aligned} \tag{40}
$$

Proof. Differentiating the functional $I_3(t)$ using $(10)_1, (10)_2$ and integrating by parts, we obtain

$$
\begin{aligned}
I_3'(t) = &\,\rho_z \int_0^L z_t^2 dx + \rho_u \int_0^L u_t^2 dx - \alpha_3 \int_0^L u_x^2 dx - 2\alpha_2 \int_0^L u_x z_x dx \\
&- \alpha_1 \int_0^L z_x^2 dx - \beta \int_0^L \theta u_x dx - \mu \int\limits_0^L u\,\varphi\,(x,1,t)\, dx.
\end{aligned} \tag{41}
$$

Note that

$$
\begin{aligned}
&-\alpha_3 \int_0^L u_x^2 dx - 2\alpha_2 \int_0^L u_x z_x dx - \alpha_1 \int_0^L z_x^2 dx \\
&= -\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L u_x^2 dx - \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx.
\end{aligned} \tag{42}
$$

So, (41) becomes

$$I_3'(t) = \rho_z \int_0^L z_t^2 dx + \rho_u \int_0^L u_t^2 dx - \beta \int_0^L \theta u_x dx - \mu \int_0^L u\, \varphi(x,1,t)\, dx$$
$$- \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L u_x^2 dx - \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx. \tag{43}$$

Using Young's and Poincaré's inequalities

$$-\beta \int_0^L \theta u_x dx \le \frac{1}{4}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L u_x^2 dx + C_2 \int_0^L \theta^2 dx. \tag{44}$$

$$-\mu \int_0^L u\, \varphi(x,1,t)\, dx \le \frac{1}{4}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L u_x^2\, dx + C_2 \int_0^L \varphi^2(x,1,t)\, dx. \tag{45}$$

Substituting (44) and (45) into (43), we get (40). □

Lemma 3.5. *Let $(z, u, \theta, q, \varphi)$ be the solution of (10)-(11). Then, the functional*

$$I_4(t) = -c\rho_u \int_0^L \theta \left(\int_x^L u_t(y)\, dy\right) dx, \ \forall t \ge 0,$$

satisfies, for any $\varepsilon_3, \varepsilon_4, \varepsilon_5 > 0$, the following estimate

$$I_4'(t) \le -\frac{\beta\rho_u}{2} \int_0^L u_t^2 dx + \varepsilon_3 \int_0^L u_x^2 dx + \varepsilon_4 \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx$$
$$+ \frac{\rho_u}{2\beta} \int_0^L q^2 dx + C_3 \left(1 + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4} + \frac{1}{\varepsilon_5}\right) \int_0^L \theta^2 dx + \varepsilon_5 \int_0^L \varphi^2(x,1,t)\, dx. \tag{46}$$

Proof. By differentiating $I_4(t)$, using $(10)_2$, $(10)_3$ and integrating by parts, we obtain

$$I_4'(t) = \rho_u \int_0^L q u_t dx - \beta\rho_u \int_0^L u_t^2 dx + c\alpha_3 \int_0^L \theta u_x dx$$
$$+ c\alpha_2 \int_0^L \theta z_x dx + c\beta \int_0^L \theta^2 dx + c\,\mu \int_0^L \theta \int_x^L \varphi(y,1,t)\, dy\, dx. \tag{47}$$

Using the fact that

$$c\alpha_3 \int_0^L \theta u_x dx + c\alpha_2 \int_0^L \theta z_x dx$$
$$= c\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L \theta u_x dx + \frac{c\alpha_2}{\sqrt{\alpha_1}} \int_0^L \theta \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right) dx.$$

Then, (47) can be rewritten as follows

$$I_4'(t) = \rho_u \int_0^L q u_t dx - \beta \rho_u \int_0^L u_t^2 dx + c\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L \theta u_x dx$$
$$+ \frac{c\alpha_2}{\sqrt{\alpha_1}} \int_0^L \theta \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right) dx + c\beta \int_0^L \theta^2 dx$$
$$+ c\,\mu \int_0^L \theta \int_x^L \varphi(y,1,t)\, dy\, dx. \tag{48}$$

Young's inequality leads to

$$c\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) \int_0^L \theta u_x dx \le \varepsilon_3 \int_0^L u_x^2 dx + \frac{C_3}{\varepsilon_3} \int_0^L \theta^2 dx, \tag{49}$$

$$\frac{c\alpha_2}{\sqrt{\alpha_1}} \int_0^L \theta \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right) dx \le \varepsilon_4 \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx + \frac{C_3}{\varepsilon_4} \int_0^L \theta^2 dx, \tag{50}$$

and

$$\rho_u \int_0^L q u_t dx \le \frac{\beta \rho_u}{2} \int_0^L u_t^2 dx + \frac{\rho_u}{2\beta} \int_0^L q^2 dx. \tag{51}$$

Young's and Cauchy Schwarz inequalities lead to

$$c\,\mu \int_0^L \theta \int_x^L \varphi(y,1,t)\, dy\, dx \le \varepsilon_5 \int_0^L \varphi^2(x,1,t)\, dx + \frac{C_3}{\varepsilon_5} \int_0^L \theta^2 dx. \tag{52}$$

Estimate (46) follows by substituting (49)-(52) into (48). $\square$

Lemma 3.6. *Let $(z, u, \theta, q, \varphi)$ be the solution of (10) - (11). Then, the functional*

$$I_5(t) = -c\tau_0 \int_0^L \theta \left(\int_x^L q(y)\, dy\right) dx, \ \forall t \ge 0,$$

satisfies, for any $\varepsilon_6 > 0$, the following estimate

$$I_5'(t) \le -\frac{ck}{2} \int_0^L \theta^2 dx + \varepsilon_6 \int_0^L u_t^2 dx + C_4 \left(1 + \frac{1}{\varepsilon_6}\right) \int_0^L q^2 dx. \tag{53}$$

Proof. By differentiating $I_5(t)$, using $(10)_3$, $(10)_4$ and integrating by parts, we obtain

$$I_5'(t) = -\,ck \int_0^L \theta^2 dx + \tau_0 \int_0^L q^2 dx - \tau_0 \beta \int_0^L u_t q dx$$
$$+ c \int_0^L \theta \left(\int_x^L q(y)\, dy\right) dx. \tag{54}$$

Using Young's inequality, we get

$$-\tau_0 \beta \int_0^L u_t q dx \le \varepsilon_6 \int_0^L u_t^2 dx + \frac{C_4}{\varepsilon_6} \int_0^L q^2 dx. \tag{55}$$

Young's and Cauchy Schwarz inequalities leads to

$$c\int_0^L \theta\left(\int_x^L q(y)\,dy\right)dx \le \frac{ck}{2}\int_0^L \theta^2 dx + C_4\int_0^L q^2 dx. \tag{56}$$

Inserting (55)-(56) in (54), we obtain (53). ☐

Lemma 3.7. *Let* (z,u,θ,q,φ) *be the solution of* (10)-(11). *Then, the functional*

$$I_6(t) = \tau\int_0^L\int_0^1 e^{-\tau\rho}\varphi^2(x,\rho,t)\,d\rho\,dx, \tag{57}$$

satisfies

$$I_6'(t) \le \int_0^L u_t^2\,dx - e^{-\tau}\left(\int_0^L \varphi^2(x,1,t)\,dx + \tau\int_0^L\int_0^1 \varphi^2(x,\rho,t)\,d\rho\,dx\right). \tag{58}$$

Proof. By differentiating $I_6(t)$ and using $(10)_5$, we obtain

$$I_6'(t) = \int_0^L u_t^2\,dx - e^{-\tau}\int_0^L \varphi^2(x,1,t)\,dx - \tau\int_0^L\int_0^1 e^{-\tau\rho}\varphi^2(x,\rho,t)\,d\rho\,dx.$$

Using that fact that $\varphi(x,0,t) = \psi(x,t)$ and $e^{-\tau} \le e^{-\tau\rho} \le 1$ for all $\rho \in [0,1]$, we get (58). ☐

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + 2(I_3 + I_4)(t) + N_3 I_5(t) + N_4 I_6(t), \tag{59}$$

where N, N_1, N_2, N_3, N_4 are positive constants.

Theorem 3.8. *Let* (z,u,θ,q,φ) *be the solution of* (10)-(11). *Then, there exist two positive constants* κ_1 *and* κ_2 *such that the Lyapunov functional* (59) *satisfies*

$$\kappa_1 E(t) \le \mathcal{L}(t) \le \kappa_2 E(t),\ \forall t \ge 0, \tag{60}$$

and

$$\mathcal{L}'(t) \le -\beta_1 E(t),\ \beta_1 > 0. \tag{61}$$

Proof. From (59), we have

$$\begin{aligned}|\mathcal{L}(t) - NE(t)| &\le N_1\rho_u\int_0^L |u_t u|\,dx + N_1\frac{\alpha_2}{\alpha_1}\rho_z\int_0^L |z_t u|\,dx\\
&+ N_2\rho_u\alpha_2\int_0^L \left|u_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right)\right|dx + N_2\frac{\alpha_2^2}{\alpha_1}\rho_z\int_0^L \left|z_t\left(\frac{\alpha_2}{\sqrt{\alpha_1}}u + \sqrt{\alpha_1}z\right)\right|dx\\
&+ 2\rho_u\int_0^L |uu_t|\,dx + 2\rho_z\int_0^L |zz_t|\,dx + 2c\rho_u\int_0^L \left|\theta\left(\int_x^L u_t(y)\,dy\right)\right|dx\\
&+ N_3 c\tau_0\int_0^L \left|\theta\left(\int_x^L q(y)\,dy\right)\right|dx + N_4\tau\int_0^L\int_0^1 e^{-\tau\rho}\varphi^2(x,\rho,t)\,d\rho\,dx.\end{aligned}$$

By using the Young's, Poincaré's, Cauchy-Schwarz inequalities, we obtain

$$|\mathcal{L}(t) - NE(t)| \leq \gamma E(t), \ \gamma > 0,$$

which yields

$$(N-\gamma)\, E(t) \leq \mathcal{L}(t) \leq (N+\gamma)\, E(t),$$

by choosing N (depending on N_1, N_2, N_3 and N_4) sufficiently large we obtain (60).

Now, By differentiating $\mathcal{L}(t)$, exploiting (25), (29), (34), (40), (46), (53), (58) and setting $\varepsilon_1 = \frac{1}{N_1}$, $\varepsilon_2 = \frac{1}{N_2}$, $\varepsilon_3 = \varepsilon_5 = \frac{1}{2}$, $\varepsilon_4 = \frac{1}{4}$, $\varepsilon_6 = \frac{1}{N_3}$, we get

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\rho_u \beta - |\mu| N - N_1\left(\rho_u + \frac{\alpha_2^2 \rho_z^2 N_1}{4\alpha_1^2}\right) - C_1 N_2 - 2\rho_u - 1 - N_4\right] \int_0^L u_t^2 dx \\
& - \left[\frac{N}{k} - \frac{\rho_u}{\beta} - N_3 C_4 (1+N_3)\right] \int_0^L q^2 dx - \left[\frac{\alpha_2^2 \rho_z N_2}{2\sqrt{\alpha_1}} - 1 - 2\rho_z\right] \int_0^L z_t^2 dx \\
& - \left[\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)\left(\frac{N_1}{2} + 1\right) - C_{\varepsilon_2} N_2 - 1\right] \int_0^L u_x^2 dx \\
& - \left[\frac{ckN_3}{2} - 18C_3 - 2C_2 - C_{\varepsilon_2} N_2 - C_0 N_1\right] \int_0^L \theta^2 dx \\
& - \frac{1}{2} \int_0^L \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x\right)^2 dx - \tau\, e^{-\tau} N_4 \int_0^L \int_0^1 \varphi^2(x,\rho,t)\, d\rho\, dx \\
& - \left[e^{-\tau} N_4 - 1 - 2C_2 - N_2 C_{\varepsilon_2} - N_1 C_0\right] \int_0^L \varphi^2(x,1,t)\, dx. \qquad (62)
\end{aligned}$$

Now, we select our parameters appropriately as follows:
First, we choose N_2 large enough so that

$$\frac{\alpha_2^2 \rho_z}{2\sqrt{\alpha_1}} N_2 - 1 - 2\rho_z > 0.$$

Next, we select N_1 large enough so that

$$\frac{1}{2}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right)(N_1 + 2) - C_{\varepsilon_2} N_2 - 1 > 0.$$

We take N_3 large such that

$$\frac{ckN_3}{2} - 18C_3 - 2C_2 - C_{\varepsilon_2} N_2 - C_0 N_1 > 0.$$

We pick N_4 large so that

$$e^{-\tau} N_4 - 1 - 2C_2 - N_2 C_{\varepsilon_2} - N_1 C_0 > 0.$$

We choose N large enough so that (60) remains valid, further

$$\frac{N}{k} - \frac{\rho_u}{\beta} - N_3 C_4 (1+N_3) > 0.$$

Finally, by taking $|\mu|$ so small that All these choices with the relation (62) leads to

$$\mathcal{L}'(t) \leq -\alpha_1 \int_0^L \left(z_t^2 + u_x^2 + u_t^2 + \left(\frac{\alpha_2}{\sqrt{\alpha_1}} u_x + \sqrt{\alpha_1} z_x \right)^2 \right.$$
$$\left. + \theta^2 + q^2 + \int_0^1 \varphi^2 (x, \rho, t)\, d\rho \right) dx. \tag{63}$$

On the other hand, from Eq. (24), we obtain (61). □

Now, we can state and prove the following stability result

Lemma 3.9. *Let $(z, u, \theta, q, \varphi)$ be the solution of (10) - (11). Then, for any $U_0 \in D(\mathcal{A})$, there exist two positive constants λ_1 and λ_2 such that*

$$E(t) \leq \lambda_2 e^{-\lambda_1 t}, \ \forall t \geq 0. \tag{64}$$

Proof. By using the estimation (61), we get

$$\mathcal{L}'(t) \leq -\beta_1 E(t), \ t \geq 0,$$

having in mind the equivalence of $E(t)$ and $\mathcal{L}(t)$ we infer that

$$\mathcal{L}'(t) \leq -\lambda_1 \mathcal{L}(t), \ t \geq 0, \tag{65}$$

where $\lambda_1 = \dfrac{\beta_1}{\kappa_2}$. A simple integration of (65) gives

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\lambda_1 t}, \ t \geq 0,$$

which yields the serial result (64) with $\lambda_2 = \dfrac{\mathcal{L}(0)}{\kappa_1}$ and by using the other side of the equivalence relation (60) again. The proof is complete. □

4. Numerical study

In this section, we established the existence and uniqueness of the solution to system (10) - (11), but it is challenging to determine the exact value of the solution. To address this, in the following, we will numerically solve the system in the one-dimension domain Ω of length L., allowing us to determine approximations of the solution. To this end, we will employ the Euler scheme for discretizing temporal variable and the classical finite differences method for discretizing space. To solve the discretized problem, we use a fixed point algorithm with study of their convergence. In addition, we provide an example where the numerical experiment demonstrates that the discrete energy E^n decays exponentially for different choices of the system parameters, supporting the asymptotic behavior of the discretized issue solution.

Let us introduce the functions $\widehat{z} = z_t$, $\widehat{u} = u_t$ and for any $N, m, M \in \mathbb{N}$, we introduce the nets

$$\Omega_N = \{x_i = \rho_i = ih, \ i = 0, 1, ..., N+1 \ , \text{ where } h = \frac{L}{N+1} \},$$
$$\Gamma_M = \{t_n = n\Delta t, \ n = 0, 1, ..., M+1 \ , \text{ where } \Delta t = \frac{T}{M+1} \},$$
$$\Upsilon_M = \{t_n = n\Delta t, \ n = -M', -M'+1, ..., 0 \ , \text{ with } 0 < M' < M \}.$$

such that the width of delay mesh is $\tau = M'\Delta t$.

Now, using a backward Euler scheme in time and finite differences in space, we define the following approximation of the derivatives:

$$\phi_{xx}(x_i, t_n) = \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{h^2},\ \phi_x(x_i, t_n) = \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2h},\ \phi_t(x_i, t_n) = \frac{\phi_i^n - \phi_i^{n-1}}{\Delta t},$$
$$\phi_t(x_i, t_{n-M'}) = \frac{\phi_i^{n-M'} - \phi_i^{n-M'-1}}{\Delta t},\ 1 \le i \le N,\ 1 \le n \le M.$$

where $\phi = \phi(x,t)$ be a function with second order partial derivatives. As a result, our problem consists to find $(\hat{z}, \hat{u}, \theta, q, \varphi)$ satisfying the discrete formulation of the system (10) - (11) presented by the following numerical scheme

$$\begin{cases} \frac{\rho_z}{\Delta t}(\widehat{z}_i^n - \widehat{z}_i^{n-1}) = \frac{\alpha_1}{h^2}(z_{i+1}^n - 2z_i^n + z_{i-1}^n) + \frac{\alpha_2}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \\ \frac{\rho_u}{\Delta t}(\widehat{u}_i^n - \widehat{u}_i^{n-1}) = \frac{\alpha_3}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \frac{\alpha_2}{h^2}(z_{i+1}^n - 2z_i^n + z_{i-1}^n) \\ +\frac{\beta}{2h}\left(\theta_{i+1}^n - \theta_{i-1}^n\right) - \frac{\mu}{\Delta t}\left(u_i^{n-M'} - u_i^{n-M'-1}\right), \\ \frac{c}{\Delta t}(\theta_i^n - \theta_i^{n-1}) = \frac{-1}{2h}\left(q_{i+1}^n - q_{i-1}^n\right) + \frac{\beta}{2h}(\widehat{u}_{i+1}^n - \widehat{u}_{i-1}^n), \\ \frac{\tau_0}{\Delta t}(q_i^n - q_i^{n-1}) = -q_i^n - \frac{k}{2h}\left(\theta_{i+1}^n - \theta_{i-1}^n\right), \\ \frac{\tau}{\Delta t}\left(\varphi_{i,j}^n - \varphi_{i,j}^{n-1}\right) = -\frac{1}{2h}\left(\varphi_{i,j+1}^n - \varphi_{i,j-1}^n\right), \end{cases} \tag{66}$$

where $z_i^n = z(x_i, t_n)$, $\hat{z}_i^n = z_t(x_i, t_n)$, $u_i^n = u(x_i, t_n)$, $\hat{u}_i^n = u_t(x_i, t_n)$, $\theta_i^n = \theta(x_i, t_n)$, $q_i^n = q(x_i, t_n)$, $u_i^{n-M'} = u(x_i, t_{n-M'})$, $\varphi_{i,j}^n = \varphi(x_i, \rho_j, t_n)$ for all $i = 1, 2, ..., N$, $j = 1, 2, ..., m$ and $n = 1, 2, ..., M$, with $\varphi(x_i, \rho_{m+1}, t_n) = u_i^{n-M'}$ according to (8). To simplicity our numerical calculations in our scheme, we consider the discrete boundary conditions given by

$$z_0^n = u_0^n = \theta_{N+1}^n = q_0^n = 0,\ z_{N+1}^n = z_N^n,\ u_{N+1}^n = u_N^n,\ \theta_1^n = \theta_0^n, \tag{67}$$

and initial conditions

$$\begin{cases} z_i^0 = z_0(x_i),\ u_i^0 = u_0(x_i),\ \theta_i^0 = \theta_0(x_i),\ q_i^0 = q_0(x_i) \\ \widehat{z}_i^0 = z_1(x_i) \text{ and } \widehat{u}_i^0 = u_1(x_i) \\ \varphi_{i,0}^n = \widehat{u}_i^n,\ u_i^{n-M'} = f_0(x_i, t_{n-M'}),\ \varphi_{i,j}^0 = f_0(x_i, \rho_j.t_{-M'}),\ t_{n-M'} \in \Upsilon_M, \\ \varphi_{N+1,N+1}^n = \hat{u}_{N+1}^{n-M'} \end{cases} \tag{68}$$

where

$$z_i^n = z_i^{n-1} + \Delta t \widehat{z}_i^n \text{ and } u_i^n = u_i^{n-1} + \Delta t \widehat{u}_i^n$$

for all $i = 1, 2, ..., N$ and $n = 1, 2, ..., M$.

Note that to find $\{\widehat{z}^n, \widehat{u}^n, \theta^n, q^n, \varphi^n\}$, we need to solve five coupled systems of algebraic equations. So, to solve the problem (66)-(68) at each time step we propose to consider the following fixed-point algorithm:

$$\begin{cases} \widehat{z}_i^{n,l} = \dfrac{\alpha_1 \Delta t}{\rho_z h^2}(z_{i+1}^{n,l-1} - 2z_i^{n,l-1} + z_{i-1}^{n,l-1}) + \dfrac{\alpha_2 \Delta t}{\rho_z h^2}(u_{i+1}^{n,l-1} - 2u_i^{n,l-1} + u_{i-1}^{n,l-1}) + \widehat{z}_i^{n-1}, \\ \widehat{u}_i^{n,l} = \dfrac{\alpha_3 \Delta t}{\rho_u h^2}(u_{i+1}^{n,l-1} - 2u_i^{n,l-1} + u_{i-1}^{n,l-1}) + \dfrac{\alpha_2 \Delta t}{\rho_u h^2}(z_{i+1}^{n,l-1} - 2z_i^{n,l-1} + z_{i-1}^{n,l-1}) \\ + \dfrac{\beta \Delta t}{2\rho_u h}\left(\theta_{i+1}^{n,l-1} - \theta_{i-1}^{n,l-1}\right) + \widehat{u}_i^{n-1} - \dfrac{\mu}{\rho_u}\left(u_i^{n-M'} - u_i^{n-M'-1}\right), \\ \theta_i^{n,l} = \theta_i^{n-1} - \dfrac{\Delta t}{2ch}\left(q_{i+1}^{n,l-1} - q_{i-1}^{n,l-1}\right) + \dfrac{\beta \Delta t}{2ch}(\widehat{u}_{i+1}^{n,l} - \widehat{u}_{i-1}^{n,l}), \\ (1 + \dfrac{\Delta t}{\tau_0})q_i^{n,l} = -\dfrac{k\Delta t}{2h\tau_0}\left(\theta_{i+1}^{n,l} - \theta_{i-1}^{n,l}\right) + q_i^{n-1}, \\ \varphi_{i,j}^{n,l} = \varphi_{i,j}^{n-1,l-1} - \dfrac{\Delta t}{2h\tau}\left(\varphi_{i,j+1}^{n,l-1} - \varphi_{i,j-1}^{n,l-1}\right) \end{cases} \tag{69}$$

with

$$\begin{cases} z_i^{n,0} = z_i^{n-1}, \ u_i^{n,0} = u_i^{n-1}, \ \theta_i^{n,0} = \theta_i^{n-1}, \ q_i^{n,0} = q_i^{n-1}, \ z_i^{n,l} = z_i^{n-1,l} + \Delta t \widehat{z}_i^{n,l}, \\ u_i^{n,l} = u_i^{n-1,l} + \Delta t \widehat{u}_i^{n,l}, \ \varphi_{i,j}^{n,0} = \hat{u}_{i,j}^{n-M'}, \end{cases} \tag{70}$$

for all $i, j = 1, 2, ..., N$, $n = 1, 2, ..., M$ and $l = 1, 2, ...$
At each time step, we solve the scheme (69) by an iterative procedure that was stopped when the difference between two successive iterations became smaller than a given tolerance ε.

4.1. Convergence of the proposed point fixed algorithm.

Let $\widehat{Z}^{n,l} = \left(\widehat{z}_i^{n,l}\right)_{1\le i\le N}$, $\widehat{U}^{n,l} = \left(\widehat{u}_i^{n,l}\right)_{1\le i\le N}$, $\widehat{\Theta}^{n,l} = \left(\widehat{\theta}_i^{n,l}\right)_{1\le i\le N}$, $\widehat{Q}^{n,l} = \left(\widehat{q}_i^{n,l}\right)_{1\le i\le N}$, $\vartheta^{n,l} = \left(\varphi_{i,j}^{n,l}\right)_{1\le i,j\le N}$. Then, the system (69)-(70) can be rewritten as follows

$$\begin{cases} \widehat{Z}^{n,l} = \dfrac{\alpha_1 \Delta t}{\rho_z h^2} A\, Z^{n,l-1} + \dfrac{\alpha_2 \Delta t}{\rho_z h^2} A\, U^{n,l-1} + \widehat{Z}^{n-1} \\ \widehat{U}^{n,l} = \dfrac{\alpha_3 \Delta t}{\rho_u h^2} A\, U^{n,l-1} + \dfrac{\alpha_2 \Delta t}{\rho_u h^2} A\, Z^{n,l} + \dfrac{\beta \Delta t}{2\rho_u h} B\, \Theta^{n,l-1} + \widehat{U}^{n-1} \\ -\dfrac{\mu}{\rho_u}\left(U^{n-M'} - U^{n-M'-1}\right) \\ \Theta^{n,l} = \Theta^{n-1} - \dfrac{\Delta t}{2c\,h} C\, Q^{n,l-1} + \dfrac{\beta \Delta t}{2c\,h} D\, \widehat{U}^{n,l} \\ \left(1 + \dfrac{\Delta t}{\tau_0}\right) Q^{n,l} = -\dfrac{k\,\Delta t}{2h\,\tau_0} B\, \Theta^{n,l} + Q^{n-1} \\ \vartheta^{n,l} = \vartheta^{n-1,l-1} - \dfrac{\Delta t}{2h\,\tau} E\, \vartheta^{n,l-1} \end{cases} \tag{71}$$

with

$$\begin{cases} Z^{n,0} = Z^{n-1},\ U^{n,0} = U^{n-1},\ \Theta^{n,0} = \Theta^{n-1},\ Q^{n,0} = Q^{n-1},\ Z^{n,l} = Z^{n-1,l} + \Delta t \widehat{Z}^{n,l}, \\ U^{n,l} = U^{n-1,l} + \Delta t \widehat{U}_i^{n,l},\ \vartheta^{n,0} = \widehat{U}^{n-M'} \end{cases} \tag{72}$$

where A, B, C, D are real matrices of dimensions $(n \times n)$ and E is a real matrix of dimension $\left(n^2 \times n^2\right)$ defined as follows

$$A = diag\,(1, -2, 1)\,,\ D = diag(-1, 0, 1),$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix},\ B = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} D & E_1 & \cdots & E_1 \\ E_1 & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & E_1 \\ E_1 & \cdots & E_1 & D \end{pmatrix},\ E_1 = 0_{\mathbb{R}^n \times \mathbb{R}^n}$$

Now, we are ready to state and prove the following convergence result

Lemma 4.1. *Let* $\left(\widehat{Z}^{n,l}, \widehat{U}^{n,l}, \widehat{\Theta}^{n,l}, \widehat{Q}^{n,l}, \vartheta^{n,l}\right)$ *be the solution of the system (71)-(72). Then, the fixed point algorithm proposed to solve the system (71)-(72) converges if and only if the following condition is satisfied*

$$\Delta t < min\left\{\frac{2h\,\tau}{]|E|[}, h\sqrt{\frac{\rho_u}{\alpha_3\,]|A|[}}, h\sqrt{\frac{\rho_u}{\alpha_1\,]|A|[}}, h\sqrt{\frac{\rho_z}{\alpha_1\,]|A|[}}, (\Delta t)_2\right\}, \tag{73}$$

where $(\Delta t)_2 = \dfrac{1-\sqrt{1+4r\tau_0}}{2r}$ *with* $r = \dfrac{k}{2c\,h^2}\,]|C|[\,]|B|[$ *and* $]|.|[$ *represent a subordinate matrix norm to the vector norm* $\|.\|_V$ *.*

Proof. From $(71)_1$ we can write

$$\left\|\widehat{Z}^{n,l+1} - \widehat{Z}^{n,l}\right\|_V \le \frac{\alpha_1 \Delta t}{\rho_z h^2}\ ]|A|[\ \left\|Z^{n,l} - Z^{n,l-1}\right\|_V + \frac{\alpha_2 \Delta t}{\rho_z h^2}\ ]|A|[\ \left\|U^{n,l} - U^{n,l-1}\right\|_V. \tag{74}$$

On the other hand, we use the fact that

$$\begin{cases} Z^{n,l} = Z^{n-1} + \Delta t\, \widehat{Z}^{n,l}, \\ Z^{n,l-1} = Z^{n-1} + \Delta t\, \widehat{Z}^{n,l-1}, \end{cases}$$

and

$$\begin{cases} U^{n,l} = U^{n-1} + \Delta t\, \widehat{U}^{n,l}, \\ U^{n,l-1} = U^{n-1} + \Delta t\, \widehat{U}^{n,l-1}. \end{cases}$$

Then, (74) can be rewritten as follows

$$\left\|\widehat{Z}^{n,l+1}-\widehat{Z}^{n,l}\right\|_V \leq \frac{\alpha_1(\Delta t)^2}{\rho_z h^2}\ ]|A|[\ \left\|\widehat{Z}^{n,l}-\widehat{Z}^{n,l-1}\right\|_V + \frac{\alpha_2(\Delta t)^2}{\rho_z h^2}\ ]|A|[\ \left\|\widehat{U}^{n,l}-\widehat{U}^{n,l-1}\right\|_V .$$

After $l$iterations, we get

$$\left\|\widehat{Z}^{n,l+1}-\widehat{Z}^{n,l}\right\|_V \leq \left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}\ ]|A|[\right)^l \left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V + \alpha_2 \sum_{k=0}^{k=l-1} \alpha_1^k \left(\frac{(\Delta t)^2}{\rho_z h^2}\ ]|A|[\right)^{k+1} \left\|\widehat{U}^{n,l-k}-\widehat{U}^{n,l-k-1}\right\|_V . \quad (75)$$

By using with $(71)_2$, we can easily obtain

$$\left\|\widehat{U}^{n,l+1}-\widehat{U}^{n,l}\right\|_V \leq \left(\frac{\alpha_3(\Delta t)^2}{\rho_u h^2}\ ]|A|[\right)^l \left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V + \frac{\alpha_2(\Delta t)^2}{\rho_u h^2}\ ]|A|[\ \left\|\widehat{Z}^{n,l+1}-\widehat{Z}^{n,l}\right\|_V + \frac{\beta\Delta t}{2\rho_u\, h}\ ]|B|[\ \left\|\Theta^{n,l}-\Theta^{n,l-1}\right\|_V . \quad (76)$$

Also, from $(71)_3$ and $(71)_4$, we have

$$\left\|\Theta^{n,l+1}-\Theta^{n,l}\right\|_V \leq \frac{\Delta t}{2c\, h}\ ]|C|[\ \left\|Q^{n,l}-Q^{n,l-1}\right\|_V + \frac{\beta\,\Delta t}{2c\, h}\ ]|D|[\ \left\|\widehat{U}^{n,l+1}-\widehat{U}^{n,l}\right\|_V , \quad (77)$$

and

$$\left\|Q^{n,l+1}-Q^{n,l}\right\|_V \leq \frac{\mu_1\, k\,\Delta t}{2\tau_0\, h}\ ]|B|[\ \left\|\Theta^{n,l+1}-\Theta^{n,l}\right\|_V ,\ \mu_1 = \frac{\tau_0}{1+\Delta t}. \quad (78)$$

From $(71)_5$, we can write

$$\left\|\vartheta^{n,l+1}-\vartheta^{n,l}\right\|_V \leq \left\|\vartheta^{n-1,l}-\vartheta^{n-1,l-1}\right\|_V + \frac{\Delta t}{2h\,\tau}\ ]|E|[\ \left\|\vartheta^{n,l}-\vartheta^{n,l-1}\right\|_V , \quad (79)$$

we estimate the last term of (79), every time, and replacing the result in (79), we obtain

$$\left\|\vartheta^{n,l+1}-\vartheta^{n,l}\right\|_V \leq \sum_{i=0}^{n_0} \sum_i + \sum_{i=1}^{n_0} \Gamma_i,\ n_0 \geq 1, \quad (80)$$

with

$$\sum_0 = \mu_2^l \left\|\vartheta^{n,1}-\vartheta^{n,0}\right\|_V ,\ \mu_2 = \frac{\Delta t}{2h\,\tau}\ ]|E|[,$$

and

$$\sum_i = \sum_{\sigma_1=0}^{l-1} \sum_{\sigma_2=0}^{l-\sigma_1-2} \sum_{\sigma_3=0}^{l-(\sigma_1+\sigma_2)-3} \cdots \sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1}\varsigma_j - i} \mu_2^{l-i} \left\|\vartheta^{n-i,1}-\vartheta^{n-i,0}\right\|_V ,\ i \geq 1,$$

where

$$\varsigma_j = \begin{cases} 0, & \text{if } j=0, \\ \sigma_j, & \text{otherwise}, \end{cases} \quad (81)$$

and for all $i \geq 1$

$$\Gamma_i = \sum_{\sigma_1=0}^{l-1} \sum_{\sigma_2=0}^{l-\sigma_1-2} \sum_{\sigma_3=0}^{l-(\sigma_1+\sigma_2)-3} \cdots \sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1}\varsigma_j - i} \mu_2^{\sum_{k=1}^{i}\sigma_k} \left\| \vartheta^{n-i,l-\sum_{k=1}^{i}\sigma_k-(i-1)} - \vartheta^{n-i,l-\sum_{k=1}^{i}\sigma_k - i} \right\|_V .$$

For n_0 sufficiently large such that $\sum_{k=1}^{i} \sigma_k = l - i$, the estimate (80) becomes

$$\left\|\vartheta^{n,l+1} - \vartheta^{n,l}\right\|_V \leq \sum_{i=0}^{n_0} \sum_{i} + \sum_{i=0}^{n_0} \sum_{\sigma_1=0}^{l-1} \sum_{\sigma_2=0}^{l-\sigma_1-2} \cdots \sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1}\varsigma_j - i} \mu_2^{l-i} \left\|\vartheta^{n-i,1} - \vartheta^{n-i,0}\right\|_V .$$

In order to $\left(\vartheta^{n,l}\right)_{l\geq 0}$ converge for $l \to \infty$, it is necessary that $\mu_2 < 1$, which leads to

$$\Delta t < \frac{2h\,\tau}{]|E|[}. \tag{82}$$

Now, by replacing (75) into (76), we have

$$\begin{aligned}
\left\|\widehat{U}^{n,l+1} - \widehat{U}^{n,l}\right\|_V \leq & \left(\frac{\alpha_3 (\Delta t)^2}{\rho_u h^2}\ ]|A|[\right)^l \left\|\widehat{U}^{n,1} - \widehat{U}^{n,0}\right\|_V \\
& + \alpha_2 \alpha_1^l \left(\frac{(\Delta t)^2}{\rho_z h^2}\ ]|A|[\right)^{l+1} \left\|\widehat{Z}^{n,1} - \widehat{Z}^{n,0}\right\|_V \\
& + \alpha_2^2 \sum_{k=0}^{l-1} \alpha_1^k \left(\frac{(\Delta t)^2}{\rho_z h^2}\ ]|A|[\right)^{k+2} \left\|\widehat{U}^{n,l-k} - \widehat{U}^{n,l-k-1}\right\|_V \\
& + \frac{\beta\,\Delta t}{2\rho_u h}\ ]|B|[\ \left\|\Theta^{n,l} - \Theta^{n,l-1}\right\|_V .
\end{aligned} \tag{83}$$

From (77), we get

$$\left\|\Theta^{n,l} - \Theta^{n,l-1}\right\|_V \leq \frac{\Delta t}{2c\,h}\ ]|C|[\ \left\|Q^{n,l-1} - Q^{n,l-2}\right\|_V + \frac{\beta\,\Delta t}{2c\,h}\ ]|D|[\ \left\|\widehat{U}^{n,l} - \widehat{U}^{n,l-1}\right\|_V .$$

From (78), we arrive at

$$\left\|Q^{n,l-1} - Q^{n,l-2}\right\|_V \leq \frac{\mu_1\,k\,\Delta t}{2\tau_0\,h}\ ]|B|[\ \left\|\Theta^{n,l-1} - \Theta^{n,l-2}\right\|_V ,$$

then,

$$\begin{aligned}
\left\|\Theta^{n,l} - \Theta^{n,l-1}\right\|_V & \leq \gamma_0 \left\|\Theta^{n,l-1} - \Theta^{n,l-2}\right\|_V + \frac{\beta\,\Delta t}{2c\,h}\ ]|D|[\ \left\|\widehat{U}^{n,l} - \widehat{U}^{n,l-1}\right\|_V \\
& \ \vdots \\
& \leq \gamma_0^{l-1} \left\|\Theta^{n,1} - \Theta^{n,0}\right\|_V + \frac{\beta\,\Delta t}{2c\,h}\ ]|D|[\sum_{k=0}^{l-2} \gamma_0^k \left\|\widehat{U}^{n,l-k} - \widehat{U}^{n,l-k-1}\right\|_V ,
\end{aligned}$$

where

$$\gamma_0 = \frac{k\mu_1 (\Delta t)^2}{2c\,h^2 \tau_0}\ ]|C|[\,]|B|[\,.$$

Therefore, the estimate (83) becomes

$$\begin{aligned}
&\left\|\widehat{U}^{n,l+1}-\widehat{U}^{n,l}\right\|_V \\
&\le \left(\left(\frac{\alpha_3(\Delta t)^2}{\rho_u h^2}]|A|[\right)^l+\left(\frac{\alpha_2}{\alpha_1}\right)^2\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}]|A|[\right)^{l+1}\right)\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V \\
&+\frac{\alpha_2}{\alpha_1}\left(\frac{\alpha_1(\Delta t)^2}{\rho_u h^2}]|A|[\right)^{l+1}\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V+\frac{\beta\,\Delta t}{2\rho_u h}]|B|[\ \gamma_0^{l-1}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V \\
&+\sum_{k=0}^{l-2}\left(\left(\frac{\alpha_2}{\alpha_1}\right)^2\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}]|A|[\right)^{k+2}+\frac{1}{c\,\rho_u}\left(\frac{\beta\,\Delta t}{2h}\right)^2]|B|[]|D|[\gamma_0^k\right)\left\|\widehat{U}^{n,l-k}-\widehat{U}^{n,l-k-1}\right\|_V.
\end{aligned}$$

To reduce this last estimation, we use the following notation

$$\begin{aligned}
\left\|\widehat{U}^{n,l+1}-\widehat{U}^{n,l}\right\|_V &\le \chi_{1,l}\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V+\chi_{2,l}\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V \\
&+\chi_{3,l}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V+\sum_{\sigma_1=0}^{l-2}\xi_{\sigma_1}\left\|\widehat{U}^{n,l-\sigma_1}-\widehat{U}^{n,l-\sigma_1-1}\right\|_V, \qquad (84)
\end{aligned}$$

where

$$\begin{aligned}
\chi_{1,l} &= \left(\left(\frac{\alpha_3(\Delta t)^2}{\rho_u h^2}]|A|[\right)^l+\left(\frac{\alpha_2}{\alpha_1}\right)^2\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}]|A|[\right)^{l+1}\right), \\
\chi_{2,l} &= \frac{\alpha_2}{\alpha_1}\left(\frac{\alpha_1(\Delta t)^2}{\rho_u h^2}]|A|[\right)^{l+1}, \\
\chi_{3,l} &= \frac{\beta\,\Delta t}{2\rho_u h}]|B|[\ \gamma_0^{l-1}, \\
\xi_{\sigma_1} &= \left(\left(\frac{\alpha_2}{\alpha_1}\right)^2\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}]|A|[\right)^{\sigma_1+2}+\frac{1}{c\,\rho_u}\left(\frac{\beta\,\Delta t}{2h}\right)^2]|B|[]|D|[\gamma_0^{\sigma_1}\right).
\end{aligned}$$

By using the same iterative process as in the estimation of (79), we can estimate the last term of (84) as follows

$$\begin{aligned}
\left\|\widehat{U}^{n,l+1}-\widehat{U}^{n,l}\right\|_V &\le \left(\chi_{1,l}+\sum_{i=1}^{n_0-1}\Upsilon_i\right)\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V \\
&+\left(\chi_{2,l}+\sum_{i=1}^{n_0-1}\Psi_i\right)\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V \\
&+\left(\chi_{3,l}+\sum_{i=1}^{n_0-1}\Phi_i\right)\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V+\sum_{i=1}^{n_0}\Lambda_i,
\end{aligned}$$

with

$$\Upsilon_i=\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1}\varsigma_j-(i+1)}\chi_{1,l-\sum_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\xi_{\sigma_k}$$

$$\Psi_i = \sum_{\sigma_1=0}^{l-2} \sum_{\sigma_2=0}^{l-\sigma_1-3} \cdots \sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1} \varsigma_j-(i+1)} \chi_{2,l-\sum_{k=1}^{i} \sigma_k - i} \cdot \prod_{k=1}^{i} \xi_{\sigma_k},$$

$$\Phi_i = \sum_{\sigma_1=0}^{l-2} \sum_{\sigma_2=0}^{l-\sigma_1-3} \cdots \sum_{\sigma_i=0}^{l-\sum_{j=0}^{i-1} \varsigma_j-(i+1)} \chi_{3,l-\sum_{k=1}^{i} \sigma_k - i} \cdot \prod_{k=1}^{i} \xi_{\sigma_k},$$

$$\Lambda_i = \sum_{\sigma_1=0}^{l-2} \sum_{\sigma_2=0}^{l-\sigma_1-3} \cdots \sum_{\sigma_i=0}^{l-\sum_{j=0}^{i} \varsigma_j-(i+1)} \prod_{i=1}^{n_0} \xi_{\sigma_i} \left\| U^{n,l-\sum_{i=1}^{n_0} \sigma_i-(n_0-1)} - U^{n,l-\sum_{i=1}^{n_0} \sigma_i - n_0} \right\|_V,$$

$$\xi_{\sigma_k} = \left(\left(\frac{\alpha_2}{\alpha_1}\right)^2 \left(\frac{\alpha_1 (\Delta t)^2}{\rho_z h^2} \,]|A|[\right)^{\sigma_k+2} + \frac{1}{c\,\rho_u} \left(\frac{\beta\,\Delta t}{2h}\right)^2]|B|[]|D|[\gamma_0^{\sigma_k} \right),\ k = \overline{1,i},$$

and ς_j is defined by (81).

Until $\sum_{i=0}^{n_0} \sigma_i = l - n_0 - 1,\ (l > n_0 + 1)$,

$$\begin{aligned}\left\| \widehat{U}^{n,l+1} - \widehat{U}^{n,l} \right\|_V \le & \left(\chi_{1,l} + \sum_{i=1}^{n_0-1} \Upsilon_i \right) \left\| \widehat{U}^{n,1} - \widehat{U}^{n,0} \right\|_V \\ & + \left(\chi_{2,l} + \sum_{i=1}^{n_0-1} \Psi_i \right) \left\| \widehat{Z}^{n,1} - \widehat{Z}^{n,0} \right\|_V \\ & + \left(\chi_{3,l} + \sum_{i=1}^{n_0-1} \Phi_i \right) \left\| \Theta^{n,1} - \Theta^{n,0} \right\|_V \\ & + \sum_{i=0}^{n_0} \sum_{\sigma_1=0}^{l-2} \sum_{\sigma_2=0}^{l-\sigma_1-3} \cdots \sum_{\sigma_i=0}^{l-\sum_{j=0}^{i} \varsigma_j-(i+1)} \prod_{i=1}^{n_0} \xi_{\sigma_i} \left\| \widehat{U}^{n,1} - \widehat{U}^{n,0} \right\|_V.\end{aligned}$$

In order to $\left(\widehat{U}^{n,l}\right)_{l\ge 0}$ converge, it's necessary and sufficient to the following conditions hold

$$\left\{ \begin{array}{l} \dfrac{\alpha_3 (\Delta t)^2}{\rho_u h^2} \,]|A|[< 1 \\ \dfrac{\alpha_1 (\Delta t)^2}{\rho_u h^2} \,]|A|[< 1 \\ \dfrac{\alpha_1 (\Delta t)^2}{\rho_z h^2} \,]|A|[< 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \Delta t < h\sqrt{\dfrac{\rho_u}{\alpha_3 \,]|A|[}} \\ \Delta t < h\sqrt{\dfrac{\rho_u}{\alpha_1 \,]|A|[}} \\ \Delta t < h\sqrt{\dfrac{\rho_z}{\alpha_1 \,]|A|[}} \end{array} \right. \tag{85}$$

and

$$\gamma_0 < 1.$$

This last condition is equivalent to

$$r \frac{(\Delta t)^2}{(\tau_0 + \Delta t)} < 1, \tag{86}$$

with

$$r = \frac{k}{2c\,h^2}\,]|C|[]|B|[\,,$$

the inequation (86) holds if and only if

$$\Delta t \in](\Delta t)_1\,,(\Delta t)_2[\,,$$

where

$$(\Delta t)_1 = \frac{1-\sqrt{1+4r\tau_0}}{2r} < 0, \quad (\Delta t)_2 = \frac{1+\sqrt{1+4r\tau_0}}{2r}.$$

Because $\Delta t > 0$, Therefore, (86) holds if and only if

$$\Delta t \in]0,(\Delta t)_2[\,. \tag{87}$$

By the same technique, we can easily obtain

$$\begin{aligned}\left\|\widehat{Z}^{n,l+1}-\widehat{Z}^{n,l}\right\|_V \leq & \left(\chi_{4,l}+\sum_{i=1}^{n_0-1}\Psi_i^{'}\right)\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V \\ & + \left(\chi_{5,l}+\sum_{i=1}^{n_0-1}\sum{}_i^{'}\right)\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V \\ & + \left(\sum_{i=1}^{n_0-1}\Phi_i^{'}\right)\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V \\ & + \sum_{i=0}^{n_0}\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i}\varsigma_j-(i+1)}\prod_{i=1}^{n_0}\acute{\xi}_{\sigma_i}\left\|\widehat{U}^{n,2}-\widehat{U}^{n,1}\right\|_V,\end{aligned}$$

with

$$\acute{\xi}_{\sigma_k} = \left(\frac{\alpha_2}{\alpha_1}\right)\left(\frac{\alpha_1(\Delta t)^2}{\rho_z h^2}\,]|A|[\right)^{\sigma_k+1},\ k=\overline{1,i},$$

$$\sum{}_i^{'} = \sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)}\chi_{1,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\acute{\xi}_{\sigma_k},$$

$$\Psi_i^{'} = \sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)}\chi_{2,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\acute{\xi}_{\sigma_k},$$

$$\Phi_i^{'} = \sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)}\chi_{3,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\acute{\xi}_{\sigma_k},$$

we deduce that under the same conditions (85) on Δt , the sequence $\left(\widehat{Z}^{n,l}\right)_{l\geq 0}$ converges.

From (77)

$$\begin{aligned}
&\left\|\Theta^{n,l}-\Theta^{n,l-1}\right\|_V \\
&\leq \gamma_0^{l-1}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V+\frac{\beta\,\Delta t}{2c\,h}\,]|D|[\,\sum_{\sigma_1=0}^{l-2}\gamma_0^{\sigma_1}\left\|\widehat{U}^{n,l-\sigma_1}-\widehat{U}^{n,l-\sigma_1-1}\right\|_V \\
&\leq \gamma_0^{l-1}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V+\sum_{\sigma_1=0}^{l-2}\xi''_{\sigma_1}\left\|\widehat{U}^{n,l-\sigma_1}-\widehat{U}^{n,l-\sigma_1-1}\right\|_V,
\end{aligned}$$

where

$$\xi''_{\sigma_1}=\gamma_0^{\sigma_1}\frac{\beta\,\Delta t}{2c\,h}\,]|D|[\,.$$

By using the same technique as in the estimation of the last term of (84), we deduce that

$$\begin{aligned}
&\left\|\Theta^{n,l}-\Theta^{n,l-1}\right\|_V\leq \gamma_0^{l-1}\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V \\
&+\sum_{i=1}^{n_0-1}\Big(\textstyle\sum''_i\left\|\widehat{U}^{n,1}-\widehat{U}^{n,0}\right\|_V+\Psi''_i\left\|\widehat{Z}^{n,1}-\widehat{Z}^{n,0}\right\|_V+\Phi''_i\left\|\Theta^{n,1}-\Theta^{n,0}\right\|_V\Big) \\
&+\sum_{i=1}^{n_0}\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\delta_1-3}\cdots\sum_{\sigma_i=0}^{n_0+1-\sum\limits_{j=i-1}^{n_0}\sigma_j-(i+1)}\prod_{i=1}^{n_0}\xi''_{\sigma_i}\left\|\widehat{U}^{n,2}-\widehat{U}^{n,1}\right\|_V
\end{aligned}$$

with

$$\xi''_{\sigma_k}=\gamma_0^{\sigma_k}\frac{\beta\,\Delta t}{2c\,h}\,]|D|[\,,\ \ k=\overline{1,i},$$

$$\sum''_i=\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)}\chi_{1,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\xi''_{\sigma_k},$$

$$\Psi''_i=\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)}\chi_{2,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\xi''_{\sigma_k},$$

$$\Phi''_i=\sum_{\sigma_1=0}^{l-2}\sum_{\sigma_2=0}^{l-\sigma_1-3}\cdots\sum_{\sigma_i=0}^{l-\sum\limits_{j=0}^{i-1}\varsigma_j-(i+1)}\chi_{3,l-\sum\limits_{k=1}^{i}\sigma_k-i}\cdot\prod_{k=1}^{i}\xi''_{\sigma_k}.$$

In order to the sequence $\left(\Theta^{n,l}\right)_{l\geq 0}$ converges, it's necessary and sufficient that $\gamma_0<1$. From (78), if $\left(\Theta^{n,l}\right)_{l\geq 0}$ converges, then, $\left(Q^{n,l}\right)_{l\geq 0}$ converges under the same condition on $\left(\Theta^{n,l}\right)_{l\geq 0}$. Therefore, from (82), (85) and (87), the fixed point iterative scheme converges if and only if Δt verifies (73). □

4.2. Approximation of the discrete energy. To approximate the continuous energy (24), we use the trapezoidal quadrature formula to compute the integral $I = \int_0^L f(x)dx$

$$I_N = \sum_{i=0}^{N+1} a_i f(x_i) \simeq I,$$

where the weights $\{a_i\}_{i=1}^{N}$ are given by $a_0 = a_{N+1} = \frac{h}{2}$ and for $i = 1, 2, ..., N$, $a_i = h$. Concerning the trapezoidal quadrature formula in tow dimensional case to compute the last part of (24), we use the following approximation

$$\begin{aligned}\int_0^L \int_0^1 f(x,y)dydx \simeq &\frac{h^2}{4}\left(f(x_0,y_0) + f(x_0,y_{N+1}) + f(x_{N+1},y_0) + f(x_{N+1},y_{N+1})\right)\\ &+ \frac{h^2}{2}\sum_{i=1}^{N}\left(f(x_i,y_0) + f(x_i,y_{N+1})\right)\\ &+ \frac{h^2}{2}\sum_{j=1}^{N}\left(f(x_0,y_j) + f(x_{N+1},y_j)\right)\\ &+ h^2\sum_{i=1}^{N}\sum_{j=1}^{N} f(x_i,y_j).\end{aligned}$$

Therefore, the discrete energy at the time step t_n of system (66)-(68) is given by

$$\begin{aligned}E^n =&\frac{1}{2}\sum_{i=0}^{N+1} a_i[\rho_z \left(\hat{z}_i^n\right)^2 + \rho_u \left(\hat{u}_i^n\right)^2 + c\left(\theta_i^n\right)^2 + \frac{\tau_0}{k}\left(q_i^n\right)^2 + \alpha_1\left((z_x)_i^n\right)^2 \\ &+ \alpha_3\left((u_x)_i^n\right)^2 + 2b\,(u_x)_i^n\,(z_x)_i^n] \\ &+ \frac{|\mu|\,\tau}{2}[\frac{h^2}{4}\left(\left(\varphi_{0,0}^n\right)^2 + \left(\varphi_{0,N+1}^n\right)^2 + \left(\varphi_{N+1,0}^n\right)^2 + \left(\varphi_{N+1,N+1}^n\right)^2\right) \\ &+ \frac{h^2}{2}\sum_{j=1}^{N}\left(\varphi_{0,j}^n\right)^2 + \frac{h^2}{2}\sum_{j=1}^{N}\left(\varphi_{N+1,j}^n\right)^2 + \frac{h^2}{2}\sum_{i=1}^{N}\left(\varphi_{i,0}^n\right)^2 \\ &+ \frac{h^2}{2}\sum_{i=1}^{N}\left(\varphi_{i,N+1}^n\right)^2 + h^2\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\varphi_{i,j}^n\right)^2],\end{aligned} \tag{88}$$

with

$$\begin{aligned}\hat{u}_i^n &= u_t\left(x_i, t_n\right), \quad (u_x)_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2h}, \\ \hat{z}_i^n &= z_t\left(x_i, t_n\right), \quad (z_x)_i^n = \frac{z_{i+1}^n - z_{i-1}^n}{2h}.\end{aligned}$$

which is a discretization of the continuous energy (24).

4.3. Numerical illustration. In the table below, we consider five different choices of delay's weight and we note that the asymptotic behavior was reached in the last case even if in the first four cases the weights of delay are not considerable. This shows the great influence of the delay on the stability of this type of system throughout time.

Case	Weight of delay	Iterations time	Asymptotic behavior
1	$\mu = 0.9$	200	Unreached
2	$\mu = 0.03$	200	Unreached
3	$\mu = 0.005$	200	Unreached
4	$\mu = 0.0004$	200	Unreached
5	$\mu = 0.00008$	200	Reached

Table1. Asymptotic behavior for different cases of delay's weight.

In the next, we describe the following numerical example where the asymptotic behavior was reached, that is the case when $\mu_0 = 0.00008$ and for different choices of the system parameters with the condition (2) holds.

For this numerical test, we choose the following different values for the coefficients of the system

$$\rho_z = 3.5,\ \alpha_3 = 0.2,\ \mu = 0.8,\ \alpha_2 = 0.1,\ \beta = 0.05,\ \tau = 0.1$$
$$\rho_u = 1.5,\ \alpha_1 = 0.1,\ c = 4,\ \tau_0 = 0.06,\ c = 1.5,\ k = 10^{-5}$$

We run our code for the following discretization parameters: $N = 100$, $m = 150$, $M = 200$, $L = 1$, $T = 1$ and take $\varepsilon = 10^{-5}$. With the following initial conditions

$$z_0\left(x\right) = 10^{-2}\left(x^3 - \frac{3}{2}x^2\right),\ z_1\left(x\right) = \frac{1}{8}\left(2x^2 - 4x\right),\ u_0\left(x\right) = \frac{1}{8}\left(2x^2 - 2x\right),$$
$$u_1\left(x\right) = 0,\ \theta_0\left(x\right) = 0,\ q_0\left(x\right) = \frac{1}{5}x^3 e^{-\frac{3}{2}x^2},$$
$$f_0\left(x, t-\tau\right) = \frac{1}{2}.10^{-3}\cos(x).\cos\left(\frac{1}{10\pi}\left(t-\tau\right)\right)$$

with the above parameters choice, we deduce that

$$\frac{2h\,\tau}{]|E|[} = 4.901960,\ h\sqrt{\frac{\rho_u}{\alpha_3\,]|A|[}} = 0.013424,\ h\sqrt{\frac{\rho_u}{\alpha_1\,]|A|[}} = 0.018985,$$
$$h\sqrt{\frac{\rho_z}{\alpha_1\,]|A|[}} = 0.029000,\ \left(\Delta t\right)_2 = 34.766946.$$

with

$$]|\mathcal{M}|[\ = \max_i \sum_j \left|\mathcal{M}_{ij}\right|.$$

Note that

$$dt = 0.005 < 0.013424 = min\left\{\frac{2h\,\tau}{]|E|[}, h\sqrt{\frac{\rho_u}{\alpha_3\,]|A|[}}, h\sqrt{\frac{\rho_u}{\alpha_1\,]|A|[}}, h\sqrt{\frac{\rho_z}{\alpha_1\,]|A|[}}, \left(\Delta t\right)_2\right\}.$$

which confirms the convergence of the algorithm.
Here are the evolution in time of the solutions $z, u, \theta, q, \varphi\left(x, 1, t\right)$ and of the discrete energy. In above numerical test, the condition (73) holds and graphics presented in the Figures 1,5 show the evolution in time of the approximations solutions z, u, θ, q and $\varphi(x, 1, t)$ on the interval $[0, T]$, for different choices of the system parameters and

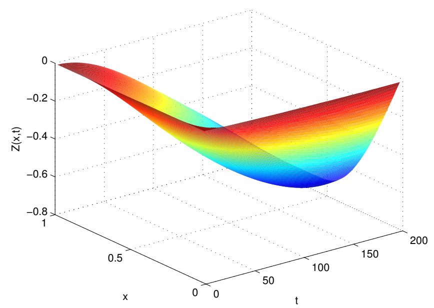

FIGURE 1.
Evolution in time of the function z

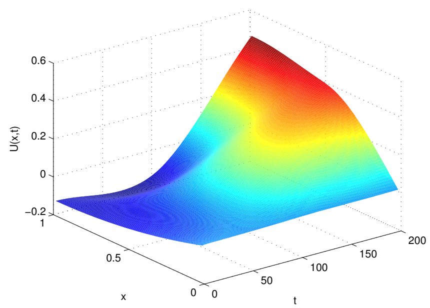

FIGURE 2.
Evolution in time of the function u

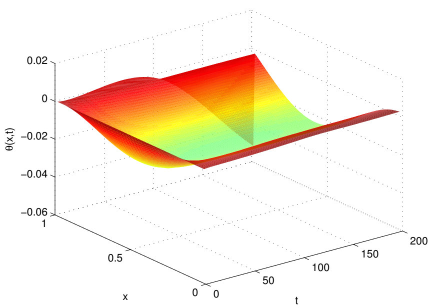

FIGURE 3.
Evolution in time of the function θ

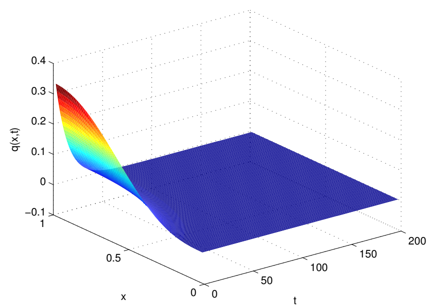

FIGURE 4.
Evolution in time of the function q

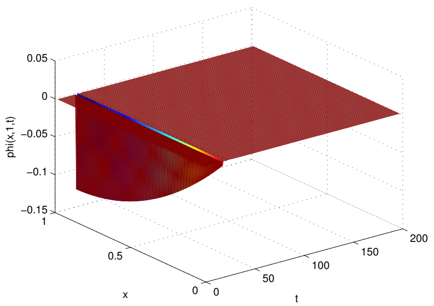

FIGURE 5.
Evolution in time of the function $\varphi(x,1,t)$

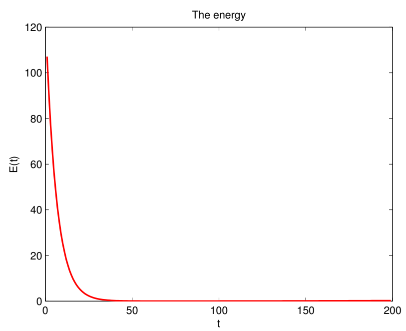

FIGURE 6.
Evolution in time of the discrete energy

of the initial data. Furthermore, the Figure 6 shows that the approximate energy (88) decays in an exponential manner which confirms the main theoretical result obtained.

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(Chaima Boulkheloua) Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS), University of 20 August 1955 Skikda, Algeria
E-mail address: c.boulkheloua@univ-skikda.dz_boulkchaima33@gmail.com

(Houssem Eddine Khochemane) Ecole normale supérieure d'enseignement technologique-Skikda-Algeria
E-mail address: khochmanehoussem@hotmail.com

(Lamine Bouzettouta) Laboratory of Applied Mathematics and History and Didactics of Mathematics (LAMAHIS), University of 20 August 1955 Skikda, Algeria
E-mail address: lami_750000@yahoo.fr