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Multi-Valued Fixed Point Results with Feng-Liu Type Integral Inequality and Application to Dynamical Programming

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Abstract. In this paper, we obtain a fixed point result for multi-valued mappings satisfying Feng-Liu type integral inequality in the framework of partial metric space without using Pompeiu-Hausdorff distance. Our result extends an existing result in the integral setting for partial metric space. A common fixed point result and a coupled fixed point result of integral type are also obtained. The results are supported by suitable examples. An application is given to functional equations arising in dynamical programming.

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1. Introduction

Partial metric space extends the notion of metric space by allowing for a non-zero self distance between the points. This concept was pioneered by S. G. Matthews in 1994 [23] while investigating the denotational semantics of dataflow networks. His work signified the applicability of the Banach contraction mapping theorem in a broader context of partial metrics for program verification applications. Following this development, many authors have derived different fixed point results in partial metric space (refer to [2, 3, 11, 14, 16, 20, 26]).

Fixed point theory for multi-valued mappings was initiated by Nadler [24] in 1969 by derivation of fixed point results considering complete metric space for multi-valued contraction mapping. In 2006, Feng and Liu [13] proved a fixed point result for multi-valued contractive mappings which is a generalization of Nadler's fixed point theorem. In [17], Jaleli and Samet derived a new type of contraction mapping in the framework of generalized metric space by introducing a set of functions θ from $(0,\infty) \to (1,\infty)$ with some specific conditions. Later in 2015, Altun and Minak [1] established a new contraction type multi-valued mapping in metric space considering Jaleli and Samet's technique for single valued mappings by adding a weaker condition on θ. Based on these, Nguyen and Phuong [25] extended the result of Feng and Liu [13] for multi-valued mappings using a wider class of functions θ by replacing the intervals by $[0,\infty)$ and $[1,\infty)$.

The theory of dynamic programming originates from the domain of multistage decision processes, where the emergence of certain functional equations plays a pivotal role for its development. In [7, 8], Bellman first derived the existence of solutions for a certain class of functional equations through the application of Banach fixed point theorem. Later Baskaran and Subrahmanyam [5], Belbas [6], Bhakta and Mitra [9],

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Bhakta and Choudhury [10], Liu [21], Kaliaj [18], Rasham et al.[27] and others established the existence, uniqueness and iterative approximation of solution for different types of functional equations by using fixed point results.

Motivated by these works, in this paper, we establish an extension of the fixed point result by Nguyen and Phuong [25], considering partial metric space in the integral settings. Furthermore, we derive a common fixed point result and a coupled fixed point result satisfying Feng and Liu inequality for multi-valued mappings. The results are obtained without using Pompeiu-Hausdorff distance. An illustrative application is presented which demonstrates the utilization of the common fixed point result to the existence of solution for a system of functional equations that emerges within certain types of continuous multistage decision processes.

2. Preliminaries

Before proceeding to the main findings, several fundamental definitions and related results are presented.

Definition 2.1. [23] For a non-empty set X, let $p : X \times X \to [0, \infty)$ be a mapping satisfying the following axioms :
P_0 : $0 \leq p(x, x) \leq p(x, y)$,
P_1 : $p(x, x) = p(x, y) = p(y, y)$ if and only if $x = y$,
P_2 : $p(x, y) = p(y, x)$,
P_3 : $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$,
for all $x, y, z \in X$.
Then p is called the partial metric and the pair (X, p) forms a partial metric space.

Some examples are as follows.

(*i*) If $X = \mathbb{R}^+ \cup \{0\}$, $p(x, y) = 1 + |x - y|$ for all $x, y \in X$, then (X, p) is a partial metric space.

(*ii*) Let $X = \{0, 1, 4\}$ and $p(x, y) = \frac{1}{4}|x - y| + \frac{1}{2}\max\{x, y\}$ for all $x, y \in X$. Then p is a partial metric on X (refer to [3]).

Definition 2.2. [23] Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X, p) is said to be Cauchy if and only if $\lim\limits_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(ii) A sequence $\{x_n\}$ in (X, p) converges to y in X if and only if $\lim\limits_{n\to\infty} p(x_n, y) = \lim\limits_{n\to\infty} p(x_n, x_n) = p(y, y)$.

Definition 2.3. [23] A partial metric space (X, p) is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in (X, p) converges to a point y in X, that is, $p(y, y) = \lim\limits_{n,m\to\infty} p(x_n, x_m)$.

For a partial metric space (X, p), suppose $P(X)$ denotes the collection of all subsets of X and $CB^p(X)$ signifies the set containing all non-empty closed and bounded subsets of X relative to the partial metric p.

Let (X, p) be a partial metric space. A point $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \to P(X)$ if $x \in Tx$.

Definition 2.4. [19] Let (X,p) be a partial metric space, $C \subset X$ and $f : C \to \mathbb{R}^+$ be a mapping on C. Then f is called a lower semi-continuous mapping on C whenever

$$\lim_{n\to\infty} p(x_n,x) = p(x,x) \text{ implies } f(x) \leq \lim_{n\to\infty} \inf f(x_n).$$

3. Main Results

We consider the family Θ of all functions $\theta : [0,\infty) \to [1,\infty)$ satisfying the following conditions (refer to [17, 25]):
(Θ_1) θ is non-decreasing,
(Θ_2) for each sequence $\{t_n\} \subset [0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$.
Also, let Φ be the family of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying
(Φ_1) ϕ is a continuous mapping,
(Φ_2) ϕ is Lebesgue integrable and summable,
(Φ_3) for every $\epsilon > 0$, $\int_0^\epsilon \phi(t)\,dt > 0$.
Using the above (θ,ϕ) functions, first we derive the following multi-valued fixed point result considering the Feng-Liu type integral inequality. Our result extends the fixed point result of Nguyen and Phuong [25] in the integral setting for partial metric space.

Theorem 3.1. *Let (X,p) be a complete partial metric space and $T : X \to CB^p(X)$ be a multi-valued mapping such that $x \to p(x,Tx)$ is lower semi-continuous. Assume that there exist real numbers $a,s,r \in (0,\infty)$ with $s > r$, $l \in (0,\infty]$, $k \in [0,1)$ and a function $\theta \in \Theta$, $\phi \in \Phi$ such that*
(A_1) $\int_0^{\theta(t)} \phi(t)\,dt \geq 1$ *for all* $t \geq 0$.
(A_2) *For any sequence $\{\lambda_n\}$ in $[0,\infty)$,* $\lim_{n\to\infty} \int_0^{\theta(\lambda_n)} \phi(t)\,dt = 1$ *if and only if* $\lim_{n\to\infty} \lambda_n = 0^+$.
(A_3) $\lim_{\mu\to 0^+} \dfrac{\int_0^{\theta(\mu)} \phi(t)\,dt - 1}{(\int_0^{\mu} \phi(t)\,dt)^r} = l.$
If for any $x \in X$ with $p(x,Tx) > 0$, there is $y \in X$ such that

$$\int_0^{p(x,y)} \phi(t)\,dt \leq a\left(\int_0^{p(x,Tx)} \phi(t)\,dt\right)^s$$

$$\text{and } \int_0^{\theta(p(y,Ty))} \phi(t)\,dt \leq \left(\int_0^{\theta(p(x,Tx))} \phi(t)\,dt\right)^k,$$

then there exists $u \in X$ such that $u \in Tu$.

Proof. Let $A = \{x \in X : p(x,Tx) > 0\}$. If $A = \emptyset$, then T has a fixed point in X. Suppose $A \neq \emptyset$. Let $x_0 \in A$, then there exists $x_1 \in X$ such that

$$\int_0^{p(x_0,x_1)} \phi(t)\,dt \leq a\left(\int_0^{p(x_0,Tx_0)} \phi(t)\,dt\right)^s$$

$$\text{and } \int_0^{\theta(p(x_1,Tx_1))} \phi(t)\,dt \leq \left(\int_0^{\theta(p(x_0,Tx_0))} \phi(t)\,dt\right)^k.$$

If $x_1 \notin A$, then x_1 is a fixed point of T. Suppose $x_1 \in A$. Then there exists $x_2 \in X$ such that

$$\int_0^{p(x_1,x_2)} \phi(t)\,dt \le a\left(\int_0^{p(x_1,Tx_1)} \phi(t)\,dt\right)^s$$

$$\text{and} \quad \int_0^{\theta(p(x_2,Tx_2))} \phi(t)\,dt \le \left(\int_0^{\theta(p(x_1,Tx_1))} \phi(t)\,dt\right)^k.$$

If $x_2 \notin A$, then x_2 is a fixed point of T. If $x_2 \in A$, then continuing in the above way, we can generate a sequence $\{x_n\}$ in X such that $x_n \in A$ with

$$\int_0^{p(x_n,x_{n+1})} \phi(t)\,dt \le a\left(\int_0^{p(x_n,Tx_n)} \phi(t)\,dt\right)^s \tag{1}$$

$$\text{and} \quad \int_0^{\theta(p(x_{n+1},Tx_{n+1}))} \phi(t)\,dt \le \left(\int_0^{\theta(p(x_n,Tx_n))} \phi(t)\,dt\right)^k \quad \text{for all } n \in \mathbb{N}\cup\{0\}.$$

Using (A_1),

$$\begin{aligned} 1 &\le \int_0^{\theta(p(x_{n+1},Tx_{n+1}))} \phi(t)\,dt \le \left(\int_0^{\theta(p(x_n,Tx_n))} \phi(t)\,dt\right)^k \\ &\le \left(\int_0^{\theta(p(x_{n-1},Tx_{n-1}))} \phi(t)\,dt\right)^{k^2} \le \ldots\ldots \le \left(\int_0^{\theta(p(x_0,Tx_0))} \phi(t)\,dt\right)^{k^{n+1}}. \end{aligned} \tag{2}$$

Since $k \in [0,1)$, $\displaystyle\lim_{n\to\infty} \int_0^{\theta(p(x_n,Tx_n))} \phi(t)\,dt = 1$.
By (A_2),

$$\lim_{n\to\infty} p(x_n,Tx_n) = 0^+. \tag{3}$$

Then by (A_3), there exist $r \in (0,\infty)$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\int_0^{\theta(p(x_n,Tx_n))} \phi(t)\,dt - 1}{\left(\int_0^{p(x_n,Tx_n)} \phi(t)\,dt\right)^r} = l\,.$$

Suppose $l < \infty$, then there is some $k_0 \in \mathbb{N}$ such that

$$\left|\frac{\int_0^{\theta(p(x_n,Tx_n))} \phi(t)\,dt - 1}{\left(\int_0^{p(x_n,Tx_n)} \phi(t)\,dt\right)^r} - l\right| \le \frac{l}{2} \quad \text{for all } n \ge k_0,$$

$$\text{i.e.,} \quad \frac{\int_0^{\theta(p(x_n,Tx_n))} \phi(t)\,dt - 1}{\left(\int_0^{p(x_n,Tx_n)} \phi(t)\,dt\right)^r} \ge \frac{l}{2} \quad \text{for all } n \ge k_0,$$

and so,

$$\left(\int_0^{p(x_n,Tx_n)} \phi(t)\,dt\right)^r \le \frac{2}{l}\left(\int_0^{\theta(p(x_n,Tx_n))} \phi(t)\,dt - 1\right).$$

Again, suppose $l = \infty$. Let $M > 0$. There exists $k_1 \in \mathbb{N}$ such that

$$\frac{\int_0^{\theta(p(x_n,Tx_n))} \phi(t)\,dt - 1}{\left(\int_0^{p(x_n,Tx_n)} \phi(t)\,dt\right)^r} \geq M \quad \text{for all } n \geq k_1.$$

Thus, for $l \in (0, \infty]$, there exist $C > 0$ and $k_2 \in \mathbb{N}$ such that

$$\left(\int_0^{p(x_n,Tx_n)} \phi(t)\,dt\right)^r \leq C\left(\int_0^{\theta(p(x_n,Tx_n))} \phi(t)\,dt - 1\right) \text{ for all } n \geq k_2,$$

and so,

$$\left(\int_0^{p(x_n,Tx_n)} \phi(t)\,dt\right)^r \leq C\left[\left(\int_0^{\theta(p(x_0,Tx_0))} \phi(t)\,dt\right)^{k^n} - 1\right]. \quad \text{(by (2))}$$

Since $\lim_{n\to\infty} \left[\left(\int_0^{\theta(p(x_0,Tx_0))} \phi(t)\,dt\right)^{k^n} - 1\right] = 0,$

so, $\lim_{n\to\infty} \left(\int_0^{p(x_n,Tx_n)} \phi(t)\,dt\right)^r = 0.$

Thus, there exists $k_3 \in \mathbb{N}$ such that

$$\int_0^{p(x_n,Tx_n)} \phi(t)\,dt \leq \frac{1}{n^{1/r}} \text{ for all } n \geq k_3. \tag{4}$$

From (1) and (4), we get,

$$\int_0^{p(x_n,x_{n+1})} \phi(t)\,dt \leq \frac{a}{n^{s/r}} \text{ for all } n \geq k_3.$$

Now, for $m > n \geq k_3$,

$$\int_0^{p(x_n,x_m)} \phi(t)\,dt \leq \int_0^{p(x_n,x_{n+1})} \phi(t)\,dt + \int_0^{p(x_{n+1},x_{n+2})} \phi(t)\,dt + \ldots$$
$$\ldots + \int_0^{p(x_{m-1},x_m)} \phi(t)\,dt$$
$$= \sum_{i=n}^{m-1} \int_0^{p(x_i,x_{i+1})} \phi(t)\,dt \leq \; a \sum_{i=n}^{\infty} \frac{1}{i^{s/r}}\,.$$

Since $s > r$, $\lim_{n\to\infty} \int_0^{p(x_n,x_m)} \phi(t)\,dt = 0$, and so, using (Φ_3), $p(x_n, x_m) \to 0$ as $m, n \to \infty$, i.e., $\{x_n\}$ is a Cauchy sequence in (X, p) and so, converges to some $u \in X$. Therefore,

$$\lim_{n\to\infty} p(x_n, u) = p(u, u) = \lim_{n\to\infty} p(x_n, x_n).$$

Since $x \to p(x, Tx)$ is lower semi-continuous and $x_n \to u$ as $n \to \infty$, we have,

$p(u, Tu) \leq \lim_{n\to\infty} \inf p(x_n, Tx_n)$

i.e., $p(u, Tu) = 0 \quad$ (by (3))

and so, $u \in Tu$. $\square$

The following example exhibits the above Theorem.

Example 3.1. Let $X = [0, \infty)$, and define a complete partial metric p on X by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $T : X \to CB^p(X)$ be defined by

$$Tx = \begin{cases} [0, x], & \text{if } x \in [0, 1) \\ [nx, (n+2)x], & \text{if } x \in [n, n+2),\ n = 1, 3, 5, \dots \end{cases} \tag{5}$$

Assume $\theta(t) = 2^{\sqrt{t}}$ with $r = \frac{1}{4}$, $a = 1 = s$, $k = \frac{1}{3}$ and $\phi(t) = t^2 + t + \frac{1}{6}$ for all $t \in \mathbb{R}^+ \cup \{0\}$.

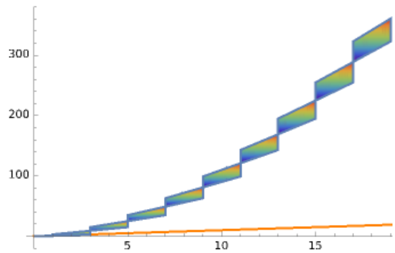

FIGURE 1. Graphical representation of fixed point.

Case 1: For $x \in [0, 1)$, $p(x, Tx) = x$.
We take $y = 0$. Then

$$\int_0^{p(x,y)} \phi(t)\, dt = \int_0^x (t^2 + t + \frac{1}{6})\, dt = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}$$

and

$$a\left(\int_0^{p(x,Tx)} \phi(t)\, dt\right)^s = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}.$$

Also,

$$\int_0^{\theta(p(y,Ty))} \phi(t)\, dt = \int_0^{\theta(0)} (t^2 + t + \frac{1}{6})\, dt = 1$$

and

$$\left(\int_0^{\theta(p(x,Tx))} \phi(t)\, dt\right)^k = \left(\frac{2^{3\sqrt{x}}}{3} + \frac{2^{2\sqrt{x}}}{2} + \frac{2^{\sqrt{x}}}{6}\right)^{1/3}.$$

Case 2: For $x \in [n, n+2)$, where $n = 1, 3, 5...$, we have, $p(x, Tx) = nx$.
For $y = 0$,

$$\int_0^{p(x,y)} \phi(t)\, dt = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}$$

and

$$a\left(\int_0^{p(x,Tx)} \phi(t)\, dt\right)^s = \frac{n^3x^3}{3} + \frac{n^2x^2}{2} + \frac{nx}{6}.$$

Also,

$$\int_0^{\theta(p(y,Ty))} \phi(t)\, dt = 1,$$

and

$$\left(\int_0^{\theta(p(x,Tx))}\phi(t)\,dt\right)^k=\left(\frac{2^{3\sqrt{nx}}}{3}+\frac{2^{2\sqrt{nx}}}{2}+\frac{2^{\sqrt{nx}}}{6}\right)^{1/3}.$$

Clearly, all the conditions of Theorem 3.1 are satisfied and so, T has fixed point. Here, all the points lying on $[0,3]$ are the fixed points of T which can be seen from the graphical representation of multi-valued mapping T, i.e., (5) and $Tx=x$ (the orange line) as shown in Figure 1.

Remark 3.1. Theorem 3.1 does not guarantee the uniqueness of the fixed point which is clear from the above example.

Next we establish a common fixed point result.

Theorem 3.2. *Let (X,p) be a complete partial metric space and $S,T:X\to CB^p(X)$ are multi-valued mappings such that $x\to p(x,Sx)$ and $x\to p(x,Tx)$ are lower semi-continuous. Assume that there exist real numbers $a,s,r\in(0,\infty)$ with $s>r$, $l\in(0,\infty]$, $k\in[0,1)$ and $\theta\in\Theta$, $\phi\in\Phi$ satisfying (A_1), (A_2), (A_3) of Theorem 3.1. If for any $x\in X$ with $\max\{p(x,Sx),p(x,Tx)\}>0$, there is $y\in X$ such that*

$$\int_0^{p(x,y)}\phi(t)\,dt\le a\left(\int_0^{\frac{1}{2}[p(x,Sx)+p(x,Tx)]}\phi(t)\,dt\right)^s$$

$$\text{and}\quad\int_0^{\theta(\frac{1}{2}[p(y,Sy)+p(y,Ty)])}\phi(t)\,dt\le\left(\int_0^{\theta(\frac{1}{2}[p(x,Sx)+p(x,Tx)])}\phi(t)\,dt\right)^k,$$

then there exists $u\in X$ such that $u\in CF(S,T)$, where $CF(S,T)$ denotes the set of common fixed points of S and T.

Proof. Let $A=\{x\in X:\max\{p(x,Sx),p(x,Tx)\}>0\}$. If $A=\emptyset$, then clearly $x\in CF(S,T)$. Suppose $A\neq\emptyset$. Let $x_0\in A$. Then there exists $x_1\in X$ such that

$$\int_0^{p(x_0,x_1)}\phi(t)\,dt\le a\left(\int_0^{\frac{1}{2}[p(x_0,Sx_0)+p(x_0,Tx_0)]}\phi(t)\,dt\right)^s$$

$$\text{and}\quad\int_0^{\theta(\frac{1}{2}[p(x_1,Sx_1)+p(x_1,Tx_1)])}\phi(t)\,dt\le\left(\int_0^{\theta(\frac{1}{2}[p(x_0,Sx_0)+p(x_0,Tx_0)])}\phi(t)\,dt\right)^k.$$

For $x_1\in A$, there exists $x_2\in X$ such that

$$\int_0^{p(x_1,x_2)}\phi(t)\,dt\le a\left(\int_0^{\frac{1}{2}[p(x_1,Sx_1)+p(x_1,Tx_1)]}\phi(t)\,dt\right)^s$$

$$\text{and}\quad\int_0^{\theta(\frac{1}{2}[p(x_2,Sx_2)+p(x_2,Tx_2)])}\phi(t)\,dt\le\left(\int_0^{\theta(\frac{1}{2}[p(x_1,Sx_1)+p(x_1,Tx_1)])}\phi(t)\,dt\right)^k.$$

Continuing in the above way, we can generate a sequence $\{x_n\}$ in X such that for $x_n\in X$, there exists $x_{n+1}\in X$ with

$$\int_0^{p(x_n,x_{n+1})}\phi(t)\,dt\le a\left(\int_0^{\frac{1}{2}[p(x_n,Sx_n)+p(x_n,Tx_n)]}\phi(t)\,dt\right)^s\tag{6}$$

$$\text{and} \int_0^{\theta(\frac{1}{2}[p(x_{n+1},Sx_{n+1})+p(x_{n+1},Tx_{n+1})])} \phi(t)\,dt$$
$$\leq \left(\int_0^{\theta(\frac{1}{2}[p(x_n,Sx_n)+p(x_n,Tx_n)])} \phi(t)\,dt\right)^k \quad \text{for all } n \in \mathbb{N}\cup\{0\}. \tag{7}$$

By (A_1) and (A_2), it can be shown that

$$\lim_{n\to\infty}[p(x_n,Sx_n)+p(x_n,Tx_n)] = 0,$$
$$\text{i.e.,} \lim_{n\to\infty} p(x_n,Sx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} p(x_n,Tx_n) = 0. \tag{8}$$

Proceeding similar to the previous theorem, we can show that there exist $k_3,\ k_4 \in \mathbb{N}$ such that

$$\int_0^{p(x_n,Sx_n)} \phi(t)\,dt \leq \frac{1}{n^{1/r}} \text{ for all } n \geq k_3 \tag{9}$$

and

$$\int_0^{p(x_n,Tx_n)} \phi(t)\,dt \leq \frac{1}{n^{1/r}} \text{ for all } n \geq k_4. \tag{10}$$

Taking $n_0 = \min\{k_3, k_4\}$ and using (6), (9), (10) we get,

$$\int_0^{p(x_n,x_{n+1})} \phi(t)\,dt \leq a\left(\int_0^{\frac{1}{2}p(x_n,Sx_n)} \phi(t)\,dt + \int_0^{\frac{1}{2}p(x_n,Tx_n)]} \phi(t)\,dt\right)^s$$
$$\leq \frac{a2^s}{n^{s/r}} \text{ for all } n \geq n_0. \tag{11}$$

Now, for $m > n \geq n_0$,

$$\int_0^{p(x_n,x_m)} \phi(t)\,dt \leq \sum_{i=n}^{m-1} \int_0^{p(x_i,x_{i+1})} \phi(t)\,dt \leq\ a2^s \sum_{i=n}^{\infty} \frac{1}{i^{s/r}}.$$

Since $s > r$, $\lim_{n\to\infty} \int_0^{p(x_n,x_m)} \phi(t)\,dt = 0$, and so, from (Φ_3), $p(x_n,x_m) \to 0$ as $m,n \to \infty$, i.e., $\{x_n\}$ is a Cauchy sequence in (X,p). Let $x_n \to u\ \in X$ as $n \to \infty$. Since $x \to p(x,Sx)$ and $x \to p(x,Tx)$ are lower semi-continuous, as in Theorem 3.1, we can show that u is a common fixed point of S and T. □

Example 3.2. Let $X = \{0,1,2\}$, and define a complete partial metric p on X by $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. Let $S,T : X \to CB^p(X)$ be defined by

$$Sx = \{x\} \text{ for all } x \in X \quad \text{and } Tx = \begin{cases} \{0\}, & \text{if } x = 0 \\ \{1\}, & \text{otherwise}. \end{cases}$$

Assume $\theta(t) = e^t$ with $r = \frac{1}{4}$, $a = 1 = s$, $k = \frac{1}{2}$ and $\phi(t) = t^2 + t + \frac{1}{6}$ for all $t \in \mathbb{R}^+ \cup \{0\}$.

We have,

$$p(2,S2) = 2,\ p(1,S1) = 1,\ p(0,S0) = 0,$$
$$p(2,T2) = 2,\ p(1,T1) = 1,\ p(0,T0) = 0.$$

Case 1: For $x = 2$, we take $y = 1$. Then

$$\int_0^{p(x,y)} \phi(t)\,dt = \int_0^2 \phi(t)\,dt = 5\,,$$

$$a\left\{\int_0^{\frac{1}{2}[p(x,Sx)+p(x,Tx)]} \phi(t)\,dt\right\}^s = \int_0^2 \phi(t)\,dt = 5\,,$$

and $$\int_0^{\theta(\frac{1}{2}[p(y,Sy)+p(y,Ty)])} \phi(t)\,dt = \int_0^{\theta(1)} \phi(t)\,dt = \frac{e^3}{3} + \frac{e^2}{2} + \frac{e}{6} = 10.84\,,$$

$$\left(\int_0^{\theta(\frac{1}{2}[p(x,Sx)+p(x,Tx)])} \phi(t)\,dt\right)^k = \left(\int_0^{\theta(2)} \phi(t)\,dt\right)^{\frac{1}{2}} = \left(\frac{e^6}{3} + \frac{e^4}{2} + \frac{e^2}{6}\right)^{\frac{1}{2}} = 12.76\,.$$

Case 2: For $x = 1$, taking $y = 0$, we get,

$$\int_0^{p(x,y)} \phi(t)\,dt = \int_0^1 \phi(t)\,dt = 1\,,$$

$$a\left\{\int_0^{\frac{1}{2}[p(x,Sx)+p(x,Tx)]} \phi(t)\,dt\right\}^s = \int_0^1 \phi(t)\,dt = 1\,,$$

and $$\int_0^{\theta(\frac{1}{2}[p(y,Sy)+p(y,Ty)])} \phi(t)\,dt = \int_0^{\theta(0)} \phi(t)\,dt = 1\,,$$

$$\left(\int_0^{\theta(\frac{1}{2}[p(x,Sx)+p(x,Tx)])} \phi(t)\,dt\right)^k = \left(\int_0^{\theta(1)} \phi(t)\,dt\right)^{\frac{1}{2}} = \left(\frac{e^3}{3} + \frac{e^2}{2} + \frac{e}{6}\right)^{\frac{1}{2}} = 3.29\,.$$

Case 3: For $x = 0$, let $y = 0$.

Now, $$\int_0^{p(x,y)} \phi(t)\,dt = 0\,,$$

$$a\left(\int_0^{\frac{1}{2}[p(x,Sx)+p(x,Tx)]} \phi(t)\,dt\right)^s = 0$$

and $$\int_0^{\theta(\frac{1}{2}[p(y,Sy)+p(y,Ty)])} \phi(t)\,dt = \int_0^{\theta(0)} \phi(t)\,dt = 1\,,$$

$$\left(\int_0^{\theta(\frac{1}{2}[p(x,Sx)+p(x,Tx)]} \phi(t)\,dt\right)^k = \left(\int_0^{\theta(0)} \phi(t)\,dt\right)^{\frac{1}{2}} = 1\,.$$

Thus, all the conditions of Theorem 3.2 are satisfied and clearly 0 and 1 are the common fixed points of S and T here.

The concept of coupled fixed point was introduced by Guo and Lakshmikantham in 1987 [15], in the context of coupled quasi-solutions of the initial value problems for ordinary differential equations. In a partial metric space (X, p), an element $(x, y) \in X \times X$ is said to be a coupled fixed point of $T : X \times X \to P(X)$ if $x \in T(x, y)$ and $y \in T(y, x)$. In the following, we derive a coupled fixed point theorem using integral inequalities.

Theorem 3.3. *Let (X, p) be a complete partial metric space and $T : X \times X \to CB^p(X)$ be a multi-valued mapping such that $(x, y) \to p(x, T(x, y))$ and $(y, x) \to$*

$p(y, T(y, x))$ are lower semi-continuous functions. Assume that there exist real numbers $a, s, r \in (0, \infty)$ with $s > r$, $l \in (0, \infty]$, $k \in [0, 1)$ and $\theta \in \Theta$, $\phi \in \Phi$ satisfying the conditions (A_1), (A_2), (A_3) of Theorem 3.1.
If for any $(x, y) \in X \times X$ with $\max\{p(x, T(x, y)), p(y, T(y, x))\} > 0$, there is $(u, v) \in X \times X$ such that

$$\int_0^{\frac{1}{2}[p(x,u)+p(y,v)]} \phi(t)\,dt \le a\left(\int_0^{\frac{1}{2}[p(x,T(x,y))+p(y,T(y,x))]} \phi(t)\,dt\right)^s \tag{12}$$

and $$\int_0^{\theta(\frac{1}{2}[p(u,T(u,v))+p(v,T(v,u))])} \phi(t)\,dt \le \left(\int_0^{\theta(\frac{1}{2}[p(x,T(x,y))+p(y,T(y,x))])} \phi(t)\,dt\right)^k, \tag{13}$$

then there exists $(w, z) \in X \times X$ such that $w \in T(w, z)$ and $z \in T(z, w)$.

Proof. Let $A = \{(x, y) \in X \times X : \max\{p(x, T(x, y)), p(y, T(y, x))\} > 0\}$. If $A = \emptyset$, then T has a coupled fixed point in $X \times X$. Let $A \neq \emptyset$ and $(x_0, y_0) \in A$. Then there exists $(x_1, y_1) \in X \times X$ such that

$$\int_0^{\frac{1}{2}[p(x_0,x_1)+p(y_0,y_1)]} \phi(t)\,dt \le a\left(\int_0^{\frac{1}{2}[p(x_0,T(x_0,y_0))+p(y_0,T(y_0,x_0))]} \phi(t)\,dt\right)^s$$

and

$$\int_0^{\theta(\frac{1}{2}[p(x_1,T(x_1,y_1))+p(y_1,T(y_1,x_1))])} \phi(t)\,dt \le \left(\int_0^{\theta(\frac{1}{2}[p(x_0,T(x_0,y_0))+p(y_0,T(y_0,x_0))])} \phi(t)\,dt\right)^k.$$

If $(x_1, y_1) \notin A$, then (x_1, y_1) is a coupled fixed point. Let $(x_1, y_1) \in A$. Then there exists $(x_2, y_2) \in X \times X$ such that

$$\int_0^{\frac{1}{2}[p(x_1,x_2)+p(y_1,y_2)]} \phi(t)\,dt \le a\left(\int_0^{\frac{1}{2}[p(x_1,T(x_1,y_1))+p(y_1,T(y_1,x_1))]} \phi(t)\,dt\right)^s$$

and

$$\int_0^{\theta(\frac{1}{2}[p(x_2,T(x_2,y_2))+p(y_2,T(y_2,x_2))])} \phi(t)\,dt \le \left(\int_0^{\theta(\frac{1}{2}[p(x_1,T(x_1,y_1))+p(y_1,T(y_1,x_1))])} \phi(t)\,dt\right)^k.$$

Continuing in the above way, we can generate sequences $\{x_n\}$ and $\{y_n\}$ in X such that $(x_n, y_n) \in A$ with

$$\int_0^{\frac{1}{2}[p(x_n,x_{n+1})+p(y_n,y_{n+1})]} \phi(t)\,dt \le a\left(\int_0^{\frac{1}{2}[p(x_n,T(x_n,y_n))+p(y_n,T(y_n,x_n))]} \phi(t)\,dt\right)^s \tag{14}$$

$$\text{and} \int_0^{\theta(\frac{1}{2}[p(x_{n+1},T(x_{n+1},y_{n+1}))+p(y_{n+1},T(y_{n+1},x_{n+1}))])} \phi(t)\,dt$$
$$\leq \left(\int_0^{\theta(\frac{1}{2}[p(x_n,T(x_n,y_n))+p(y_n,T(y_n,x_n))])} \phi(t)\,dt\right)^k. \quad (15)$$

Proceeding as in Theorem 3.1, it can be shown that

$$\lim_{n\to\infty}[p(x_n,T(x_n,y_n))+p(y_n,T(y_n,x_n))]=0,$$
$$\text{i.e.,} \lim_{n\to\infty} p(x_n,T(x_n,y_n))=0, \quad \text{and} \quad \lim_{n\to\infty} p(y_n,T(y_n,x_n))=0 \quad (16)$$

and there exist $k_3, k_4 \in \mathbb{N}$ such that

$$\int_0^{p(x_n,T(x_n,y_n))} \phi(t)\,dt \leq \frac{1}{n^{1/r}} \text{ for all } n \geq k_3 \quad (17)$$
$$\text{and} \int_0^{p(y_n,T(y_n,x_n))} \phi(t)\,dt \leq \frac{1}{n^{1/r}} \text{ for all } n \geq k_4. \quad (18)$$

Take $n_0 = \min\{k_3, k_4\}$ and using (14), (17), (18), we get,

$$\int_0^{\frac{1}{2}[p(x_n,x_{n+1})+p(y_n,y_{n+1})]} \phi(t)\,dt \leq a(\int_0^{\frac{1}{2}p(x_n,T(x_n,y_n))} \phi(t)\,dt$$
$$+\int_0^{\frac{1}{2}p(y_n,T(y_n,x_n))} \phi(t)\,dt)^s$$
$$\leq \frac{a2^s}{n^{s/r}} \text{ for all } n \geq n_0.$$

For $m > n \geq n_0$,

$$\int_0^{\frac{1}{2}p(x_n,x_m)} \phi(t)\,dt \leq \int_0^{\frac{1}{2}p(x_n,x_{n+1})} \phi(t)\,dt + \int_0^{\frac{1}{2}p(x_{n+1},x_{n+2})} \phi(t)\,dt + \ldots$$
$$\ldots + \int_0^{\frac{1}{2}p(x_{m-1},x_m)} \phi(t)\,dt$$

$$\leq \int_0^{\frac{1}{2}[p(x_n,x_{n+1})+p(y_{n+1},y_{n+2})]} \phi(t)\,dt + \int_0^{\frac{1}{2}[p(x_{n+1},x_{n+2})+p(y_{n+1},y_{n+2})]} \phi(t)\,dt + \ldots$$
$$\ldots + \int_0^{\frac{1}{2}[p(x_{m-1},x_m)+p(y_{m-1},y_m)]} \phi(t)\,dt$$

$$= \sum_{i=n}^{m-1} \int_0^{\frac{1}{2}[p(x_i,x_{i+1})+p(y_i,y_{i+1})]} \phi(t)\,dt \leq 2^s a \sum_{i=n}^{\infty} \frac{1}{i^{s/r}}.$$

Therefore, $\lim_{n\to\infty} \int_0^{\frac{1}{2}p(x_n,x_m)} \phi(t)\,dt = 0$, and so, from (Φ_3), $p(x_n,x_m) \to 0$ as $m,n \to \infty$, i.e., $\{x_n\}$ is a Cauchy sequence in (X,p) and converges to some $w \in X$.

Similarly, $\{y_n\}$ is a Cauchy sequence in (X, p) converging to some $z \in X$. Therefore,

$$\lim_{n\to\infty} p(x_n, w) = p(w, w) = \lim_{n\to\infty} p(x_n, x_n),$$
$$\text{and } \lim_{n\to\infty} p(y_n, z) = p(z, z) = \lim_{n\to\infty} p(y_n, y_n).$$

Since $(x, y) \to p(x, T(x, y))$ and $(y, x) \to p(y, T(y, x))$ are lower semi-continuous and $x_n \to w$, $y_n \to z$, using (16), we have,

$$p(w, T(w, z)) \leq \lim_{n\to\infty} p(x_n, T(x_n, y_n)) = 0 \text{ i.e., } w \in T(w, z)$$
$$\text{and } p(z, T(z, w)) \leq \lim_{n\to\infty} p(y_n, T(y_n, x_n)) = 0 \text{ i.e., } z \in T(z, w).$$

Thus, (w, z) is a coupled fixed point of T. $\square$

We illustrate the above theorem by the following example.

Example 3.3. Let $X = [0, 1]$ with a complete partial metric $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $T : X \times X \to CB^p(X)$ be defined by

$$T(x, y) = \left\{\frac{x + y}{2}\right\} \quad \text{for all} \quad x, y \in X.$$

Assume $\theta(t) = e^t$ with $r = \frac{1}{2}$, $a = 1 = s$, $k = \frac{1}{3}$ and $\phi(t) = t^2 + t + \frac{1}{6}$ for all $t \in \mathbb{R}^+ \cup \{0\}$.

For any $(x, y) \in X \times X$, we take $(u, v) = (0, 0)$.

$$\int_0^{\frac{1}{2}[p(x,u)+p(y,v)]} \phi(t)\, dt = \int_0^{\frac{1}{2}(x+y)} (t^2 + t + \frac{1}{6})\, dt, \tag{19}$$

$$a\left(\int_0^{\frac{1}{2}[p(x,T(x,y))+p(y,T(y,x))]} \phi(t)\, dt\right)^s = \int_0^{\frac{1}{2}[\max\{x,\frac{x+y}{2}\}+\max\{y,\frac{x+y}{2}\}]} (t^2 + t + \frac{1}{6})\, dt. \tag{20}$$

$$\text{Again, } \int_0^{\theta(\frac{1}{2}[p(u,T(u,v))+p(v,T(v,u))])} \phi(t)\, dt = \int_0^{\theta(0)} (t^2 + t + \frac{1}{6})\, dt = 1, \tag{21}$$

$$\left(\int_0^{\theta(\frac{1}{2}[p(x,T(x,y))+p(y,T(y,x))])} \phi(t)\, dt\right)^k = \left(\int_0^{\theta(\frac{1}{2}[\max\{x,\frac{x+y}{2}\}+\max\{y,\frac{x+y}{2}\}])} \phi(t)\, dt\right)^{\frac{1}{2}}. \tag{22}$$

The two graphs depict the conditions of Theorem 3.3. In Figure 2, the brown and blue surfaces correspond to (20) and (19), that is, the RHS and LHS of condition (12) respectively. Similarly, in Figure 3, the red and green surfaces correspond to (22) and (21), that is, the RHS and LHS of condition (13) respectively. Consequently, all the conditions of Theorem 3.3 are satisfied. Thus, T possesses coupled fixed points, which are clearly the set of points $\{(x, x) : x \in [0, 1]\}$ here.

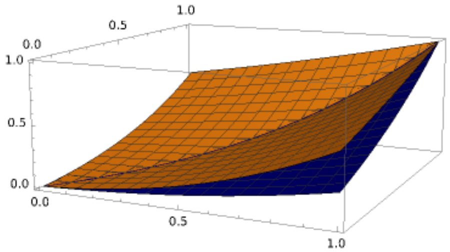

FIGURE 2.

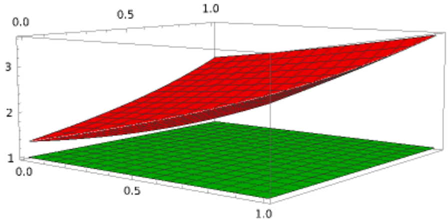

FIGURE 3.

4. Application to functional equations arising in dynamical programming

In this section, we find the existence of a common solution for a system of functional equations arising in dynamic programming with the help of Theorem 3.2. For Banach spaces U and V, let $W \subset U$ be a state space and $D \subset V$ denote the decision space. Let $C(W)$ denote the space of all continuous real valued functions on W. Then $C(W)$ is a partial metric space with respect to the partial metric $p(u,v) = \sup_{x\in W} |u(x) - v(x)|$, $u, v \in C(W)$. We consider the following system of functional equations:

$$\begin{aligned} F(x) &= \sup_{y\in D}\{H(x,y,h(\mu(x,y))\} \\ \text{and } G(x) &= \sup_{y\in D}\{K(x,y,g(\mu(x,y))\} \quad \text{for all } x \in W, \end{aligned} \tag{23}$$

where x and y represent the state and decision vectors respectively, μ denote the transformations of the process, H and K are bounded functions from $W \times D \times \mathbb{R}$ to $\mathbb{R}$, $F(x)$ and $G(x)$ denote the maximum return functions with the initial state x and $h, g \in C(W)$.

For $X = C(W)$, let $S, T : X \to CB^p(X)$ be two multi-valued mappings defined by:

$$\begin{aligned} Sh(x) &= \{u : u(x) = \sup_{y\in D}\{H(x,y,h(\mu(x,y)))\}\} \\ \text{and } Tg(x) &= \{z : z(x) = \sup_{y\in D}\{K(x,y,g(\mu(x,y)))\}\}, \end{aligned} \tag{24}$$

where $x \in W$ and $h, g \in X$.

Theorem 4.1. *Let $H, K : W \times D \times \mathbb{R} \to \mathbb{R}$ be two bounded functions and for $X = C(W)$, let $S, T : X \to CB^p(X)$ be two multi-valued mappings defined by (24). Assume that the following conditions hold:*
for each $h \in X$, there exists $g \in X$ such that

(i) $\int_0^{|h(x)-g(x)|} \phi(t)\,dt \le a \left(\int_0^{\Delta_1(x,y,h)} \phi(t)\,dt\right)^s$,

(ii) $\int_0^{e^{\Delta_2(x,y,g)}} \phi(t)\,dt \le \left(\int_0^{e^{\Delta_1(x,y,h)}} \phi(t)\,dt\right)^k$,
where $\Delta_1(x,y,h) = \frac{1}{2}\{|H(x,y,h(\mu(x,y)))| + |K(x,y,h(\mu(x,y)))| - 2|h(x)|\}$,
$\Delta_2(x,y,g) = \frac{1}{2}\{|H(x,y,g(\mu(x,y)))| + |K(x,y,g(\mu(x,y)))| + 2|g(x)|\}$

for $x \in W,\ y \in D,\ a, s \in (0,\infty),\ k \in [0,1)$ *and* $\phi \in \Phi$ *satisfying* $(A_1),\ (A_2),\ (A_3)$ *of Theorem 3.1.*

Then there exists a solution of the system of functional equations (23).

Proof. Since H and K are bounded, there exists $M > 0$ such that

$$\sup\{|H(x,y,t)|, |K(x,y,t)| : (x,y,t) \in W \times D \times \mathbb{R}\} \leq M.$$

Let $h \in X$ and $u \in Sh,\ v \in Th$ be arbitrary.
Then $u(x) = \sup_{y \in D}\{H(x,y,h(\mu(x,y))\}$ and $v(x) = \sup_{y \in D}\{K(x,y,h(\mu(x,y))\}$
and so,

$$u(x) \geq H(x,y,h(\mu(x,y))) \text{ and } v(x) \geq K(x,y,h(\mu(x,y))) \text{ for all } y \in D. \tag{25}$$

Now, for $x \in W$,

$$\begin{aligned}\Delta_1(x,y,h) &\leq \frac{1}{2}\{|u(x)| + |v(x)| - 2|h(x)|\} \quad (\text{using } (25))\\ &= \frac{1}{2}\{|u(x)| - |h(x)| + |v(x)| - |h(x)|\}.\end{aligned}$$

$$\begin{aligned}\text{Thus,}\ \ \Delta_1(x,y,h) &\leq \frac{1}{2}\{\sup_{x \in W}|u(x) - h(x)| + \sup_{x \in W}|v(x) - h(x)|\}\\ &= \frac{1}{2}\{p(u,h) + p(v,h)\}.\end{aligned} \tag{26}$$

From (i), we get,

$$\begin{aligned}\int_0^{|h(x)-g(x)|} \phi(t)\,dt &\leq a\left(\int_0^{\Delta_1(x,y,h)} \phi(t)\,dt\right)^s\\ &\leq a\left(\int_0^{\frac{1}{2}\{p(u,h)+p(v,h)\}} \phi(t)\,dt\right)^s. \quad (\text{using } (26))\end{aligned}$$

Since u and v are arbitrary, so,

$$\int_0^{|h(x)-g(x)|} \phi(t)\,dt \leq a\left(\int_0^{\frac{1}{2}\{\inf_{u \in Sh} p(u,h) + \inf_{v \in Th} p(v,h)\}} \phi(t)\,dt\right)^s$$

$$\text{i.e.,}\ \int_0^{p(h,g)} \phi(t)\,dt \leq a\left(\int_0^{\frac{1}{2}\{p(Sh,h)+p(Th,h)\}} \phi(t)\,dt\right)^s.$$

Again let $g \in X$ and $w \in Sg,\ z \in Tg$ be arbitrary. Then for $x \in W$,

$$w(x) = \sup_{y \in D}\{H(x,y,g(\mu(x,y))\} \text{ and } z(x) = \sup_{y \in D}\{K(x,y,g(\mu(x,y))\}$$

and so,

$$w(x) \geq H(x,y,g(\mu(x,y))) \text{ and } z(x) \geq K(x,y,g(\mu(x,y))) \text{ for all } y \in D. \tag{27}$$

Again from (ii),

$$\int_0^{e^{\Delta_2(x,y,g)}} \phi(t)\,dt \leq \left(\int_0^{e^{\Delta_1(x,y,h)}} \phi(t)\,dt\right)^k.$$

So, $\displaystyle\int_0^{e^{\sup_{y\in D}\Delta_2(x,y,g)}}\phi(t)\,dt \le \left(\int_0^{e^{\sup_{y\in D}\Delta_1(x,y,h)}}\phi(t)\,dt\right)^k,$

i.e., $\displaystyle\int_0^{e^{\frac{1}{2}\{|w(x)|+|z(x)|+2|g(x)|\}}}\phi(t)\,dt \le \left(\int_0^{e^{\frac{1}{2}\{|u(x)|+|v(x)|-2|h(x)|\}}}\phi(t)\,dt\right)^k.$

$$\begin{aligned}\text{Now, } \int_0^{e^{\frac{1}{2}\{|w(x)-g(x)|+|z(x)-g(x)|\}}}\phi(t)\,dt &\le \int_0^{e^{\frac{1}{2}\{|w(x)|+|z(x)|+2|g(x)|\}}}\phi(t)\,dt\\ &\le \left(\int_0^{e^{\frac{1}{2}\{|u(x)-h(x)|+|v(x)-h(x)|\}\}}}\phi(t)\,dt\right)^k.\end{aligned}$$

Again,

$$\sup_{x\in W}\int_0^{e^{\frac{1}{2}\{|w(x)-g(x)|+|z(x)-g(x)|\}}}\phi(t)\,dt \le \sup_{x\in W}\left(\int_0^{e^{\frac{1}{2}\{|u(x)-h(x)|+|v(x)-h(x)|\}\}}}\phi(t)\,dt\right)^k,$$

i.e., $\displaystyle\int_0^{e^{\frac{1}{2}\{p(w,g)+p(z,k)\}}}\phi(t)\,dt \le \left(\int_0^{e^{\frac{1}{2}\{p(u,h)+p(v,h)\}}}\phi(t)\,dt\right)^k.$

Since u, v, w and z are arbitrary, so,

$$\begin{aligned}&\int_0^{e^{\frac{1}{2}\{\inf_{w\in Sg}p(w,g)+\inf_{z\in Tg}p(z,g)\}}}\phi(t)\,dt\\ &\qquad\le \left(\int_0^{e^{\frac{1}{2}\{\inf_{u\in Sh}p(u,h)+\inf_{v\in Th}p(v,h)\}}}\phi(t)\,dt\right)^k,\end{aligned}$$

$$\text{i.e., } \int_0^{\theta[\frac{1}{2}\{p(Sg,g)+p(Tg,g)\}]}\phi(t)\,dt \le \left(\int_0^{\theta[\frac{1}{2}\{p(Sh,h)+p(Th,h)\}]}\phi(t)\,dt\right)^k.$$

Thus, S and T satisfy all the conditions of Theorem 3.2 with $\theta(t) = e^t$, $t \in [0,\infty)$. Hence, S and T have a common fixed point and thus the system (23) has a solution. □

5. Conclusion

We have deduced fixed point results for multi-valued mappings satisfying Feng-Liu type integral inequality in partial metric space. In all the results, the completeness of partial metric space is being considered. The validity of our results in case of incomplete partial metric space is a scope for further research. In [22], Majid et al. obtained some fixed point results for monotone multi-valued mappings considering partially ordered complete D^*-metric space using integral type contractive conditions. Similar investigation can be done for the integral type mappings depicted in this paper in the context of partially ordered D^*-metric space. Moreover, the applicability of our results in case of system of fractional order integral inclusions is another area of further discussion.

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