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Ground State Solutions for a Kirchhoff Type Equation Involving p-Biharmonic Operator with Exponential Growth non-linearity

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Abstract. In this article, we study the following non local weighted problem

$$g\Big(\int_B (w(x)|\Delta u|^{\frac{N}{2}})dx\Big)\Delta(w(x)|\Delta u|^{\frac{N}{2}-2}\Delta u) = |u|^{q-2}u + f(x,u) \quad \text{in } B, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B,$$

where B is the unit ball in $\mathbb{R}^N$ and $w(x)$ is a singular weight of logarithm type. The non-linearity is a combination of a reaction source $f(x,u)$ which is critical in view of exponential inequality of Adams' type and a polynomial function. The Kirchhoff function g is positive and continuous. The energy function loses compactness in the critical case. To remedy this, we introduce a new asymptotic condition for non-linearity and go through an intermediate problem. Using the Nehari manifold method, the quantitative deformation lemma and results from degree theory, we establish the existence of a ground-state solution.

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1. Introduction and Main results

In this paper, we consider the non local fourth order weighted elliptic equation:

$$(P)\quad \begin{cases} g\big(\int_B (v_\beta(x)|\Delta u|^{\frac{N}{2}})dx\big)\Delta(v_\beta(x)|\Delta u|^{\frac{N}{2}-2}\Delta u) = |u|^{q-2}u + f(x,u) & \text{in } B \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where $B = B(0,1)$ is the unit open ball in $\mathbb{R}^N$, $q > N$, $f(x,t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{\alpha t^{\frac{N}{(N-2)(1-\beta)}}\}$ as $|t| \to +\infty$, for some $\alpha > 0$ uniformly with respect to $x \in B$. The weight $v_\beta(x)$ is given by

$$v_\beta(x) = \left(\log \frac{e}{|x|}\right)^{\beta(\frac{N}{2}-1)}, \; \beta \in (0,1)\cdot \tag{1.2}$$

The Kirchhoff function g is positive, continuous and verifies some mild conditions.

The study of Kirchhoff problems was initiated in 1883, when Kirchhoff [22] studied the following equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = f(x,u), \tag{1.3}$$

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where ρ, P_0, h, E, L represent physical quantities. This model extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. We call (1.3) a nonlocal problem since the equation contains an integral over $[0, L]$ which makes the study of it interesting. After Lions in his pioneering work [25] presented an abstract functional analysis framework to (1.3). We mention that non-local problems also arise in other areas, for instance, biological systems where the function u describes a process that depends on the average of itself (for example, population density), see for instance [4, 5] and its references.

Recently, Trudinger-Moser inequalities [29, 31] have been extended to the Sobolev space with logarithmic weight

$$W^{1,N}_{0,rad}(B,\rho) = \text{closure}\{u \in C^{\infty}_{0,rad}(B) | \int_B |\nabla u|^N \varrho(x)dx < \infty\}.$$

The result was established by Calanchi and Ruff, that is

Theorem 1.1. [7]

(i) Let $\beta \in [0,1)$ and let ϱ given by $\varrho(x) = \left(\log \frac{1}{|x|}\right)^{\beta(N-1)}$, then

$$\int_B e^{|u|^{\gamma}} dx < +\infty, \forall u \in W^{1,N}_{0,rad}(B,\varrho), \text{ if and only if } \gamma \le \gamma_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta}$$

and

$$\sup_{\substack{u \in W^{1,N}_{0,rad}(B,\varrho) \\ \int_B |\nabla u|^N w(x)dx \le 1}} \int_B e^{\alpha |u|^{\gamma_{N,\beta}}} dx < +\infty \ \Leftrightarrow \ \alpha \le \alpha_{N,\beta} = N[\omega^{\frac{1}{N-1}}_{N-1}(1-\beta)]^{\frac{1}{1-\beta}}$$

where ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$ and N' is the Hölder conjugate of N.

(ii) Let ϱ given by $\varrho(x) = \left(\log \frac{e}{|x|}\right)^{N-1}$, then

$$\int_B exp\{e^{|u|^{\frac{N}{N-1}}}\}dx < +\infty, \ \forall \ u \in W^{1,N}_{0,rad}(B,\varrho)$$

and

$$\sup_{\substack{u \in W^{1,N}_{0,rad}(B,\varrho) \\ \|u\|_{\varrho} \le 1}} \int_B exp\{\beta e^{\omega^{\frac{1}{N-1}}_{N-1} |u|^{\frac{N}{N-1}}}\}dx < +\infty \ \Leftrightarrow \ \beta \le N,$$

where ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$ and N' is the Hölder conjugate of N and B is the unit ball in $\mathbb{R}^N$.

This result has allowed the study of elliptic problems with logarithmic weights involving exponential growth nonlinearities in the sense of Theorem 1.1 (see[1, 6, 8, 9, 11, 13, 15, 19, 21, 35]).

To give some motivation for our study, we give a brief overview of Adam's inequalities in a bounded domain of $\mathbb{R}^N$. We then proceed to discuss the extension of these inequalities to second-order Sobolev spaces with logarithmic weights.

The notion of critical exponential growth was extended to higher order Sobolev spaces by Adams' [2]. More precisely, Adams' proved the following result, for $m \in \mathbb{N}$

and Ω an open bounded set of $\mathbb{R}^N$ such that $m < N$, there exists a positive constant $C_{m,N}$ such that

$$\sup_{u\in W_0^{m,\frac{N}{m}}(\Omega),|\nabla^m u|_{\frac{N}{m}}\leq 1}\int_\Omega e^{\beta_0|u|^{\frac{N}{N-m}}}\ dx \leq C_{m,N}|\Omega|,$$

where $W_0^{m,\frac{N}{m}}(\Omega)$ denotes the m^{th}-order Sobolev space, $\nabla^m u$ denotes the m^{th}-order gradient of u, namely

$$\nabla^m u := \begin{cases} \Delta^{\frac{m}{2}} u, & \text{if } m \text{ is even} \\ \\ \nabla\Delta^{\frac{m-1}{2}} u, & \text{if } m \text{ odd} \end{cases}$$

and

$$\beta_0 = \beta_0(m,N) := \frac{N}{\omega_{N-1}}\begin{cases} \Big[\dfrac{\pi^{\frac{N}{2}}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{N-m}{2})}\Big]^{\frac{N}{N-m}}, & \text{if } m \text{ is even} \\ \\ \Big[\dfrac{\pi^{\frac{N}{2}}2^m\Gamma(\frac{m+1}{2})}{\Gamma(\frac{N-m+1}{2})}\Big]^{\frac{N}{N-m}}, & \text{if } m \text{ odd.} \end{cases}$$

Lately, there has been an extension of work concerning Adams' inequalities into Sobolev spaces incorporating logarithmic weights. Specifically, Wang and Zhu [34] established the following result.

Theorem 1.2. [34] *Let $\beta \in (0,1)$ and let $\omega_\beta(x) = (1-\log|x|)^\beta$, then*

$$\sup_{u\in W_{0,rad}^{2,2}(B_1,\omega_\beta),\|u\|\leq 1}\int_{B_1} e^{\alpha|u|^{\frac{2}{1-\beta}}}\ dx < \infty \ \Leftrightarrow\ \alpha \leq \alpha_\beta = 4[8\pi^2(1-\beta)]^{\frac{1}{1-\beta}},$$

where B_1 is the unit ball of $\mathbb{R}^4$, $W_{0,rad}^{2,2}(B_1,\omega_\beta)$ denotes the weighted Sobolev space of radial functions given by

$$W_{0,rad}^{2,2}(B_1,\omega_\beta) = closure\Big\{u \in C_{0,rad}^\infty(B_1)\ |\ \int_B \omega_\beta(x)|\Delta u|^2\ dx < \infty\Big\},$$

endowed with the norm $\|u\|_{W_{0,rad}^{2,2}(B_1,\omega_\beta)} = \Big(\int_{B_1}\omega_\beta(x)|\Delta u|^2\ dx\Big)^{\frac{1}{2}}$.

As an application of Theorem 1.2, Dridi and Jaidane [16] considered the following problem

$$\begin{cases} \Delta(\omega_\beta(x)\Delta u) - \Delta u + V(x)u &= f(x,u) \text{ in } B_1 \\ u = \frac{\partial u}{\partial n} &= 0 \text{ on } \partial B_1, \end{cases}$$

where $B_1 = B(0,1)$ is the unit open ball in $\mathbb{R}^4$, $f(x,t)$ is continuous in $B_1\times\mathbb{R}$ and behaves like $e^{\alpha t^{\frac{2}{1-\beta}}}$ as $t\to+\infty$, for some $\alpha > 0$, and the potential V is positive and continuous on $\overline{B_1}$ and bounded away from zero in B_1. The authors demonstrated the existence of a nontrivial weak solution to the mentioned problem by employing the Mountain Pass Theorem in conjunction with the logarithmic Adams inequality.

Similarly, Jaidane [20] applied these techniques to investigate the following Kirchhoff-type biharmonic problem

$$\begin{cases} L(u) &= f(x,u) \quad \text{in} \quad B_1 \\ u = \dfrac{\partial u}{\partial n} &= 0 \qquad\quad \text{on} \quad \partial B_1, \end{cases}$$

where

$$L(u) = m\big(\int_{B_1} (\omega_\beta |\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2)dx\big)\big[\Delta(\omega_\beta(x)|\Delta u|^2 - \Delta u + V(x)u\big],$$

m is a Kirchhoff function satisfying some mild conditions and the nonlinearity has exponential critical growth in the sense of Theorem 1.2.
For $g(t) = 1$ and $N = 4$, Dridi et al. treated a similar problem [13]. The authors proved the existence of a solution using the Nehari method.

Now we'll introduce our workspace. We denotes by $W^{2,\frac{N}{2}}_{0,rad}(B, v_\beta)$ the weighted Sobolev space of radial functions given by

$$\mathbf{W} := W^{2,\frac{N}{2}}_{0,rad}(B,\omega) = \text{closure}\Big\{u \in C^\infty_{0,rad}(B) \mid \int_B v_\beta(x)|\Delta u|^{\frac{N}{2}}\, dx < \infty\Big\},$$

with respect to the norm

$$\|u\|_{W_0^{2,\frac{N}{2}}(B,w)} = \big(\int_B v_\beta(x)|\Delta u|^{\frac{N}{2}} dx + \int_B |\nabla u|^{\frac{N}{2}} + \int_B |u|^{\frac{N}{2}} dx\big)^{\frac{2}{N}}.$$

The space $\mathbf{W}$ is endowed with the norm

$$\|u\| = \left(\int_B v_\beta(x)|\Delta u|^{\frac{N}{2}} dx\right)^{\frac{1}{2}}.$$

According to Drabek et al. and Kufner in [12, 24], $\mathbf{W}$ is a Banach and reflexive space.

Motivated by previous studies, we are investigating the existence of ground-state solutions. This exploration focuses in particular on scenarios where the non-linear terms exhibit critical exponential growth, as defined in the Adams inequalities [36].

Theorem 1.3. [36] *Let $\beta \in (0,1)$ and w be given by (1.2), then*

$$\sup_{\substack{u\in W^{2,\frac{N}{2}}_{0,rad}(B,w) \\ \int_B v_\beta(x)|\Delta u|^{\frac{N}{2}} dx \le 1}} \int_B e^{\alpha |u|^{\frac{N}{(N-2)(1-\beta)}}} dx < +\infty$$

$$\Leftrightarrow \alpha \le \alpha_\beta = N[(N-2)NV_N]^{\frac{2}{(N-2)(1-\beta)}}(1-\beta)^{\frac{1}{(1-\beta)}}, \tag{1.4}$$

where V_N is the volume of the unit ball B in $\mathbb{R}^N$.

Let $\gamma := \dfrac{N}{(N-2)(1-\beta)}$. According to inequality (1.7), we will say that f has critical growth at infinity if there exists some $\alpha_0 > 0$,

$$\lim_{s\to+\infty} \frac{|f(x,s)|}{e^{\alpha s^\gamma}} = 0, \ \forall\ \alpha \text{ such that } \alpha_0 < \alpha \text{ and } \lim_{s\to+\infty} \frac{|f(x,s)|}{e^{\alpha s^\gamma}} = +\infty, \ \forall\ \alpha < \alpha_0. \tag{1.5}$$

Now we define the Kirchhoff function g and give the conditions on it . The function g is continuous in $\mathbb{R}^+$ and verifies :

(G_1)**:** g is increasing with $g(0) = g_0 > 0$;

(G_2)**:** $\dfrac{g(t)}{t}$ is nonincreasing for $t>0$.

The assmption (G_2) implies that $\dfrac{g(t)}{t}\leq g(1)$ for all $t\geq 1$.
From (G_1) and (G_2), we can get

$$G(t+s)\geq G(t)+G(s)\ \forall\ s,t\geq 0 \text{ where } G(t)=\int_0^t g(s)ds \tag{1.6}$$

and

$$g(t)\leq g(1)+g(1)t, \quad \forall\ t\geq 0\cdot \tag{1.7}$$

As a consequence, we get

$$G(t)\leq g(1)t+\frac{g(1)}{2}t^2, \quad \forall\ t\geq 0\cdot \tag{1.8}$$

Moreover, we have that

$$\frac{2}{N}G(t)-\frac{1}{N}g(t)t \text{ is nondecreasing and positive for } t>0. \tag{1.9}$$

Consequently, one has for all $q\geq N$,

$$t\mapsto \frac{2}{N}G(t)-\frac{1}{q}g(t)t, \text{ is nondecreasing and positive for } t>0. \tag{1.10}$$

A typical example of a function g fulfilling the conditions (G_1) and (G_2) is given by

$$g(t)=g_0+at,\ g_0,a>0.$$

Another example is given by $g(t)=1+\ln(1+t)$.

Furthermore, we suppose that $f(x,t)$ has critical growth and satisfies the following hypothesis:

(A_1) $f:B\times\mathbb{R}\to\mathbb{R}$ is continuous and radial in x.

(A_2) There exist $\theta>q>N$ such that we have

$$0<\theta F(x,t)\leq tf(x,t),\forall(x,t)\in\ B\times\mathbb{R}\setminus\{0\}$$

where

$$F(x,t)=\int_0^t f(x,s)ds.$$

(A_3) For each $x\in B$, $t\mapsto\dfrac{f(x,t)}{|t|^{q-1}}$ is increasing for $t\in\ \mathbb{R}\setminus\{0\}$.

(A_4) $\lim\limits_{t\to 0}\dfrac{|f(x,t)|}{|t|^{\frac{N}{2}-1}}=0$.

(A_5) There exist p such that $p>q>N$ and $C_p>1$ such that

$$sgn(t)f(x,t)\geq C_p|t|^{p-1}, \quad \text{for all } (x,t)\in B\times\mathbb{R},$$

where $sgn(t)=1$ if $t>0$, $sgn(t)=0$ if $t=0$, and $sgn(t)=-1$ if $t<0$.

Remark 1.1. The conditions (A_2) and (A_3) imply that $t\mapsto\dfrac{f(x,t)}{t^{N-1}}$ is increasing for $t>0$.

We give an example of such nonlinearity. The nonlinearity $f(x,t)=C_p|t|^{p-\frac{N}{2}}t+|t|^{p-\frac{N}{2}}t\exp(\alpha_0|t|^{\gamma})$ satisfies the assumptions (A_1), (A_2), (A_3) ,(A_4) and (A_5).

We will consider the following definition of solutions.

Definition 1.1. We say that a function $u \in \mathbf{W}$ is a weak solution to problem (1.1) if

$$g(\|u\|^{\frac{N}{2}})\Big(\int_B \big(v_\beta(x)\,|\Delta u|^{\frac{N}{2}-2}\,\Delta u\,\Delta\varphi dx\big) = \int_B |u|^{q-2}u\varphi\,dx + \int_B f(x,u)\,\varphi\,dx,\ \forall\,\varphi \in \mathbf{W}.$$

Let $\mathcal{J} : \mathbf{W} \to \mathbb{R}$ be the energy functional given by

$$\mathcal{J}(u) = \frac{2}{N}G(\|u\|^{\frac{N}{2}}) - \frac{1}{q}\int_B |u|^q\,dx - \int_B F(x,u)\,dx, \tag{1.11}$$

where

$$F(x,t) = \int_0^t f(x,s)ds.$$

Note that, by the hypothesis (A_4), for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$|f(x,t)| \le \varepsilon |t|^{\frac{N}{2}-1},\ \forall\ 0 < |t| \le \delta_0,\ \text{uniformly in } x \in B. \tag{1.12}$$

Moreover, since f is critical at infinity, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\forall t \ge C_\varepsilon\ |f(x,t)| \le \varepsilon \exp(\ \alpha |t|^\gamma) \text{ with } \alpha > \alpha_0 \text{ uniformly in } x \in B. \tag{1.13}$$

In particular, we obtain for $r > 2$,

$$|f(x,t)t| \le \frac{\varepsilon}{C_\varepsilon^{q-1}}|t|^r \exp(\alpha\ |t|^\gamma) \text{ with } \alpha > \alpha_0 \text{ uniformly in } x \in B. \tag{1.14}$$

Hence, using (1.12), (1.13), (1.14) and the continuity of f, for every $\varepsilon > 0$, for every $r > 2$, there exist positive constants C and c such that

$$|f(x,t)| \le \varepsilon|t|^{\frac{N}{2}-1} + C|t|^{r-1}e^{\alpha\ |t|^\gamma},\ \forall\ (x,t) \in B \times \mathbb{R},\ \alpha > \alpha_0. \tag{1.15}$$

It follows from (1.15) and (A_2), that for all $\varepsilon > 0$, there exists $C > 0$ such that

$$F(x,t) \le \frac{1}{N}\varepsilon|t|^{\frac{N}{2}} + C|t|^r e^{\alpha\ |t|^\gamma},\ \text{for all } t \in \mathbb{R},\ \alpha > \alpha_0. \tag{1.16}$$

So, by (1.11) and (1.16) the functional $\mathcal{J}$ given by (1.11), is well defined. Moreover, by standard arguments, $\mathcal{J} \in C^1(\mathbf{W}, \mathbb{R})$. It is standard to check that critical points of $\mathcal{J}$ are precisely weak solutions of (1.1). Moreover, we have

$$\begin{aligned}\langle \mathcal{J}'(u), \varphi\rangle = \mathcal{J}'(u)\varphi =& g(\|u\|^{\frac{N}{2}})\Big(\int_B \big(v_\beta(x)\ |\Delta u|^{\frac{N}{2}-2}\ \Delta u\ \Delta\varphi dx\big) \\ &- \int_B |u|^{q-2}u\varphi\ dx - \int_B f(x,u)\ \varphi\ dx\ ,\ \forall\ \varphi \in \mathbf{W}\cdot\end{aligned}$$

where $\langle .,. \rangle$ denotes the duality between $\mathbf{W}$ and its dual space $\mathbf{W}^*$.

Our objective is to find solutions that minimise the bound energy $\mathcal{J}$ among all possible solutions to the problem (P). To achieve this goal, we define the Nehari set as follows

$$\mathcal{N} := \{u \in \mathbf{W} : \langle \mathcal{J}'(u), u\rangle = 0, u \neq 0\}$$

and we are looking for a minimization of the energy function $\mathcal{J}$ through the following minimization problem:

$$m = \inf_{u \in \mathcal{N}} \mathcal{J}(u)\cdot$$

To our best knowledge, there are no results for solutions to the non local weighted p-biharmonic equation with critical exponential nonlinearity combined with a polynomial term on the weighted Sobolev space $\mathbf{W}$.

Now, we give our main result as follows:

Theorem 1.4. *Let $f(x,t)$ be a function that has a critical growth at $+\infty$. Suppose that (A_1), (A_2), (A_3), (A_4) , (A_5), (G_1) and (G_2) are satisfied. There exists $\delta > 0$ such that problem (P) has a radial solution with minimal energy provided*

$$C_p > \max\left\{1, \frac{N\tau}{2}\left((\tau - \frac{N}{2p})|w_p|_p^p \frac{pN}{g_0(q-N)}\left(\frac{2(\alpha_0+\delta)}{\alpha_\beta}\right)^{\frac{N}{2\gamma}}\right)^{\frac{2p-N}{N}}\right\} \tag{1.17}$$

where $\tau = \dfrac{2g(1)}{Ng_0} + \dfrac{g(1)}{Ng_0^2}\dfrac{pq}{p-q}m_p$ *with* $m_p = \inf\limits_{u\in\mathcal{N}_p} J_p(u) > 0$,

$$J_p(u) := \frac{2}{N}G(\|u\|^{\frac{N}{2}}) - \frac{1}{p}\int_B |u|^p dx$$

and

$$\mathcal{N}_p := \{u \in \mathbf{W}, u \neq 0 \text{ and } \langle J_p'(u), u\rangle = 0\}.$$

$|w_p|_p$ *denote the norm of w_p in the Lebesgue space $L^p(B)$.*

Generally, exploring fourth-order partial differential equations is regarded as an intriguing subject. The interest in examining these equations has been sparked by their applications in various fields such as micro-electro-mechanical systems, phase field models of multi-phase systems, thin film theory, surface diffusion on solids, interface dynamics, and flow in Hele-Shaw cells, as referenced in [10, 18, 27].

This work is structured as follows: Section 2 covers essential preliminary knowledge about functional space and preliminary results. In Section 3, we present some key technical lemmas. Section 4 delves into studying an auxiliary problem crucial for proving our main result. Section 5 is dedicated to proving Theorem 1.4.

Additionally, it's important to note that the constant C might vary from one line to another, and occasionally, we index the constants to demonstrate their variations. Furthermore, we'll use the notation $|u|_p$ to denote the norm in the Lebesgue space $L^p(B)$.

2. Weighted Sobolev Space setting and embedding results

The space $\mathbf{W}$ is a Banach and reflexive space with the norm

$$\|u\| = \left(\int_B v_\beta(x)|\Delta u|^{\frac{N}{2}}dx\right)^{\frac{1}{2}}.$$

We also have the continuous embedding

$$\mathbf{W} \hookrightarrow L^t(B) \text{ for all } t \geq \frac{N}{2}.$$

Moreover, $\mathbf{W}$ is compactly embedded in $L^t(B)$ for all $t \geq 1$. In fact, we have

Lemma 2.1.

(i) *Let u be a radially symmetric function in $C^\infty_{0,rad}(B)$. Then, we have*

(i) [36]

$$\begin{aligned}|u(x)| &\leq \left(\frac{N}{\alpha_\beta}(|\log(\frac{e}{|x|}|-1)\right)^{\frac{1}{\gamma}}\left(\int_B w_\beta(x)|\Delta u|^{\frac{N}{2}}dx\right)^{\frac{2}{N}}\\ &\leq \left(\frac{N}{\alpha_\beta}(|\log(\frac{e}{|x|}|-1)\right)^{\frac{1}{\gamma}}\|u\|\cdot\end{aligned}$$

(*ii*) $\displaystyle\int_B e^{|u(x)|^\gamma} dx < +\infty, \ \forall \ u \in W^{2,\frac{N}{2}}_{0,rad}(B).$

(iii) *The following embedding is continuous*

$$\mathbf{W} \hookrightarrow L^t(B) \text{ for all } t \geq \frac{N}{2}\cdot$$

(vi) $\mathbf{W}$ *is compactly embedded in* $L^t(B)$ *for all* $t \geq 1\cdot$

Proof. (*i*) see [36].
(*ii*) From (*i*) and using the identity $\log(\frac{e}{|x|}) - |\log(|x|)| = 1 \ \forall x \in B$, we get

$$|u(x)|^\gamma \leq \frac{N}{\alpha_\beta}\left| |\log(\frac{e}{|x|} - 1) \right| \|u\|^\gamma \ \leq \frac{N}{\alpha_\beta}\big(1 + \big|\log(|x|)\big|\big) \ \|u\|^\gamma.$$

Hence, using the fact that the function $r \mapsto r^{N-1} e^{\frac{\|u\|^\gamma (1+|\log r|)}{\alpha_\beta}}$ is increasing, we get

$$\int_{|x|<1} e^{|u|^\gamma} dx \leq NV_N \int_0^1 r^{N-1} e^{\frac{N\|u\|^\gamma (1+|\log r|)}{\alpha_\beta}} dr \leq NV_N e^{\frac{N\|u\|^\gamma}{\alpha_\beta}} < +\infty.$$

Then (*ii*) follows by density.
(*iii*) and (*iv*). Since $v_\beta(x) \geq 1$, then following embedding are continuous

$$\mathbf{W} \hookrightarrow W^{2,2}_{0,rad}(B, w) \hookrightarrow W^{2,2}_{0,rad}(B) \hookrightarrow L^t(B) \ \forall t \geq \frac{N}{2}.$$

We also have, by Rellich-Kondrachov, the following compact injection

$$W^{2,\frac{N}{2}}_{0,rad}(B) \hookrightarrow\hookrightarrow L^t(B) \ \forall t \geq 1.$$

This concludes the lemma. □

Remark 2.1. According to (*ii*), it will be said that f has subcritical growth at infinity if

$$\lim_{|s|\to+\infty} \frac{|f(x,s)|}{e^{\alpha s^\gamma}} = 0, \ \forall \ \alpha > 0. \tag{2.18}$$

3. Some technical lemmas

In the following we assume, unless otherwise stated, that the function f satisfies the conditions (A_1) to (A_4) and the function g satisfies (G_1) and (G_2) . Let $u \in \mathbf{W}$ with $u \not\equiv 0$ a.e. in the ball B, and we define the function $\Upsilon_u : [0, \infty) \to \mathbb{R}$ as

$$\Upsilon_u(t) = \mathcal{J}(tu). \tag{3.19}$$

It's clear that $\Upsilon'_u(t) = 0$ is equivalent to $tu \in \mathcal{N}$.
In the next results, we show that $\mathcal{N}$ is not empty and that $\mathcal{J}$, restricted to $\mathcal{N}$, is bounded from below.

Lemma 3.1. (i) *For each* $u \in \mathbf{W}$ *with* $u \neq 0$, *there exists an unique* $t_u > 0$, *such that* $t_u u \in \mathcal{N}$. *In particular, the set* $\mathcal{N}$ *is nonempty and* $\mathcal{J}(u) > 0$, *for every* $u \in \mathcal{N}$.

(ii) *For all* $t \geq 0$ *with* $t \neq t_u$, *we have*

$$\mathcal{J}(tu) < \mathcal{J}(t_u u)\cdot$$

Proof. (i) Note that since $q \geq N$, we have

$$\lim_{|t|\to 0} \frac{|t|^{q-1}}{|t|^{\frac{N}{2}-1}} = 0,$$

$$\lim_{|t|\to \infty} \frac{|t|^{q-1}}{|t|^{r-1}} = 0, \text{ for all } r \in (q,\infty),$$

Then for any $\epsilon > 0$, there exists a positive constant $C_1 = C_1(\varepsilon)$ such that

$$|t|^{q-1} \leq \epsilon |t|^{\frac{N}{2}-1} + C_1 |t|^{r-1} \text{ for all } t \in \mathbb{R}. \tag{3.20}$$

From (1.15) and (1.16), for all $\epsilon > 0$, there exist positive constants $C_1' = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$ such that

$$f(x,t)t \leq \epsilon |t|^{\frac{N}{2}} + C_1' |t|^r \exp(\alpha |t|^\gamma) \text{ for all } \alpha > \alpha_0, r > q. \tag{3.21}$$

and

$$F(x,t) \leq \frac{1}{N}\epsilon |t|^{\frac{N}{2}} + C_2 |t|^r \exp(\alpha |t|^\gamma) \text{ for all } \alpha > \alpha_0, r > q. \tag{3.22}$$

Now, given $u \in \mathbf{W}$ fixed with $u \neq 0$. From (3.22), (3.20) and (1.9), for all $\varepsilon > 0$, we have

$$\begin{aligned}
\Upsilon_u(t) = \mathcal{J}(tu) &= \frac{2}{N} G(|t|^{\frac{N}{2}} \|u\|^{\frac{N}{2}}) - \frac{1}{q}\int_B |tu|^q \, dx - \int_B F(x,tu)\, dx \\
&\geq \frac{1}{N} g(|t|^{\frac{N}{2}} \|u\|^{\frac{N}{2}}) |t|^{\frac{N}{2}} \|u\|^{\frac{N}{2}} - \frac{\epsilon}{q}|t|^{\frac{N}{2}} \int_B |u|^{\frac{N}{2}} \, dx - C_1 \frac{|t|^r}{q} \int_B |u|^r - \int_B F(x,tu)\, dx \\
&\geq \frac{g_0}{N} |t|^{\frac{N}{2}} \|u\|^{\frac{N}{2}} - \frac{1}{N}\epsilon |t|^{\frac{N}{2}} \int_B |u|^{\frac{N}{2}} dx - \frac{\epsilon}{q} |t|^{\frac{N}{2}} \int_B |u|^{\frac{N}{2}} \, dx \\
&\quad - C_1 \frac{|t|^r}{q} \int_B |u|^r - C_1' \int_B |tu|^r \exp(\alpha t |u|^\gamma).dx
\end{aligned} \tag{3.23}$$

Using the Hölder inequality, with $a, a' > 1$ such that $\frac{1}{a} + \frac{1}{a'} = 1$, and Sobolev embedding Lemma 2.1, we get

$$\begin{aligned}
\Upsilon_u(t) \geq& \frac{g_0}{N} |t|^{\frac{N}{2}} \|u\|^{\frac{N}{2}} - C_4' \frac{\epsilon}{q} |t|^{\frac{N}{2}} \|u\|^{\frac{N}{2}} - C_5 \frac{|t|^r}{q} \|u\|^r - C_3 \frac{1}{N} \epsilon |t|^{\frac{N}{2}} \|u\|^{\frac{N}{2}} \\
&- C_1 \left(\int_B |tu|^{a' r} dx \right)^{\frac{1}{a'}} \left(\int_B \exp(\alpha t a |u|^\gamma) dx \right)^{\frac{1}{a}} \\
\geq& \left(\frac{g_0}{N} - \epsilon (\frac{1}{N} C_3 + C_4' \frac{1}{q}) \right) \|tu\|^{\frac{N}{2}} - C_4 \frac{|t|^r}{q} \|u\|^r \\
&- \left(\int_B \exp\left(\alpha a \|tu\|^\gamma (\frac{|u|}{\|u\|})^\gamma\right) dx \right)^{\frac{1}{a}} C_5 \|tu\|^r.
\end{aligned}$$

By (1.5), the last integral is finte provided $t > 0$ is chosen small enough such that $\alpha a \|tu\|^\gamma \leq \alpha_\beta$. Then,

$$\Upsilon_u(t) \geq \left(\frac{g_0}{N} - \epsilon (\frac{1}{N} C_3 + C_4' \frac{1}{q}) \right) \|tu\|^{\frac{N}{2}} - C_6 \|tu\|^r \text{ with } \alpha a \|tu\|^\gamma \leq \alpha_\beta \text{ and } \alpha > \alpha_0$$

holds. Choosing $\epsilon > 0$ such that $\frac{g_0}{N} - \epsilon(\frac{1}{N}C_3 + C_4'\frac{1}{q}) > 0$ and since $r > N$, we obtain,

$$\Upsilon_u(t) > 0 \text{ for small } t > 0. \tag{3.24}$$

Now, from (A_2), we can derive that there exist $C_5, C_6 > 0$ such that

$$F(x,t) \geq C_5|t|^\theta - C_6. \tag{3.25}$$

Then, by using (1.7) and (3.25), we get

$$\Upsilon_u(t) = \mathcal{J}(tu) \leq \frac{2g(1)}{N}|t|^{\frac{N}{2}}\|u\|^{\frac{N}{2}} + \frac{g(1)}{N}|t|^N\|u\|^N - C_5'|t|^\theta|u|_\theta^\theta - C_6|B|.$$

Since $\theta > N$, we obtain

$$\Upsilon_u(t) \to -\infty \text{ as } t \to +\infty. \tag{3.26}$$

Hence, from (3.24) and (3.25), there exists at least one $t_u > 0$ such that $\Upsilon_u'(t_u) = 0$, i.e. $t_u u \in \mathcal{N}$.

Now we will show the uniqueness of t_u. Let $s > 0$ such that $su \in \mathcal{N}$ and suppose that $s \neq t_u$. Without loss of generality, we can assume that $s > t_u$. So we have $\langle \mathcal{J}'(t_u u), t_u u\rangle = 0$ and $\langle \mathcal{J}'(su), su\rangle = 0$, then

$$\frac{g(\|su\|^{\frac{N}{2}})}{|s|^{\frac{N}{2}}\|u\|^{\frac{N}{2}}} = \frac{1}{\|u\|^N}\Big(\int_B |su|^{q-N}|u|^N dx + \int_B \frac{f(x,su)}{(su)^{N-1}}|u|^N dx\Big), \tag{3.27}$$

$$\frac{g(\|t_u u\|^{\frac{N}{2}})}{|t|_u^{\frac{N}{2}}\|u\|^{\frac{N}{2}}} = \frac{1}{\|u\|^N}\Big(\int_B |t_u u|^{q-N}|u|^N dx + \int_B \frac{f(x,t_u u)}{(t_u u)^{N-1}}|u|^N dx\Big)\cdot \tag{3.28}$$

Combining (3.27), (3.28), we get

$$\begin{aligned}&\frac{g(\|su\|^{\frac{N}{2}})}{|s|^{\frac{N}{2}}\|u\|^{\frac{N}{2}}} - \frac{g(\|t_u u\|^{\frac{N}{2}})}{|t_u|^{\frac{N}{2}}\|u\|^{\frac{N}{2}}} \geq \\ &= \frac{1}{\|u\|^N}\Big(\int_B ((|su|^{q-N} - |t_u u|^{q-N})|u|^N + (\frac{f(x,su)}{(su)^{N-1}} - \frac{f(x,t_u u)}{(t_u u)^{N-1}})|u|^N)dx\Big)\cdot\end{aligned}$$

Clearly, according to (A_3), Remark 1.1 and (G_2), the left-hand side of the last equality is negative for $t_u > s$ while the right-hand side is positive, which is a contradiction. This contradict the fact that $s > t_u$. The case $t_u > s > 0$ is similar and we omit it. Then, $s = t_u$.
(ii) Follows from (i) , since $\mathcal{J}(t_u u) = \max_{t\geq 0} \Upsilon_u(t)$. □

Lemma 3.2. *Assume that $(A_1) - (A_4)$ hold. Then for any $u \in \mathbf{W}$ with $u \neq 0$ such that $\langle \mathcal{J}'(u), u\rangle \leq 0$, the unique maximum point of Υ_u on $\mathbb{R}_+$ satisfies $0 < t_u \leq 1$.*

Proof. Since $t_u u \in \mathcal{N}$, we have

$$\frac{g(\|t_u u\|^{\frac{N}{2}})}{|t|_u^{\frac{N}{2}}\|u\|^{\frac{N}{2}}} = \frac{1}{\|u\|^N}\Big(\int_B |t_u u|^{q-\frac{N}{2}}|u|^{\frac{N}{2}} dx + \int_B \frac{f(x,t_u u)}{(t_u u)^{\frac{N}{2}-1}}|u|^N dx\Big)\cdot \tag{3.29}$$

Furthermore, since $\langle \mathcal{J}'(u), u\rangle \leq 0$, we have

$$\frac{g(\|u\|^{\frac{N}{2}})}{\|u\|^{\frac{N}{2}}} \leq \frac{1}{\|u\|^N}\Big(\int_B |u|^{q-\frac{N}{2}}|u|^{\frac{N}{2}} dx + \int_B \frac{f(x,u)}{(u)^{\frac{N}{2}-1}}|u|^{\frac{N}{2}} dx\Big).$$

Then by (3.29), we have

$$\begin{aligned}
&\frac{g(\|t_u u\|^{\frac{N}{2}})}{|t|_u^{\frac{N}{2}}\|u\|^{\frac{N}{2}}} - \frac{g(\|u\|^{\frac{N}{2}})}{\|u\|^{\frac{N}{2}}} \geq \\
&\geq \frac{1}{\|u\|^N}\int_B \Big((|t_u u|^{q-\frac{N}{2}} - |u|^{q-\frac{N}{2}} + \frac{f(x,t_u u)}{(t_u u)^{\frac{N}{2}-1}} - \frac{f(x,u)}{(u)^{\frac{N}{2}-1}})\Big)|u|^{\frac{N}{2}}\,dx. \qquad (3.30)
\end{aligned}$$

Obviously, from (G_2) the left hand side of (3.30) is negative for $t_u > 1$ whereas the right hand side is positive, which is a contradiction. Therefore $0 < t_u \leq 1$. □

In the sequel, we prove that sequences in $\mathcal{N}$ cannot converge to 0.

Lemma 3.3. *For all $u \in \mathcal{N}$,*
(i) there exists $\kappa > 0$ such that $\|u\| \geq \kappa$;
(ii) $\mathcal{J}(u) \geq (\frac{1}{N} - \frac{1}{q})g_0\|u\|^{\frac{N}{2}}$.

Proof. (i) We argue by contradiction. Suppose that there exists a sequence $\{u_n\} \subset \mathcal{N}$ such that $u_n \to 0$ in $\mathbf{W}$. Since $\{u_n\} \subset \mathcal{N}$, then $\langle \mathcal{J}'(u_n), u_n\rangle = 0$. Hence, it follows from (3.21), (3.22) and the radial Lemma 2.1 that

$$\begin{aligned}
g_0\|u_n\|^{\frac{N}{2}} < g(\|u_n\|^{\frac{N}{2}})\|u_n\|^{\frac{N}{2}} &= \int_B |u|^q dx + \int_B f(x,u_n)u_n dx \\
&\leq 2\epsilon \int_B |u_n|^{\frac{N}{2}} dx + C_1\int_B |u_n|^r dx + C_1'\int_B |u_n|^r \exp(\alpha|u_n|^\gamma)dx \\
&\leq \epsilon C_6\|u_n\|^{\frac{N}{2}} + C_7\|u\|^r + C_1\int_B |u_n|^r\exp(\alpha|u_n|^\gamma)dx. \qquad (3.31)
\end{aligned}$$

Let $a > 1$ with $\frac{1}{a} + \frac{1}{a'} = 1$. Since $u_n \to 0$ in $\mathbf{W}$, for n large enough, we get $\|u_n\| \leq (\frac{\alpha_\beta}{\alpha a})^{\frac{1}{\gamma}}$. From Hölder inequality, (1.11) and again the radial Lemma 2.1, we have

$$\begin{aligned}
\int_B |u_n|^r\exp(\alpha|u_n|^\gamma)dx &\leq \Big(\int_B |u_n|^{ra'}dx\Big)^{\frac{1}{a'}}\Big(\int_B \exp\big(\alpha a\|u\|^\gamma(\frac{|u|}{\|u\|})^\gamma\big)dx\Big)^{\frac{1}{a}} \\
&\leq C_7\Big(\int_B |u_n|^{ra'}dx\Big)^{\frac{1}{a'}} \leq C_8\|u_n\|^r.
\end{aligned}$$

Combining (3.31) with the last inequality, for n large enough, we obtain

$$g_0\|u_n\|^{\frac{N}{2}} \leq \epsilon C_6\|u_n\|^{\frac{N}{2}} + C_8\|u_n\|^r. \qquad (3.32)$$

Choose suitable $\epsilon > 0$ such that $g_0 - \epsilon C_6 > 0$. Since $N < r$, then (3.32) contradicts the fact that $u_n \to 0$ in $\mathbf{W}$.

(ii) Given $u \in \mathcal{N}$, by the definition of $\mathcal{N}$, (1.9) and (A_3), we obtain

$$\begin{aligned}
\mathcal{J}(u) &= \mathcal{J}(u) - \frac{1}{q}\langle \mathcal{J}'(u), u\rangle \\
&= \frac{2}{N}G(\|u\|^{\frac{N}{2}}) - \frac{1}{q}g(\|u\|^{\frac{N}{2}})\|u\|^{\frac{N}{2}} + \Big(\int_B \frac{1}{q}f(x,u)u - F(x,u)dx\Big) + \frac{1}{q}|u|_q^q \\
&\geq (\frac{1}{N} - \frac{1}{q})g_0\|u\|^{\frac{N}{2}}.
\end{aligned}$$

Lemma 3.3 implies that $\mathcal{J}(u) > 0$ for all $u \in \mathcal{N}$. □

As a consequence, $\mathcal{J}$ is bounded by below in $\mathcal{N}$, and therefore $m := \inf_{u\in\mathcal{N}} \mathcal{J}(u)$ is well-defined.

In the following lemma we prove that if the minimum of $\mathcal{J}$ on $\mathcal{N}$ is realized at some $u \in \mathcal{N}$, then u is a critical point of $\mathcal{J}$.

Lemma 3.4. *If $u_0 \in \mathcal{N}$ satisfies $\mathcal{J}(u_0) = m$, then $\mathcal{J}'(u_0) = 0$.*

Proof. We argue by contradiction. We assume that $\mathcal{J}'(u_0) \neq 0$. By the continuity of $\mathcal{J}'$, there exist $\iota, \delta \geq 0$ such that

$$\|\mathcal{J}'(v)\|_{\mathbf{W}^*} \geq \iota \text{ for all } \ v \text{ such that } \|v - u_0\| \leq \delta. \tag{3.33}$$

Let $D = (1-\tau, 1+\tau) \subset \mathbb{R}$ with $\tau \in (0, \dfrac{\delta}{4\|u_0\|})$ and define $h : D \to \mathbf{W}$, by

$$h(\rho) = \rho u_0, \rho \in D\cdot$$

By virtue of $u_0 \in \mathcal{N}$, $\mathcal{J}(u_0) = m$ and Lemma 3.1, it is clear that

$$\bar{m} := \max_{\partial D} \mathcal{J} \circ h < m \text{ and } \mathcal{J}(h(\rho)) < m, \ \forall \ \rho \neq 1. \tag{3.34}$$

Let $\epsilon := \min\{\frac{m-\bar{m}}{2}, \frac{\iota\delta}{16}\}$, $S_r := B(u_0, r), r \geq 0$ and $\mathcal{J}^a := \mathcal{J}^{-1}(]-\infty, a])$. According to the quantitative deformation Lemma [[33], Lemma 2.3], there exists a deformation $\eta \in C(\mathbf{W}, \mathbf{W})$ such that:

(1) $\eta(v) = v$, if $v \notin \mathcal{J}^{-1}([m-\epsilon, m+\epsilon]) \cap S_\delta$

(2) $\eta\left(\mathcal{J}^{m+\epsilon} \cap S_{\frac{\delta}{2}}\right) \subset \mathcal{J}^{m-\epsilon}$,

(3) $\mathcal{J}(\eta(v)) \leq \mathcal{J}(v)$, for all $v \in \mathbf{W}$.

By lemma 3.1 (ii), we have $\mathcal{J}(h(\rho)) \leq m$. In addition, we have,

$$\|h(\rho) - u_0\| = \|(\rho-1)u_0\| \leq \frac{\delta}{4}, \ \forall \rho \in D\cdot$$

Then $h(\rho) \in S_{\frac{\delta}{2}}$ for $\rho \in \bar{D}$. Therefore, it follows from (2) that

$$\max_{\rho\in\bar{D}} \mathcal{J}(\eta(h(\rho)) \leq m - \epsilon. \tag{3.35}$$

In the sequel, we will prove that $\eta(h(D)) \cap \mathcal{N}$ is nonempty. In such case, due to the definition of m, this contradicts (3.35). To do this, we first define

$$\bar{h}(\rho) := \eta(h(\rho)),$$

$$\Upsilon_0(\rho) = \langle \mathcal{J}'(h(\rho)), u_0 \rangle,$$

and

$$\Upsilon_1(\rho) := (\frac{1}{\rho}\langle \mathcal{J}'(\bar{h}(\rho), (\bar{h}(\rho))\rangle.$$

We have that for $\rho \in \overline{D}$,

$$\mathcal{J}(h(\rho)) \leq \overline{m} < m - \varepsilon.$$

Indeed, for all $\rho \in D$, $\mathcal{J}(h(\rho)) \leq m + \varepsilon$. In addition, $\|h(\rho) - u_0\| = \|(\rho-1)u_0\| \leq \frac{\delta}{2}$, $\forall \ \rho \in D$. So $h(\overline{D}) \subset S_{\frac{\delta}{2}}$ and then by (2), we get

$$\mathcal{J}(\eta(h(\rho)) = \mathcal{J}(\overline{h}(\rho)) \leq m - \varepsilon \ \forall \ \rho \in \overline{D}.$$

Therefore, $\bar{h}(\rho) = \eta(h(\rho)) = \rho u_0$. Hence,

$$\Upsilon_0(\rho) = \Upsilon_1(\rho), \forall \rho \in \overline{D}. \tag{3.36}$$

On one hand, we have that $\rho = 1$ is the unique critical point of Υ_0. So by degree theory, we get that $d^0(\Upsilon_0, D, 0) = 1$. On the other hand, from (3.36), we deduce that $d^0(\Upsilon_1, D, 0) = 1$. Consequently, there exists $\overline{\rho} \in D$ such that $\overline{h}(\overline{\rho}) \in \mathcal{N}$. This implies that

$$m \leq \mathcal{J}(\overline{h}(\overline{\rho})) = \mathcal{J}(\eta(h(\overline{\rho})).$$

This contradicts (3.35) and finish the proof of the Lemma. □

4. The auxiliary problem (P_a)

In this section, in order to prove our existence result , we consider the auxiliary problem

$$(P_a) \begin{cases} g\big(\int_B (v_\beta(x)|\Delta u|^{\frac{N}{2}})dx\big)\Delta(v_\beta(x)|\Delta u|^{\frac{N}{2}-2}\Delta u) &= |u|^{p-2}u \quad \text{in} \quad B \\ u &= \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial B, \end{cases} \tag{4.37}$$

where p is the constant that appear in the hypothesis (A_5). The energy J_p associated to problem (4.37) is given by

$$J_p(u) := \frac{2}{N} G(\|u\|^{\frac{N}{2}}) - \frac{1}{p}\int_B |u|^p dx.$$

We introduce the Nehari manifold associeted to J_p that is

$$\mathcal{N}_p := \{u \in \mathbf{W}, u \neq 0 \text{ and } \langle J_p'(u), u\rangle = 0\}.$$

Let $m_p = \inf_{\mathcal{N}_p} J_p(u) > 0$, we have the following results for J_p.

By following the proof in [13], we can easily see that we have the following results.

Lemma 4.1. *Given $u \in \mathbf{W}, u \neq 0$, there exists a unique $t > 0$ such that $tu \in \mathcal{N}_p$. In addition, t satisfies*

$$J_p(tu) = \max_{s \geq 0} J_p(su). \tag{4.38}$$

As a consequence, we have

Corollary 4.2. *Let $u \in \mathbf{W}, u \neq 0$. Then $u \in \mathcal{N}_p$ if and only if $J_p(tu) = \max_{s\geq 0} J_p(su)$.*

Furthermore, proving the subsequent lemmas is quite straightforward.

Lemma 4.3. *For all $u \in \mathcal{N}_p$,*
(i) there exists $\kappa_0 > 0$ such that $\|u\| \geq \kappa_0$;
(ii) $\mathcal{J}_p(u) \geq g_0(\frac{2}{N} - \frac{1}{p})|u|_p^p$.

Lemma 4.4. *There exists $w_p \in \mathcal{N}_p$ such that $J_p(w_p) = m_p$.*

Proof. Let sequence $(w_n) \subset \mathcal{N}_p$ satisfy $\lim_{n\to+\infty} J_p(w_n) = m_p$. It is clearly that (w_n) is bounded by Lemma 4.3. Then, up to a subsequence, there exists $w_p \in \mathbf{W}$ such that

$$\begin{array}{ll} w_n \rightharpoonup w_p & \text{in } \mathbf{W}, \\ w_n \to w_p & \text{in } L^t(B),\ \forall t \geq \frac{N}{2}, \\ w_n \to w_p & \text{a.e. in } B. \end{array} \tag{4.39}$$

We claim that $w_p \neq 0$. Suppose, by contradiction, $w_p = 0$. From the definition of $\mathcal{N}_p$ and (4.39) , we have that $\lim_{n\to+\infty} \|w_n\|^{\frac{N}{2}} = 0$, which contradicts Lemma 4.3. Hence, $w_p \neq 0$.

From the continuity of g, the lower semi continuity of norm and (4.39), it follows that

$$g(\|w_p\|^{\frac{N}{2}})\|w_p\|^{\frac{N}{2}} \leq \liminf_{n\to+\infty} g(\|w_n\|^{\frac{N}{2}})\|w_n\|^{\frac{N}{2}}\cdot \tag{4.40}$$

On the other hand, by using $\langle J'(w_n), w_n\rangle = 0$ and (4.39), we have

$$\liminf_{n\to+\infty} g(\|w_n\|^{\frac{N}{2}})\|w_n\|^{\frac{N}{2}} = \liminf_{n\to+\infty} \int_B |w_n|^p dx = \int_B |w_p|^p dx. \tag{4.41}$$

From (4.40) and (4.41) we deduce that $\langle J_p'(w_p), w_p\rangle \leq 0$. Then, as in Lemma 3.2 this implies that there exists $s_u \in (0, 1]$ such that $s_u w_p \in \mathcal{N}_p$. Thus, by the lower semi continuity of norm, (1.9) and (4.39), we get that

$$\begin{aligned} m_p &\leq J_p(s_u w_p) = J(s_u w_p) - \frac{1}{N}\langle J_p'(s_u w_p), s_u w_p\rangle \\ &= \frac{2}{N} G(\|s_u w_p\|^{\frac{N}{2}}) - \frac{1}{N} g(\|s_u w_p\|^{\frac{N}{2}})\|s_u w_p\|^{\frac{N}{2}} + \big(\frac{1}{N} - \frac{1}{p}\big) s_u^p \int_B |w_p|^p dx \\ &\leq J_p(w_p) - \frac{1}{N}\langle J_p'(w_p), w_p\rangle \\ &= \frac{2}{N} G(\|w_p\|^{\frac{N}{2}}) - \frac{1}{p}\int_B |w_p|^p dx - \frac{1}{N} g(\|w_p\|^{\frac{N}{2}})\|w_p\|^{\frac{N}{2}} + \frac{1}{N}\int_B |w_p|^p dx \\ &\leq \liminf_{n\to+\infty} \Big[\frac{2}{N} G(\|w_n\|^{\frac{N}{2}}) - \frac{1}{p}\int_B |w_n|^p dx\Big] \\ &\quad - \liminf_{n\to+\infty} \Big[\frac{1}{N} g(\|w_n\|^{\frac{N}{2}})\|w_p\|^{\frac{N}{2}} + \frac{1}{N}\int_B |w_n|^p dx\Big] \\ &\leq \liminf_{n\to+\infty} \big[J_p(w_n) - \frac{1}{N}\langle J_p'(w_n), w_n\rangle\big] = m_p. \end{aligned}$$

Therefore, this leads us to the result: $J_p(w_p) = m_p$, fulfilling the intended conclusion. □

5. Proof of Theorem 1.2

Next, we'll establish a fundamental estimate for level m. This will serve as a valuable tool in obtaining an appropriate bound on the norm of a minimizing sequence for m within $\mathcal{N}$.

Lemma 5.1. *If $(u_n) \subset \mathcal{N}$ is a minimizing sequence for m, then*

$$\limsup_{n\to+\infty} \|u_n\|^{\frac{N}{2}} \leq m \frac{qN}{g_0(q-N)}. \tag{5.42}$$

Proof. Let $(u_n) \subset \mathcal{N}$ is a minimizing sequence for m. Using (A_2) and (1.10), we obtain that

$$\begin{aligned}
m + o_n(1) = \mathcal{J}(u_n) &= \big(\mathcal{J}(u_n) - \frac{1}{q}\langle \mathcal{J}'(u_n), u_n\rangle\big)\\
&= \big(\frac{2}{N}G(\|u_n\|^{\frac{N}{2}}) - \frac{1}{q}g(\|u_n\|^{\frac{N}{2}})\|u_n\|^{\frac{N}{2}} + \frac{1}{q}\int_B \big(f(x,u_n)u_n - qF(x,u_n)\big)dx\big)\\
&\quad + \frac{1}{q}\int_B |u_n|^q dx\\
&> (\frac{1}{N} - \frac{1}{q})g(\|u_n\|^{\frac{N}{2}}\|)\|u_n\|^{\frac{N}{2}}\\
&> \frac{q-N}{qN}g_0\|u_n\|^{\frac{N}{2}}.
\end{aligned}$$

Therefore (5.42) holds. □

Let α_0 be the real number that appears in the equation (1.5). Then there exists $\delta > 0$ such that $\alpha = \alpha_0 + \delta$. We arrive at the estimate for level m.

Lemma 5.2. *Assume that* $(A_1) - (A_5)$ *and* (1.17) *are satisfied. It holds that*

$$m \le g_0 \frac{q-N}{qN}\left(\frac{\alpha_\beta}{2(\alpha_0+\delta)}\right)^{\frac{N}{2\gamma}}. \tag{5.43}$$

Proof. From Lemma 4.4, there exists $w_p \in \mathcal{N}_p$ such that $J_p(w_p) = m_p$ and $J_p'(w_p) = 0$. Consequently, using (1.9) we get

$$\frac{2}{N}G(\|w_p\|^{\frac{N}{2}}) - \frac{1}{p}\int_B |w_p|^p \, dx = m_p \tag{5.44}$$

and

$$g_0\|w_p\|^{\frac{N}{2}} < g(\|w_p\|^{\frac{N}{2}})\|w_p\|^{\frac{N}{2}} = \int_B |w_p|^p \, dx. \tag{5.45}$$

Note that by using (5.44), (5.45), (1.10) and the fact that $p > q > N$, we have

$$(\frac{1}{q} - \frac{1}{p})|w_p|_p^p = \frac{1}{q}g(\|w_p\|^{\frac{N}{2}})\|w_p\|^{\frac{N}{2}} - \frac{2}{N}G(\|w_p\|^{\frac{N}{2}}) + m_p \le m_p.$$

So,

$$|w_p|_p^p < \frac{pq}{p-q}m_p. \tag{5.46}$$

According to (A_5) and (5.45), we have $\langle \mathcal{J}'(w_p), w_p\rangle \le 0$ which, with lemma 3.2, gives that there exists a unique $s \in (0,1)$ such that $sw_p \in \mathcal{N}$. Using (A_5), (5.44), (5.45), (1.8) and (5.46), we obtain

$$\begin{aligned}
m &\le \mathcal{J}(sw_p) \le \frac{2g(1)s^{\frac{N}{2}}}{N}\|w_p\|^{\frac{N}{2}} + \frac{g(1)s^N}{N}\|w_p\|^N - \frac{C_p s^p}{p}|w_p|_p^p\\
&\le \frac{2g(1)s^{\frac{N}{2}}}{N}\|w_p\|^{\frac{N}{2}} + \frac{g(1)s^{\frac{N}{2}}}{N}\|w_p\|^N - \frac{C_p s^p}{p}|w_p|_p^p\\
&\le \frac{2g(1)s^{\frac{N}{2}}}{Ng_0}|w_p|_p^p + \frac{g(1)s^{\frac{N}{2}}}{Ng_0^2}|w_p|_p^{2p} - \frac{C_p s^p}{p}|w_p|_p^p\\
&= \left((\frac{2g(1)}{Ng_0} + \frac{g(1)}{Ng_0^2}|w_p|_p^p)s^{\frac{N}{2}} - \frac{C_p s^p}{p}\right)|w_p|_p^p
\end{aligned}$$

$$\leq \max_{\xi>0}\left((\frac{2g(1)}{Ng_0}+\frac{g(1)}{Ng_0^2}|w_p|_p^p)\xi^{\frac{N}{2}}-\frac{C_p\xi^p}{p}\right)|w_p|_p^p$$
$$\leq \max_{\xi>0}\left((\frac{2g(1)}{Ng_0}+\frac{g(1)}{Ng_0^2}\frac{pq}{p-q}m_p)\xi^{\frac{N}{2}}-\frac{C_p\xi^p}{p}\right)|w_p|_p^p.$$

By some simple algebraic calculations, we get

$$m \leq (\frac{N\tau}{2C_p})^{\frac{\frac{N}{2}}{p-\frac{N}{2}}}\left(\tau-\frac{N}{2p}\right)|w_p|_p^p. \tag{5.47}$$

Thus, by using (5.47), we obtain

$$m < (\frac{N\tau}{2C_p})^{\frac{\frac{N}{2}}{p-\frac{N}{2}}}\left(\tau-\frac{N}{2p}\right)(\frac{pq}{p-q})m_p. \tag{5.48}$$

Therefore, by (1.17) and (5.48), we get that (5.43) is valid. □

The result below gives us some compactness properties of minimising sequences.

Lemma 5.3. *If $(u_n) \subset \mathcal{N}$ is a minimizing sequence for m, then there exists $u \in \mathbf{W}$ such that*

$$\int_B f(x,u_n)u_n dx \to \int_B f(x,u)u dx$$

and

$$\int_B F(x,u_n)dx \to \int_B F(x,u)dx.$$

Proof. We must prove the first limit, since the second one is analogous. For this, we use (1.15) and introduce the following function $k(u_n(x))$ given by

$$k(u_n(x)) := \varepsilon|u_n|^{\frac{N}{2}}\,dx + C|u_n|^q\exp(\alpha|u_n|^\gamma).$$

It's clear that is sufficient to prove that $k(u_n(x))$ is convergent in $L^1(B)$. We have

$$\int_B f(x,u_n)\ u_n dx \leq \varepsilon\int_B |u_n|^{\frac{N}{2}}\,dx + C\int_B |u_n|^q\exp(\alpha|u_n|^\gamma)dx$$
$$= \int_B k(u_n(x))\ dx, \quad \text{for all } \alpha > \alpha_0 \text{ and } q > N. \tag{5.49}$$

First note that

$$|u_n|^{\frac{N}{2}} \to |u|^{\frac{N}{2}} \text{ in } L^1(B). \tag{5.50}$$

Considering $s, s' > 1$ such that $\frac{1}{s}+\frac{1}{s'}=1$ and s close to 1, we get

$$|u_n|^q \to |u|^q \text{ in } L^{s'}(B). \tag{5.51}$$

On the other hand, by (5.42), we have

$$m > \frac{q-N}{qN}g_0\limsup_{n\to+\infty}\|u_n\|^{\frac{N}{2}} \tag{5.52}$$

which, together with Lemma 5.2 leads to the following estimation $\limsup_{n\to+\infty}\|u_n\|^\gamma < \frac{\alpha_\beta}{2(\alpha_0+\delta)}$.

Now choosing $\alpha = \alpha_0 + \delta,\ \delta > 0$, we have that

$$\begin{aligned}\int_B \exp(\alpha s |u_n|^\gamma)dx &\le \int_B \exp\big(s(\alpha_0+\delta)\|u_n\|^\gamma (\frac{|u_n|}{\|u_n\|})^\gamma\big)dx \\ &\le \int_B \exp\big(\frac{s}{2}\alpha_\beta (\frac{|u_n|}{\|u_n\|})^\gamma\big)dx. \end{aligned}\tag{5.53}$$

Since $s > 1$ and is sufficiently close to 1, we get $\frac{s}{2}\alpha_\beta \le \alpha_\beta$. Then it follows by (1.4) that there is $M > 0$ such that

$$\int_B \exp(\alpha s |u_n|^\gamma)dx \le M. \tag{5.54}$$

Since

$$\exp(\alpha|u_n|^\gamma) \to \exp(\alpha|u|^\gamma) \text{ a.e in } \ B, \tag{5.55}$$

from (5.53) and [[32], Lemma 4.8], we get that

$$\exp(\alpha|u_n|^\gamma) \rightharpoonup \exp(\alpha|u|^\gamma) \ \text{ in } \ L^s(B). \tag{5.56}$$

Then, using (5.50), (5.51), (5.54), (5.56) and the Hölder inequality, we get

$$\begin{aligned}&\int_B \big(k(u_n(x)) - k(w(x))\big)dx = \varepsilon \int_B \big(|u_n|^{\frac{N}{2}} - |u|^{\frac{N}{2}}\big)dx \\ &\quad + C\int_B \big(|u_n|^q - |u|^q\big)\exp(\alpha|u_n|^\gamma)dx + C\int_B |u|^q\big(\exp(\alpha|u_n|^\gamma) - \exp(\alpha|u|^\gamma)\big)dx \\ &\le \varepsilon \int_B \big(|u_n|^{\frac{N}{2}} - |u|^{\frac{N}{2}}\big)dx + C\Big(\int_B \big(|u_n|^q - |u|^q\big)^{s'}dx\Big)^{\frac{1}{s'}}\Big(\int_B \exp(s\alpha|u|^\gamma)dx\Big)^{\frac{1}{s}} \\ &\quad + C\int_B |u|^q\big(\exp(\alpha|u_n|^\gamma) - \exp(\alpha|u|^\gamma)\big)dx \\ &\le \varepsilon \int_B \big(|u_n|^{\frac{N}{2}} - |u|^{\frac{N}{2}}\big)dx + CM\Big(\int_B \big(|u_n|^q - |u|^q\big)^{s'}dx\Big)^{\frac{1}{s'}} \\ &+ C\int_B |u|^q\big(\exp(\alpha|u_n|^\gamma) - \exp(\alpha|u|^\gamma)\big)dx \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

We conclude that

$$\int_B f(x,u_n)\ u_n dx \to \int_B f(x,u)\ u dx. \tag{5.57}$$

□

In the following, we give an additional important result that will be used to prove our main result.

Lemma 5.4. *Assume that the conditions (A_1), (A_2) and (A_3) are satisfied. Then, for each $x \in B$, we have*

$$tf(x,t) - qF(x,t) \ \textit{is increasing for } t > 0 \ \ \textit{and decreasing for } t < 0.$$

In particular, $tf(x,t) - qF(x,t) > 0$ for all $(x,t) \in B \times \mathbb{R} \setminus \{0\}$.

Proof. Assume that $0 < t < s$. For each $x \in B$, we have

$$\begin{aligned} tf(x,t) - qF(x,t) &= \frac{f(x,t)}{t^{q-1}}t^q - qF(x,s) + q\int_t^s f(x,\nu)d\nu \\ &< \frac{f(x,t)}{s^{q-1}}t^q - qF(x,s) + \frac{f(x,s)}{s^{q-1}}(s^q - t^q) \\ &= sf(x,s) - qF(x,s)\cdot \end{aligned}$$

The proof in the case $t < s < 0$ is similar.
The assertion $tf(x,t) - qF(x,t) > 0$ for all $(x,t) \in B \times \mathbb{R} \setminus \{0\}$ comes from (A_2). □

By the following lemma, we prove that the minimum of $\mathcal{J}$ on $\mathcal{N}$ is achieved in some $w_0 \in \mathcal{N}$.

Lemma 5.5. *There exists $w_0 \in \mathcal{N}$ such that $\mathcal{J}(w_0) = m$.*

Proof. Let sequence $(w_n) \subset \mathcal{N}$ satisfying $\lim_{n\to+\infty} \mathcal{J}(w_n) = m$. It is clearly that (w_n) is bounded by Lemma 4.3. Then, up to a subsequence, there exists $w_0 \in \mathbf{W}$ such that

$$\begin{array}{ll} w_n \rightharpoonup w_0 & \text{in } \mathbf{W}, \\ w_n \to w_0 & \text{in } L^t(B),\ \forall t \geq \frac{N}{2}, \\ w_n \to w_0 & \text{a.e. in } B. \end{array} \tag{5.58}$$

We claim that $w_0 \neq 0$. Suppose, by contradiction, $w_0 = 0$. From the definition of $\mathcal{N}$ and (5.58) , we have that $\lim_{n\to+\infty} \|w_n\|^{\frac{N}{2}} = 0$, which contradicts Lemma 3.3. Hence, $w_0 \neq 0$.

From the lower semi continuity of norm, the continuity of g and (5.58), it follows that

$$g(\|w_0\|^{\frac{N}{2}})\|w_0\|^{\frac{N}{2}} - \lim_{n\to\infty}\int_B |w_0|^q\, dx \leq \liminf_{n\to+\infty} \big(g(\|w_n\|^{\frac{N}{2}})\|w_n\|^{\frac{N}{2}} - \int_B |w_n|^q\, dx\big)\cdot \tag{5.59}$$

On the other hand, by using $\langle \mathcal{J}'(w_n), w_n\rangle = 0$ and (5.58), we have

$$\begin{aligned} \liminf_{n\to+\infty} g(\|w_n\|^{\frac{N}{2}})\|w_n\|^{\frac{N}{2}} &= \liminf_{n\to+\infty}\int_B (f(x,w_n)w_n + |w_n|^q)dx \\ &= \int_B (f(x,w_0)w_0 + |w_0|^q)dx. \end{aligned} \tag{5.60}$$

From (5.59) and (5.60) we deduce that $\langle \mathcal{J}'(w_0), w_0\rangle \leq 0$. Then, as in Lemma 3.2 this implies that there exists $s \in (0,1]$ such that $sw_0 \in \mathcal{N}$. Thus, by the lower semi continuity of norm, (1.8), Lemma 5.4 and Lemma 5.3, we get that

$$\begin{aligned} m \leq \mathcal{J}(sw_0) &= \mathcal{J}(sw_0) - \frac{1}{q}\langle \mathcal{J}'(sw_0), sw_0\rangle \\ &= \frac{2}{N}G(\|sw_0\|^{\frac{N}{2}}) - \frac{1}{q}g(\|sw_0\|^{\frac{N}{2}})\|sw_0\|^{\frac{N}{2}} \\ &\quad + \frac{1}{q}\int_B \big(f(x,sw_0)sw_0 - qF(x,sw_0)\big)dx \end{aligned}$$

$$
\begin{aligned}
&< \frac{2}{N}G(\|w_0\|^{\frac{N}{2}}) - \frac{1}{q}g(\|w_0\|^{\frac{N}{2}})\|w_0\|^{\frac{N}{2}} + \frac{1}{q}\int_B \big(f(x,w_0)w_0 - qF(x,w_0)\big)dx \\
&\leq \liminf_{n\to+\infty}\Big[\frac{2}{N}G(\|w_n\|^{\frac{N}{2}}) - \int_B F(x,w_n)dx\Big] \\
&\quad - \liminf_{n\to+\infty}\Big[\frac{1}{q}g(\|w_n\|^{\frac{N}{2}})\|w_n\|^{\frac{N}{2}} - \frac{1}{q}\int_B f(x,w_n)w_n dx\Big] \\
&\leq \liminf_{n\to+\infty}\big[\mathcal{J}(w_n) - \frac{1}{q}\langle \mathcal{J}'(w_n), w_n\rangle\big] = m.
\end{aligned}
$$

Therefore, we get that $\mathcal{J}(sw_0) = m$, which is the desired conclusion. □

Proof of Theorem 1.4. From Lemma 5.5 there exists w_0 such that $\mathcal{J}(w_0) = m$. Now, by Lemma 3.4, we deduce that $\mathcal{J}'(w_0) = 0$. So, w_0 is a solution to problem (P). □

Remark 5.1. In the sub-critical case, our energy does not lose its compactness. In this case, we can have an analogous result only with the conditions (G_1), (G_2), (H_1), (H_2), (H_3) and (H_4) and without going through the auxiliary problem.

So, we can announce the following theorem

Theorem 5.6. *Let $f(x,t)$ be a function that verifies (2.18), (H_1), (H_2), (H_3) and (H_4). Assume that the condition (G_1) and (G_1) hold. Then problem (P) has a radial solution with minimal energy.*

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