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Fault-tolerant Resolvability of Double Antiprism and Its Related Graphs

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Abstract. Consider a graph, $\Gamma = (V, E)$, where $V(\Gamma)$ and $E(\Gamma)$ are referred to be its vertex and edge sets respectively. Two vertices k_1 and k_2 in Γ are said to be resolved by a vertex k, if $d(k, k_1) \neq d(k, k_2)$ in Γ. Then, a subset $R \subseteq V(\Gamma)$ with this property, i.e., every pair of different vertices in Γ can be resolved by at least one member of R, is said to be a resolving set (RS) for Γ. The smallest cardinality set R with resolving characteristic is called the metric basis (MB) for Γ, and the MB set cardinality is the metric dimension (MD) for Γ, denoted by $dim_v(\Gamma)$. A resolving set R_f for Γ is said to have the property of fault-tolerance or said to be FTRS (fault-tolerant resolving set) if the property of resolving holds in $R_f - \{k\}$ for every k in R_f. The FTMD of Γ is the minimum cardinality of a FTRS, denoted $dim_f(\Gamma)$. These are also known as the resolvability parameters for the Γ. We introduce the concept of independence in FTRSs for graphs and derive several results and observations for the same in this manuscript. We also consider three almost similar families of infinite convex polytopes and investigate their FTRSs as well as FTMD.

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1. Introduction and Preliminaries

Throughout this article, all graphs taken under consideration are planar, connected, non-trivial, undirected, and simple. In order to carry out the basic of graph theory, we are following the book [23]. Consider a graph, denoted by $\Gamma = (V, E)$, where $V(\Gamma)$ and $E(\Gamma)$ are referred to be its vertex and edge sets respectively. The totality of distinct edges in the shortest length path between two different vertices k_1 and k_2 in $V(\Gamma)$, is referred to as the distance $(d(k_1, k_2))$ between k_1 and k_2 in Γ. The number of distinct edges touches a vertex k in Γ is called as the degree (valency) of a vertex k (denoted by d_k). We adopt the following symbols, i.e., K_m, P_m, and C_m to represent the complete graphs, path graphs, and cycle graphs of order $m \geq 3$. Now, suppose $R = \{c^1, c^2, c^3, ..., c^t\}$ be a subset consisting of t number of vertices in $V(\Gamma)$ and $k \in V$. Then, the metric coordinate (metric code) $\ss(k|R)$ of k corresponding to R is the t-tuple $(d(k, c^1), d(k, c^2), d(k, c^3), ..., d(k, c^t))$. Two vertices c_1 and c_2 in Γ are said to resolved by a vertex k, if $d(k, c_1) \neq d(k, c_2)$ in Γ. Then, a subset R in $V(\Gamma)$ with this property, i.e., every pair of different vertices in Γ can be resolved by at least one member of R, is said to be a resolving set (RS) for Γ. The smallest cardinality set R with resolving characteristic is called the metric basis (MB) for Γ, and the MB set cardinality is the metric dimension (MD) for Γ, denoted by $dim_v(\Gamma)$.

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The concept of MD was introduced independently by Slater [19] and Harary & Melter [5]. Since then, the problem of MD has received a lot of attention. The notions of RS and MD have proven useful in various areas such as robotic navigation, the structure of chemical compounds, combinatorial optimization, image processing & pattern recognization, connected joins in networks, pharmaceutical chemistry, game theory, etc. for these see [9, 13, 14, 17].

Hernando et al. [7] computed the MD of Fan graph F_n $(n \geq 7)$, and proved that $dim(F_n) = \lfloor \frac{2n+2}{5} \rfloor$. In [2], Buczkowski et al. proved that for Wheel graph W_n $(n \geq 7)$, the MD is $\lfloor \frac{2n+2}{5} \rfloor$. Tomescu and Javaid [21] proved that $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$, for the Jahangir graph J_{2n} $(n \geq 4)$. After that, the MD for several well-known graphs have been investigated such as prism graph, convex polytopes, antiprism graph, unicyclic graph, Petersen graph, flower graph, some regular graphs, etc. see [11, 16, 17, 18] and references therein.

Recent advancements in the definition of MD have paved the path for a new related concept called fault tolerance. Assume that a network has n-processing elements (or units). To create a self-stable fault-tolerant system, it is necessary that if any single unit fails (or crashes), another chain of units (with the exception of the faulty element) will replace the initial (or original) chain. As a result, the nature of this fault tolerance enables the machine to keep operating, possibly at a reduced pace, rather than crashing completely.

A RS R_f is called fault-tolerant (FT) if $R_f \setminus u^i$ is also a RS, for all $u^i \in R_f$, and therefore the minimum cardinality of such R_f is referred to as FTMD of Γ, denoted by $fdim(\Gamma)$. If $|R_f| = fdim(\Gamma)$, then R_f is known as the *fault-tolerant metric basis* (FTMB) for Γ. Fault-tolerant structures have been used successfully in engineering and computer science [6]. Slater initiated the study of fault-tolerant sets in [20]. In [8], Hernando et al. proposed the idea of FTMD. They investigated the fault tolerance in trees and presented a significant result for FTMD in the form of an lower bound using MD, independent of graph choice, which is $dim(\Gamma)(1+2.5^{dim(\Gamma)-1}) \leq fdim(\Gamma)$.

The notion of FTMD is an interesting concept and has been studied by many researchers. The FTMD for P_n, C_n, and K_n are as follows:

Proposition 1.1. [4] *For $n \geq 3$, we have $fdim(P_n) = 2$, $fdim(C_n) = 3$, and $fdim(K_n) = n$.*

In [15], Raza et al. computed the FTMD of some classes of convex polytopes. Voronov in [22] investigated the FTMD of the king's graph. For more work on the FTMD, see references in [1, 6]. A subset R^i of $V(\Gamma)$ is an independent set for Γ if no two vertices in R^i are adjacent.

The independence in RSs was first introduced by Chartrand et al. [3]. They characterize all connected non-trivial graphs Γ of order n with independent resolving numbers 1, $n-2$, and $n-1$.

Likewise resolving sets, in this work, we study the independence in FTRSs, and obtain it for some known graphs. We obtain the FTMD for three closely related classes of convex polytopes, viz., double antiprism $\mathbb{A}_n$, S_n, and T_n [10]. We locate FTRS of minimum cardinality in them. We conclude the article with some open problems regarding the independence of FTRSs. For the double antiprism $\mathbb{A}_n$, S_n, and T_n, Imran et al. in [10], proved the following:

Proposition 1.2. *$dim(\mathbb{A}_n) = dim(S_n) = dim(T_n) = 3$, where $n \geq 6$ is a positive integer.*

By the definition of FTRS and Proposition 1.2, we again have the following Proposition.

Proposition 1.3. *$fdim(\mathbb{A}_n) = fdim(S_n) = fdim(T_n) \leq 4$, where $n \geq 6$ is a positive integer.*

2. Independent Fault-Tolerant Resolving Sets

Independent sets (or stable sets, ISs for short) in graphs are the most extensively studied concepts in gralabelph theory. The *maximum independent sets* (MISs) are ISs with maximum cardinality, and these ISs have received attention in the recent past. The *vertex independence number* (or *independence number*) of a graph Γ, denoted by $\beta(\Gamma)$, is the cardinality of MIS in Γ. There are also several ISs of minimum cardinality which are of interest with respect to theoretical and applied points of view.

A *maximal independent set* of vertices is an IS of vertices that are not properly contained in any other IS of vertices. The minimum cardinality of a maximal independent set is denoted by $i(\Gamma)$. This parameter is also known as the independent domination number because it has the smallest cardinality of an IS of vertices that dominates all the vertices of Γ.

In [3], Chartrand et al. explored the independence in resolving sets and provided some significant observations and results. We can see that some graphs consist of ISs R_f^i with the property that $R_f^i - \{v^i\}$ is a RS for every v^i in R_f^i. Therefore, this paper aims to detect the existence of such ISs in graphs and, if they exist, to study the minimum possible cardinality of such a set.

An *independent fault-tolerant resolving set* (IFTRS) R_f^i in a simple connected graph, Γ is (1) independent and (2) FTRS. The cardinality of a minimum IFTRS in Γ is the independent fault-tolerant resolving number, denoted by ifr(Γ). Suppose Γ be a non-trivial connected graph with $|V(\Gamma)| = n$, containing an IFTRS. Since every IFTRS is an FTRS, so it follows that

$$1 + dim(\Gamma) \leq fdim(\Gamma) \leq ifr(\Gamma) \leq 1 + \beta(\Gamma) \leq n \tag{1}$$

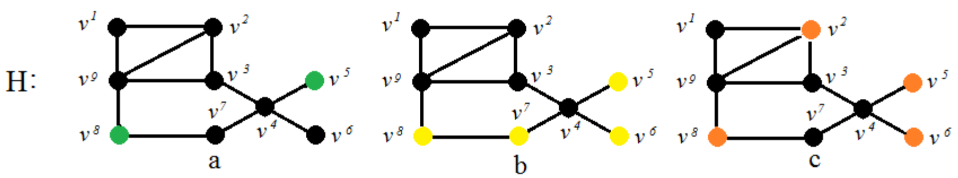

FIGURE 1. Graph H.

To explain this concept, consider a graph H in Fig. 1(a). The set $R = \{v^5, v^8\}$ (with green vertices) is a basis for H and so $dim(H) = 2$. Next, the set $R_f = \{v^5, v^6, v^7, v^8\}$ (with yellow vertices) in Fig. 1(b) is the FTRS set for H and therefore $fdim(H) = 4$. However, R_f is not an IFTRS for H. The set $R_f^i = \{v^2, v^5, v^6, v^8\}$

(with orange vertices) in Fig. 1(c) is an IFTRS. Now, the co-ordinates of each vertex of H with respect to R_f^i are $ß_f(v^1|R_f^i) = (1,4,4,2)$, $ß_f(v^2|R_f^i) = (0,3,3,2)$, $ß_f(v^3|R_f^i) = (1,2,2,2)$, $ß_f(v^4|R_f^i) = (2,1,1,2)$, $ß_f(v^5|R_f^i) = (3,0,2,3)$, $ß_f(v^6|R_f^i) = (3,2,0,3)$, $ß_f(v^7|R_f^i) = (3,2,2,1)$, $ß_f(v^8|R_f^i) = (2,3,3,0)$, $ß_f(v^9|R_f^i) = (1,3,3,1)$. The codes with respect to the FTRS R_f are called as the fault-tolerant metric codes (FTMC), denoted by $ß_f(v^j|R_f)$. A case-by-case analysis shows that H contains no 3-element IFTRS (or FTRS) and so $\text{ifr}(H) = 4$. The set $\{v^1, v^3, v^5, v^6, v^7\}$ is a MIS of H and therefore $\beta(H) = 5$. Thus the graph H of Fig. 1 has $\beta(H) = 5$, $dim(H) = 2$, $fdim(H) = 4$, and $\text{ifr}(H) = 4$.

Likewise independent resolving set (IRS), all graphs do not have IFTRS, as a result, $\text{ifr}(\Gamma)$ is not defined for all graphs Γ. For example, the only ISs of the complete graph K_n; $n \geq 3$ are the singleton sets. Hence, $\text{ifr}(K_n)$ is not defined for $n \geq 1$. In [3], Chartrand et al. consider three regular graphs viz., the Petersen graph P, $K_{3,3}$, and Q_3 (see Fig. 2). For these graphs they found that, $\text{ir}(P)$=$\text{ir}(Q_3) = 3$ and for $K_{3,3}$, $\text{ir}(K_{3,3})$ does not exist.

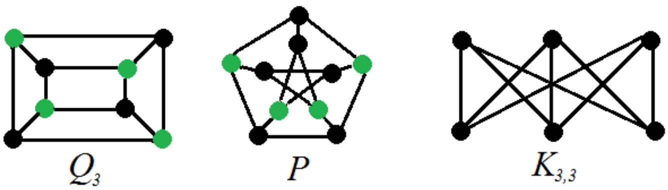

FIGURE 2. Regular graphs.

Similarly, for the IFTRS, we find that $\text{ifr}(K_{3,3})$ does not exist and $\text{ifr}(P)$=$\text{ifr}(Q_3) = 4$. The green vertices in Fig. 2 represent the minimum IFTRS for both P and Q_3. Now, we have some results and observations regarding IRS and IFTRS.

Proposition 2.1. *$\Gamma = P_n$; $n \geq 3$ iff $\text{ifr}(P_n) = 2$.*

Proof. The proof is the same as for the FTMD of paths in [8]. □

Suppose $V(C_n) = \{v^1, v^2, v^3, ..., v^n\}$ denotes the set of vertices in the cycle graph C_n. Then for IFTRS for C_n, we have:

Proposition 2.2. *For cycle graph C_n; $n \geq 6$, we have $\text{ifr}(C_n) = 3$.*

Proof. Consider $R_f^1 = \{v^1, v^3, v^5\}$ and $R_f^2 = \{v^1, v^3, v^6\}$. Then, from Lemma 2 in [12], we find that, for $n \geq 6$ and $n \neq 8$, R_f^1 is the IFTRS for C_n and R_f^2 is the IFTRS of C_n for $n = 8$. Therefore, $\text{ifr}(C_n) = 3$ for $n \geq 6$. □

Proposition 2.3. *Every graph with IFTRS has IRS.*

By the definition of IFTRS, we see that Proposition 2.3 is trivial. But the converse of the Proposition 2.3 is not true. For example, suppose C_5, A_{10}, and B_{12} are three graphs with 5, 10, and 12 vertices, as shown in Fig. 3. We find that $\text{ir}(\Gamma)$ is defined

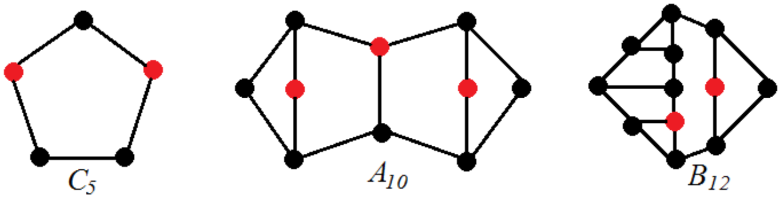

FIGURE 3. Graphs with IRS.

for these three graphs (vertices in red color represent IRS), but ifr(Γ) is not, where $\Gamma = C_5$, A_{10}, and B_{12}.

If $d(x, d) = d(x, c)$, $\forall\ x \in V(\Gamma) - \{c, d\}$, then the vertices d and c are said to be *distance similar* (or distance equivalent) in Γ. Let $N(z)$ (open neighbourhood) be the set of vertices adjacent to z in Γ, and let $N[z] = N(z) \cup \{z\}$ (closed neighbourhood). Then, in a non-trivial connected graph Γ, two vertices x and c are distance equivalent iff (1) $xc \in E(\Gamma)$ and $N[x] = N[c]$ or (2) $xc \notin E(\Gamma)$ and $N(x) = N(c)$. Moreover, the distance similarity is an equivalence relation on $V(\Gamma)$. Then we have the following observation.

Observation 1. *In a connected graph* Γ*, if* D *is a distance similar equivalence class with* $|D| = w \geq 2$*, then every FTRS of* Γ *contains all the vertices from* D*.*

If D is a distance similar equivalence class of Γ, then either the subgraph $\langle D \rangle$ induced by D is complete in Γ or D is an independent set in Γ. Thus, we observe the following:

Observation 2. *Let* Γ *be a graph and let* D *with* $|D| \geq 3$ *be a distance similar equivalence class in* Γ*. Then ifr(*Γ*) is not defined if* D *is not independent in* Γ*.*

For observation 2, we find that the converse is not true. For instance, suppose $\Gamma = K_{3,3}$ with partite sets A_1 and A_2 (see Fig. 2). Then, we find that ifr(Γ) is not defined. On the other side, A_1 and A_2 are the only two independent distance similar equivalence classes in $\Gamma = K_{3,3}$.

Determining the FTRS for a complicated and large graph or we can say for every graph, is always a challenging problem. However, several researchers have made efforts to obtain FTRS as well as FTMD for certain graph families. There are numerous graph families for which FTRS and FTMD have yet to be investigated. So, in this direction, we in this paper consider three almost similar graph families of convex polytopes and study their FTRS and FTMD.

3. Minimum Fault-Tolerant Number of $\mathbb{A}_n$

In this section, we take a well-known graph, denoted by $\mathbb{A}_n$ and shown in Fig. 4, and investigates its several properties including its FTMD.

The graph of $\mathbb{A}_n$, which is also known as double antiprism graph, comprises of $4n + 2$ faces, i.e., $4n$ triangular faces, one n sided face, and an outer face. The total number of vertices present in $\mathbb{A}_n$ are $3n$ and total number of edges present are $7n$. We adopt two symbols to denote sets of edges and vertices, viz., $E(\mathbb{A}_n)$ and $V(\mathbb{A}_n)$.

These two sets are shown as follows:

$$V(\mathbb{A}_n) \quad = \quad \{j_r^{\bar{q}} : 1 \leq r \leq 3; 1 \leq \bar{q} \leq n\}$$

$$E(\mathbb{A}_n) \quad = \quad \{j_r^{\bar{q}} j_r^{\bar{q}+1}, j_1^{\bar{q}} j_2^{\bar{q}}, j_2^{\bar{q}} j_3^{\bar{q}}, j_2^{\bar{q}} j_1^{\bar{q}+1}, j_3^{\bar{q}} j_2^{\bar{q}+1} : 1 \leq r \leq 3; 1 \leq \bar{q} \leq n\}$$

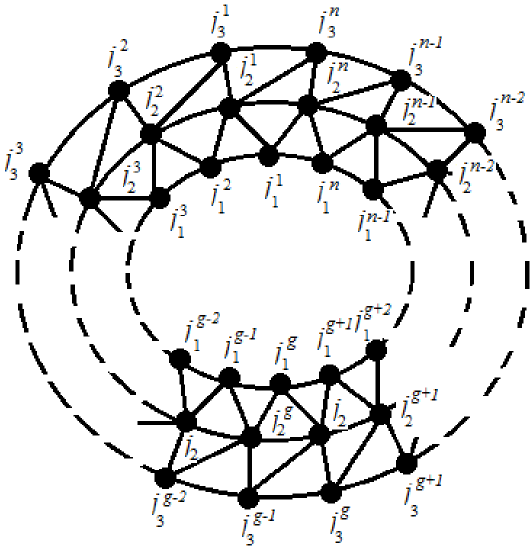

FIGURE 4. Double Antiprism $\mathbb{A}_n$.

For the vertices present in $\{j_1^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *first cycle* vertices in $\mathbb{A}_n$; for vertices in $\{j_2^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *second cycle* vertices in $\mathbb{A}_n$; and the vertices in $\{j_3^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *third cycle* vertices in $\mathbb{A}_n$. In the following result, we investigate the FTMD of $\mathbb{A}_n$.

Theorem 3.1. *$fdim(\mathbb{A}_n) = 4$, where $n \geq 6$ is a positive integer.*

Proof. To complete the proof, we need to show that the FTMD is 4, for the graph of convex polytope $\mathbb{A}_n$. This can be achieved by considering two cases directly depending upon the even ($n \equiv 0(mod\ 2)$) and odd ($n \equiv 1(mod\ 2)$) nature of the integer n. For n, we first consider the even case and then later the odd case.

Case(I) $n \equiv 0(mod\ 2)$
For this, we set $n = 2\bar{c}$, $\bar{c} \in \mathbb{N}$ and $\bar{c} \geq 3$. Assuming the set $R_f = \{j_1^1, j_1^2, j_1^{\bar{c}+1}, j_1^{\bar{c}+2}\} \subseteq V(\mathbb{A}_n)$.

Claim: The set R_f is a FTRS for $\mathbb{A}_n$.
In order to get this, we give FTMC to every vertex of $\mathbb{A}_n$ with respect to R_f.

The FTMCs for members in the set $J_1 = \{j_1^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, are presented below in Table 1 and the collection of FTMCs for these vertices are denoted by the set JC_1.

Table 1. FTMCs for J_1 set vertices.

$ß_f(j_1^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+2}$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}-1$	1	$\bar{c}-\bar{q}+1$	$\bar{c}-1$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(2\leq\bar{q}\leq\bar{c}+1)$	$\bar{q}-1$	$\bar{q}-2$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+2$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+2)$	$2\bar{c}-\bar{q}+1$	$\bar{q}-2$	$\bar{q}-\bar{c}-1$	$\bar{c}-\bar{q}+2$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{c}+3\leq\bar{q}\leq 2\bar{c})$	$2\bar{c}-\bar{q}+1$	$2\bar{c}-\bar{q}+2$	$\bar{q}-\bar{c}-1$	$\bar{q}-\bar{c}-2$

The FTMCs for members in the set $J_2=\{j_2^{\bar{q}}:1\leq\bar{q}\leq n\}$, are presented below in Table 2 and the collection of FTMCs for these vertices are denoted by the set JC_2.

Table 2. FTMCs for J_2 set vertices.

$ß_f(j_2^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+2}$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}$	1	$\bar{c}-\bar{q}+1$	$\bar{c}$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(2\leq\bar{q}\leq\bar{c})$	$\bar{q}$	$\bar{q}-1$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+2$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+1)$	$2\bar{c}-\bar{q}+1$	$\bar{q}-1$	$\bar{q}-\bar{c}$	$\bar{c}-\bar{q}+2$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{c}+2\leq\bar{q}\leq 2\bar{c})$	$2\bar{c}-\bar{q}+1$	$2\bar{c}-\bar{q}+2$	$\bar{q}-\bar{c}$	$\bar{q}-\bar{c}-1$

Lastly, the FTMCs for members in the set $J_3=\{j_3^{\bar{q}}:1\leq\bar{q}\leq n\}$, are presented below in Table 3 and the collection of FTMCs for these vertices are denoted by the set JC_3.

Table 3. FTMCs for J_3 set vertices.

$ß_f(j_3^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+2}$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}+1$	2	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+2$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(2\leq\bar{q}\leq\bar{c}-1)$	$\bar{q}+1$	$\bar{q}$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+2$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c})$	$\bar{q}+1$	$\bar{q}$	2	$\bar{c}-\bar{q}+2$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+1)$	$2\bar{c}-\bar{q}+1$	$\bar{q}+1$	$\bar{q}-\bar{c}+1$	2
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{c}+2\leq\bar{q}\leq 2\bar{c}-1)$	$2\bar{c}-\bar{q}+1$	$2\bar{c}-\bar{q}+2$	$\bar{q}-\bar{c}+1$	$\bar{q}-\bar{c}$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=2\bar{c})$	2	$2\bar{c}-\bar{q}+2$	$\bar{q}-\bar{c}+1$	$\bar{q}-\bar{c}$

The total number of FTMCs listed above are equal to $|JC_1\cup JC_2\cup JC_3|=3n$. To prove that the set R_f is a FTRS for $\mathbb{A}_n$, it is first compulsory to show that, it is also a resolving set for $\mathbb{A}_n$. From all the codes presented above in JC_1, JC_2, and JC_3, one can clearly verify that all are different from one an other in at least one coordinate and are also unique. Therefore, from this fact, now it is clear that the set R_f is a resolving set for $\mathbb{A}_n$. Next, to finish the proof, we only need to show that the set R_f possesses the fault-tolerance property in it. For this, we simply adopt the definition of FTRS and prove the following four sets $R_1=\{j_1^1,j_1^2,j_1^{\bar{c}+1}\}$, $R_2=\{j_1^1,j_1^2,j_1^{\bar{c}+2}\}$, $R_3=\{j_1^1,j_1^{\bar{c}+1},j_1^{\bar{c}+2}\}$, and $R_4=\{j_1^2,j_1^{\bar{c}+1},j_1^{\bar{c}+2}\}$ (i.e., by using $R_f\setminus\{j\}$, $\forall\, j\in R_f$) to be resolving in $\mathbb{A}_n$. But on verifying manually the FTMCs with respect to the set R_f in $\mathbb{A}_n$, we find that the respective metric codes with respect to the sets R_1, R_2, R_3, and R_4 are unique and distinct for every vertex present in $\mathbb{A}_n$. From this particular fact, we find that the set R_f is the FTRS for $\mathbb{A}_n$. Therefore, we have $fdim(\mathbb{A}_n)\leq 4$.

Now, from this and Proposition 1.3, we find that $fdim(\mathbb{A}_n) = 4$, which completes the proof.

Case(II) $n \equiv 1(mod\ 2)$
For this, we set $n = 2\bar{c}+1$, $\bar{c} \in \mathbb{N}$ and $\bar{c} \geq 3$. Assuming the set
$R_f = \{j_1^1, j_1^2, j_1^{\bar{c}+1}, j_1^{\bar{c}+3}\} \subseteq V(\mathbb{A}_n)$.

Claim: The set R_f is a FTRS for $\mathbb{A}_n$.
In order to get this, we give FTMC to every vertex of $\mathbb{A}_n$ with respect to R_f. The FTMCs for members in the set $J_1 = \{j_1^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, are presented below in Table 4 and the collection of FTMCs for these vertices are denoted by the set JC_1.

TABLE 4. FTMCs for J_1 set vertices.

$ß_f(j_1^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+3}$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}-1$	1	$\bar{c}-\bar{q}+1$	$\bar{c}-1$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=2)$	$\bar{q}-1$	$\bar{q}-2$	$\bar{c}-\bar{q}+1$	$\bar{c}$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(3 \leq \bar{q} \leq \bar{c}+1)$	$\bar{q}-1$	$\bar{q}-2$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+3$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+2)$	$2\bar{c}-\bar{q}+2$	$\bar{q}-2$	$\bar{q}-\bar{c}-1$	$\bar{c}-\bar{q}+3$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+3)$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}-1$	$\bar{c}-\bar{q}+3$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{c}+4 \leq \bar{q} \leq 2\bar{c}+1)$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}-1$	$\bar{q}-\bar{c}-3$

The FTMCs for members in the set $J_2 = \{j_2^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, are presented below in Table 5 and the collection of FTMCs for these vertices are denoted by the set JC_2.

TABLE 5. FTMCs for J_2 set vertices.

$ß_f(j_2^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+3}$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}$	1	$\bar{c}-\bar{q}+1$	$\bar{c}$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(2 \leq \bar{q} \leq \bar{c})$	$\bar{q}$	$\bar{q}-1$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+3$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+1)$	$2\bar{c}-\bar{q}+2$	$\bar{q}-1$	$\bar{q}-\bar{c}$	$\bar{c}-\bar{q}+3$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+2)$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}$	$\bar{c}-\bar{q}+3$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{c}+3 \leq \bar{q} \leq 2\bar{c}+1)$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}$	$\bar{q}-\bar{c}-2$

Lastly, the FTMCs for members in the set $J_3 = \{j_3^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, are presented below in Table 6 and the collection of FTMCs for these vertices are denoted by the set JC_3.

Table 6. FTMCs for J_3 set vertices.

$ß_f(j_3^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+3}$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}+1$	2	$\bar{c}-\bar{q}+1$	$\bar{c}+1$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(2\leq\bar{q}\leq\bar{c}-1)$	$\bar{q}+1$	$\bar{q}$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c})$	$\bar{q}+1$	$\bar{q}$	2	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+1)$	$2\bar{c}-\bar{q}+2$	$\bar{q}$	$\bar{q}-\bar{c}+1$	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+2)$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}+1$	2
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{c}+3\leq\bar{q}\leq2\bar{c})$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}+1$	$\bar{q}-\bar{c}-1$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=2\bar{c}+1)$	2	$2\bar{c}-\bar{q}+3$	$\bar{c}+1$	$\bar{q}-\bar{c}-1$

The total number of FTMCs listed above are equal to $|JC_1 \cup JC_2 \cup JC_3| = 3n$. To prove that the set R_f is a FTRS for $\mathbb{A}_n$, it is first compulsory to show that, it is also a resolving set for $\mathbb{A}_n$. From all the codes presented above in JC_1, JC_2, and JC_3, one can clearly verify that all are different from one an other in at least one coordinate and are also unique. Therefore, from this fact, now it is clear that the set R_f is a resolving set for $\mathbb{A}_n$. Next, to finish the proof, we only need to show that the set R_f possesses the fault-tolerance property in it. For this, we simply adopt the definition of FTRS and prove the following four sets $R_1 = \{j_1^1, j_1^2, j_1^{\bar{c}+1}\}$, $R_2 = \{j_1^1, j_1^2, j_1^{\bar{c}+3}\}$, $R_3 = \{j_1^1, j_1^{\bar{c}+1}, j_1^{\bar{c}+3}\}$, and $R_4 = \{j_1^2, j_1^{\bar{c}+1}, j_1^{\bar{c}+3}\}$ (i.e., by using $R_f \setminus \{j\}$, $\forall\, j \in R_f$) to be resolving in $\mathbb{A}_n$. But on verifying manually the FTMCs with respect to the set R_f in $\mathbb{A}_n$, we find that the respective metric codes with respect to the sets R_1, R_2, R_3, and R_4 are unique and distinct for every vertex present in $\mathbb{A}_n$. From this particular fact, we find that the set R_f is the FTRS for $\mathbb{A}_n$. Therefore, we have $fdim(\mathbb{A}_n) \leq 4$. Now, from this and Proposition 1.3, we find that $fdim(\mathbb{A}_n) = 4$, which completes the proof for this case. □

Corollary 3.2. *The FTMD for the double antiprism $\mathbb{A}_n$ is constant.*

4. Minimum Fault-Tolerant Number of S_n

In this section, we take a graph, denoted by S_n and shown in Fig. 5, and investigates its several properties including its FTMD.

The graph of S_n, comprises of $4n+2$ faces, i.e., $2n$ triangular faces, $2n$ square faces, one n sided face, and an outer face. The total number of vertices present in S_n are $4n$ and total number of edges present are $8n$. We adopt two symbols to denote sets of edges and vertices, viz., $E(S_n)$ and $V(S_n)$. These two sets are shown as follows:

$$V(S_n) = \{j_r^{\bar{q}} : 1 \leq r \leq 4; 1 \leq \bar{q} \leq n\}$$

$$E(S_n) = \{j_r^{\bar{q}} j_r^{\bar{q}+1}, j_1^{\bar{q}} j_2^{\bar{q}}, j_2^{\bar{q}} j_3^{\bar{q}}, j_3^{\bar{q}} j_4^{\bar{q}}, j_3^{\bar{q}} j_2^{\bar{q}+1} : 1 \leq r \leq 4; 1 \leq \bar{q} \leq n\}$$

For the vertices present in $\{j_1^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *first cycle* vertices in S_n; for vertices in $\{j_2^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *second cycle* vertices in S_n; for vertices in $\{j_3^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *third cycle* vertices in S_n; and the vertices in $\{j_4^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *fourth cycle* vertices in S_n. In the following result,

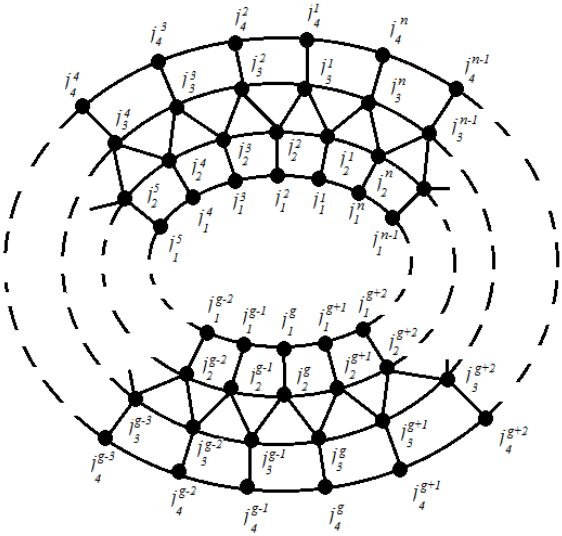

FIGURE 5. The Graph S_n.

we investigate the FTMD of S_n. In the following result, we investigate the FTMD of S_n.

Theorem 4.1. *$fdim(S_n) = 4$, where $n \geq 6$ is a positive integer.*

Proof. To complete the proof, we need to show that the FTMD is 4, for the graph of convex polytope S_n. This can be achieved by considering two cases directly depending upon the even ($n \equiv 0(mod\ 2)$) and odd ($n \equiv 1(mod\ 2)$) nature of the integer n. For n, we first consider the even case and then later the odd case.

Case(I) $n \equiv 0(mod\ 2)$
For this, we set $n = 2\bar{c}$, $\bar{c} \in \mathbb{N}$ and $\bar{c} \geq 3$. Assuming the set $R_f = \{j_1^1, j_1^2, j_1^{\bar{c}+1}, j_1^{\bar{c}+2}\} \subseteq V(S_n)$.

Claim: The set R_f is a FTRS for S_n.
In order to get this, we give FTMC to every vertex of S_n with respect to R_f.

The FTMCs for members in the set $J_1 = \{j_1^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, are presented below in Table 7 and the collection of FTMCs for these vertices are denoted by the set JC_1.

TABLE 7. FTMCs for J_1 set vertices.

$ß_f(j_1^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+2}$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q} = 1)$	$\bar{q} - 1$	1	$\bar{c} - \bar{q} + 1$	$\bar{c} - 1$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(2 \leq \bar{q} \leq \bar{c} + 1)$	$\bar{q} - 1$	$\bar{q} - 2$	$\bar{c} - \bar{q} + 1$	$\bar{c} - \bar{q} + 2$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q} = \bar{c} + 2)$	$2\bar{c} - \bar{q} + 1$	$\bar{q} - 2$	$\bar{q} - \bar{c} - 1$	$\bar{c} - \bar{q} + 2$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{c} + 3 \leq \bar{q} \leq 2\bar{c})$	$2\bar{c} - \bar{q} + 1$	$2\bar{c} - \bar{q} + 2$	$\bar{q} - \bar{c} - 1$	$\bar{q} - \bar{c} - 2$

The FTMCs for members in the set $J_2 = \{j_2^{\bar{q}} : 1 \le \bar{q} \le n\}$ are as follows $ß_f(j_2^{\bar{q}}|R_f) = ß_f(j_1^{\bar{q}}|R_f) + (1,1,1,1)$, for $1 \le \bar{q} \le n$ and the collection of FTMCs for these vertices are denoted by the set JC_2. Next, the FTMCs for members in the set $J_3 = \{j_3^{\bar{q}} : 1 \le \bar{q} \le n\}$, are presented below in Table 8 and the collection of FTMCs for these vertices are denoted by the set JC_3.

Table 8. FTMCs for J_3 set vertices.

$ß_f(j_3^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+2}$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}+1$	2	$\bar{c}-\bar{q}+2$	$\bar{c}+1$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(2 \le \bar{q} \le \bar{c})$	$\bar{q}+1$	$\bar{q}$	$\bar{c}-\bar{q}+2$	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+1)$	$2\bar{c}-\bar{q}+2$	$\bar{q}$	$\bar{q}-\bar{c}+1$	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{c}+2 \le \bar{q} \le 2\bar{c})$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}+1$	$\bar{q}-\bar{c}$

Lastly, the FTMCs for members in the set $J_4\{j_4^{\bar{q}} : 1 \le \bar{q} \le n\}$ are as follows $ß_f(j_4^{\bar{q}}|R_f) = ß_f(j_3^{\bar{q}}|R_f) + (1,1,1,1)$, for $1 \le \bar{q} \le n$ and the collection of FTMCs for these vertices are denoted by the set JC_4. The total number of FTMCs listed above are equal to $|JC_1 \cup JC_2 \cup JC_3 \cup JC_4| = 4n$. To prove that the set R_f is a FTRS for S_n, it is first compulsory to show that, it is also a resolving set for S_n. From all the codes presented above in JC_1, JC_2, JC_3, and JC_4, one can clearly verify that all are different from one an other in at least one coordinate and are also unique. Therefore, from this fact, now it is clear that the set R_f is a resolving set for S_n. Next, to finish the proof, we only need to show that the set R_f possesses the fault-tolerance property in it. For this, we simply adopt the definition of FTRS and prove the following four sets $R_1 = \{j_1^1, j_1^2, j_1^{\bar{c}+1}\}$, $R_2 = \{j_1^1, j_1^2, j_1^{\bar{c}+2}\}$, $R_3 = \{j_1^1, j_1^{\bar{c}+1}, j_1^{\bar{c}+2}\}$, and $R_4 = \{j_1^2, j_1^{\bar{c}+1}, j_1^{\bar{c}+2}\}$ (i.e., by using $R_f \setminus \{j\}$, $\forall\ j \in R_f$) to be resolving in S_n. But on verifying manually the FTMCs with respect to the set R_f in S_n, we find that the respective metric codes with respect to the sets R_1, R_2, R_3, and R_4 are unique and distinct for every vertex present in S_n. From this particular fact, we find that the set R_f is the FTRS for S_n. Therefore, we have $fdim(S_n) \le 4$. Now, from this and Proposition 1.3, we find that $fdim(S_n) = 4$, which completes the proof.

Case(II) $n \equiv 1(mod\ 2)$
For this, we set $n = 2\bar{c}+1$, $\bar{c} \in \mathbb{N}$ and $\bar{c} \ge 3$. Assuming the set $R_f = \{j_1^1, j_1^2, j_1^{\bar{c}+1}, j_1^{\bar{c}+2}\} \subseteq V(T_n)$.

Claim: The set R_f is a FTRS for T_n.
In order to get this, we give FTMC to every vertex of S_n with respect to R_f.

The FTMCs for members in the set $J_1 = \{j_1^{\bar{q}} : 1 \le \bar{q} \le n\}$, are presented below in Table 9 and the collection of FTMCs for these vertices are denoted by the set JC_1.

TABLE 9. FTMCs for J_1 set vertices.

$ß_f(j_1^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+2}$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}-1$	1	$\bar{c}-\bar{q}+1$	$\bar{c}$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(2\leq\bar{q}\leq\bar{c}+1)$	$\bar{q}-1$	$\bar{q}-2$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+2$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+2)$	$2\bar{c}-\bar{q}+2$	$\bar{q}-2$	$\bar{q}-\bar{c}-1$	$\bar{c}-\bar{q}+2$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{c}+3\leq\bar{q}\leq 2\bar{c}+1)$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}-1$	$\bar{q}-\bar{c}-2$

The FTMCs for members in the set $J_2=\{j_2^{\bar{q}}:1\leq\bar{q}\leq n\}$ are as follows $ß_f(j_2^{\bar{q}}|R_f)=ß_f(j_1^{\bar{q}}|R_f)+(1,1,1,1)$, for $1\leq\bar{q}\leq n$ and the collection of FTMCs for these vertices are denoted by the set JC_2. Next, the FTMCs for members in the sets $\{j_3^{\bar{q}}:1\leq\bar{q}\leq n\}$ are shown presented below in Table 10 and the collection of FTMCs for these vertices are denoted by the set JC_3.

TABLE 10. FTMCs for J_3 set vertices.

$ß_f(j_3^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+2}$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}+1$	2	$\bar{c}-\bar{q}+2$	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(2\leq\bar{q}\leq\bar{c})$	$\bar{q}+1$	$\bar{q}$	$\bar{c}-\bar{q}+2$	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+1)$	$2\bar{c}-\bar{q}+3$	$\bar{q}$	$\bar{q}-\bar{c}+1$	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{c}+2\leq\bar{q}\leq 2\bar{c}+1)$	$2\bar{c}-\bar{q}+3$	$2\bar{c}-\bar{q}+4$	$\bar{q}-\bar{c}+1$	$\bar{q}-\bar{c}$

Lastly, the FTMCs for members in the set $J_4\{j_4^{\bar{q}}:1\leq\bar{q}\leq n\}$ are as follows $ß_f(j_4^{\bar{q}}|R_f)=ß_f(j_3^{\bar{q}}|R_f)+(1,1,1,1)$, for $1\leq\bar{q}\leq n$ and the collection of FTMCs for these vertices are denoted by the set JC_4. The total number of FTMCs listed above are equal to $|JC_1\cup JC_2\cup JC_3\cup JC_4|=4n$. To prove that the set R_f is a FTRS for S_n, it is first compulsory to show that, it is also a resolving set for S_n. From all the codes presented above in JC_1, JC_2, JC_3, and JC_4, one can clearly verify that all are different from one an other in at least one coordinate and are also unique. Therefore, from this fact, now it is clear that the set R_f is a resolving set for S_n. Next, to finish the proof, we only need to show that the set R_f possesses the fault-tolerance property in it. For this, we simply adopt the definition of FTRS and prove the following four sets $R_1=\{j_1^1,j_1^2,j_1^{\bar{c}+1}\}$, $R_2=\{j_1^1,j_1^2,j_1^{\bar{c}+2}\}$, $R_3=\{j_1^1,j_1^{\bar{c}+1},j_1^{\bar{c}+2}\}$, and $R_4=\{j_1^2,j_1^{\bar{c}+1},j_1^{\bar{c}+2}\}$ (i.e., by using $R_f\setminus\{j\}$, $\forall\ j\in R_f$) to be resolving in S_n. But on verifying manually the FTMCs with respect to the set R_f in S_n, we find that the respective metric codes with respect to the sets R_1, R_2, R_3, and R_4 are unique and distinct for every vertex present in S_n. From this particular fact, we find that the set R_f is the FTRS for S_n. Therefore, we have $fdim(S_n)\leq 4$. Now, from this and Proposition 1.3, we find that $fdim(S_n)=4$, which completes the proof. □

Corollary 4.2. *The FTMD for S_n is constant.*

5. Minimum Fault-Tolerant Number of T_n

In this section, we take a graph, denoted by T_n and shown in Fig. 6, and also investigates its several properties including its FTMD.

The graph of T_n, comprises of $4n+2$ faces, i.e., $3n$ triangular faces, n pentagonal faces, one n sided face, and an outer face. The total number of vertices present in T_n are $4n$ and total number of edges present are $8n$. We adopt two symbols to denote sets of edges and vertices, viz., $E(T_n)$ and $V(T_n)$. These two sets are shown as follows:

$$V(T_n) = \{j_r^{\bar{q}} : 1 \leq r \leq 4; 1 \leq \bar{q} \leq n\}$$

$$E(T_n) = \{j_r^{\bar{q}} j_r^{\bar{q}+1}, j_1^{\bar{q}} j_2^{\bar{q}}, j_2^{\bar{q}} j_3^{\bar{q}}, j_3^{\bar{q}} j_4^{\bar{q}}, j_2^{\bar{q}} j_1^{\bar{q}+1}, j_3^{\bar{q}} j_2^{\bar{q}+1} : r = 1, 2, 4; 1 \leq \bar{q} \leq n\}$$

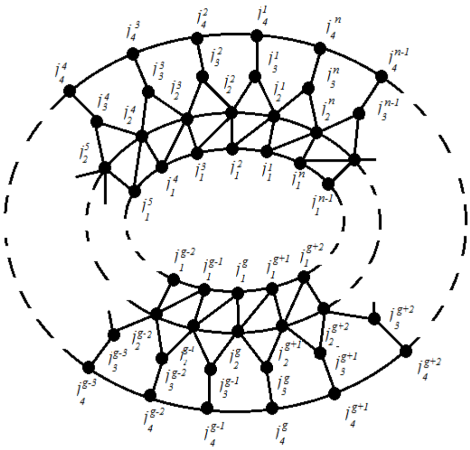

FIGURE 6. The Graph T_n

For the vertices present in $\{j_1^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *first cycle* vertices in T_n; for vertices in $\{j_2^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *second cycle* vertices in T_n; for vertices in $\{j_3^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *third cycle* vertices in T_n; and the vertices in $\{j_4^{\bar{q}} : 1 \leq \bar{q} \leq n\}$, we name them *fourth cycle* vertices in T_n. In the following result, we investigate the FTMD of T_n. In the following result, we investigate the FTMD of T_n.

Theorem 5.1. *$fdim(T_n) = 4$, where $n \geq 6$ is a positive integer.*

Proof. To complete the proof, we need to show that the FTMD is 4, for the graph of convex polytope T_n. This can be achieved by considering two cases directly depending upon the even ($n \equiv 0(mod\ 2)$) and odd ($n \equiv 1(mod\ 2)$) nature of the integer n. For n, we first consider the even case and then later the odd case.

Case(I) $n \equiv 0(mod\ 2)$
For this, we set $n = 2\bar{c}$, $\bar{c} \in \mathbb{N}$ and $\bar{c} \geq 3$. Assuming the set $R_f = \{j_1^1, j_1^2, j_1^{\bar{c}+1}, j_1^{\bar{c}+2}\} \subseteq V(T_n)$.

Claim: The set R_f is a FTRS for T_n.
In order to get this, we give FTMC to every vertex of T_n with respect to R_f.

The FTMCs for members in the sets $J_1 = \{j_1^{\bar{q}} : 1 \leq \bar{q} \leq n\}$ are presented below in Table 11 and the collection of FTMCs for these vertices are denoted by the set JC_1.

TABLE 11. FTMCs for J_1 set vertices.

$ß_f(j_1^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+2}$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}-1$	1	$\bar{c}-\bar{q}+1$	$\bar{c}-1$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(2 \leq \bar{q} \leq \bar{c}+1)$	$\bar{q}-1$	$\bar{q}-2$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+2$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+2)$	$2\bar{c}-\bar{q}+1$	$\bar{q}-2$	$\bar{q}-\bar{c}-1$	$\bar{c}-\bar{q}+2$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{c}+3 \leq \bar{q} \leq 2\bar{c})$	$2\bar{c}-\bar{q}+1$	$2\bar{c}-\bar{q}+2$	$\bar{q}-\bar{c}-1$	$\bar{q}-\bar{c}-2$

The FTMCs for members in the sets $J_2 = \{j_2^{\bar{q}} : 1 \leq \bar{q} \leq n\}$ are presented below in Table 12 and the collection of FTMCs for these vertices are denoted by the set JC_2.

TABLE 12. FTMCs for J_2 set vertices.

$ß_f(j_2^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+2}$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}$	1	$\bar{c}-\bar{q}+1$	$\bar{c}$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(2 \leq \bar{q} \leq \bar{c})$	$\bar{q}$	$\bar{q}-1$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+2$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+1)$	$2\bar{c}-\bar{q}+1$	$\bar{q}-1$	$\bar{q}-\bar{c}$	$\bar{c}-\bar{q}+2$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{c}+2 \leq \bar{q} \leq 2\bar{c})$	$2\bar{c}-\bar{q}+1$	$2\bar{c}-\bar{q}+2$	$\bar{q}-\bar{c}$	$\bar{q}-\bar{c}-1$

The FTMCs for members in the sets $J_3 = \{j_3^{\bar{q}} : 1 \leq \bar{q} \leq n\}$ are presented below in Table 13 below and the collection of FTMCs for these vertices are denoted by the set JC_3.

TABLE 13. FTMCs for J_3 set vertices.

$ß_f(j_3^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+2}$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}+1$	2	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+2$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(2 \leq \bar{q} \leq \bar{c}-1)$	$\bar{q}+1$	$\bar{q}$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+2$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c})$	$\bar{q}+1$	$\bar{q}$	2	$\bar{c}-\bar{q}+2$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+1)$	$2\bar{c}-\bar{q}+1$	$2\bar{c}-\bar{q}+2$	$\bar{q}-\bar{c}+1$	2
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{c}+2 \leq \bar{q} \leq 2\bar{c}-1)$	$2\bar{c}-\bar{q}+1$	$2\bar{c}-\bar{q}+2$	$\bar{q}-\bar{c}+1$	$\bar{q}-\bar{c}$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=2\bar{c})$	2	$2\bar{c}-\bar{q}+2$	$\bar{q}-\bar{c}+1$	$\bar{q}-\bar{c}$

Lastly, the FTMCs for members in the set $J_4 = \{j_4^{\bar{q}} : 1 \leq \bar{q} \leq n\}$ are as follows $ß_f(j_4^{\bar{q}}|R_f) = ß_f(j_3^{\bar{q}}|R_f) + (1,1,1,1)$, for $1 \leq \bar{q} \leq n$ and the collection of FTMCs for these vertices are denoted by the set JC_4. The total number of FTMCs listed above are equal to $|JC_1 \cup JC_2 \cup JC_3 \cup JC_4| = 4n$. To prove that the set R_f is a FTRS for T_n, it is first compulsory to show that, it is also a resolving set for T_n. From all the codes presented above in JC_1, JC_2, JC_3, and JC_4, one can clearly verify that all are different from one an other in at least one coordinate and are also unique. Therefore, from this fact, now it is clear that the set R_f is a resolving set for T_n.

Next, to finish the proof, we only need to show that the set R_f possesses the fault-tolerance property in it. For this, we simply adopt the definition of FTRS and prove the following four sets $R_1 = \{j_1^1, j_1^2, j_1^{\bar{c}+1}\}$, $R_2 = \{j_1^1, j_1^2, j_1^{\bar{c}+2}\}$, $R_3 = \{j_1^1, j_1^{\bar{c}+1}, j_1^{\bar{c}+2}\}$, and $R_4 = \{j_1^2, j_1^{\bar{c}+1}, j_1^{\bar{c}+2}\}$ (i.e., by using $R_f \setminus \{j\}$, $\forall\ j \in R_f$) to be resolving in T_n. But on verifying manually the FTMCs with respect to the set R_f in T_n, we find that the respective metric codes with respect to the sets R_1, R_2, R_3, and R_4 are unique and distinct for every vertex present in T_n. From this particular fact, we find that the set R_f is the FTRS for T_n. Therefore, we have $fdim(T_n) \leq 4$. Now, from this and Proposition 1.3, we find that $fdim(T_n) = 4$, which completes the proof.

Case(II) $n \equiv 1(mod\ 2)$
For this, we set $n = 2\bar{c} + 1$, $\bar{c} \in \mathbb{N}$ and $\bar{c} \geq 3$. Assuming the set
$R_f = \{j_1^1, j_1^2, j_1^{\bar{c}+1}, j_1^{\bar{c}+3}\} \subseteq V(T_n)$.

Claim: The set R_f is a FTRS for T_n.
In order to get this, we give FTMC to every vertex of T_n with respect to R_f.

The FTMCs for members in the sets $J_1 = \{j_1^{\bar{q}} : 1 \leq \bar{q} \leq n\}$ are presented below in Table 14 and the collection of FTMCs for these vertices are denoted by the set JC_1.

Table 14. FTMCs for J_1 set vertices.

$ß_f(j_1^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+3}$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}-1$	1	$\bar{c}-\bar{q}+1$	$\bar{q}+\bar{c}-2$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=2)$	$\bar{q}-1$	$\bar{q}-2$	$\bar{c}-\bar{q}+1$	$\bar{q}+\bar{c}-2$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(2 \leq \bar{q} \leq \bar{c}+1)$	$\bar{q}-1$	$\bar{q}-2$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+3$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+2)$	$2\bar{c}-\bar{q}+2$	$\bar{q}-2$	$\bar{q}-\bar{c}-1$	$\bar{c}-\bar{q}+3$
$ß_f(j_1^{\bar{q}}\|R_f)$:$(\bar{c}+3 \leq \bar{q} \leq 2\bar{c}+1)$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}-1$	$\bar{q}-\bar{c}-3$

The FTMCs for members in the sets $J_2 = \{j_2^{\bar{q}} : 1 \leq \bar{q} \leq n\}$ are presented below in Table 15 and the collection of FTMCs for these vertices are denoted by the set JC_2.

Table 15. FTMCs for J_2 set vertices.

$ß_f(j_2^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+3}$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}$	1	$\bar{c}-\bar{q}+1$	$\bar{c}$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(2 \leq \bar{q} \leq \bar{c})$	$\bar{q}$	$\bar{q}-1$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+3$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+1)$	$2\bar{c}-\bar{q}+2$	$\bar{q}-1$	$\bar{q}-\bar{c}$	$\bar{c}-\bar{q}+3$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+2)$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}$	$\bar{c}-\bar{q}+3$
$ß_f(j_2^{\bar{q}}\|R_f)$:$(\bar{c}+3 \leq \bar{q} \leq 2\bar{c}+1)$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}$	$\bar{q}-\bar{c}-2$

The FTMCs for members in the sets $J_3 = \{j_3^{\bar{q}} : 1 \leq \bar{q} \leq n\}$ are presented below in Table 16 and the collection of FTMCs for these vertices are denoted by the set JC_3.

TABLE 16. FTMCs for J_3 set vertices.

$ß_f(j_3^{\bar{q}}\|R_f)$	j_1^1	j_1^2	$j_1^{\bar{c}+1}$	$j_1^{\bar{c}+3}$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=1)$	$\bar{q}+1$	2	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+2$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(2 \leq \bar{q} \leq \bar{c}-1)$	$\bar{q}+1$	$\bar{q}$	$\bar{c}-\bar{q}+1$	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c})$	$\bar{q}+1$	$\bar{q}$	2	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+1)$	$2\bar{c}-\bar{q}+2$	$\bar{q}$	$\bar{q}-\bar{c}+1$	$\bar{c}-\bar{q}+3$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=\bar{c}+2)$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}+1$	2
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{c}+2 \leq \bar{q} \leq 2\bar{c})$	$2\bar{c}-\bar{q}+2$	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}+1$	$\bar{q}-\bar{c}$
$ß_f(j_3^{\bar{q}}\|R_f)$:$(\bar{q}=2\bar{c}+1)$	2	$2\bar{c}-\bar{q}+3$	$\bar{q}-\bar{c}$	$\bar{q}-\bar{c}-1$

Lastly, the FTMCs for members in the set $J_4 = \{j_4^{\bar{q}} : 1 \leq \bar{q} \leq n\}$ are as follows $ß_f(j_4^{\bar{q}}|R_f) = ß_f(j_3^{\bar{q}}|R_f) + (1,1,1,1)$, for $1 \leq \bar{q} \leq n$ and the collection of FTMCs for these vertices are denoted by the set JC_4. The total number of FTMCs listed above are equal to $|JC_1 \cup JC_2 \cup JC_3 \cup JC_4| = 4n$. To prove that the set R_f is a FTRS for T_n, it is first compulsory to show that, it is also a resolving set for T_n. From all the codes presented above in JC_1, JC_2, JC_3, and JC_4, one can clearly verify that all are different from one an other in at least one coordinate and are also unique. Therefore, from this fact, now it is clear that the set R_f is a resolving set for T_n. Next, to finish the proof, we only need to show that the set R_f possesses the fault-tolerance property in it. For this, we simply adopt the definition of FTRS and prove the following four sets $R_1 = \{j_1^1, j_1^2, j_1^{\bar{c}+1}\}$, $R_2 = \{j_1^1, j_1^2, j_1^{\bar{c}+3}\}$, $R_3 = \{j_1^1, j_1^{\bar{c}+1}, j_1^{\bar{c}+3}\}$, and $R_4 = \{j_1^2, j_1^{\bar{c}+1}, j_1^{\bar{c}+3}\}$ (i.e., by using $R_f \setminus \{j\}$, $\forall\, j \in R_f$) to be resolving in T_n. But on verifying manually the FTMCs with respect to the set R_f in T_n, we find that the respective metric codes with respect to the sets R_1, R_2, R_3, and R_4 are unique and distinct for every vertex present in T_n. From this particular fact, we find that the set R_f is the FTRS for T_n. Therefore, we have $fdim(T_n) \leq 4$. Now, from this and Proposition 3, we find that $fdim(T_n) = 4$, which completes the proof. □

Corollary 5.2. *The FTMD for T_n is constant.*

6. Conclusion

We investigated the presence of independent fault-tolerant resolving sets in graphs and obtained some basic results comparing the independence of the resolving sets in this paper. We proved that $fdim(\mathbb{A}_n) = fdim(S_n) = fdim(T_n) = 4$, for the double antiprism $\mathbb{A}_n$, two convex polytopes S_n, and T_n. We end this section by posing a question as an open problem regarding IFTRS that derives naturally from the article.

Open Problem: *Is ifr($\mathbb{A}_n$) =ifr(S_n) =ifr(T_n) $= 4$?*

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