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On Schur inequality and Schur functions

Marius Rădulescu, Sorin Rădulescu, and Petrus Alexandrescu

Abstract. The statement of the Schur's inequality is the following:
Theorem. Let x, y, z be nonnegative real numbers. Then for every $r > 0$ the following inequality holds:
$x^r (x-y)(x-z) + y^r (y-z)(y-x) + z^r (z-x)(z-y) \geq 0.$
In case the exponent r is an even number the above inequality holds for every x, y, z real numbers.
The goal of the paper is to introduce the notion of Schur function and to prove some properties of Schur functions. Our results represent generalizations of the Schur's inequality.

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1. Introduction

The following inequality is known as the Schur inequality.

Theorem. Let x, y, z be nonnegative real numbers. Then for every $r > 0$ the following inequality holds:

$$x^r (x-y)(x-z) + y^r (y-z)(y-x) + z^r (z-x)(z-y) \geq 0 \tag{1.1}$$

Equality holds if and only if $x = y = z$ or if two of x, y, z are equal and the third is zero.

In case the exponent r is an even number then inequality (1.1) holds for every x, y, z real numbers.

One of the reasons for which Schur's inequality is studied is its applications to geometric programming.

Geometric programming is a part of nonlinear programming where both the objective function and constraints are polynomials with positive coefficients (posinomials), that is

$$P(x_1, x_2, ..., x_n) = \sum_{|\alpha| \leq m} a_\alpha x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a n-dimensional vector with components natural numbers, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ and all coefficients a_α are nonnegative numbers.

Expanding terms in (1) we get

$$\sum x^{r+2} + xyz \left(\sum x^{r-1} \right) \geq \sum x^{r+1} y + \sum x^{r+1} z$$

Therefore Schur's inequality is equivalent to an inequality between two posinomials.

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In the next paragraph we define the notion of Schur function and we study various properties of Schur functions.

2. Properties of Schur functions

In the present paragraph we shall denote by D a subset of $\mathbf{R}$ containing at least two elements. For every map $f : D \to \mathbf{R}$ we shall denote by $S(f, x, y, z)$ the sum $\sum f(x)(x-y)(x-z)$ that is

$$S(f, x, y, z) = f(x)(x-y)(x-z) + f(y)(y-z)(y-x) + f(z)(z-x)(z-y)$$

Definition 2.1. Let D be a subset of $\mathbf{R}$ containing at least two elements and $f : D \to \mathbf{R}$ be a map. We say that f is a Schur function on D if

$$\sum f(x)(x-y)(x-z) \geq 0 \qquad \text{for every} \qquad x, y, z \in D \tag{2.1}$$

We denote by $\mathcal{S}(D)$ the set of all Schur functions defined on D.

Proposition 2.2. The following assertions hold:

1^0. If $f \in \mathcal{S}(D)$ then $f \geq 0$.

2^0 If $f \in \mathcal{S}(D)$ and for some $a, b \in D, a < b$, we have $f(a) = f(b) = 0$ then $f(x) = 0$ for every $x \in [a, b] \cap D$.

3^0. If there exists $a, b \in D,\ a < b$, such that $\frac{a+b}{2} \in D$ and

$$f\left(\frac{a+b}{2}\right) > 2f(a) + 2f(b) \tag{2.2}$$

then $f \notin \mathcal{S}(D)$.

Proof. Note that the map $S(f, x, y, z)$ is a symmetric map. In order to prove 1^0 consider $f \in \mathcal{S}(D)$ and take two distinct elements of $D,\ x, y \in D$. Then $0 \leq S(f, x, y, y) = f(x)(x-y)^2$, hence $.f \geq 0$

To prove 2^0. let $f \in \mathcal{S}(D), a, b \in D,\ a < b,\ f(a) = f(b) = 0$. Then

$0 \leq S(f, x, a, b) = f(x)(x-a)(x-b)$ for every $x \in [a, b] \cap D$. Suppose that there exists $x_0 \in [a, b] \cap D$ such that $f(x_0) > 0$. This implies $(x_0 - a)(x_0 - b) < 0$, hence $S(f, x_0, a, b) < 0$. We have obtained a contradiction. It follows that $f = 0$ on $[a, b] \cap D$.

If f satisfies (2.2) then

$$S\left(f, \frac{a+b}{2}, a, b\right) = \frac{(a-b)^2}{4}\left(2f(a) + 2f(b) - f\left(\frac{a+b}{2}\right)\right) < 0 \tag{2.3}$$

hence $f \notin \mathcal{S}(D)$. Thus we proved assertion 3^0 .

Theorem 2.3. Let $f : D \to \mathbf{R}$ be a map. Then the following assertions are equivalent:

1^0. f is a Schur map on D.

2^0. $f(\alpha x + \beta y) \leq \frac{f(x)}{\alpha} + \frac{f(y)}{\beta}$, for every $x, y \in D$,and $\alpha, \beta \in (0, 1),\ \alpha + \beta = 1$,such that $\alpha x + \beta y \in D$.

Proof. Let S be the map defined in the beginning of the section, $x, y, z \in D$, $x < z < y,\ \alpha, \beta \in (0, 1),\ \alpha + \beta = 1,\ z = \alpha x + \beta y$. Then one can easily see that the following equality holds:

$$S(f,x,y,z) = \alpha\beta\,(x-y)^2\left[\frac{f(x)}{\alpha}+\frac{f(y)}{\beta}-f(z)\right] \tag{2.4}$$

Then the equivalence of the assertions from the statement of the above theorem follows at once from identity (2.4).

Corollary 2.4. Let $f : D \to \mathbf{R}$ be a map. Suppose that there exist two positive constants m, M such that:

$$0 < m \le f(x) \le M \le 4m \quad \text{for every } x \in D$$

Then f is a Schur map on D.

Proof. Let $x, y \in D, \alpha, \beta \in (0,1)$, $\alpha+\beta = 1$ such that $\alpha x + \beta y \in D$ Then

$$f(\alpha x + \beta y) \le M \le 4m \le \frac{m}{\alpha\beta} = \frac{m}{\alpha} + \frac{m}{\beta} \le \frac{f(x)}{\alpha} + \frac{f(y)}{\beta}$$

By the preceding theorem it follows that f is a Schur map on D.

Corollary 2.5. Let $f : D \to \mathbf{R}$, be a map. Suppose that there exist two positive constants m, M such that:

$$0 < m \le f(x) \le M \qquad \text{for every } x \in D$$

For every $a \ge 0$ consider the map $f_a : D \to \mathbf{R}$, $f_a(x) = f(x) + a$, $x \in D$.

Then for every $a \ge \max\left(\frac{M-4m}{3}, 0\right)$ the map f_a is a Schur map on D.

Proof. Note that $a \ge \max\left(\frac{M-4m}{3}, 0\right)$ implies that

$$0 < m + a \le f_a(x) \le M + a \le 4(m+a) \quad \text{for every } x \in D$$

By the preceding corollary f_a is a Schur map on D.

In the following we shall give a definition of a quasiconvex map which is a slight more general than the classical one.

Definition 2.6. A map $f : D \to \mathbf{R}$ is quasiconvex if

$$\begin{aligned} f(\alpha x + \beta y) &\le \max(f(x), f(y)) \text{ for every } x, y \in D, \\ \alpha, \beta &\in (0,1), \alpha + \beta = 1 \text{ such that } \alpha x + \beta y \in D \end{aligned}$$

Recall that in the classical definition of a quasiconvex map one supposes that the set D is convex.

Corollary 2.7. The following assertions hold:

1^0. Every nonnegative quasiconvex map is a Schur map.

2^0. Every nonnegative map which is a sum of two nonnegative monotone maps is a Schur map.

Proof. Let $x, y \in D, \alpha, \beta \in (0,1), \alpha + \beta = 1$ such that $\alpha x + \beta y \in D$. If f is a nonnegative quasiconvex map then

$$f(\alpha x + \beta y) \le \max(f(x), f(y)) \le \frac{f(x)}{\alpha} + \frac{f(y)}{\beta}$$

By theorem 2.3., it follows that f is a Schur map on D.

Suppose that $f = u_1 + u_2$, $u_i \ge 0$, u_i monotone $i =$ 1,2. Then one can easily see that

$$u_i(\alpha x + \beta y) \le \max(u_i(x), u_i(y)) \le u_i(x) + u_i(y) \le \frac{u_i(x)}{\alpha} + \frac{u_i(y)}{\beta} \quad i = 1, 2.$$

By theorem 2.3., f is a Schur map on D.

Theorem 2.8. Let $f : D \to \mathbf{R}_+$ be a map with the property

$$\begin{aligned} f(\alpha x+\beta y) &\leq \left(\sqrt{f(x)}+\sqrt{f(y)}\right)^2 \text{ for every } x,y\in D, \\ \alpha,\beta &\in (0,1), \alpha+\beta=1 \text{ such that } \alpha x+\beta y\in D \end{aligned}$$

Then f is a Schur map on D.

Proof. The assertion of the theorem follows at once from the inequalities:

$$f(\alpha x+\beta y)\leq\left(\sqrt{f(x)}+\sqrt{f(y)}\right)^2\leq\frac{f(x)}{\alpha}+\frac{f(y)}{\beta}$$

for every $x,y\in D,\alpha,\beta\in(0,1),\alpha+\beta=1$ such that $\alpha x+\beta y\in D$.

Theorem 2.9. Let $f : D\to\mathbf{R}_+$ be a map and M a positive constant. Suppose that he following inequality holds:

$$\begin{aligned} f(\alpha x+\beta y)-\frac{f(x)}{\alpha}-\frac{f(y)}{\beta} &\leq M \quad \text{for every } x,y\in D, \\ \alpha,\beta &\in (0,1), \alpha+\beta=1 \text{ such that } \alpha x+\beta y\in D \end{aligned}$$

For every $a\geq 0$ consider the map $f_a : D\to\mathbf{R}$, $f_a(x)=f(x)+a$, $x\in D$.

Then for every $a\geq\frac{M}{3}$ we have $f_a\in\mathcal{S}(D)$.

Proof. Let $a\geq\frac{M}{3}$. Then for every $x,y\in D,\alpha,\beta\in(0,1),\alpha+\beta=1$ such that $\alpha x+\beta y\in D$, we have

$$f_a(\alpha x+\beta y)-\frac{f_a(x)}{\alpha}-\frac{f_a(y)}{\beta}=$$

$$=f(\alpha x+\beta y)-\frac{f(x)}{\alpha}-\frac{f(y)}{\beta}+a\left(1-\frac{1}{\alpha\beta}\right)\leq M+a(1-4)=M-3a\leq 0$$

By theorem 2.3., it follows that f_a is a Schur map on D.

Theorem 2.10. Let $f : D\to\mathbf{R}_+$ be a map. For every $a\geq 0$ consider the map $f_a : D\to\mathbf{R}$, $f_a(x)=f(x)+a$, $x\in D$. If $f\notin\mathcal{S}(D)$, then there exists $a_0>0$ such that for every $a\in(0,a_0)$ we have $f_a\notin\mathcal{S}(D)$.

Proof. Suppose that $f\notin\mathcal{S}(D)$. Then by theorem 2.3., there exist $x_0,y_0\in D,\alpha_0,\beta_0\in(0,1),\alpha_0+\beta_0=1$, such that $\alpha_0x_0+\beta_0y_0\in D$ and

$$A(f)=f(\alpha_0x_0+\beta_0y_0)-\frac{f(x_0)}{\alpha_0}-\frac{f(y_0)}{\beta_0}>0$$

Take $a_0=\frac{A(f)}{\frac{1}{\alpha_0\beta_0}-1}$. Note that if $a\in[0,a_0)$ then

$$\begin{aligned} A(f_a) &= f_a(\alpha_0x_0+\beta_0y_0)-\frac{f_a(x_0)}{\alpha_0}-\frac{f_a(y_0)}{\beta_0}= \\ &= A(f)+a\left(1-\frac{1}{\alpha_0\beta_0}\right)>A(f)+a_0\left(1-\frac{1}{\alpha_0\beta_0}\right)=0 \end{aligned}$$

Consequently $f_a\notin\mathcal{S}(D)$.

Proposition 2.11. Let $f(x)=\left(x^2-1\right)^2, x\in\mathbf{R}$,$\psi$: $[0,1]\to\mathbf{R}$, $\psi(t)=\frac{(1-t)(1+t)^2}{1+t+t^2+t^3+t^4}, t\in[0,1]$. Denote

$Q = \left\{(\alpha,\beta)\in \mathbf{R}^2 : \alpha,\beta\in(0,1), \alpha+\beta=1\right\}, \gamma_0 = \max\limits_{(\alpha,\beta)\in Q}[\psi(\alpha)+\psi(\beta)]$. For every $x,y\in\mathbf{R}$, $(\alpha,\beta)\in Q$ define

$g(x,y,\alpha,\beta) = f(\alpha x+\beta y) - \frac{f(x)}{\alpha} - \frac{f(y)}{\beta}$

Then the following inequalities hold:

$$\frac{36}{31} \leq \gamma_0 < 2$$

$$g(x,y,\alpha,\beta) \leq \gamma_0 + 1 \leq 3$$

Proof. Note that f is decreasing on $(-\infty,-1]$ and increasing on $[-1,0]$. Hence the restriction of f to $(-\infty,0]$ is quasiconvex. Thus if $x,y\leq 0$ and $(\alpha,\beta)\in Q$ then

$$\begin{aligned} g(x,y,\alpha,\beta) &= f(\alpha x+\beta y) - \frac{f(x)}{\alpha} - \frac{f(y)}{\beta} \leq \\ &\leq \max(f(x), f(y)) - \frac{f(x)}{\alpha} + \frac{f(y)}{\beta} \leq 0 \end{aligned}$$

Since f is decreasing on $[0,1]$ and increasing on $(1,\infty]$ it follows that the restriction of f to $[0,\infty)$ is quasiconvex. Thus if $x,y\geq 0$ and $(\alpha,\beta)\in Q$ then $g(x,y,\alpha,\beta)\leq 0$.

Now we shall consider the case $x\geq 0, y\leq 0$. Let $z=-y$. We shall prove that

$$g(x,-z,\alpha,\beta) \leq 1+\gamma_0 < 3$$

Consider the maps: $\phi_1(t,x) = \frac{t^5-1}{t}x^4 + \frac{2(1-t^3)}{t}x^2 - \frac{1}{t},\ t\in(0,1), x\in\mathbf{R}$,
$\phi_2(x) = ax^4+bx^2+c, \quad x\in\mathbf{R}$. Note that $a<0$ and $b>0$ implies that

$$\phi_2(x) \leq \phi_2\left(\sqrt{-\frac{b}{2a}}\right) = \frac{4ac-b^2}{4a} \text{ for every } x\in\mathbf{R}$$

If in the preceding inequality we take $a=\frac{t^5-1}{t},\ b=\frac{2(1-t^3)}{t},\ t\in(0,1)$, we obtain

$$\phi_1(t,x) \leq \phi_2\left(\sqrt{-\frac{b}{2a}}\right) = \frac{4ac-b^2}{4a} = \frac{t^2\left(2-t^2-t^3\right)}{t^5-1} = \psi(t)-1 \qquad (2.5)$$

Let $x,z\geq 0$. Then

$$g(x,-z,\alpha,\beta) = f(\alpha x-\beta z) - \frac{f(x)}{\alpha} - \frac{f(-z)}{\beta} =$$

$$= (\alpha x-\beta z)^4 - \frac{x^4}{\alpha} - \frac{z^4}{\beta} - 2(\alpha x-\beta z)^2 + \frac{2x^2}{\alpha} + \frac{2z^2}{\beta} + 1 - \frac{1}{\alpha} - \frac{1}{\beta} =$$

$$= \left(\alpha^4-\frac{1}{\alpha}\right)x^4 + \left(\frac{2}{\alpha}-2\alpha^2\right)x^2 - \frac{1}{\alpha} + \left(\beta^4-\frac{1}{\beta}\right)z^4 + \left(\frac{2}{\beta}-2\beta^2\right)z^2 - \frac{1}{\beta} +$$

$$+8\alpha^2\beta^2x^2z^2 - 4\alpha^3\beta x^3z - 4\alpha\beta^3xz^3 - 2\alpha^2\beta^2x^2z^2 + 4\alpha\beta xz + 1 =$$

$$= \phi_1(\alpha,x) + \phi_1(\beta,z) - 4\alpha\beta xz(\alpha x-\beta z)^2 - 2\alpha^2\beta^2x^2z^2 + 4\alpha\beta xz + 1 \leq$$

$$\begin{aligned} &\leq \phi_1(\alpha, x) + \phi_1(\beta, z) + 3 \leq \psi(\alpha) - 1 + \psi(\beta) - 1 + 3 = \\ &= \psi(\alpha) + \psi(\beta) + 1 \leq \gamma_0 + 1 \end{aligned}$$

Thus we proved that $x \geq 0, y \leq 0$ implies $g(x, y, \alpha, \beta) \leq \gamma_0 + 1 \leq 3$. Analogously one can prove the same inequality for $x \leq 0, y \geq 0$.

Proposition 2.12. Let $f(x) = \left(x^2 - 1\right)^2, x \in \mathbf{R}$. For every $a \geq 0$ consider the map $f_a : \mathbf{R} \to \mathbf{R}$,

$f_a(x) = f(x) + a, x \in \mathbf{R}$. Then the following assertions hold:

1^0. For every $a \in [0, \frac{1}{3})$, $f_a \notin \mathcal{S}(\mathbf{R})$.

2^0 For every $a \in [1, \infty)$, $f_a \in \mathcal{S}(\mathbf{R})$

3^0 For every $a \in \mathbf{R}$, f_a is not quasiconvex and is not the sum of two positive monotone functions.

Proof. To prove 1^0 take $\alpha_0 = \beta_0 = \frac{1}{2}, x_0 = -1, y_0 = 1$.Then $\alpha_0 x_0 + \beta_0 y_0 = 0$,

$A(f) = f(\alpha_0 x_0 + \beta_0 y_0) - \frac{f(x_0)}{\alpha_0} - \frac{f(y_0)}{\beta_0} = f(0) - 2f(-1) - 2f(1) = 1 > 0$ and $a_0 = \frac{A(f)}{\frac{1}{\alpha_0\beta_0} - 1} = \frac{1}{3}$.

By theorem 2.10. $f_a \notin \mathcal{S}(\mathbf{R})$ for every $a \in [0, \frac{1}{3})$.The assertion from 2^0 follows at once from theorem 2.9**.** and proposition 2.11.

Note that f_a is not monotone and there does not exist $u_0 \in \mathbf{R}$ such that f_a is decreasing on $(-\infty, u_0]$ and increasing on $[u_0, \infty)$. Therefore f_a is not quasiconvex.

Proposition 2.13. Let $\phi : \mathbf{R} \to \mathbf{R}$ be a map with the property that $f \circ \phi \in \mathcal{S}(\mathbf{R})$ for every $f \in \mathcal{S}(\mathbf{R})$. Then ϕ is monotone.

Proof. Suppose contrary that ϕ is not a monotone map. Then there exist $x < y < z$ such that

$\max(\phi(x), \phi(z)) < \phi(y)$. Let $\lambda \in (\max(\phi(x), \phi(z)), \phi(y))$. For every $a, b > 0$ consider the map

$$f_{a,b}(t) = \begin{cases} a \text{ if } t \in (-\infty, \lambda] \\ b \text{ if } t \in (\lambda, +\infty) \end{cases}$$

Note that all maps $f_{a,b}$ are monotone, hence they are Schur maps. Since $y \in (x, z)$ it follows that there exist $\alpha, \beta \in (0, 1), \alpha + \beta = 1$ such that $y = \alpha x + \beta z$. By hypothesis $f_{a,b} \circ \phi \in \mathcal{S}(\mathbf{R})$.Therefore

$$b = (f_{a,b} \circ \phi)(y) = f_{a,b}(\phi(\alpha x + \beta z)) \leq \frac{f_{a,b}(\phi(x))}{\alpha} + \frac{f_{a,b}(\phi(z))}{\beta} = \frac{a}{\alpha} + \frac{a}{\beta} = \frac{a}{\alpha\beta}$$

By the preceding inequality it follows that $\alpha\beta \leq \frac{a}{b}$ for every $a, b > 0$. The contradiction we have obtained proves that the map ϕ must be monotone.

Proposition 2.14. Let $f : \left[0, \frac{2}{3}\right] \to \mathbf{R}$, $f(x) = x - x^2, x \in \left[0, \frac{2}{3}\right]$. Then $f \in \mathcal{S}\left(\left[0, \frac{2}{3}\right]\right)$.

Proof. Let $\alpha, \beta \in (0, 1), \alpha + \beta = 1, g(x, y, \alpha, \beta) = f(\alpha x + \beta y) - \frac{f(x)}{\alpha} - \frac{f(y)}{\beta}, x, y \in \left[0, \frac{2}{3}\right]$.

Note that

$$\begin{aligned} -\alpha\beta g(x, y, \alpha, \beta) &= \left(\alpha^3\beta - \beta\right)x^2 + \left(\beta - \alpha^2\beta\right)x + \\ &+ \left(\alpha\beta^3 - \alpha\right)y^2 + \left(\alpha - \alpha\beta^2\right)y + 2\alpha^2\beta^2 xy \end{aligned}$$

Note that the coefficients of x^2 and y^2 from the right hand side of the above identity, that is $\alpha^3\beta-\beta$ and $\alpha\beta^3-\alpha$ are strictly negative. Moreover, the matrix

$$\begin{pmatrix} \alpha^3\beta-\beta & \alpha^2\beta^2 \\ \alpha^2\beta^2 & \alpha\beta^3-\alpha \end{pmatrix}$$

is negative definite. Consequently the map $h_{\alpha,\beta}(x,y)=-\alpha\beta g(x,y,\alpha,\beta)$ is concave. Hence

$$-\alpha\beta g(x,y,\alpha,\beta)\geq \min\left(h_{\alpha,\beta}(0,0),h_{\alpha,\beta}\left(0,\frac{2}{3}\right),h_{\alpha,\beta}\left(\frac{2}{3},0\right),h_{\alpha,\beta}\left(\frac{2}{3},\frac{2}{3}\right)\right)$$

Since all the arguments of min are nonnegative it follows that $g(x,y,\alpha,\beta)\leq 0$. This implies that $f\in\mathcal{S}\left(\left[0,\frac{2}{3}\right]\right)$.

Proposition 2.15. Let D be a subset of $\mathbf{R}$ with more than three elements and $f:D\to\mathbf{R}$ be an increasing Schur map. Consider $x,y,z\in D$ distinct elements. Then $S(f,x,y,z)=0$ if and only if one of the following situations occurs:

1^0. All x,y,z are equal, that is $x=y=z$.

2^0. Two of x,y,z are equal and the third is a zero of f.

3^0. All three x,y,z are zeros of f.

Proof. Without any loss of generality we may suppose that $x\geq y\geq z$. Denote

$$A(f,x,y,z)=(x-y)\left[f(x)(x-y)+(f(x)-f(y))(y-z)\right]$$

$$B(f,x,y,z)=f(z)(x-z)(y-z)$$

Note that $S(f,x,y,z)=A(f,x,y,z)+B(f,x,y,z)$. and $A(f,x,y,z)\geq 0$ and $B(f,x,y,z)\geq 0$.

Consequently $S(f,x,y,z)=0$ implies that $A(f,x,y,z)=B(f,x,y,z)=0$.We shall study two cases.

Case 1. $x=y$. in this case we have $A(f,x,y,z)=0$.From $B(f,x,y,z)=0$ it follows that $y=z$ or $f(z)=0$.

Thus case 1. reduces to situation 1^0 or 2^0.

Case 2. $x>y$. From $A(f,x,y,z)=0$ it follows that $f(x)=0$ and $(f(x)-f(y))(y-z)=0$.

Hence $f(y)=0$ or $y=z$. From $B(f,x,y,z)=0$ it follows that $f(z)=0$ or $y=z$.

(Marius Rădulescu) Institute of Mathematical Statistics and Applied Mathematics
Casa Academiei Române,Calea 13 Septembrie nr.13, Bucharest 5, RO-050711, Romania
E-mail address: mradulescu@csm.ro

(Sorin Rădulescu) Institute of Mathematical Statistics and Applied Mathematics
Casa Academiei Române,Calea 13 Septembrie nr.13, Bucharest 5, RO-050711, Romania
E-mail address: mradulescu@csm.ro

(Petrus Alexandrescu) Institute of Sociology
Casa Academiei Române, Calea 13 Septembrie nr. 13, Bucharest 5, RO-050711, Romania
E-mail address: alexandr@insoc.ro