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On Location of Zeros of a Quaternionic Polynomial with Restricted Coefficients Using Matrix Method

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Abstract. In this paper, we consider the connection between zeros of a quaternionic polynomial and left eigen values of its corresponding companion matrix. As an application of which, we determine the region containing the zeros of some special quaternionic polynomials with restricted coefficients. Our result generalizes some previously proved results in this direction. Furthermore, we provide numerical examples and graphical representation to demonstrate the superior precision of our result and its corollary to some previously established results.

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1. Introduction

The study of the distribution of zeros of a polynomial in a circular or annular region has a long and illustrious history in mathematics. This study has been the inspiration for much theoretical research occupying its own place both within and outside of mathematics. Various authors have extensively studied problems involving polynomials and the location of their zeros over the past few decades including their applications. There are various research papers concerning an upper bound for the moduli of all the zeros of a polynomial when its coefficients are restricted with special conditions. The first contribution to this subject was made by Gauss and Cauchy. One of the classical results concerning the zeros and their regional location of a polynomial with restricted coefficients is known as Eneström-Kakeya theorem.

In 1829, Cauchy [8] proved that if $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then all the zeros of $P(z)$ lie in

$$|z| < 1 + M, \tag{1}$$

where $M = \max\left\{\frac{|a_v|}{|a_n|} : v = 0, 1, 2, ...n-1\right\}$.
In 1890, Gustav Eneström [8] proved the following result by considering a polynomial with real, positive, monotone coefficients.

Theorem 1.1. *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 \geq 0, \tag{2}$$

then $P(z)$ has all its zeros in $|z| \leq 1$.

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Further, Aziz and Zargar [1] proved the following result which is an extension of Theorem 1.1 by relaxing the hypothesis.

Theorem 1.2. *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n with real coefficients such that for some positive numbers k and η with $k \geq 1$ and $0 < \eta \leq 1$, $ka_n \geq a_{n-1} \geq \dots. \geq a_1 \geq \eta a_0 \geq 0$, then all the zeros of $P(z)$ lie in*

$$|z + k - 1| \leq \frac{ka_n + 2a_0(1-\eta)}{a_n}. \tag{3}$$

Also, Nwaeze [13] proved the following result which is a generalization of Theorem 1.2.

Theorem 1.3. *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that for some real numbers λ and ρ, $\lambda + a_n \geq a_{n-1} \geq a_{n-2} \dots. \geq a_1 \geq a_0 - \rho$, then all the zeros of $P(z)$ lie in*

$$\left| z + \frac{\lambda}{a_n} \right| \leq \frac{1}{|a_n|} \left\{ a_n + \lambda - a_0 + \rho + |\rho| + |a_0| \right\}. \tag{4}$$

Recently, Zargar et al. [16] proved the following result which generalizes as well as extends Theorem 1.3 by considering a polynomial with complex coefficients.

Theorem 1.4. *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n with complex coefficients, $a_v = \alpha_v + i\beta_v$ for $v = 0, 1, ..., n$ such that for some real numbers κ,λ,τ, and ρ*

$$\lambda + \alpha_n \geq \alpha_{n-1} \geq \dots\dots \geq \alpha_1 \geq \alpha_0 - \rho$$

and

$$\kappa + \beta_n \geq \beta_{n-1} \geq \dots. \geq \beta_1 \geq \beta_0 - \tau,$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{\lambda + i\kappa}{a_n} \right| \leq \frac{1}{|a_n|} \Bigg\{ \alpha_n + \beta_n + \lambda + \kappa - (\alpha_0 + \beta_0) + \tau + \rho + |\tau| + |\rho| + |\alpha_0| + |\beta_0| \Bigg\}. \tag{5}$$

2. Preliminary knowledge

In order to introduce the framework in which we will work, let us introduce some preliminaries on quaternions and regular functions of a quaternionic variable which will have useful and interesting consequences. Quaternions are essentially a generalization of complex numbers to four dimensions(one real and three imaginary parts) which was first studied and developed by Sir Rowan William Hamilton in 1843. This number system is denoted by $\mathbf{H}$ in honor of Hamilton. We use the notation $\mathbf{H} = \{\alpha + \beta i + \gamma j + \delta k : \alpha, \beta, \gamma, \delta \in \mathbf{R}\}$ where i, j, k satisfy $i^2 = j^2 =$

$k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. It is a non-commutative division ring. Every element $q = \alpha + \beta i + \gamma j + \delta k \in \mathbf{H}$ is composed of real part $Re(q) = \alpha$ and imaginary part $Im(q) = \beta i + \gamma j + \delta k$. The conjugate of q is denoted by $\bar{q}$ and is defined as $\bar{q} = \alpha - \beta i - \gamma j - \delta k$ and the norm of q is $|q| = \sqrt{q\bar{q}} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The inverse of each non zero element q of $\mathbf{H}$ is given by $q^{-1} = |q|^{-2}\bar{q}$. For $r > 0$, we define the ball $B(0, r) = \{q \in \mathbf{H} : |q| < r\}$. By B we denote the open unit ball in $\mathbf{H}$ centered at the origin, i.e., $B = \left\{q = \alpha + \beta i + \gamma j + \delta k : \alpha^2 + \beta^2 + \gamma^2 + \delta^2 < 1\right\}$ and by S the unit sphere of purely imaginary quaternions, i.e., $S = \left\{q = \beta i + \gamma j + \delta : \beta^2 + \gamma^2 + \delta^2 = 1\right\}$. We represent the indeterminate for a quaternionic polynomial as q. Since $\mathbf{H}$ being a non-commutative division ring, aq^n and $a_0qa_1q.....qa_n$, where $a = a_0a_1....a_n$ are different. Hence, we adopt the standard that polynomials have the indeterminate on the left and coefficients on the right. That is, a quaternionic polynomial P of degree n in the variable q and coefficients a_v for $v = 0, 1, ...n - 1, n$, is given by $P(q) = \sum_{v=0}^{n} q^v a_v$. It is known that these are the only polynomials in the quaternion-valued functions of a quaternionic variable to satisfy the regularity conditions, and hence their behavior resembles very closely to that of regular functions of a complex variable. Two quaternionic polynomials of this kind can be multiplied according to the convolution product (Cauchy multiplication rule), i.e., for $P_1(q) = \sum_{i=0}^{n} q^i a_i$ and $P_2(q) = \sum_{j=0}^{m} q^j b_j$, we define $(P_1 * P_2)(q) = \sum_{\substack{i=0,1,2,,,n \\ j=0,1,2...m}} q^{i+j} a_i b_j$.

The quaternion Companion Matrix: The $n \times n$ companion matrix of a monic quaternionic polynomial of the form $f(q) = q^n + q^{n-1}a_1 + ... + qa_{n-1} + a_n$, is given by

$$\begin{pmatrix} 0 & 0 & 0 & ... & 0 & -a_n \\ 1 & 0 & 0 & ... & 0 & -a_{n-1} \\ 0 & 1 & 0 & ... & 0 & -a_n \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & ... & 1 & -a_1 \end{pmatrix},$$

whereas, the $n \times n$ companion matrix for a monic quaternion polynomial of the form $g(q) = q^n + a_1q^{n-1} + ... + a_{n-1}q + a_n$ is given by

$$\begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ -a_n & -a_{n-1} & -a_{n-2} & .. & . & -a_1 \end{pmatrix}.$$

Right Eigenvalue: Given an $n \times n$ matrix $A = (a_{\mu\nu})$ of quaternions, $\lambda \in \mathbf{H}$ is called the right eigen value of A, if $Ax = x\lambda$ for some non zero eigenvector $x =$

$[x_1, x_2, ...x_n]^T$ of quaternions.
Left Eigenvalue: Given an $n \times n$ matrix $A = (a_{\mu\nu})$ of quaternions, $\lambda \in \mathbf{H}$ is called the left eigen value of A, if $Ax = \lambda x$ for some non zero eigenvector $x = [x_1, x_2, ...x_n]^T$ of quaternions.

For complex case, concerning the location of the eigenvalues, the famous Geršgorin theorem due to Marden [8] which can be stated as;

Theorem 2.1. *All the eigenvalues of a $n \times n$ complex matrix $A = (a_{\mu\nu})$ are contained in the union of n Geršgorin discs defined by* $D_\mu = \left\{ z \in \mathbf{C} : |z - a_{\mu\mu}| \le \sum_{\nu=1, \nu\neq\mu}^{n} |a_{\mu\nu}| \right\}$.

Recently, Dar et al. [3] proved the following quaternion version of Geršgorin Theorem.

Theorem 2.2. *All the left eigenvalues of a $n \times n$ matrix $A = (a_{\mu\nu})$ of quaternions lie in the union of the n Geršgorin balls defined by $B_\mu = \{q \in \mathbf{H} : |q - a_{\mu\mu}| \le \rho_\mu(A)\}$, where $\rho_\mu(A) = \sum_{\nu=1,\nu\neq\mu}^{n} |a_{\mu\nu}|$.*

In the same paper, they considered the quaternionic polynomial with coefficients on left side and gave connection between its zeros and the left eigenvalues of its corresponding companion matrix by proving the following result.

Theorem 2.3. *Let $f(q) = q^n + a_{n-1}q^{n-1} + ... + a_1 q + a_0$ be a quaternionic polynomial with quaternion coefficients and q be quaternionic variable, then for any diagonal matrix $D = diag(d_1, d_2, d_3, ..., d_{n-1}, d_n)$, where $d_1, d_2, ..., d_n$ are positive real numbers, the left eigenvalues of $D^{-1}C_p D$ and the zeros of $f(q)$ are same.*

And very recently Rather et al. [14] proved that the above relation holds true for quaternionic polynomials with coefficients on right side. From this, it was concluded that the zeros of quaternionic polynomial and left eigenvalues of its corresponding companion matrix are same irrespective of the position of its coefficients.

In the past few years, a series of papers related to regular functions of a quaternionic variable has been published (see for example, [9], [15], [4]). Many remarkable advancements have been made by using the structure of the zeros of polynomials, for example, a topological proof of the Fundamental Theorem of Algebra was established in ([5]-[7]). We point out that the Fundamental Theorem of Algebra for regular polynomials with coefficients in $\mathbf{H}$ was already proved by Niven (see [11], [12]), by using different techniques. This leads to the complete identification of the zeros of polynomials in terms of their factorization and hence it becomes an interesting perspective to think about the regions containing some or all the zeros of a regular polynomial of quaternionic variable. The proof of Eneström-Kakeya Theorem is done using the Triangle inequality and Maximum Modulus Theorem. It is clear that both the above mentioned facts hold true for the functions of a quaternionic variable. Therefore, Carney et al. [2] extended the Enestöm-Kakeya Theorem to a function of a quaternion variable. More precisely, they proved

Theorem 2.4. *If $P(q) = \sum_{v=0}^{n} q^v a_v$ is a quaternionic polynomial of degree n with real coeficients such that*

$$a_n \ge a_{n-1} \ge ... \ge a_1 \ge a_0 \ge 0,$$

then all the zeros of $P(q)$ lie in $|q| \leq 1$.

Further, they [2] also proved the following result by considering a quaternionic polynomial with monotone increasing real parts and imaginary parts.

Theorem 2.5. *If $P(q) = \sum_{v=0}^{n} q^v a_v$ is a quaternionic polynomial of degree n with real coefficients such that*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0,$$

then all the zeros of $P(q)$ lie in

$$|q| \leq \frac{(|a_0| - a_0 + a_n)}{|a_n|}. \tag{6}$$

Milovanović et al. [10] generalized Theorem 2.4 and Theorem 2.5 by proving the following result.

Theorem 2.6. *If $P(q) = \sum_{v=0}^{n} q^v a_v$ is a quaternionic polynomial of degree n with real coefficients such that*

$$a_n \leq a_{n-1} \leq .... \leq a_\lambda \geq ... \geq a_1 \geq a_0,$$

then all the zeros of $P(q)$ lie in

$$|q| \leq \frac{(2a_\lambda + |a_0| - a_0 - a_n)}{|a_n|}. \tag{7}$$

Because of the restriction on the coefficients that they should be real and monotonic, the results discussed above are applicable to a small class of polynomials, so it is interesting to look for the results without any restriction on the coefficients and applicable to every quaternionic polynomial with quaternion or complex or real coefficients. In this direction, Dar et al. [3] proved various results concerning the location of the zeros of quaterionic polynomials with quaternion coefficients without any restriction on the coefficients and besides, Cauchy's theorem is also extended to quaternion setting by proving the following result.

Theorem 2.7. *If $f(q) = q^n + q^{n-1}a_1 + ...qa_{n-1} + a_n$ is a quaternion polynomial with quaternion coefficients and q is quaternionic variable, then all the zeros of $f(q)$ lie inside the ball*

$$|q| \leq 1 + \max_{1 \leq \nu \leq n} |a_\nu|. \tag{8}$$

Now, in view of Theorem 2.1 and the fact that the zeros of a quaternion polynomial and the left eigenvalues of corresponding companion matrix are same, very recently, Rather et al. [14] proved some results concerning the location of the zeros of quaternion polynomials which is a refinement of Theorem 2.7.

Theorem 2.8. *Let $f(q) = q^n + q^{n-1}a_1 + ...qa_{n-1} + a_n$ be a quaternionic polynomial with quaternion coefficients and q is quaternionic variable. If $\alpha_2 \geq \alpha_3 \geq ... \geq \alpha_n$ are ordered positive numbers,*

$$\alpha_\nu = \frac{|a_{\nu|}}{r^\nu}, \ \nu = 2, 3...n \tag{9}$$

where r is a positive real number, then all the zeros of $f(q)$ lie in the union of balls

$$\{q \in \mathbf{H} : |q| \leq r(1+\alpha_2)\} \tag{10}$$

and

$$\{q \in \mathbf{H} : |q + a_1| \leq r\}. \tag{11}$$

In the same paper, as an application of Theorem 2.8, Rather et al. [14] proved the following result concerning the location of the zeros of the quaternionic polynomials with quaternion coefficients.

Theorem 2.9. *Let $f(q) = q^n + q^{n-1}a_1 + ...qa_{n-1} + a_n$ be a quaternionic polynomial with quaternion coefficients and q is quaternionic variable. If $\alpha_2 \geq \alpha_3 \geq ... \geq \alpha_n$ are ordered positived numbers,*

$$\alpha_\nu = \frac{|a_\nu|}{r^\nu}, \ \nu = 2, 3...n \tag{12}$$

where r is a positive real number, then all the zeros of $f(q)$ lie in the union of balls

$$\{q \in \mathbf{H} : |q| \leq r(1+\beta)\} \tag{13}$$

and

$$\{q \in \mathbf{H} : |q + a_1| \leq r\} \tag{14}$$

where

$$\beta = \alpha_2 - \frac{\delta_2}{1+\alpha_2} - \frac{\delta_3}{(1+\alpha_2)^2} - .... - \frac{\delta_n}{(1+\alpha_2)^{n-1}}, \tag{15}$$

$\delta_\nu = \alpha_\nu - \alpha_{\nu+1}$, $\nu = 2, 3, ...n$ *and* $\alpha_{n+1} = 0$.

3. Main Results

The main goal of this paper is to determine the regions containing all the zeros of a quaternionic polynomial with some restrictions on its coefficients which produces various generalizations of some previously established results by using the relation between zeros of a quaternionic polynomial and left eigen values of its corresponding companion matrix. More precisely, we prove the following result which generalizes Theorem 2.8 by considering a quaternionic polynomial.

Theorem 3.1. *Let $f(q) = q^n + q^{n-1}a_1 + ...qa_{n-1} + a_n$ be a quaternionic polynomial with quaternion coefficients and q be a quaternionic variable. If $\alpha_2 \leq \alpha_3 \leq ..... \leq \alpha_\lambda \geq .. \geq \alpha_n$ are positive numbers with*

$$\alpha_\nu = \frac{|a_\nu|}{r^\nu}, \ \nu = 2, 3...n \tag{16}$$

where r is a positive real number, then all the zeros of $f(q)$ lie in the union of balls

$$\{q \in \mathbf{H} : |q| \leq r(1+\alpha_\lambda)\} \tag{17}$$

and

$$\{q \in \mathbf{H} : |q + a_1| \leq r\}. \tag{18}$$

Remark 3.1. If we set $\lambda = 2$, Theorem 3.1 becomes Theorem 2.8.

Remark 3.2. Taking $\lambda = n$, Theorem 3.1 yields following interesting result which is applicable to a special class of quaternionic polynomial.

Corollary 3.2. *Let $f(q) = q^n + q^{n-1}a_1 + ...qa_{n-1} + a_n$ be a quaternionic polynomial with quaternion coefficients and q be a quaternionic variable. If $\alpha_2 \leq \alpha_3 \leq ..... \leq \alpha_n$ are positive numbers with*

$$\alpha_\nu = \frac{|a_\nu|}{r^\nu}, \ v = 2, 3...n \tag{19}$$

where r is a positive real number, then all the zeros of $f(q)$ lie in the union of balls

$$\{q \in \mathbf{H} : |q| \leq r(1 + \alpha_n)\} \tag{20}$$

and

$$\{q \in \mathbf{H} : |q + a_1| \leq r\}. \tag{21}$$

Remark 3.3. If we consider a complex polynomial, Theorem 3.1 reduces to

Corollary 3.3. *Let $f(z) = z^n + a_1 z^{n-1} + ... + a_{n-1}z + a_n$ be a complex polynomial with complex coefficients and z be a complex variable. If $\alpha_2 \leq \alpha_3 \leq ..... \leq \alpha_\lambda \geq .. \geq \alpha_n$ are positive numbers with*

$$\alpha_\nu = \frac{|a_\nu|}{r^\nu}, \ v = 2, 3...n \tag{22}$$

where r is a positive real number, then all the zeros of $f(z)$ lie in the union of discs

$$\{z \in \mathbf{C} : |z| \leq r(1 + \alpha_\lambda)\} \tag{23}$$

and

$$\{z \in \mathbf{C} : |z + a_1| \leq r\}. \tag{24}$$

4. Proof of the Theorem

Proof of Theorem 3.1. Let $f(q) = q^n + q^{n-1}a_1 + ...qa_{n-1} + a_n$ be a quaternionic polynomial with quaternion coefficients and q be a quaternionic variable. Then, by definition, the companion matrix of the quaternion polynomial $f(q)$ is given by

$$C_f = \begin{pmatrix} 0 & 0 & 0 & ... & 0 & -a_n \\ 1 & 0 & 0 & ... & 0 & -a_{n-1} \\ 0 & 1 & 0 & ... & 0 & -a_n \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & ... & 1 & -a_1 \end{pmatrix}.$$

We take a diagonal matrix $T = diag(r^{n-1}, r^{n-2}, ..., r, 1)$, where r is a positive real number, then

$$T^{-1}C_fT = \begin{pmatrix} 0 & 0 & 0 & ... & 0 & \frac{-a_n}{r^{n-1}} \\ r & 0 & 0 & ... & 0 & \frac{-a_{n-1}}{r^{n-2}} \\ 0 & r & 0 & ... & 0 & \frac{-a_{n-2}}{r^{n-3}} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & ... & r & -a_1 \end{pmatrix}.$$

Applying Theorem 2.2 to $T^{-1}C_fT$, it follows that all the left eigenvalues of $T^{-1}C_fT$ lie in the union of balls

$$\begin{aligned} |q| &\leq \left|\frac{a_n}{r^{n-1}}\right| = r\frac{|a_n|}{r^n} < r + r\frac{|a_n|}{r^n}, \\ |q| &\leq r + \left|\frac{a_{n-1}}{r^{n-2}}\right| < r + r\frac{|a_{n-1}|}{r^{n-1}}, \\ &. \\ &. \\ &. \\ |q| &\leq r + \left|\frac{a_2}{r^{n-1}}\right| < r + r\frac{|a_2|}{r^1} \end{aligned}$$

and

$$|q + a_1| \leq r.$$

Using the relation $\alpha_\nu = \frac{|a_\nu|}{r^\nu}$, $\nu = 2, 3...n$, given by (16) to the above inequalities, all the left eigenvalues of $T^{-1}C_fT$ lie in the union of balls

$$\begin{aligned} |q| &\leq r(1 + \alpha_n), \\ |q| &\leq r(1 + \alpha_{n-1}), \\ &. \\ &. \\ &. \\ |q| &\leq r(1 + \alpha_2) \end{aligned}$$

and

$$|q + a_1| \leq r.$$

Applying the fact $\alpha_2 \leq \alpha_3 \leq ..... \leq \alpha_\lambda \geq .. \geq \alpha_n$, it follows that all left eigenvalues of $T^{-1}C_fT$ lie in the union of balls

$$|q| \leq r(1 + \alpha_\lambda) \ and \ |q + a_1| \leq r.$$

Then, by Theorem 2.3, all the zeros of $f(q)$ lie in the union of balls

$$|q| \leq r(1 + \alpha_\lambda) \ and \ |q + a_1| \leq r,$$

and this completes the proof of Theorem 3.1. □

Example 4.1. Let $f(q) = q^4 + q^2 0.5j + q(\frac{2}{8} + \frac{3}{8}i + \frac{1}{2}j + \frac{1}{8}k) + (\frac{1}{8} + \frac{3}{16}k)$ be a monic quaternionic polynomial where $a_0 = 1$, $a_1 = 0$, $a_2 = 0.5j$ and $a_3 = \frac{2}{8} + \frac{3}{8}i + \frac{1}{2}j + \frac{1}{8}k$ and $a_4 = \frac{1}{8} + \frac{3}{16}k$.
For $r = \frac{1}{2}$, and using the relation $\alpha_\nu = \frac{|a_\nu|}{r^\nu}$, $v = 2, 3...n$, we have $\alpha_2 = 2$, $\alpha_3 = \sqrt{30} = 5.47722558$ and $a_4 = \sqrt{13} = 3.6055128$ such that $\alpha_2 \leq \alpha_3 \geq \alpha_4$. It is vividly seen that Theorem 2.8 and Theorem 2.6 are not applicable to this quaternionic polynomial and hence our result (Theorem 3.1) is applicable to a larger class of polynomials with quaternionic coefficients.
Using Theorem 3.1, the zeros of $f(q)$ will lie in the union of the balls

$$\{q \in \mathbf{H} : |q| \leq 0.5(1 + \alpha_3)\}$$

and

$$\{q \in \mathbf{H} : |q + a_1| \leq 0.5\},$$

which implies that the zeros of $f(q)$ wil lie in the union of the balls

$$\{q \in \mathbf{H} : |q| \leq 3.23861279\}$$

and

$$\{q \in \mathbf{H} : |q| \leq 0.5\}.$$

Example 4.2. Let $f_1(q) = q^4 + q^2\frac{1}{11}j + q2i + \frac{1}{10}k$ be a monic quaternionic polynomial which holds Theorem 3.1.
For $r = 2$, using Theorem 3.1, all the zeros of $f_1(q)$ lie in union of the balls $|q| \leq 2.35$ and $|q| \leq 2$. Whereas, using Theorem 2.7, the zeros of $f_1(q)$ will lie in the ball

$$\{q \in \mathbf{H} : |q| \leq (1 + |a_3|) = 3\}.$$

This shows that our result (Theorem 3.1) gives more accurate zero region of the polynomial $f_1(q)$ than that of Theorem 2.7.

Moreover, we are interested to notice that Corollary 3.3 significantly improves zero bound than its counterparts given by inequality (22) of Corollary 3.3, inequality (8)(Theorem 2.7) due to Dar et al., and (7)(Theorem 2.6) due to Milovanović et al. by making use of WOLFRAM MATHEMATICA.

Example 4.3. We consider the complex polynomial $p(z) = z^4 + z^3 + 3z^2 + \frac{1}{2}z + \frac{1}{3}$.
For $r = 2$, and using the relation $\alpha_\nu = \frac{|a_\nu|}{r^\nu}$, $v = 2, 3...n$, we have, $\alpha_2 = 0.3$, $\alpha_3 = 0.03125$ and $\alpha_4 = 0.0104166667$ such that $\alpha_2 \geq \alpha_3 \geq \alpha_4$. Hence, using Corollary 3.3, all zeros of $p(z)$ lie in union of $|z| \leq 2.6$ and $|z + 1| \leq 2$.
Using Theorems 2.6 and 2.7, it is concluded that all the zeros of $p(z)$ lie respectively in and $|z| \leq 5$ and $|z| \leq 4$ showing that the bound given by Corollary 3.3 improves most and the respective sizes of zero regions are depicted in the figure below.

It is of interest to examine the area-percentage of improvements of the bound of Corollary 3.3 over Theorem 2.7 due to Dar et al. and Theorem 2.6 due to Milovanović et al. are respectively 55.53% and 71.54%.

5. Conclusion

The regular functions of a quaternionic variable have been introduced and intensively studied in the past few years and it has been proven to be fertile topic in analysis and its rapid development has been largely driven by the applications to operator theory. Thus, it becomes an interesting perspective to think about the regions containing all the zeros of a quaternionic polynomial. In this paper, we prove a result which gives an improved the bound of the zeros of a special quaternionic polynomial whose moduli of the coefficients satisfy a suitable inequality.

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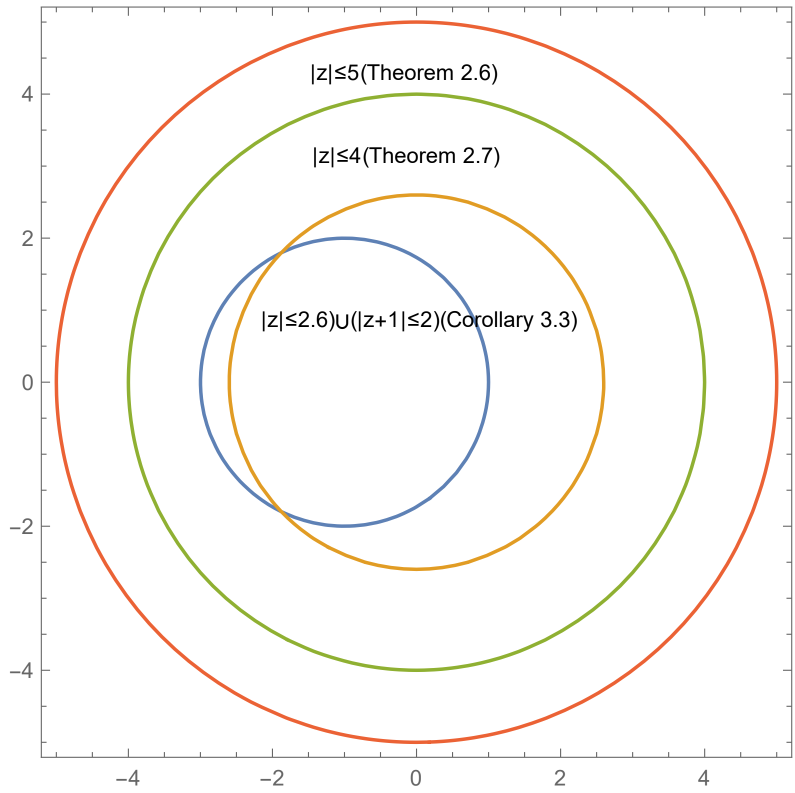

FIGURE 1. Comparison of the zero regions given by inequality (22) of Corollary 3.3, inequalities (8)(Theorem 2.7) due to Dar et al., and (7)(Theorem 2.6) due to Milovanović et al..

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