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Hypo-q-Norms on a Cartesian Product of Normed Linear Spaces

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Abstract. In this paper we introduce the hypo-q-norms on a Cartesian product of normed linear spaces. A representation of these norms in terms of bounded linear functionals of norm one, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar Shisha-Mond, Birnacki et al. and other Grüss type inequalities are also given.

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1. Introduction

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. On $\mathbb{K}^n$ endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$ and the unit sphere

$$\mathbb{S}(\|\cdot\|_n) := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \mid \|\lambda\|_n = 1\}.$$

As an example of such norms we should mention the usual *p-norms*

$$\|\lambda\|_{n,p} := \begin{cases} \max\{|\lambda_1|, \dots, |\lambda_n|\} & \text{if } p = \infty; \\ \\ \left(\sum_{k=1}^n |\lambda_k|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases} \tag{1}$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\lambda\|_{n,2} = \left(\sum_{k=1}^n |\lambda_k|^2\right)^{\frac{1}{2}}.$$

It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following *p-norms*:

$$\|\mathbf{x}\|_{n,p} := \begin{cases} \max\{\|x_1\|, \dots, \|x_n\|\} & \text{if } p = \infty; \\ \\ \left(\sum_{k=1}^n \|x_k\|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty); \end{cases} \tag{2}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

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Following [6], for a given norm $\|\cdot\|_n$ on $\mathbb{K}^n$, we define the functional $\|\cdot\|_{h,n} : E^n \to [0,\infty)$ given by

$$\|\mathbf{x}\|_{h,n} := \sup_{\lambda \in \mathbb{S}\left(\|\cdot\|_n\right)} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{3}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

It is easy to see, by the properties of the norm $\|\cdot\|$, that:

(i) $\|\mathbf{x}\|_{h,n} \geq 0$ for any $\mathbf{x} \in E^n$;

(ii) $\|\mathbf{x} + \mathbf{y}\|_{h,n} \leq \|\mathbf{x}\|_{h,n} + \|\mathbf{y}\|_{h,n}$ for any $\mathbf{x}$, $\mathbf{y} \in E^n$;

(iii) $\|\alpha \mathbf{x}\|_{h,n} = |\alpha| \|\mathbf{x}\|_{h,n}$ for each $\alpha \in \mathbb{K}$ and $\mathbf{x} \in E^n$;

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n. This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n.

If by $\mathbb{S}_{n,p}$ with $p \in [1,\infty]$ we denote the spheres generated by the p-norms $\|\cdot\|_{n,p}$ on $\mathbb{K}^n$, then we can obtain the following *hypo-q-norms* on E^n :

$$\|\mathbf{x}\|_{h,n,q} := \sup_{\lambda \in \mathbb{S}_{n,p}} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{4}$$

with $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ if $p > 1$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$.

For $p = 2$, we have the Euclidean sphere in $\mathbb{K}^n$, which we denote by $\mathbb{S}_n$, $\mathbb{S}_n = \left\{ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \left| \sum_{i=1}^{n} |\lambda_i|^2 = 1 \right. \right\}$ that generates the *hypo-Euclidean norm* on E^n, i.e.,

$$\|\mathbf{x}\|_{h,e} := \sup_{\lambda \in \mathbb{S}_n} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|. \tag{5}$$

Moreover, if $E = H$, H is a inner product space over $\mathbb{K}$, then the *hypo-Euclidean norm* on H^n will be denoted simply by

$$\|\mathbf{x}\|_e := \sup_{\lambda \in \mathbb{S}_n} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|. \tag{6}$$

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and $n \in \mathbb{N}$, $n \geq 1$. In the Cartesian product $H^n := H \times \cdots \times H$, for the $n-$tuples of vectors $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in H^n$, we can define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^{n} \langle x_j, y_j \rangle, \qquad \mathbf{x},\ \mathbf{y} \in H^n, \tag{7}$$

which generates the Euclidean norm $\|\cdot\|_2$ on H^n, i.e.,

$$\|\mathbf{x}\|_2 := \left(\sum_{j=1}^{n} \|x_j\|^2 \right)^{\frac{1}{2}}, \qquad \mathbf{x} \in H^n. \tag{8}$$

The following result established in [6] connects the usual Euclidean norm $\|\cdot\|$ with the hypo-Euclidean norm $\|\cdot\|_e$.

Theorem 1.1 (Dragomir, 2007, [6]). *For any* $\mathbf{x} \in H^n$ *we have the inequalities*

$$\frac{1}{\sqrt{n}} \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_e \leq \|\mathbf{x}\|_2 , \tag{9}$$

i.e., $\|\cdot\|_2$ *and* $\|\cdot\|_e$ *are equivalent norms on* H^n.

The following representation result for the hypo-Euclidean norm plays a key role in obtaining various bounds for this norm:

Theorem 1.2 (Dragomir, 2007, [6]). *For any* $\mathbf{x} \in H^n$ *with* $\mathbf{x} = (x_1, \ldots, x_n)$, *we have*

$$\|\mathbf{x}\|_e = \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle x, x_j \rangle|^2 \right)^{\frac{1}{2}} . \tag{10}$$

Motivated by the above results, in this paper we introduce the hypo-q-norms on a Cartesian product of normed linear spaces. A representation of these norms in terms of bounded linear functionals of norm one, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar Shisha-Mond, Birnacki et al. and other Grüss type inequalities are also given.

2. General Results

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. We denote by E^* its dual space endowed with the norm $\|\cdot\|$ defined by

$$\|f\| := \sup_{\|x\|=1} |f(x)| < \infty, \text{ where } f \in E^*.$$

We have the following representation result for the *hypo-q-norms* on E^n.

Theorem 2.1. *Let* $(E, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$. *For any* $\mathbf{x} \in E^n$ *with* $\mathbf{x} = (x_1, \ldots, x_n)$, *we have the representation*

$$\|\mathbf{x}\|_{h,n,q} = \sup_{\|f\|=1} \left\{ \left(\sum_{j=1}^{n} |f(x_j)|^q \right)^{1/q} \right\}, \tag{11}$$

where p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathbf{x}\|_{h,n,1} = \sup_{\|f\|=1} \left\{ \sum_{j=1}^{n} |f(x_j)| \right\} \tag{12}$$

and

$$\|\mathbf{x}\|_{h,n,\infty} = \|\mathbf{x}\|_{n,\infty} = \max_{j \in \{1,\ldots,n\}} \{\|x_j\|\} . \tag{13}$$

In particular,

$$\|\mathbf{x}\|_{h,e} = \sup_{\|f\|=1} \left\{ \left(\sum_{j=1}^{n} |f(x_j)|^2 \right)^{1/2} \right\}. \tag{14}$$

Proof. Using Hölder's discrete inequality for p, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\left(\sum_{j=1}^{n}|\alpha_j|^p\right)^{1/p}\left(\sum_{j=1}^{n}|\beta_j|^q\right)^{1/q},$$

which implies that

$$\sup_{\|\alpha\|_p=1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\|\beta\|_q \tag{15}$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$.

For $(\beta_1,\ldots,\beta_n)\neq 0$, consider $\alpha=(\alpha_1,\ldots,\alpha_n)$ with

$$\alpha_j:=\frac{\overline{\beta_j}\,|\beta_j|^{q-2}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}$$

for those j for which $\beta_j\neq 0$ and $\alpha_j=0$, for the rest.

We observe that

$$\begin{aligned}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|&=\left|\sum_{j=1}^{n}\frac{\overline{\beta_j}\,|\beta_j|^{q-2}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}\beta_j\right|=\frac{\sum_{j=1}^{n}|\beta_j|^q}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}\\&=\left(\sum_{j=1}^{n}|\beta_j|^q\right)^{1/q}=\|\beta\|_q\end{aligned}$$

and

$$\begin{aligned}\|\alpha\|_p^p&=\sum_{j=1}^{n}|\alpha_j|^p=\sum_{j=1}^{n}\frac{\left|\overline{\beta_j}\,|\beta_j|^{q-2}\right|^p}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}=\sum_{j=1}^{n}\frac{\left(|\beta_j|^{q-1}\right)^p}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}\\&=\sum_{j=1}^{n}\frac{|\beta_j|^{qp-p}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}=\sum_{j=1}^{n}\frac{|\beta_j|^q}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}=1.\end{aligned}$$

Therefore, by (15) we have the representation

$$\sup_{\|\alpha\|_p=1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\|\beta\|_q \tag{16}$$

for any $\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{K}^n$.

By Hahn-Banach theorem, we have for any $u\in E$, $u\neq 0$ that

$$\|u\|=\sup_{\|f\|=1}|f(u)|. \tag{17}$$

Let $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{K}^n$ and $\mathbf{x}\in E^n$ with $\mathbf{x}=(x_1,\ldots,x_n)$. Then by (17) we have

$$\left\|\sum_{j=1}^{n}\alpha_jx_j\right\|=\sup_{\|f\|=1}\left|f\left(\sum_{j=1}^{n}\alpha_jx_j\right)\right|=\sup_{\|f\|=1}\left|\sum_{j=1}^{n}\alpha_jf(x_j)\right|. \tag{18}$$

By taking the supremum in this equality we have

$$\begin{aligned}\sup_{\|\alpha\|_p=1}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| &= \sup_{\|\alpha\|_p=1}\left(\sup_{\|f\|=1}\left|\sum_{j=1}^{n}\alpha_j f(x_j)\right|\right)\\ &= \sup_{\|f\|=1}\left(\sup_{\|\alpha\|_p=1}\left|\sum_{j=1}^{n}\alpha_j f(x_j)\right|\right) = \sup_{\|f\|=1}\left(\sum_{j=1}^{n}|f(x_j)|^q\right)^{1/2},\end{aligned}$$

where for the last equality we used the representation (16).

This proves (11).

Using the properties of the modulus, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \max_{j\in\{1,\ldots,n\}}|\alpha_j|\sum_{j=1}^{n}|\beta_j|$$

which implies that

$$\sup_{\|\alpha\|_\infty=1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \|\beta\|_1 \tag{19}$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$.

For $(\beta_1,\ldots,\beta_n)\neq 0$, consider $\alpha=(\alpha_1,\ldots,\alpha_n)$ with $\alpha_j := \frac{\overline{\beta_j}}{|\beta_j|}$ for those j for which $\beta_j \neq 0$ and $\alpha_j = 0$, for the rest.

We have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \left|\sum_{j=1}^{n}\frac{\overline{\beta_j}}{|\beta_j|}\beta_j\right| = \sum_{j=1}^{n}|\beta_j| = \|\beta\|_1$$

and

$$\|\alpha\|_\infty = \max_{j\in\{1,\ldots,n\}}|\alpha_j| = \max_{j\in\{1,\ldots,n\}}\left|\frac{\overline{\beta_j}}{|\beta_j|}\right| = 1$$

and by (19) we get the representation

$$\sup_{\|\alpha\|_\infty=1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \|\beta\|_1 \tag{20}$$

for any $\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{K}^n$.

By taking the supremum in the equality (18) we have

$$\begin{aligned}\sup_{\|\alpha\|_\infty=1}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| &= \sup_{\|\alpha\|_\infty=1}\left(\sup_{\|f\|=1}\left|\sum_{j=1}^{n}\alpha_j f(x_j)\right|\right)\\ &= \sup_{\|f\|=1}\left(\sup_{\|\alpha\|_\infty=1}\left|\sum_{j=1}^{n}\alpha_j f(x_j)\right|\right) = \sup_{\|f\|=1}\left(\sum_{j=1}^{n}|f(x_j)|\right),\end{aligned}$$

where for the last equality we used the equality (20), which proves the representation (12).

Finally, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\sum_{j=1}^{n}|\alpha_j|\max_{j\in\{1,\dots,n\}}|\beta_j|$$

which implies that

$$\sup_{\|\alpha\|_1=1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\|\beta\|_\infty \tag{21}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$.

For $(\beta_1,\dots,\beta_n)\neq 0$, let $j_0\in\{1,...,n\}$ such that $\|\beta\|_\infty=\max_{j\in\{1,\dots,n\}}|\beta_j|=|\beta_{j_0}|$. Consider $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_{j_0}=\frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}$ and $\alpha_j=0$ for $j\neq j_0$. For this choice we get

$$\sum_{j=1}^{n}|\alpha_j|=\frac{\left|\overline{\beta_{j_0}}\right|}{|\beta_{j_0}|}=1 \text{ and } \left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\left|\frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}\beta_{j_0}\right|=|\beta_{j_0}|=\|\beta\|_\infty,$$

therefore by (21) we obtain the representation

$$\sup_{\|\alpha\|_1=1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\|\beta\|_\infty \tag{22}$$

for any $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{K}^n$.

By taking the supremum in the equality (18) and by using the equality (22), we have

$$\begin{aligned}\sup_{\|\alpha\|_1=1}\left\|\sum_{j=1}^{n}\alpha_jx_j\right\| &= \sup_{\|\alpha\|_1=1}\left(\sup_{\|f\|=1}\left|\sum_{j=1}^{n}\alpha_jf(x_j)\right|\right)\\ &= \sup_{\|f\|=1}\left(\sup_{\|\alpha\|_1=1}\left|\sum_{j=1}^{n}\alpha_jf(x_j)\right|\right)=\sup_{\|f\|=1}\left(\max_{j\in\{1,\dots,n\}}|f(x_j)|\right)\\ &= \max_{j\in\{1,\dots,n\}}\left(\sup_{\|f\|=1}|f(x_j)|\right)=\max_{j\in\{1,\dots,n\}}\{\|x_j\|\},\end{aligned}$$

which proves (13). For the last equality we used the property (17). □

Corollary 2.2. *With the assumptions of Theorem 2.1 we have for $q\geq 1$ that*

$$\frac{1}{n^{1/q}}\|\mathbf{x}\|_{n,q}\leq\|\mathbf{x}\|_{h,n,q}\leq\|\mathbf{x}\|_{n,q} \tag{23}$$

for any any $\mathbf{x}\in E^n$.

In particular, we have

$$\frac{1}{\sqrt{n}}\|\mathbf{x}\|_2\leq\|\mathbf{x}\|_{h,e}\leq\|\mathbf{x}\|_2 \tag{24}$$

for any any $\mathbf{x}\in E^n$.

Proof. Let $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $f \in E^*$ with $\|f\| = 1$, then for $q \geq 1$

$$\left(\sum_{j=1}^{n} |f(x_j)|^q\right)^{1/q} \leq \left(\sum_{j=1}^{n} (\|f\| \|x_i\|)^q\right)^{1/q} = \|f\| \left(\sum_{j=1}^{n} \|x_i\|^q\right)^{1/q} = \|f\| \|\mathbf{x}\|_{n,q}$$

and by taking the supremum over $\|f\| = 1$, we get the second inequality in (23).

By the properties of complex numbers, we have

$$\max_{j\in\{1,\dots,n\}} \{|f(x_j)|\} \leq \left(\sum_{j=1}^{n} |f(x_j)|^q\right)^{1/q}$$

and by taking the supremum over $\|f\| = 1$, we get

$$\sup_{\|f\|=1} \left(\max_{j\in\{1,\dots,n\}} \{|f(x_j)|\}\right) \leq \sup_{\|f\|=1} \left(\sum_{j=1}^{n} |f(x_j)|^q\right)^{1/q} \tag{25}$$

and since

$$\begin{aligned}\sup_{\|f\|=1} \left(\max_{j\in\{1,\dots,n\}} \{|f(x_j)|\}\right) &= \max_{j\in\{1,\dots,n\}} \left\{\sup_{\|f\|=1} |f(x_j)|\right\} \\ &= \max_{j\in\{1,\dots,n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty},\end{aligned}$$

then by (25) we get

$$\|\mathbf{x}\|_{n,\infty} \leq \|\mathbf{x}\|_{h,n,q} \text{ for any } \mathbf{x} \in E^n. \tag{26}$$

Since

$$\left(\sum_{j=1}^{n} \|x_j\|^q\right)^{1/q} \leq \left(n \|\mathbf{x}\|_{n,\infty}^q\right)^{1/q} = n^{1/q} \|\mathbf{x}\|_{n,\infty}$$

then also

$$\frac{1}{n^{1/q}} \|\mathbf{x}\|_{n,q} \leq \|\mathbf{x}\|_{n,\infty} \text{ for any } \mathbf{x} \in E^n. \tag{27}$$

By utilizing the inequalities (26) and (27) we obtain the first inequality in (23). □

Remark 2.1. In the case of inner product spaces the inequality (24) has been obtained in a different and more difficult way [6] by employing the rotation-invariant normalized positive Borel measure on the unit sphere.

Corollary 2.3. *With the assumptions of Theorem 2.1 we have for $r \geq q \geq 1$ that*

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{r-q}{rq}} \|\mathbf{x}\|_{h,n,r} \tag{28}$$

for any any $\mathbf{x} \in E^n$.

In particular, for $q \geq 2$ we have

$$\|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{h,e} \leq n^{\frac{q-2}{2q}} \|\mathbf{x}\|_{h,n,q} \tag{29}$$

and for $1 \leq q \leq 2$ we have

$$\|\mathbf{x}\|_{h,e} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{2-q}{2q}} \|\mathbf{x}\|_{h,e} \tag{30}$$

for any any $\mathbf{x} \in E^n$.

Proof. We use the following elementary inequalities for the nonnegative numbers a_j, $j = 1, ..., n$ and $r \geq q > 0$ (see for instance [8])

$$\left(\sum_{j=1}^{n} a_j^r\right)^{1/r} \leq \left(\sum_{j=1}^{n} a_j^q\right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^{n} a_j^r\right)^{1/r}. \tag{31}$$

Let $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $f \in E^*$ with $\|f\| = 1$, then for $r \geq q \geq 1$ we have

$$\left(\sum_{j=1}^{n} |f(x_j)|^r\right)^{1/r} \leq \left(\sum_{j=1}^{n} |f(x_j)|^q\right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^{n} |f(x_j)|^r\right)^{1/r}. \tag{32}$$

By taking the supremum over $f \in E^*$ with $\|f\| = 1$ and using Theorem 2.1, we get (28). □

Remark 2.2. If we take $q = 1$ in (28), then we get

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,1} \leq n^{\frac{r-1}{r}} \|\mathbf{x}\|_{h,n,r} \tag{33}$$

for any any $\mathbf{x} \in E^n$.

In particular, for $r = 2$ we get

$$\|\mathbf{x}\|_{h,e} \leq \|\mathbf{x}\|_{h,n,1} \leq \sqrt{n} \|\mathbf{x}\|_{h,e} \tag{34}$$

for any any $\mathbf{x} \in E^n$.

3. Some Reverse Inequalities

Recall the following reverse of Cauchy-Buniakowski-Schwarz inequality [4] (see also [5, Theorem 5. 14])

Lemma 3.1. *Let a, $A \in \mathbb{R}$ and $\mathbf{z} = (z_1, \dots, z_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ be two sequences of real numbers with the property that:*

$$ay_j \leq z_j \leq Ay_j \quad \text{for each} \quad j \in \{1, \dots, n\}. \tag{35}$$

Then for any $\mathbf{w} = (w_1, \dots, w_n)$ a sequence of positive real numbers, one has the inequality

$$0 \leq \sum_{j=1}^{n} w_j z_j^2 \sum_{j=1}^{n} w_j y_j^2 - \left(\sum_{j=1}^{n} w_j z_j y_j\right)^2 \leq \frac{1}{4}(A-a)^2 \left(\sum_{j=1}^{n} w_j y_j^2\right)^2. \tag{36}$$

The constant $\frac{1}{4}$ is sharp in (36).

O. Shisha and B. Mond obtained in 1967 (see [9]) the following counterparts of (CBS)- inequality (see also [5, Theorem 5.20 & 5.21]).

Lemma 3.2. *Assume that $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are such that there exists a, A, b, B with the property that:*

$$0 \leq a \leq a_j \leq A \quad \text{and} \quad 0 < b \leq b_j \leq B \quad \text{for any } j \in \{1, \dots, n\} \tag{37}$$

then we have the inequality

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}}\right)^2 \sum_{j=1}^{n} a_j b_j \sum_{j=1}^{n} b_j^2. \tag{38}$$

and

Lemma 3.3. *Assume that* $\mathbf{a}$, $\mathbf{b}$ *are nonnegative sequences and there exists* γ, Γ *with the property that*

$$0 \leq \gamma \leq \frac{a_j}{b_j} \leq \Gamma < \infty \quad \textit{for any} \quad j \in \{1, \dots, n\}. \tag{39}$$

Then we have the inequality

$$0 \leq \left(\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2\right)^{\frac{1}{2}} - \sum_{j=1}^{n} a_j b_j \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2. \tag{40}$$

We have the following result:

Theorem 3.4. *Let* $(E, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$ *and* $\mathbf{x} \in E^n$ *with* $\mathbf{x} = (x_1, \dots, x_n)$. *Then we have*

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \frac{1}{4} n \|\mathbf{x}\|_{n,\infty}^2, \tag{41}$$

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \|\mathbf{x}\|_{h,n,1} \|\mathbf{x}\|_{n,\infty} \tag{42}$$

and

$$0 \leq \|\mathbf{x}\|_{h,e} - \frac{1}{\sqrt{n}} \|\mathbf{x}\|_{h,n,1} \leq \frac{1}{4} \sqrt{n} \|\mathbf{x}\|_{n,\infty}. \tag{43}$$

Proof. Let $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and put $R = \max_{j \in \{1,\dots,n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. If $f \in E^*$ with $\|f\| = 1$ then $|f(x_j)| \leq \|f\| \|x_j\| \leq R$ for any $j \in \{1, ..., n\}$.

If we write the inequality (36) for $z_j = |f(x_j)|$, $w_j = y_j = 1$, $A = R$ and $a = 0$, we get

$$0 \leq n \sum_{j=1}^{n} |f(x_j)|^2 - \left(\sum_{j=1}^{n} |f(x_j)|\right)^2 \leq \frac{1}{4} n^2 R^2$$

for any $f \in E^*$ with $\|f\| = 1$.

This implies that

$$\sum_{j=1}^{n} |f(x_j)|^2 \leq \frac{1}{n} \left(\sum_{j=1}^{n} |f(x_j)|\right)^2 + \frac{1}{4} n R^2 \tag{44}$$

for any $f \in E^*$ with $\|f\| = 1$.

By taking the supremum in (44) over $f \in E^*$ with $\|f\| = 1$ we get (41).

If we write the inequality (38) for $a_j = |f(x_j)|$, $b_j = 1$, $b = B = 1$, $a = 0$ and $A = R$, then we get

$$0 \leq n \sum_{j=1}^{n} |f(x_j)|^2 - \left(\sum_{j=1}^{n} |f(x_j)|\right)^2 \leq n R \sum_{j=1}^{n} |f(x_j)|,$$

for any $f \in E^*$ with $\|f\| = 1$.

This implies that

$$\sum_{j=1}^{n} |f(x_j)|^2 \le \frac{1}{n}\left(\sum_{j=1}^{n} |f(x_j)|\right)^2 + R\sum_{j=1}^{n} |f(x_j)|, \tag{45}$$

for any $f \in E^*$ with $\|f\| = 1$.

By taking the supremum in (45) over $f \in E^*$ with $\|f\| = 1$ we get (42).

Finally, if we write the inequality (40) for $a_j = |f(x_j)|$, $b_j = 1$, $b = B = 1$, $\gamma = 0$ and $\Gamma = R$ we have

$$0 \le \left(n\sum_{j=1}^{n} |f(x_j)|^2\right)^{\frac{1}{2}} - \sum_{j=1}^{n} |f(x_j)| \le \frac{1}{4}nR,$$

for any $f \in E^*$ with $\|f\| = 1$.

This implies that

$$\left(\sum_{j=1}^{n} |f(x_j)|^2\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{n}}\sum_{j=1}^{n} |f(x_j)| + \frac{1}{4}\sqrt{n}R, \tag{46}$$

for any $f \in E^*$ with $\|f\| = 1$.

By taking the supremum in (46) over $f \in E^*$ with $\|f\| = 1$ we get (43). □

Further, we recall the *Čebyšev's inequality* for *synchronous* n-tuples of vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, namely if $(a_j - a_k)(b_j - b_k) \ge 0$ for any j, $k \in \{1, ..., n\}$, then

$$\frac{1}{n}\sum_{j=1}^{n} a_j b_j \ge \frac{1}{n}\sum_{j=1}^{n} a_j \frac{1}{n}\sum_{j=1}^{n} b_j. \tag{47}$$

In 1950, Biernacki et al. [1] obtained the following discrete version of Grüss' inequality:

Lemma 3.5. *Assume that* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{b} = (b_1, \dots, b_n)$ *are such that there exists real numbers* a, A, b, B *with the property that:*

$$a \le a_j \le A \quad \text{and} \quad b \le b_j \le B \quad \text{for any } j \in \{1, \dots, n\}. \tag{48}$$

Then

$$\begin{aligned} \left|\frac{1}{n}\sum_{j=1}^{n} a_j b_j - \frac{1}{n}\sum_{j=1}^{n} a_j \frac{1}{n}\sum_{j=1}^{n} b_j\right| &\le \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\left(1 - \frac{1}{n}\left\lceil\frac{n}{2}\right\rceil\right)(A-a)(B-b) \\ &= \frac{1}{n^2}\left\lceil\frac{n^2}{4}\right\rceil(A-a)(B-a) \le \frac{1}{4}(A-a)(B-b), \end{aligned} \tag{49}$$

where $\lceil x \rceil$ *gives the largest integer less than or equal to* x.

The following result also holds:

Theorem 3.6. *Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$ and $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$. Then for $q, r \geq 1$ we have*

$$\begin{aligned}\|\mathbf{x}\|_{h,n,q+r}^{q+r} &\leq \frac{1}{n}\|\mathbf{x}\|_{h,n,q}^{q}\|\mathbf{x}\|_{h,n,r}^{r} + \frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil\|\mathbf{x}\|_{n,\infty}^{q+r} \\ &\leq \frac{1}{n}\|\mathbf{x}\|_{h,n,q}^{q}\|\mathbf{x}\|_{h,n,r}^{r} + \frac{1}{4}n\|\mathbf{x}\|_{n,\infty}^{q+r}. \end{aligned} \tag{50}$$

Proof. Let $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and put $R = \max_{j\in\{1,\ldots,n\}}\{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. If $f \in E^*$ with $\|f\| = 1$ then $|f(x_j)| \leq \|f\|\|x_j\| \leq R$ for any $j \in \{1, ..., n\}$.

If we take into the inequality (49) $a_j = |f(x_j)|^q$, $b_j = |f(x_j)|^r$, $a = 0$, $A = R^q$, $b = 0$ and $B = R^r$, then we get

$$\left|\frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q+r} - \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q}\frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{r}\right| \leq \frac{1}{n^2}\left\lceil\frac{n^2}{4}\right\rceil R^{q+r}. \tag{51}$$

On the other hand, since the sequences $\{a_j\}_{j=1,\ldots,n}$, $\{b_j\}_{j=1,\ldots,n}$ are synchronous, then by (47) we have

$$0 \leq \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q+r} - \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q}\frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{r}.$$

Using (51) we then get

$$\sum_{j=1}^{n}|f(x_j)|^{q+r} \leq \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q}\sum_{j=1}^{n}|f(x_j)|^{r} + \frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil R^{q+r} \tag{52}$$

for any $f \in E^*$ with $\|f\| = 1$.

By taking the supremum in (52), we get

$$\begin{aligned}&\sup_{\|f\|=1}\left\{\sum_{j=1}^{n}|f(x_j)|^{q+r}\right\} \\ &\leq \frac{1}{n}\sup_{\|f\|=1}\left\{\sum_{j=1}^{n}|f(x_j)|^{q}\sum_{j=1}^{n}|f(x_j)|^{r}\right\} + \frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil R^{q+r} \\ &\leq \frac{1}{n}\sup_{\|f\|=1}\left\{\sum_{j=1}^{n}|f(x_j)|^{q}\right\}\sup_{\|f\|=1}\left\{\sum_{j=1}^{n}|f(x_j)|^{r}\right\} + \frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil R^{q+r},\end{aligned}$$

which proves the first inequality in (50).

The second part of (50) is obvious. □

Corollary 3.7. *With the assumptions of Theorem 3.6 and if $r \geq 1$, then we have*

$$\|\mathbf{x}\|_{h,n,2r}^{2r} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,r}^{2r} + \frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil\|\mathbf{x}\|_{n,\infty}^{2r} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,r}^{2r} + \frac{1}{4}n\|\mathbf{x}\|_{n,\infty}^{2r}. \tag{53}$$

In particular, for $r = 1$ we get

$$\|\mathbf{x}\|_{h,e}^{2} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,1}^{2} + \frac{1}{n}\left\lceil\frac{n^2}{4}\right\rceil\|\mathbf{x}\|_{n,\infty}^{2} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,1}^{2} + \frac{1}{4}n\|\mathbf{x}\|_{n,\infty}^{2}. \tag{54}$$

The first inequality in (54) is better than the second inequality in (41).

4. Reverse Inequalities Via Forward Difference

For an n-tuple of complex numbers $\mathbf{a} = (a_1, \ldots, a_n)$ with $n \geq 2$ consider the $(n-1)$-tuple built by the aid of forward differences $\Delta\mathbf{a} = (\Delta a_1, ..., \Delta a_{n-1})$ where $\Delta a_k := a_{k+1} - a_k$ where $k \in \{1, ..., n-1\}$. Similarly, if $\mathbf{x} = (x_1, \ldots, x_n) \in E^n$ is an n-tuple of vectors we also can consider in a similar way the $(n-1)$-tuple $\Delta\mathbf{x} = (\Delta x_1, ..., \Delta x_{n-1})$.

We obtained the following Grüss' type inequalities in terms of forward differences:

Lemma 4.1. *Assume that $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ are n-tuples of complex numbers. Then*

$$\left| \frac{1}{n} \sum_{j=1}^{n} a_j b_j - \frac{1}{n} \sum_{j=1}^{n} a_j \frac{1}{n} \sum_{j=1}^{n} b_j \right| \tag{55}$$

$$\leq \begin{cases} \frac{1}{12}\left(n^2-1\right) \|\Delta\mathbf{a}\|_{n-1,\infty} \|\Delta\mathbf{b}\|_{n-1,\infty}, & [7], \\ \\ \frac{1}{6}\frac{n^2-1}{n} \|\Delta\mathbf{a}\|_{n-1,\alpha} \|\Delta\mathbf{b}\|_{n-1,\beta} \text{ where } \alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, & [2], \\ \\ \frac{1}{2}\left(1 - \frac{1}{n}\right) \|\Delta\mathbf{a}\|_{n-1,1} \|\Delta\mathbf{b}\|_{n-1,1}, & [3]. \end{cases}$$

The constants $\frac{1}{12}$, $\frac{1}{6}$ and $\frac{1}{2}$ are best possible in (55).

The following result also holds:

Theorem 4.2. *Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$ and $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$. Then for $q, r \geq 1$ we have*

$$\|\mathbf{x}\|_{h,n,q+r}^{q+r} \leq \frac{1}{n} \|\mathbf{x}\|_{h,n,q}^{q} \|\mathbf{x}\|_{h,n,r}^{r} \tag{56}$$

$$+ \begin{cases} \frac{1}{12} qr \left(n^2-1\right) n \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta\mathbf{x}\|_{n-1,\infty}^{2}, \\ \\ \frac{1}{6}\left(n^2-1\right) qr \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta\mathbf{x}\|_{h,n-1,\alpha} \|\Delta\mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2}(n-1) qr \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta\mathbf{x}\|_{h,n-1,1}^{2}. \end{cases}$$

Proof. Let $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $f \in E^*$ with $\|f\| = 1$. If we take into the inequality (55) $a_j = |f(x_j)|^q$, $b_j = |f(x_j)|^r$, then we get

$$\left| \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^{q+r} - \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^q \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^r \right| \tag{57}$$

$$\leq \begin{cases} \frac{1}{12}\left(n^2-1\right) \max_{j=1,\ldots,n-1} \left|\Delta |f(x_j)|^q\right| \max_{j=1,\ldots,n-1} \left|\Delta |f(x_j)|^r\right|, \\ \\ \frac{1}{6}\frac{n^2-1}{n} \left(\sum_{j=1}^{n-1} \left|\Delta |f(x_j)|^q\right|^\alpha\right)^{1/\alpha} \left(\sum_{j=1}^{n-1} \left|\Delta |f(x_j)|^r\right|^\beta\right)^{1/\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2}\left(1 - \frac{1}{n}\right) \sum_{j=1}^{n-1} \left|\Delta |f(x_j)|^q\right| \sum_{j=1}^{n-1} \left|\Delta |f(x_j)|^r\right|. \end{cases}$$

We use the following elementary inequality for powers $p \geq 1$

$$\left|a^p - b^p\right| \leq pR^{p-1}\left|a-b\right|$$

where $a, b \in [0, R]$.

Put $R = \max_{j\in\{1,\ldots,n\}}\{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. Then for any $f \in E^*$ with $\|f\| = 1$ we have $|f(x_j)| \leq \|f\|\,\|x_j\| \leq R$ for any $j \in \{1, \ldots, n\}$.

Therefore

$$\begin{aligned} \left|\Delta\left|f(x_j)\right|^q\right| &= \left|\left|f(x_{j+1})\right|^q - \left|f(x_j)\right|^q\right| \leq qR^{q-1}\left|\left|f(x_{j+1})\right| - \left|f(x_j)\right|\right| \\ &\leq qR^{q-1}\left|f(x_{j+1}) - f(x_j)\right| = qR^{q-1}\left|f(\Delta x_j)\right| \end{aligned} \tag{58}$$

for any $j = 1, \ldots, n-1$, where $\Delta x_j = x_{j+1} - x_j$ is the forward difference.

On the other hand, since the sequences $\{a_j\}_{j=1,\ldots,n}$, $\{b_j\}_{j=1,\ldots,n}$ are synchronous, then we have

$$0 \leq \frac{1}{n}\sum_{j=1}^{n}\left|f(x_j)\right|^{q+r} - \frac{1}{n}\sum_{j=1}^{n}\left|f(x_j)\right|^q\frac{1}{n}\sum_{j=1}^{n}\left|f(x_j)\right|^r \tag{59}$$

and by the first inequality in (57) we get

$$\begin{aligned} \sum_{j=1}^{n}\left|f(x_j)\right|^{q+r} &\leq \frac{1}{n}\sum_{j=1}^{n}\left|f(x_j)\right|^q\sum_{j=1}^{n}\left|f(x_j)\right|^r \\ &\quad + \frac{1}{12}\left(n^2-1\right)nqR^{q-1}\max_{j=1,\ldots,n-1}\left|f(\Delta x_j)\right| rR^{r-1}\max_{j=1,\ldots,n-1}\left|f(\Delta x_j)\right| \\ &= \frac{1}{n}\sum_{j=1}^{n}\left|f(x_j)\right|^q\sum_{j=1}^{n}\left|f(x_j)\right|^r \\ &\quad + \frac{1}{12}\left(n^2-1\right)nqrR^{q+r-2}\left(\max_{j=1,\ldots,n-1}\left|f(\Delta x_j)\right|\right)^2 \end{aligned} \tag{60}$$

for any $f \in E^*$ with $\|f\| = 1$.

Taking the supremum over $f \in E^*$ with $\|f\| = 1$ in (60) we get the first branch in the inequality (56).

We also have, by (58), that

$$\begin{aligned} \left(\sum_{j=1}^{n-1}\left|\Delta\left|f(x_j)\right|^q\right|^\alpha\right)^{1/\alpha} &\leq \left[\left(qR^{q-1}\right)^\alpha\sum_{j=1}^{n-1}\left|f(\Delta x_j)\right|^\alpha\right]^{1/\alpha} \\ &= qR^{q-1}\left(\sum_{j=1}^{n-1}\left|f(\Delta x_j)\right|^\alpha\right)^{1/\alpha} \end{aligned}$$

and, similarly,

$$\left(\sum_{j=1}^{n-1}\left|\Delta\left|f(x_j)\right|^r\right|^\beta\right)^{1/\beta} \leq rR^{r-1}\left(\sum_{j=1}^{n-1}\left|f(\Delta x_j)\right|^\beta\right)^{1/\beta}$$

where $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

By the second inequality in (57) and by (59) we have

$$\begin{aligned}\sum_{j=1}^{n}|f(x_j)|^{q+r} &\leq \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q}\sum_{j=1}^{n}|f(x_j)|^{r}\\&+\frac{1}{6}\left(n^2-1\right)\left(\sum_{j=1}^{n-1}|\Delta|f(x_j)|^{q}|^{\alpha}\right)^{1/\alpha}\left(\sum_{j=1}^{n-1}|\Delta|f(x_j)|^{r}|^{\beta}\right)^{1/\beta}\\&\leq \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q}\sum_{j=1}^{n}|f(x_j)|^{r}\\&+\frac{1}{6}\left(n^2-1\right)qrR^{q+r-2}\left(\sum_{j=1}^{n-1}|f(\Delta x_j)|^{\alpha}\right)^{1/\alpha}\left(\sum_{j=1}^{n-1}|f(\Delta x_j)|^{\beta}\right)^{1/\beta}\end{aligned}$$

for any $f\in E^*$ with $\|f\|=1$, where $\alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1.$

Taking the supremum over $f\in E^*$ with $\|f\|=1$ in (**??**) we get the second branch in the inequality (56).

We also have, by (58), that

$$\sum_{j=1}^{n-1}|\Delta|f(x_j)|^{q}|\leq qR^{q-1}\sum_{j=1}^{n-1}|f(\Delta x_j)|$$

and

$$\sum_{j=1}^{n-1}|\Delta|f(x_j)|^{r}|\leq rR^{r-1}\sum_{j=1}^{n-1}|f(\Delta x_j)|.$$

By the third inequality in (57) and by (59) we have

$$\begin{aligned}\sum_{j=1}^{n}|f(x_j)|^{q+r} &\leq \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q}\sum_{j=1}^{n}|f(x_j)|^{r}\\&+\frac{1}{2}(n-1)\sum_{j=1}^{n-1}|\Delta|f(x_j)|^{q}|\sum_{j=1}^{n-1}|\Delta|f(x_j)|^{r}|\\&\leq \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q}\sum_{j=1}^{n}|f(x_j)|^{r}\\&+\frac{1}{2}(n-1)qrR^{q+r-2}\sum_{j=1}^{n-1}|f(\Delta x_j)|\sum_{j=1}^{n-1}|f(\Delta x_j)|\end{aligned}\tag{61}$$

for any $f\in E^*$ with $\|f\|=1$.

Taking the supremum over $f\in E^*$ with $\|f\|=1$ in (61) we get the third branch in the inequality (56). □

Corollary 4.3. *With the assumptions of Theorem 4.2 and if $r \geq 1$, then we have*

$$\|\mathbf{x}\|_{h,n,2r}^{2r} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,r}^{2r} + \begin{cases} \frac{1}{12}r^2\left(n^2-1\right)n\|\mathbf{x}\|_{n,\infty}^{2r-2}\|\Delta\mathbf{x}\|_{n-1,\infty}^2, \\ \\ \frac{1}{6}r^2\left(n^2-1\right)\|\mathbf{x}\|_{n,\infty}^{2r-2}\|\Delta\mathbf{x}\|_{h,n-1,\alpha}\|\Delta\mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \frac{1}{2}r^2(n-1)\|\mathbf{x}\|_{n,\infty}^{2r-2}\|\Delta\mathbf{x}\|_{h,n-1,1}^2. \end{cases} \tag{62}$$

In particular, for $r = 1$ we get

$$\|\mathbf{x}\|_{h,e}^2 \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,1}^2 + \begin{cases} \frac{1}{12}\left(n^2-1\right)n\|\Delta\mathbf{x}\|_{n-1,\infty}^2, \\ \\ \frac{1}{6}\left(n^2-1\right)\|\Delta\mathbf{x}\|_{h,n-1,\alpha}\|\Delta\mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \frac{1}{2}(n-1)\|\Delta\mathbf{x}\|_{h,n-1,1}^2. \end{cases} \tag{63}$$

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