Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 52(2), 2025, Pages 321–344, DOI: 10.52846/ami.v52i2.1944
ISSN: 1223-6934

Pseudo Almost Periodic Solution with Measure: Application for some Impulsive Stochastic Neural Networks Model

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Abstract. This paper is concerned with the existence, uniqueness and the exponential stability of double measure r-mean pseudo almost periodic solution for a class of recurrent neural networks for $r \geq 2$ by employing the fixed point theorem and differential inequality. Finally, we give an example to confirm the reliability and feasibility of our findings.

2020 *Mathematics Subject Classification.* Primary 60H15; Secondary 34C27.
Key words and phrases. r-mean pseudo almost periodic functions, measure theory, impulsive stochastic neural networks, fixed point Theorem.

1. Introduction

In the last decades, since stochastic modeling has an interesting role in engineering, social science, finance and physics, stochastic differential equations has been widely studied (see[11, 19, 20, 21, 22, 23]). A lot of attention has been given to the qualitative properties such as uniqueness, existence and stability for stochastic differential equations. Many authors considered the stability, uniqueness and existence of almost periodic solutions for impulsive differential systems in abstract space. The stability and the existence of piecewise almost periodic solutions of impulsive differential systems with time-varying delay has been established by Satmov (see [13]). Recently, recurrent neural networks (RNN) have attracted considerable attentions, this is due to the fact that they are a class of important mathematic models which are widely used in many areas, such as classification of patterns, associative memories, parallel computation, solving certain optimization problems, and so on [4, 5]. RNN are an attractive tool for both practical applications and for the modeling of biological nerve nets, but their successful application requires an understanding of their dynamical properties, in particular, their stability. Many authors studied the dynamic behaviors of neural networks with time delays (see [2, 5, 7, 16]).
In [8], Dingshi and Yusen studied the existence of periodic measures for the impulsive stochastic equations to the following neural networks

$$\begin{cases} dy_i(s) = \Big[- a_i(s)y_i(s) + \sum_{j=1}^{n} a_{ij}(s)\varphi_j(y_j(s)) + f_i(s)\Big]ds \\ \ + \sum_{j=1}^{n} \phi_{ij}(s, y_i(s))dB_j(s),\ s \geq 0,\ s \neq s_k,\ k \in \mathbb{N}, \\ y_i(s_k^+) = I_{k,i}(y(s_k)),\ s = s_k, \\ y_i(s_0) = \psi_i,\ i = 1, 2...n, \end{cases}$$

Received May 09, 2024. Accepted August 9, 2025.

Motivated by the above description, in this paper, we investigate the uniqueness, existence and stability of almost periodic solution of the following stochastic differential equation

$$\begin{cases} dy_i(s) = \Big[- a_i(s)y_i(s) + \sum_{j=1}^{n} b_{ij}(s)\varphi_j(y_j(s)) + \sum_{j=1}^{n} f_{ij}(s)\Big] ds \\ \ + \sum_{j=1}^{n} \phi_{ij}(s, y_j(s))dB_j(s),\ s \geq 0,\ s \neq s_k,\ k \in \mathbb{N}, \\ y_i(s_k^+) = I_k(y_i(s_k)),\ s = s_k, \\ y_i(s_0) = \psi_i,\ i = 1, 2...n, \end{cases} \tag{1}$$

where $y_i(s)$ corresponds to the state of the ith unit at time s, $a_i(s) \geq 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $\varphi_j(y_j(s))$ denotes the activation functions of the jth unit at time s, n corresponds to the number of units in a neural network, $(b_{ij}(s))_{n\times n}$ is connection matrice, $f_{ij}(s)$ is the external bias on the ith unit, for all $j \in \{1, \cdots, n\}$, $\phi_{ij}(\cdot, \cdot) : [0, +\infty[\times L^r(P, \mathcal{K}) \to L^r(P, \mathcal{H}))$, for each $k \in \mathbb{N}$, $I_k(.) : L^r(P, \mathcal{K}) \to L^r(P, \mathcal{H})$, $B(s) = (B_1(s), \cdots, B_n(s))$ is an n-dimensional Brownian motion defined on $(P, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ and the initial condition $\psi = (\psi_1, \cdots, \psi_n)$. $\phi, \varphi, \psi, I_k, s_k$ satisfy suitable conditions that will be established later. Moreover, the notations $x(t_k^+)$ represent the right-hand side limits of $y(\cdot)$ at s_k, and $\underline{a_i} = \inf_{s\in\mathbb{R}} a(s)$. The paper is organized as follows: In section 2, we shall introduce the necessary preliminary results, notations and definitions needed in the later sections. In section 3, we mention some useful and important properties on a concept of mean-value of uniformly almost periodic functions. Making use of these properties, we prove the existence and uniqueness of p-mean (α, β)-pseudo almost periodic mild solutions for (1). In section 4, we prove the exponential stability of the solution of our problems (1). At last, in section 5, an example is given to illustrate our results.

2. Preliminaries

In this section we shall give some preliminary results which will be used in the sequel. The involvement of these results is stated with reference to [3, 9, 10, 14, 15, 17, 18]. Throughout this paper, we shall introduce the following notations

- $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ and $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ are real separable Hilbert spaces.
- $(\Omega, \mathcal{F}, P)$ is a complete probability space.
- Let $L^r(P, \mathcal{H})$ be a Banach space defined by $L^r(P, \mathcal{H}) := \{Y : \mathcal{H}$-valued random variable$\}$ with the norm $\|Y\|_r = \Big(\int_\Omega \mathbb{E}\|Y\|^r dP\Big)^{\frac{1}{r}}$.
- $\mathcal{L}(\mathcal{K}, \mathcal{H}) = \{Y : \mathcal{K} \to \mathcal{H}$ linear bounded operators$\}$. It is equipped with the usual operator norm $\|\cdot\|$.
- $L_2^0(\mathcal{K}, \mathcal{H}) = \{Y : \mathcal{K} \to \mathcal{H}$ Q-Hilbert-Schmidt operators$\}$ with the norm $\|Y\|^2_{L_2^0} = Tr(YQY^*) < \infty$.
- $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P)$ is a filtered probability space, where $\mathcal{F}_s = \sigma\{B(x) - B(y);\ x, y \leq s\}$ and $(B(s), s \in \mathbb{R})$ is two-sided standard Wiener process.
- Denoting $\mathcal{B}$ the Lebesgue $\sigma-$field of $\mathbb{R}$ and by $\mathcal{M} = \{\alpha :$ positive measure on $\mathcal{B}$; $\alpha(\mathbb{R}) = +\infty$ and $\alpha([x, y]) < +\infty,\ \forall\ x, y \in \mathbb{R},\ x < y\}$. Throughout this paper, we take $\alpha, \beta \in \mathcal{M}$ and satisfies the following hypotheses:

(H_0): $\limsup_{r\to+\infty} \frac{\alpha([-r,r])}{\beta([-r,r])} < \infty$.

(H_1) for all $\tau \in \mathbb{R}$, there exist $\gamma > 0$ and a bounded interval I such that: $\alpha(a+\tau; a \in \mathcal{A}) \leq \gamma\alpha(\mathcal{A})$, when $\mathcal{A} \in \mathcal{B}$ satisfy $\mathcal{A} \cap I = \emptyset$.

Let $Y : \mathbb{R} \longrightarrow L^r(P,\mathcal{K})$ a stochastic process. If there exists $l \in \mathbb{R}_+^*$ such that $\mathbb{E}\|Y(t)\|^r \leq l$, then, Y is named to be stochastically bounded in r-th mean sense. It is named to be stochastically continuous in r-th mean sense if $\lim\limits_{t\to s} \mathbb{E}\|Y(t)-Y(s)\|^r = 0$. We denoted by $\mathbf{BC}(\mathbb{R}, L^r(P,\mathcal{H}))$ the space of any stochastically bounded and continuous processes. One can show that $\mathbf{BC}(\mathbb{R}, L^r(P,\mathcal{H}))$ is a Banach space with $\|Y\|_r = \sup_{s\in\mathbb{R}} \Big(\mathbb{E}\|Y(s)\|^r\Big)^{\frac{1}{r}}$. Let S be the set consisting of all real sequences $\{s_i\}_{i\in\mathbb{N}}$ such that $\delta = \inf_{i\in\mathbb{N}}(s_{i+1}-s_i) > 0$, $\lim_{i\to\infty} s_i = \infty$. For $\{s_i\}_{i\in\mathbb{N}} \in S$, let $\mathbf{PC}(\mathbb{R}, L^r(P,\mathcal{H})) = \Big\{\varphi : \mathbb{R} \to L^r(P,\mathcal{H})$ stochastically bounded piecewise continuous functions ; $\varphi(\cdot)$ stochastically continuous at s for $s \notin \{s_i\}_{i\in\mathbb{N}}$ and $\varphi(s_i) = \varphi(s_i^+)$ for all $i \in \mathbb{N}\Big\}$ and

$\mathbf{PC}(\mathbb{R}\times L^r(P,\mathcal{K}), L^r(P,\mathcal{H})) = \Big\{\varphi : \mathbb{R}\times L^r(P,\mathcal{K}) \to L^r(P,\mathcal{H})$ stochastically bounded piecewise continuous functions ; for any $x \in L^r(P,\mathcal{K}), \varphi(.,x) \in \mathbf{PC}(\mathbb{R}, L^r(P,\mathcal{H}))$ and for any $t \in \mathbb{R}, \varphi(t,.)$ stochastically continuous at $x \in L^r(P,\mathcal{K})\Big\}$.

A continuous stochastic process $Y : \mathbb{R} \longrightarrow L^r(P,\mathcal{H})$ is called r-mean almost periodic if for any $\zeta > 0$ there exists $c(\zeta) > 0$ such that for each $\theta \in \mathbb{R}$, therefore, there exists $\xi \in [\theta, \theta + c(\zeta)]$ satisfying: $\sup_{s\in\mathbb{R}} \mathbb{E}\|Y(s+\xi) - Y(s)\|^r < \zeta$.

The set of all such processes will be denoted by $\mathbf{AP}(\mathbb{R}, L^r(P,\mathcal{H}))$.

A sequence of a continuous stochastic process $Y_n : \mathbb{R} \longrightarrow L^r(P,\mathcal{H})$ is called r-mean almost periodic if for any $\zeta > 0$ there exists $c = c(\zeta) > 0$ such that for $i \in \mathbb{N}$, there exists at least one number $k \in [i, i+c]$, satisfying: $\mathbb{E}\|Y_{n+k} - Y_n\|^r < \zeta,\ n \in \mathbb{N}$.

The set of all such processes will be denoted by $\mathbf{AP}(\mathbb{N}, L^r(P,\mathcal{H}))$.

Now, let $\varphi : \mathbb{R} \times L^r(P,\mathcal{K}) \longrightarrow L^r(P,\mathcal{H})$ be a continuous process. φ is called almost periodic in $s \in \mathbb{R}$ uniformly in $Y \in T$, where $T \in L^r(P,\mathcal{K})$ is a compact, if for any $\zeta > 0$ there exists $c(\zeta,T) > 0$ such that for each $\theta \in \mathbb{R}$, therefore, there exists $\xi \in [\theta, \theta + c(\zeta,T)]$ verified: $\sup_{s\in\mathbb{R}} \mathbb{E}\|\varphi(s+\xi,Y) - \psi(s,Y)\|^r < \zeta$ for all stochastic process $Y : \mathbb{R} \longrightarrow T$. The set of all such processes is denoted by $\mathbf{AP}(\mathbb{R} \times L^r(P,\mathcal{K}), L^r(P,\mathcal{H}))$. Now, we define:

$l^\infty(\mathbb{N}, L^r(P,\mathcal{H})) = \Big\{Y : \mathbb{N} \to L^r(P,\mathcal{H}) : \|Y\| = \sup\limits_{n\in\mathbb{N}}(\mathbb{E}\|Y(n)\|^r)^{1/r}\Big\}$,

$\mathbf{PAP_0}(\mathbb{N}, L^r(P, L^r(P,\mathcal{H})), \alpha, \beta) = \Big\{Y \in l^\infty(\mathbb{N},\mathcal{H}) : \lim\limits_{b\to\infty} \frac{1}{\beta([-b,b])}\sum\limits_{i=0}^{k} \mathbb{E}\|Y(i)\|^r\alpha(i) = 0\Big\}$, and $\mathbf{PAP}(\mathbb{N}, L^r(P,\mathcal{H}), \alpha, \beta) = \Big\{\{Y_k\}_{k\in\mathbb{Z}} \in l^\infty(\mathbb{N},\mathcal{H}); Y_k = Y_k^1 + Y_k^2; Y_k^1 \in \mathbf{AP}(\mathbb{N}, L^r(P,\mathcal{H})), Y_k^2 \in \mathbf{PAP_0}(\mathbb{N}, L^r(P,\mathcal{H}), \alpha, \beta)\Big\}$, $\{Y_k\}_{k\in\mathbb{N}}$ is called r-mean (α,β)-pseudo almost periodic. Let $\{s_i\}_{i\in\mathbb{N}} \in S$ and $\varphi \in \mathbf{PC}(\mathbb{R}, L^r(P,\mathcal{H}))$. φ is called r-mean almost periodic if the following conditions are satisfied

(1) $\{s_k^i = s_{i+k} - s_k\}$, $i \in \mathbb{N}$, is equipotentially almost periodic, that is, for any $\zeta > 0$, there exists a relatively dense set Q_ζ of $\mathbb{R}$ such that for each $t \in Q_\zeta$ there is an integer $p \in \mathbb{N}$ such that for all $|s_{k+p} - s_k - t| < \zeta$ for all $k \in \mathbb{N}$.

(2) For any $\zeta > 0$, there exists a nonnegative number $\bar{\delta} = \bar{\delta}(\zeta)$ such that if the points s' and s'' belong to a same interval of continuity of φ and $|s' - s''| < \bar{\delta}$, then $\mathbb{E}\|\varphi(s') - \varphi(s'')\|^r < \zeta$.
(3) For each $\zeta > 0$, $\exists$ a relatively dense set $\omega(\zeta) \in \mathbb{R}$ such that if $t \in \omega(\zeta)$ then, $\mathbb{E}\|\varphi(s) - \varphi(s+t)\|^r < \zeta$ for all $t \in \mathbb{R}$ satisfying the condition $|s - s_k| > \zeta, k \in \mathbb{N}$.

The number t is called ζ-translation number of φ. We denote by $\mathbf{AP_T}(\mathbb{R}, L^r(P, \mathcal{H}))$ the space of any r-mean almost periodic functions. One can show that $\mathbf{AP_T}(\mathbb{R}, L^r(P, \mathcal{H}))$ is a Banach space with $\|Y\|_\infty = \sup_{s\in\mathbb{R}} \Big(\mathbb{E}\|Y(s)\|^r\Big)^{\frac{1}{r}}$. Let

$$\mathbf{UPC}(\mathbb{R}, L^r(P, \mathcal{H})) = \Big\{Y \in \mathbf{PC}(\mathbb{R}, L^r(P, \mathcal{H})); Y \text{satisfies the condition (2)}\Big\}.$$

The function $\varphi \in \mathbf{PC}(\mathbb{R} \times L^r(P, \mathcal{K}), L^r(P, \mathcal{H}))$ is said to be r-mean almost periodic in $s \in \mathbb{R}$ uniformly in $x \in L^r(P, \mathcal{K})$ if, for every compact subset $K \subseteq L^r(P, \mathcal{K})$, $\{\varphi(., x); x \in K\}$ is uniformly bounded, and given $\zeta > 0$, there exists a relatively dense subset ω_ζ such that $\|\varphi(s + t, x) - \varphi(s, x)\|^r < \zeta$ for all $x \in K, t \in \omega_\zeta$ and $s \in \mathbb{R}$ satisfying $|s - s_k| > \zeta$.

The set of all such processes is denoted by $\mathbf{AP_T}(\mathbb{R} \times L^r(P, \mathcal{K}), L^r(P, \mathcal{H}))$. Denote
$\mathbf{PC^0_T}(\mathbb{R}, L^r(P, \mathcal{H})) = \Big\{\varphi \in \mathbf{PC}(\mathbb{R}, L^r(P, \mathcal{H})); \lim\limits_{s\to\infty} \mathbb{E}\|\varphi(s)\|^r = 0\Big\}$,

$\mathbf{PAP^0_T}(\mathbb{R}, L^r(P, \mathcal{H}), \alpha, \beta) = \big\{\varphi : \mathbf{PC}(\mathbb{R}, L^r(P, \mathcal{H}); \lim\limits_{b\to\infty} \frac{1}{\beta([-b,b])} \int_{-b}^{b} \mathbb{E}\|\varphi(s)\|^r d\alpha(s) = 0\big\}$.

$\mathbf{PAP^0_T}(\mathbb{R} \times L^r(P, \mathcal{K}), L^r(P, \mathcal{H}), \alpha, \beta) = \Big\{\varphi : \mathbf{PC}(\mathbb{R} \times L^r(P, \mathcal{K}), L^r(P, \mathcal{H});$
$\lim\limits_{b\to\infty} \dfrac{1}{\beta([-b, b])} \displaystyle\int_{-b}^{b} \mathbb{E}\|\varphi(s, x)\|^r d\alpha(s) = 0$ uniformly with respect to $x \in K$,
where K is an arbitrary compact subset of $L^r(P, \mathcal{K})\Big\}$.

$$\mathbf{PAP_T}(\mathbb{R}, L^r(P, \mathcal{H}), \alpha, \beta) = \Big\{\phi \in \mathbf{PC}(\mathbb{R}, l^r(\mathbb{R}, \mathcal{H}));$$
$$\phi = \varphi + \psi; \varphi \in \mathbf{AP_T}(\mathbb{R}, L^r(P, \mathcal{H})), \psi \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P, \mathcal{H}), \alpha, \beta)\Big\}.$$

One can show that $\mathbf{PAP_T}(\mathbb{R}, L^r(P, \mathcal{H}), \alpha, \beta)$ is a Banach space with the sup norm $\|.\|_\infty$. Then, the set $\mathbf{PAP^0_T}(\mathbb{R}, L^r(P, \mathcal{H}), \alpha, \beta)$ is translation invariant of $\mathbf{PC}(\mathbb{R}, L^r(P, \mathcal{H}))$ and $\mathbf{PC^0_T}(\mathbb{R}, L^r(P, \mathcal{H})) \subset \mathbf{PAP^0_T}(\mathbb{R}, L^r(P, \mathcal{H}), \alpha, \beta)$.
Let a sequence of functions $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathbf{PAP^0_T}(\mathbb{R}, L^r(P, \mathcal{H}), \alpha, \beta)$. If φ_k converges uniformly to φ, then $\varphi \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P, \mathcal{H}), \alpha, \beta)$.
$\mathbf{PAP_T}(\mathbb{R} \times L^r(P, \mathcal{K}), L^r(P, \mathcal{H}), \alpha, \beta) = \Big\{\varphi : \mathbf{PC}(\mathbb{R} \times L^r(P, \mathcal{K}), L^r(P, \mathcal{H}); \phi = \varphi + \psi;$
$\varphi \in \mathbf{AP_T}(\mathbb{R} \times L^r(P, \mathcal{K}), L^r(P, \mathcal{H})), \psi \in \mathbf{PAP^0_T}(\mathbb{R} \times L^r(P, \mathcal{K}), L^r(P, \mathcal{H}), \alpha, \beta)\Big\}$.
φ is called r-mean (α, β)-pseudo almost periodic.

Lemma 2.1. [12] *Suppose that* $\varphi \in \mathbf{PAP_T}(\mathbb{R} \times L^r(P, \mathcal{K}), L^r(P, \mathcal{H}), \alpha, \beta)$ *and that the following conditions hold:*
(1) $\{\varphi(s, x) : s \in \mathbb{R}, x \in K\}$ *is bounded for every bounded subset* $K \in L^r(P, \mathcal{K})$.
(2) $\varphi(s, .)$ *is uniformly continuous in each bounded subset of* $L^r(P, \mathcal{K})$ *uniformly in* $s \in \mathbb{R}$.

If $\psi(.) \in \mathbf{PAP_T}(\mathbb{R}, L^r(P, \mathcal{K}), \alpha, \beta)$ *such that* $R(\psi) \subset L^r(P, \mathcal{K})$, *then* $\varphi(., \psi(.)) \in \mathbf{PAP_T}(\mathbb{R}, L^r(P, \mathcal{H}), \alpha, \beta)$.

Lemma 2.2. [12] *Suppose that the sequence of vector valued functions* $\{I_k\}_{k\in\mathbb{N}}$ *is* (α,β)*-pseudo almost periodic, i.e., for any* $x \in L^r(P,\mathcal{H}), \{I_k(x), k \in \mathbb{N}\}$ *is a* (α,β)*-pseudo almost periodic sequence. Assume* $\{I_k(x), k \in \mathbb{N}\}$ *is bounded for every bounded subset* $K \subset L^r(P,\mathcal{H}), I_k(x)$ *is uniformly continuous in* $x \in L^r(P,\mathcal{H})$ *uniformly in* $\mathbb{N}$*. If* $\psi \in \mathbf{PAP_T}(\mathbb{R}, L^r(P,\mathcal{H}),\alpha,\beta) \cap \mathbf{UPC}(\mathbb{R}, L^r(P,\mathcal{H}))$ *such that* $R(\psi) \subset L^r(P,\mathcal{K})$*, then,* $I_k(\psi(s_k))$ *is* (α,β)*-pseudo almost periodic.*

Let $h : \mathbb{R} \to \mathbb{R}^+$ be a continuous function such that $h(s) \geq 1$ for all $s \in \mathbb{R}$ and $\lim_{|s|\to\infty} h(s) = \infty$. Define

$$\mathbf{PC^0_h}(\mathbb{R}, L^r(P,\mathcal{H})) = \Big\{\varphi \in \mathbf{PC}(\mathbb{R}, L^r(P,\mathcal{H}); \lim_{|s|\to\infty} \frac{\mathbb{E}\|\varphi(s)\|^r}{h(s)} = 0\Big\}$$

which is a Banach space with the norm $\|\varphi\|_h = \sup_{s\in\mathbb{R}} \frac{\mathbb{E}\|\varphi(s)\|^r}{h(s)}$.

Lemma 2.3. [12] *A set* $C \subseteq \mathbf{PC^0_h}(\mathbb{R}, L^r(P,\mathcal{H}))$ *is relatively compact if and only if the following conditions are verified*
(i) $\lim_{|s|\to\infty} \frac{\mathbb{E}\|\varphi(s)\|^r}{h(s)} = 0$ *uniformly for* $\varphi \in C$.
(ii) $C(s) = \{\varphi(s) : \varphi \in C\}$ *is relatively compact in* $L^r(P,\mathcal{H})$ *for every* $s \in \mathbb{R}$.
(iii) The set C *is equicontinuous on each interval* (s_k, s_{k+1}) $(k \in \mathbb{N})$.

Lemma 2.4. [12] *Assume that* $f \in \mathbf{AP}_T(\mathbb{R}, L^r(P,\mathcal{H}),\alpha,\beta)$*, the sequence* $\{y_i\}_{i\in\mathbb{N}} \in \mathbf{AP}_T(\mathbb{R}, L^r(P,\mathcal{H}),\alpha,\beta)$*, and* $\{t_i^j\}$*,* $j \in \mathbb{N}$ *are equipotentially almost periodic. Then, for each* $\varepsilon > 0$*, there exist relatively dense sets* Ω_ε *of* $\mathbb{R}$ *and* Ω_ε *of* $\mathbb{N}$ *such that*

- $\mathbb{E}\|g(t+s) - g(t)\|^r < \varepsilon$ *for all* $t \in \mathbb{R}, |t - t_i| > \varepsilon, s \in \Omega_\varepsilon$ *and* $i \in \mathbb{N}$.
- $\|\Gamma(t+u, s+u) - \Gamma(t,s)\|^r < \varepsilon$ *for all* $t, s \in \mathbb{R}, |t-s| > 0, |s - t_i| > \varepsilon, |t - t_i| > \varepsilon, u \in \Omega_\varepsilon$ *and* $i \in \mathbb{N}$.
- $\mathbb{E}\|y_{i+q} - y_i\|^r < \varepsilon$ *for all* $q \in \Omega_\varepsilon$ *and* $i \in \mathbb{N}$.
- $\mathbb{E}\|y_i^q - u\|^r < \varepsilon$ *for all* $q, u \in \Omega_\varepsilon$ *and* $i \in \mathbb{N}$.

Lemma 2.5. [6]
(i) *If* $f, g \in \mathbf{PAP_T}(\mathbb{R}, L^r(P,\mathcal{H}),\alpha,\beta)$*. Then* $f \times g \in \mathbf{PAP_T}(\mathbb{R}, L^r(P,\mathcal{H}),\alpha,\beta)$.
(ii) *If* $f \in \mathbf{AP_T}(\mathbb{R}, L^r(P,\mathcal{H}))$ *and* $g \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P,\mathcal{H}))$*. Then* $f \times g \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P,\mathcal{H}),\alpha,\beta)$.

3. Main results

In this section, we establish some results for the existence, uniqueness, and the global exponential stability of the pseudo almost periodic solution of (1). For convenience, we introduce the following notations and norms:
$\sup_{s\in\mathbb{R}}(\mathbb{E}\|b_{ij}(s)\|^r) = b_{ij}$, $\sup_{s\in\mathbb{R}}(\mathbb{E}\|f_{ij}(s)\|^r) = f_{ij}$, $\|y(t)\|^r = \sum_{i=1}^n \sup_{t\in\mathbb{R}}(\|y_i(t)\|^r)$.
We first introduce the notion of mild solution to system (3.1) – (3.2).

An $\mathcal{F}_s$-progressively measurable process $\{y_i(s)\}_{s\in\mathbb{R}}$ is called a mild solution of system (1) if, for any $s \in \mathbb{R}, s > s_0, s \neq s_k, k \in \mathbb{N}$,

$$\begin{aligned} y_i(s) &= V(s,s_0)\psi_i + \int_{s_0}^s V(s,\tau)\Big[\sum_{j=1}^n b_{ij}(\tau)\varphi_j(y_j(\tau)) + \sum_{j=1}^n f_{ij}(\tau)\Big]d\tau \\ &\quad + \int_{s_0}^s V(s,\tau)\sum_{j=1}^n \phi_{ij}(\tau, y_j(\tau))dB_j(\tau) + \sum_{s_k<s} V(s,s_k)I_k(y_i(s_k)), \end{aligned} \tag{2}$$

where $V(s,s_0)=e^{-\int_{s_0}^{s}a_i(\tau)d\tau}$. Before giving the main result in this third part of the paper, we make the following assumptions:
$(\mathbf{A}_1)$: $a_i(s)$ is continuously almost periodic function, $b_{ij},\ f_{ij}$ are $(\alpha,\beta)-\mathbf{PAP_T}$ functions, for all $1\le i,j\le n$.
$(\mathbf{A}_2)$: The functions $\phi_{ij}\in\mathbf{PAP_T}(\mathbb{R}\times L^r(P,\mathbb{K}),L^r(P,L_0^2),\alpha,\beta)$, and $\phi_{ij}(t,\cdot)$ is uniformly continuous in each bounded subset of $L^r(P,\mathbb{K})$ uniformly in $s\in\mathbb{R}$, I_k is a pseudo almost periodic sequence, $I_k(y),\ \varphi_j(y)$ are uniformly continuous in $y\in L^r(P,\mathbb{K})$ uniformly in $k\in\mathbb{N}$.
$(\mathbf{A}_3)$: There exists $\overline{\phi_{ij}}>0$, $C_k>0$ and $\overline{\varphi_j}>0$ such that for all $x,\ y\in L^r(P,\mathbb{K})$, we have: $\mathbb{E}\|\phi_{ij}(s,x)-\phi_{ij}(s,y)\|^r_{L_2^0}\le\overline{\phi_{ij}}\mathbb{E}\|x-y\|^r,\ \mathbb{E}\|I_k(x-y)\|^r\le C_k\mathbb{E}\|x-y\|^r$ and $\mathbb{E}\|\varphi_j(x-y)\|^r\le\overline{\varphi_j}\mathbb{E}\|x-y\|^r$.

Lemma 3.1. *Let the assumptions* $(\boldsymbol{A}_1)$ *be fulfilled. If* $f_{ij}\in\mathbf{PAP_T}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$ *and if the function* W_i *is defined by* $W_i(t)=\int_{-\infty}^{t}V(t,s)\sum_{j=1}^{n}f_{ij}(s)ds$, *for each* $t\in\mathbb{R}$, *then* $W_i\in PAP_T(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$.

Proof. We have that $f_{ij}\in\mathbf{PAP_T}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$, which can be decomposed as $f_{ij}=f_{ij,1}+f_{ij,2}$, where $f_{ij,1}\in\mathbf{AP_T}(\mathbb{R},L^r(P,\mathbb{K}))$ and $f_{ij,2}\in\mathbf{PAP^0_T}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$, then $W(t)$ can be decomposed as

$$\begin{aligned}W_i(t)&=\sum_{j=1}^{n}\Big[\int_{-\infty}^{t}V(t,s)f_{ij,1}(s)ds+\int_{\infty}^{t}V(t,s)f_{ij,2}(s)ds\Big]\\&=W_{i1}(t)+W_{i2}(t).\end{aligned}\tag{3}$$

Next we need to verify that $W_{i1}(t)\in\mathbf{AP_T}(\mathbb{R},L^r(P,\mathbb{K}))$ and
$W_{i2}(t)\in\mathbf{PAP^0_T}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$. Thus our proof will be split into the following three steps.
Step 1. $W_{i1}(t)\in\mathbf{UPC}(\mathbb{R},L^r(P,\mathbb{K}))$.
Let $t,t_1\in\mathbb{R}$, such that $t>t_1$, for any $\varepsilon>0$, there exist $\xi=\xi(\varepsilon)$ such that $0<t-t_1<\xi$, satisfying

$$\sum_{j=1}^{n}\Big\|f_{ij1}\Big\|^r_\infty\le\varepsilon\Big(\frac{a_i}{2}\Big)^r.$$

Using a substitution technique, we have

$$\begin{aligned}&\mathbb{E}\|W_{i1}(t)-W_{i1}(t_1)\|^r\\&\quad\le 2^{r-1}\mathbb{E}\Big(\int_{-\infty}^{t_1}\|V(t,s+t-t_1)-V(t_1,s)\|\sum_{j=1}^{n}\|f_{ij,1}(s+t-t_1)\|ds\Big)^r\\&\qquad+2^{r-1}\mathbb{E}\Big(\int_{-\infty}^{t_1}\|V(t_1,s)\|\sum_{j=1}^{n}\|f_{ij,1}(s+t-t_1)-f_{ij,1}(s)\|ds\Big)^r.\end{aligned}\tag{4}$$

By Hölder's inequality, it is easy to show that

$$\begin{aligned}&\mathbb{E}\Big(\int_{-\infty}^{t_1}\|V(t,s+t-t_1)-V(t_1,s)\|\sum_{j=1}^{n}\|f_{ij,1}(s+t-t_1)\|ds\Big)^r\\&\quad\le\Big(\int_{-\infty}^{t_1}e^{-\underline{a_i}(t_1-s)}ds\Big)^r\sup_{s\in\mathbb{R}}\sum_{j=1}^{n}\mathbb{E}\|f_{ij,1}(s)\|^r\le\frac{1}{\underline{a_i}^r}\sum_{j=1}^{n}\|f_{ij,1}\|^r_\infty,\text{ and}\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\Big(\int_{-\infty}^{t_1}\|V(t_1,s)\|\sum_{j=1}^{n}\|f_{ij,1}(s+t-t_1)-f_{ij,1}(s)\|ds\Big)^r\\&\quad\leq\Big(\int_{-\infty}^{t_1}e^{-\underline{a_i}(t_1-s)}ds\Big)^r\sum_{j=1}^{n}\sup_{s\in\mathbb{R}}\mathbb{E}\|f_{ij,1}(s)\|^r\leq\frac{1}{\underline{a_i}^r}\sum_{j=1}^{n}\|f_{ij,1}\|_\infty^r.\end{aligned}$$

Substituting these into (4) give $\mathbb{E}\|W_{i1}(t)-W_{i1}(t_1)\|^r\leq\left(\frac{2}{\underline{a_i}}\right)^r\sum_{j=1}^{n}\|f_{ij,1}\|_\infty^r<\varepsilon$.
Step 2. $W_{i1}(t)\in\mathbf{AP_T}(\mathbb{R},L^r(P,\mathbb{K}))$: Let a number l_ε such that in any interval $[\delta,\delta+l_\varepsilon]$ one finds a number τ', such that

$$\begin{aligned}\mathbb{E}\|W_{i1}(t+\tau')-W_{i1}(t)\|^r&\leq 2^{r-1}\sum_{j=1}^{n}\|f_{ij,1}\|_\infty^r\mathbb{E}\Big(\int_{-\infty}^{t}e^{-\int_s^t a_i(r+\tau')dr}-e^{-\int_s^t a_i(r)dr}ds\Big)^r\\&\quad+2^{r-1}\Big(\int_{-\infty}^{t}V(t,s)ds\Big)^{r-1}\int_{-\infty}^{t}V(t,s)\sum_{j=1}^{n}\mathbb{E}\|f_{ij,1}(s+\tau')-f_{ij,1}(s)\|^r ds.\end{aligned}$$

There exists $\theta\in]0,1[$ such that

$$\begin{aligned}&\mathbb{E}\|W_{i1}(t+\tau')-W_{i1}(t)\|^r\leq 2^{r-1}\sum_{j=1}^{n}\|f_{ij,1}\|_\infty^r\\&\times\Big[\int_{-\infty}^{t}e^{-\int_s^t a_i(r+\tau')dr+\theta|\int_s^t a_i(r)dr-\int_s^t a_i(r+\tau')dr|)}\times\Big|\int_s^t a_i(r+\tau')dr-\int_s^t a_i(r)dr\Big|\Big]^r ds\\&+2^{r-1}\Big(\int_{-\infty}^{t}e^{-\int_s^t a_i(r)dr}ds\Big)^{r-1}\times\int_{-\infty}^{t}e^{-\int_s^t a_i(r)dr}\sum_{j=1}^{n}\mathbb{E}\|f_{ij,1}(s+\tau')-f_{ij,1}(s)\|^r ds.\end{aligned}$$

Using the almost periodicity of a_i and $f_{ij,1}$, we get

$$\begin{aligned}&\mathbb{E}\|W_{i1}(t+\tau')-W_{i1}(t)\|^r\\&\leq 2^{r-1}\sum_{j=1}^{n}\|f_{ij,1}\|_\infty^r\Big(\int_{-\infty}^{t}e^{-\underline{a_i}(t-s)-\theta\varepsilon(t-s)}\varepsilon(t-s)ds\Big)^r+2^{r-1}\sum_{j=1}^{n}\|f_{ij,1}\|_\infty\varepsilon\Big(\int_{-\infty}^{t}e^{-\underline{a_i}(t-s)}\Big)^r\\&\leq 2^{r-1}\sum_{j=1}^{n}\|f_{ij,1}\|_\infty\frac{\varepsilon^r}{\underline{a_i}^{2r}}+2^{r-1}\sum_{j=1}^{n}\|f_{ij,1}\|_\infty\frac{\varepsilon}{\underline{a_i^r}}=\varepsilon 2^{r-1}\sum_{j=1}^{n}\|f_{ij,1}\|_\infty\Big(\frac{\varepsilon^{r-1}}{\underline{a_i}^{2r}}+\frac{1}{\underline{a_i^r}}\Big).\end{aligned}$$

Consequently, the function W_{i1} belongs to $\mathbf{AP_T}(\mathbb{R},L^r(P,\mathbb{K}))$.
Step 3. $W_{i2}\in\mathbf{PAP_T^0}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$. For $r>2$ and $b>0$, we have

$$\begin{aligned}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|W_{i2}(t)\|^r d\alpha(t)&\leq\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\Big\|\int_{\infty}^{t}V(t,s)\sum_{j=1}^{n}f_{ij,2}(s)ds\Big\|^r d\alpha(t)\\&\leq\frac{n^{r-1}}{\beta([-b,b])}\sum_{j=1}^{n}\int_{-b}^{b}\Big(\mathbb{E}\int_{\infty}^{t}\|V(t,s)f_{ij,2}(s)\|\,ds\Big)^r d\alpha(t).\end{aligned}$$

Using Hölder's inequality, and Fubini's Theorem we get

$$
\begin{aligned}
&\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|W_{i2}(t)\|^r d\alpha(t)\\
&\quad\leq \Big(\frac{1}{\underline{a_i}}\Big)^{r-1}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\int_{-\infty}^{t}e^{-\underline{a_i}(t-s)}\sum_{j=1}^{n}\mathbb{E}\|f_{ij,2}(s)\|^r ds d\alpha\\
&\quad\leq \Big(\frac{1}{\underline{a_i}}\Big)^{r-1}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\int_{0}^{+\infty}e^{-\underline{a_i}s}\sum_{j=1}^{n}\mathbb{E}\|f_{ij,2}(t-s)\|^r ds d\alpha(t)\\
&\quad\leq \Big(\frac{1}{\underline{a_i}}\Big)^{r-1}\int_{0}^{+\infty}\frac{e^{-\underline{a_i}s}}{\beta([-b,b])}\int_{-b}^{b}\sum_{j=1}^{n}\mathbb{E}\|f_{ij,2}(t-s)\|^r d\alpha(t) ds
\end{aligned}
$$

Moreover, we get $\left|\frac{e^{-\underline{a_i}s}}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|f_{ij,2}(t-s)\|^r d\alpha(t)\right| \leq e^{-\underline{a_i}s}\sum_{j=1}^{n}\|f_{ij,2}\|_\infty^r$. Since $f_{i2}\in \mathbf{PAP^0_T}(\mathbb{R},L^r(P,H),\alpha,\beta)$ which is translation invariant and from Lebesgue's Dominated Convergence Theorem, we deduce that:
$\lim\limits_{b\to\infty}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|W_{i2}(t)\|^r d\alpha(t)=0$. This completes the proof. □

Lemma 3.2. *Let the assumptions $(\boldsymbol{A}_1)-(\boldsymbol{A}_2)$ be fulfilled.*
If $\varphi_i, b_{ij}\in \mathbf{PAP_T}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$ and if the function Z_i is defined by:
$Z_i(t)=\int_{-\infty}^{t}V(t,s)\sum_{j=1}^{n}b_{ij}(s)\varphi_j ds=\sum_{j=1}^{n}\int_{-\infty}^{t}V(t,s)b_{ij}(s)\varphi_j ds$, for each $t\in\mathbb{R}$, then $Z_i\in PAP_T(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$.

Proof. We have that $\varphi_i, b_{ij}\in \mathbf{PAP_T}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$, which can be decomposed as $\varphi_i=\varphi_{i1}+\varphi_{i2}$ and $b_{ij}=b_{ij,1}+b_{ij,2}$ where $\varphi_{i1}, b_{ij,1}\in \mathbf{AP_T}(\mathbb{R},L^r(P,\mathbb{K}))$ and $\varphi_{i2}, b_{ij,2}\in \mathbf{PAP^0_T}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$, then $Z_i(t)$ can be decomposed as

$$
Z_i(t)=\sum_{j=1}^{n}\int_{-\infty}^{t}V(t,s)b_{ij}(s)\varphi_j ds = Z_{i1}(t)+Z_{i2}(t), \tag{5}
$$

where

$$
\begin{aligned}
Z_{i1}(t)&=\sum_{j=1}^{n}\int_{-\infty}^{t}V(t,s)b_{ij,1}(s)\varphi_{j1}ds, \quad\text{and}\\
Z_{i2}(t)&=\sum_{j=1}^{n}\Big[\int_{-\infty}^{t}V(t,s)b_{ij,1}(s)\varphi_{j2}ds+\int_{-\infty}^{t}V(t,s)b_{ij,2}(s)\varphi_{j1}ds\\
&\qquad+\int_{-\infty}^{t}V(t,s)b_{ij,2}(s)\varphi_{j2}ds\Big].
\end{aligned}
$$

Next we need to verify that $Z_{i1}(t)\in \mathbf{AP_T}(\mathbb{R},L^r(P,\mathbb{K}))$ and $Z_{i2}(t)\in \mathbf{PAP^0_T}$ $(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$. Thus our proof will be split into the following three steps.
Step 1. $Z_{i1}(t)\in \mathbf{UPC}(\mathbb{R},L^r(P,\mathbb{K}))$.
Let $t,t_1\in\mathbb{R}$, such that $t>t_1$, for any $\varepsilon>0$, there exist $\xi=\xi(\varepsilon)$ such that $0<t-t_1<\xi$, satisfying $\sum_{j=1}^{n}\Big\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r\leq\varepsilon\Big(\frac{a_i}{2}\Big)^r$.

Using a substitution technique, we have

$$
\begin{aligned}
&\mathbb{E}\|Z_{i1}(t)-Z_{i1}(t_1)\|^r\\
&=\mathbb{E}\Big\|\sum_{j=1}^{n}\Big[\int_{-\infty}^{t_1}V(t,s+t-t_1)b_{ij,1}(s+t-t_1)\varphi_{j1}-\int_{-\infty}^{t_1}V(t_1,s)b_{ij,1}(s)\varphi_{j1}ds\Big]\Big\|^r\\
&=\mathbb{E}\Big\|\sum_{j=1}^{n}\Big[\int_{-\infty}^{t_1}\Big[V(t,s+t-t_1)-V(t_1,s)\Big]b_{ij,1}(s+t-t_1)\varphi_{j1}\\
&\quad+\int_{-\infty}^{t_1}V(t_1,s)\Big[b_{ij,1}(s+t-t_1)\varphi_{j1}-b_{ij,1}(s)\varphi_{j1}\Big]ds\Big]\Big\|^r\\
&\le 2^{r-1}\mathbb{E}\Big(\int_{-\infty}^{t_1}\|V(t,s+t-t_1)-V(t_1,s)\|\sum_{j=1}^{n}\|b_{ij,1}(s+t-t_1)\varphi_{j1}\|ds\Big)^r\\
&\quad+2^{r-1}\mathbb{E}\Big(\int_{-\infty}^{t_1}\|V(t_1,s)\|\sum_{j=1}^{n}\|b_{ij,1}(s+t-t_1)\varphi_{j1}-b_{ij,1}(s)\varphi_{j1}\|ds\Big)^r\\
&=J_1(t)+J_2(t).
\end{aligned}
\tag{6}
$$

By Hölder's inequality, it is easy to show that

$$
\begin{aligned}
J_1(t)= \quad & 2^{r-1}\mathbb{E}\Big(\int_{-\infty}^{t_1}\|V(t,s+t-t_1)-V(t_1,s)\|\sum_{j=1}^{n}\|b_{ij,1}(s+t-t_1)\varphi_{j1}\|ds\Big)^r\\
\le \quad & 2^{r-1}\sum_{j=1}^{n}\Big(\int_{-\infty}^{t_1}\|V(t,s+t-t_1)-V(t_1,s)\|ds\Big)^{r-1}\\
& \times\Big(\int_{-\infty}^{t_1}\|V(t,s+t-t_1)-V(t_1,s)\|\sum_{j=1}^{n}\mathbb{E}\|b_{ij,1}(s+t-t_1)\varphi_{j1}\|^r ds\Big)\\
\le \quad & 2^{r-1}\sum_{j=1}^{n}\Big(\int_{-\infty}^{t_1}e^{-\underline{a_i}(t_1-s)}\Big)^r\sup_{s\in\mathbb{R}}\mathbb{E}\|b_{ij,1}(s)\|^r\|\varphi_{j1}\|_\infty^r\\
\le \quad & \frac{2^{r-1}}{\underline{a_i}^r}\sum_{j=1}^{n}\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
J_2(t)= \quad & 2^{r-1}\mathbb{E}\Big(\int_{-\infty}^{t_1}\|V(t_1,s)\|\sum_{j=1}^{n}\|b_{ij,1}(s+t-t_1)\varphi_{j1}-b_{ij,1}(s)\varphi_{j1}\|ds\Big)^r\\
\le \quad & 2^{r-1}\Big(\int_{-\infty}^{t_1}\|V(t,s+t-t_1)-V(t_1,s)\|ds\Big)^{r-1}\\
& \times\Big(\int_{-\infty}^{t_1}\|V(t_1,s)\|\sum_{j=1}^{n}\mathbb{E}\|b_{ij,1}(s+t-t_1)-b_{ij,1}(s)\|^r\|\varphi_{j1}\|_\infty^r ds\Big)\\
\le \quad & 2^{r-1}\Big(\int_{-\infty}^{t_1}e^{-\underline{a_i}(t_1-s)}\Big)^r\sum_{j=1}^{n}\sup_{s\in\mathbb{R}}\mathbb{E}\|b_{ij,1}(s)\|^r\|\varphi_{j1}\|_\infty^r\\
\le \quad & \frac{2^{r-1}}{\underline{a_i}^r}\sum_{j=1}^{n}\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r.
\end{aligned}
$$

Substituting these into (6) give

$$\mathbb{E}\|Z_{i1}(t)-Z_{i1}(t_1)\|^r \leq \Big(\frac{2}{\underline{a_i}}\Big)^r \sum_{j=1}^{n} \|b_{ij,1}\|_\infty^r \|\varphi_{j1}\|_\infty^r < \varepsilon.$$

Step 2. $Z_{i1}(t) \in \mathbf{AP_T}(\mathbb{R}, L^r(P,\mathbb{K}))$.
Let a number l_ε such that in any interval $[\delta, \delta+l_\varepsilon]$ one finds a number τ', such that

$$\begin{aligned}
&\mathbb{E}\|Z_{i1}(t+\tau')-Z_{i1}(t)\|^r \\
&\leq 2^{r-1}\mathbb{E}\Big\|\int_{-\infty}^{t}[V(t+\tau',s+\tau')-V(t,s)]\sum_{j=1}^{n} b_{ij,1}(s+\tau')\varphi_{j1}ds\Big\|^r \\
&\quad + 2^{r-1}\mathbb{E}\Big\|\int_{-\infty}^{t} V(t,s)\sum_{j=1}^{n}[b_{ij,1}(s+\tau')-b_{ij,1}(s)]\varphi_{j1}ds\Big\|^r \\
&\leq 2^{r-1}\sum_{j=1}^{n}\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r \mathbb{E}\Big(\int_{-\infty}^{t}\|V(t+\tau',s+\tau')-V(t,s)\|ds\Big)^r + 2^{r-1}\Big(\int_{-\infty}^{t}V(t,s)ds\Big)^{r-1} \\
&\quad \times \int_{-\infty}^{t} V(t,s)\sum_{j=1}^{n}\mathbb{E}\|b_{ij,1}(s+\tau')-b_{ij,1}(s)\|^r\|\varphi_{j1}\|_\infty^r ds \\
&\leq 2^{r-1}\sum_{j=1}^{n}\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r \mathbb{E}\Big(\int_{-\infty}^{t} e^{-\int_s^t a_i(r+\tau')dr}-e^{-\int_s^t a_i(r)dr}ds\Big)^r + 2^{r-1}\Big(\int_{-\infty}^{t}V(t,s)ds\Big)^{r-1} \\
&\quad \times \int_{-\infty}^{t} V(t,s)\sum_{j=1}^{n}\mathbb{E}\|b_{ij,1}(s+\tau')-b_{ij,1}(s)\|^r\|\varphi_{j1}\|_\infty^r ds.
\end{aligned}$$

There exists $\theta \in]0,1[$ such that

$$\begin{aligned}
&\mathbb{E}\|Z_{i1}(t+\tau')-Z_{i1}(t)\|^r \leq 2^{r-1}\sum_{j=1}^{n}\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r \\
&\times\Big[\int_{-\infty}^{t} e^{-\int_s^t a_i(r+\tau')dr+\theta|\int_s^t a_i(r)dr-\int_s^t a_i(r+\tau')dr|)} \times \Big|\int_s^t a_i(r+\tau')dr-\int_s^t a_i(r)dr\Big|\Big]^r ds \\
&+2^{r-1}\Big(\int_{-\infty}^{t} e^{-\int_s^t a_i(r)dr}ds\Big)^{r-1} \\
&\times\int_{-\infty}^{t} e^{-\int_s^t a_i(r)dr}\sum_{j=1}^{n}\mathbb{E}\|b_{ij,1}(s+\tau')\varphi_{j1}(s+\tau')-b_{ij,1}(s)\varphi_{j1}(s)\|^r ds.
\end{aligned}$$

Using the almost periodicity of a_i and $b_{ij,1}, \varphi_{j1}$ we get

$$\begin{aligned}
&\mathbb{E}\|Z_{i1}(t+\tau')-Z_{i1}(t)\|^r \\
\leq\;& 2^{r-1}\sum_{j=1}^{n}\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r\Big(\int_{-\infty}^{t} e^{-\underline{a_i}(t-s)-\theta\varepsilon(t-s)}\varepsilon(t-s)ds\Big)^r + 2^{r-1}\varepsilon\Big(\int_{-\infty}^{t} e^{-\underline{a_i}(t-s)}\Big)^r \\
\leq\;& 2^{r-1}\sum_{j=1}^{n}\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r\varepsilon^r\Big(\int_{-\infty}^{t} e^{-\underline{a_i}(t-s)}(t-s)ds\Big)^r + 2^{r-1}\frac{\varepsilon}{\underline{a_i^r}} \\
\leq\;& 2^{r-1}\sum_{j=1}^{n}\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r\frac{\varepsilon^r}{\underline{a_i}^{2r}} + 2^{r-1}\frac{\varepsilon}{\underline{a_i^r}} \\
=\;& \varepsilon\Big(2^{r-1}\sum_{j=1}^{n}\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r\frac{1}{\underline{a_i}^{2r}} + 2^{r-1}\frac{1}{\underline{a_i^r}}\Big).
\end{aligned}$$

Consequently, the function Z_{i1} belongs to $\mathbf{AP_T}(\mathbb{R}, L^r(P,\mathbb{K}))$.
Step 3. $Z_{i2} \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P,\mathbb{K}), \alpha, \beta)$.
For $r > 2$ and $b > 0$, we have

$$\begin{aligned}
&\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|Z_{i2}(t)\|^r d\alpha(t)\\
&\leq \frac{3^{r-1}}{\beta([-b,b])}\Big[\int_{-b}^{b}\mathbb{E}\Big(\int_{\infty}^{t}V(t,s)\sum_{j=1}^{n}\|b_{ij,1}(s)\|\|\varphi_{j2}\|ds\Big)^r\\
&\quad+\Big(\int_{\infty}^{t}V(t,s)\sum_{j=1}^{n}\|b_{ij,2}(s)\|\|\varphi_{j1}\|ds\Big)^r+\Big(\int_{\infty}^{t}V(t,s)\sum_{j=1}^{n}\|b_{ij,2}(s)\|\|\varphi_{j2}\|ds\Big)^r d\alpha(t)\Big].
\end{aligned}$$

Using Hölder's inequality, and Fubini's Theorem we get

$$\begin{aligned}
&\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|Z_{i2}(t)\|^r d\alpha(t)\\
&\leq \frac{3^{r-1}}{\underline{a_i}^{r-1}}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\int_{-\infty}^{t}e^{-\underline{a_i}(t-s)}\Big[\sum_{j=1}^{n}\mathbb{E}\|b_{ij,1}(s)\|^r\|\varphi_{j2}\|_\infty^r dsd\alpha\\
&\quad+\sum_{j=1}^{n}\mathbb{E}\|b_{ij,2}(s)\|^r\|\varphi_{j1}\|_\infty^r dsd\alpha+\sum_{j=1}^{n}\mathbb{E}\|b_{ij,2}(s)\|^r\|\varphi_{j2}\|_\infty^r dsd\alpha\Big].
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|Z_{i2}(t)\|^r d\alpha(t)\\
&\leq \frac{3^{r-1}}{\underline{a_i}^{r-1}}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\int_{0}^{+\infty}e^{-\underline{a_i}s}\sum_{j=1}^{n}\Big[\mathbb{E}\|b_{ij,1}(t-s)\|^r+\mathbb{E}\|b_{ij,2}(t-s)\|^r\Big]\|\varphi_{j2}\|_\infty^r\\
&\quad+\sum_{j=1}^{n}\mathbb{E}\|b_{ij,2}(t-s)\|^r\|\varphi_{j1}\|_\infty^r\Big)dsd\alpha(t)\\
&\leq \frac{3^{r-1}}{\underline{a_i}^{r-1}}\int_{0}^{+\infty}\sum_{j=1}^{n}\frac{e^{-\underline{a_i}s}}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|b_{ij,1}(t-s)\|^r\|\varphi_{j2}\|_\infty^r d\alpha(t)ds\\
&\quad+\frac{3^{r-1}}{\underline{a_i}^{r-1}}\int_{0}^{+\infty}\sum_{j=1}^{n}\frac{e^{-\underline{a_i}s}}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|b_{ij,2}(t-s)\|^r\|\varphi_{j1}\|_\infty^r d\alpha(t)ds\\
&\quad+\frac{3^{r-1}}{\underline{a_i}^{r-1}}\int_{0}^{+\infty}\sum_{j=1}^{n}\frac{e^{-\underline{a_i}s}}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|b_{ij,2}(t-s)\|^r\|\varphi_{j2}\|_\infty^r d\alpha(t)ds.
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
\Big|\frac{e^{-\underline{a_i}s}}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|b_{ij,1}(t-s)\|^r\|\varphi_{j2}\|_\infty^r d\alpha(t)\Big| &\leq e^{-\underline{a_i}s}\mathbb{E}\|b_{ij,1}\|_\infty^r\|\varphi_{j2}\|_\infty^r,\\
\Big|\frac{e^{-\underline{a_i}s}}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|b_{ij,2}(t-s)\|^r\|\varphi_{j1}\|_\infty^r d\alpha(t)\Big| &\leq e^{-\underline{a_i}s}\mathbb{E}\|b_{ij,1}\|_\infty^r\|\varphi_{j1}\|_\infty^r,\\
\Big|\frac{e^{-\underline{a_i}s}}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|b_{ij,2}(t-s)\|^r\|\varphi_{j2}\|_\infty^r d\alpha(t)\Big| &\leq e^{-\underline{a_i}s}\mathbb{E}\|b_{ij,2}\|_\infty^r\|\varphi_{j2}\|_\infty^r.
\end{aligned}$$

Since $b_{ij,1}\varphi_{j2}, b_{ij,2}\varphi_{j1}, b_{ij,2}\varphi_{j2} \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P,H), \alpha, \beta)$ which is translation invariant and from Lebesgue's Dominated Convergence Theorem, we deduce that $\lim_{b\to\infty}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|Z_{i2}(t)\|^r d\alpha(t) = 0$. This completes the proof. □

Lemma 3.3. *Let the assumptions* $(\boldsymbol{A}_1)-(\boldsymbol{A}_2)$ *be fulfilled. If* $\phi_{ij} \in \mathbf{PAP_T}(\mathbb{R}, L^r(P,\mathbb{K}), \alpha, \beta)$ *and if the function* G_i *is defined by* $G_i(t) = \sum_{j=1}^{n} \int_{-\infty}^{t} V(t,s)\phi_{ij}(s)dB_j(s)$, *for each* $t \in \mathbb{R}$, *then* $G_i \in PAP_T(\mathbb{R}, L^r(P,\mathbb{K}), \alpha, \beta)$.

Proof. We have that $\phi_{ij} \in \mathbf{PAP_T}(\mathbb{R}, L^r(P,\mathbb{K}), \alpha, \beta)$, which can be decomposed as $\phi_{ij} = \phi_{ij,1} + \phi_{ij,2}$ where $\phi_{ij,1} \in \mathbf{AP_T}(\mathbb{R}, L^r(P,\mathbb{K}))$ and $\phi_{ij,2} \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P,\mathbb{K}), \alpha, \beta)$, then $G_i(t)$ can be decomposed as

$$\begin{aligned} G_i(t) &= \sum_{j=1}^{n} \Big[\int_{-\infty}^{t} V(t,s)\phi_{ij,1}(s)dB_j(s) + \int_{\infty}^{t} V(t,s)\phi_{ij,2}(s)dB_j(s) \Big] \\ &= G_{i1}(t) + G_{i2}(t). \end{aligned} \tag{7}$$

Next we need to verify that $G_{i1}(t) \in \mathbf{AP_T}(\mathbb{R}, L^r(P,\mathbb{K}))$ and
$G_{i2}(t) \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P,\mathbb{K}), \alpha, \beta)$. Thus our proof will be split into the following three steps.

Step 1. $G_{i1}(t) \in \mathbf{UPC}(\mathbb{R}, L^r(P,\mathbb{K}))$.

Let $t, t_1 \in \mathbb{R}$, such that $t > t_1$, for any $\varepsilon > 0$, there exist $\xi = \xi(\varepsilon)$ such that $0 < t - t_1 < \xi$, satisfying: $\sum_{j=1}^{n} \|\phi_{ij,1}\|_\infty^r \leq \frac{\varepsilon}{A}$, where $A = C_r 2^{r-1}\Big(\Big(\frac{r-2}{r\underline{a_i}}\Big)^{\frac{r-2}{2}} \frac{2}{r\underline{a_i}} + \frac{1}{\underline{a_i}^r}\Big)$.
Using a substitution technique, we have

$$\begin{aligned} &\mathbb{E}\|G_{i1}(t) - G_{i1}(t_1)\|^r \\ &= \mathbb{E}\Big\| \sum_{j=1}^{n} \Big[\int_{-\infty}^{t_1} V(t, s+t-t_1)\phi_{ij,1}(s+t-t_1)dB_j(s) - \int_{-\infty}^{t_1} V(t_1,s)\phi_{ij,1}(s)dB_j(s) \Big] \Big\|^r \end{aligned}$$

$$\begin{aligned} &= \mathbb{E}\Big\| \sum_{j=1}^{n} \Big[\int_{-\infty}^{t_1} \Big[V(t, s+t-t_1) - V(t_1,s)\Big]\phi_{ij,1}(s+t-t_1) \\ &\quad + \int_{-\infty}^{t_1} V(t_1,s)\Big[\phi_{ij,1}(s+t-t_1) - \phi_{ij,1}(s)\Big]dB_j(s)\Big] \Big\|^r \\ &\leq 2^{r-1}\mathbb{E}\Big(\int_{-\infty}^{t_1} \|V(t,s+t-t_1) - V(t_1,s)\| \sum_{j=1}^{n} \|\phi_{ij,1}(s+t-t_1)\|dB_j(s)\Big)^r \\ &\quad + 2^{r-1}\mathbb{E}\Big(\int_{-\infty}^{t_1} \|V(t_1,s)\| \sum_{j=1}^{n} \|\phi_{ij,1}(s+t-t_1) - \phi_{ij,1}(s)\|dB_j(s)\Big)^r. \end{aligned} \tag{8}$$

By Itô integral and Hölder's inequality, it is easy to show that

$$\begin{aligned} &\mathbb{E}\Big(\int_{-\infty}^{t_1} \|V(t,s+t-t_1) - V(t_1,s)\| \sum_{j=1}^{n} \|\phi_{ij,1}(s+t-t_1)\|dB_j(s)\Big)^r \\ &\leq C_r\mathbb{E}\Big(\int_{-\infty}^{t_1} \|V(t,s+t-t_1) - V(t_1,s)\|^2 \sum_{j=1}^{n} \|\phi_{ij,1}(s+t-t_1)\|^2 ds\Big)^{r/2} \\ &\leq C_r\Big(\int_{-\infty}^{t_1} e^{-\frac{r}{r-2}(t_1-s)\underline{a_i}}ds\Big)^{\frac{r-2}{2}} \times \Big(\int_{-\infty}^{t_1} e^{-\frac{r}{2}(t_1-s)\underline{a_i}}ds\Big) \sum_{j=1}^{n} \sup_{s\in\mathbb{R}} \mathbb{E}\|\phi_{ij,1}(s)\|^r \\ &\leq C_r\Big(\frac{r-2}{r\underline{a_i}}\Big)^{\frac{r-2}{2}} \frac{2}{r\underline{a_i}} \sum_{j=1}^{n} \|\phi_{ij,1}\|_\infty^r \end{aligned}$$

and

$$\mathbb{E}\Big(\int_{-\infty}^{t_1}\|V(t_1,s)\|\sum_{j=1}^{n}\|\phi_{ij,1}(s+t-t_1)-\phi_{ij,1}(s)\|dB_j(s)\Big)^r$$
$$\leq C_r\Big(\int_{-\infty}^{t_1}\|V(t_1,s)\|ds\Big)^{r-1}\times\Big(\int_{-\infty}^{t_1}\|V(t_1,s)\|\mathbb{E}\sum_{j=1}^{n}\|\phi_{ij,1}(s+t-t_1)-\phi_{ij,1}(s)\|^r ds\Big)$$
$$\leq C_r\frac{1}{\underline{a_i}^r}\sum_{j=1}^{n}\|\phi_{ij,1}\|_\infty^r.$$

Substituting these into (8) give

$$\mathbb{E}\|G_{i1}(t)-G_{i1}(t_1)\|^r\leq C_r2^{r-1}\Big(\Big(\frac{r-2}{r\underline{a_i}}\Big)^{\frac{r-2}{2}}\frac{2}{r\underline{a_i}}+\frac{1}{\underline{a_i}^r}\Big)\sum_{j=1}^{n}\|\phi_{ij,1}\|_\infty^r<\varepsilon.$$

Step 2. $G_{i1}(t)\in\mathbf{AP_T}(\mathbb{R},L^r(P,\mathbb{K}))$.
Let a number l_ε such that in any interval $[\delta,\delta+l_\varepsilon]$ one finds a number τ', such that

$$\begin{aligned}
&\mathbb{E}\|G_{i1}(t+\tau')-G_{i1}(t)\|^r\\
\leq\ & 2^{r-1}\mathbb{E}\Big\|\int_{-\infty}^{t}[V(t+\tau',s+\tau')-V(t,s)]\sum_{j=1}^{n}\phi_{ij,1}(s+\tau')dB_j(s)\Big\|^r\\
&+2^{r-1}\sum_{j=1}^{n}\mathbb{E}\Big\|\int_{-\infty}^{t}V(t,s)\sum_{j=1}^{n}[\phi_{ij,1}(s+\tau')-\phi_{ij,1}(s)]dB_j(s)\Big\|^r\\
=\ &\tilde{J}_1+\tilde{J}_2.
\end{aligned}$$

Using Itô integral and Hölder's inequality, we have

$$\begin{aligned}
\tilde{J}_1 &\leq C_r2^{r-1}\Big(\int_{-\infty}^{t}|V(t+\tau',s+\tau')-V(t,s)|^2\sum_{j=1}^{n}\mathbb{E}\|\phi_{ij,1}(s+\tau')\|^2ds\Big)^{r/2}\\
&\leq C_r2^{r-1}\sum_{j=1}^{n}\|\phi_{ij,1}\|_\infty^r\mathbb{E}\Big(\int_{-\infty}^{t}|e^{-\int_s^t a_i(r+\tau')dr}-e^{-\int_s^t a_i(r)dr}|^2ds\Big)^{r/2}.
\end{aligned}$$

There exists $\theta\in]0,1[$ such that

$$\begin{aligned}
\tilde{J}_1 &\leq C_r2^{r-1}\sum_{j=1}^{n}\|\phi_{ij,1}\|_\infty^r\Big[\int_{-\infty}^{t}e^{-2\int_s^t a_i(r+\tau')dr+2\theta|\int_s^t a_i(r)dr-\int_s^t a_i(r+\tau')dr|)}\\
&\quad\times\Big|\int_s^t a_i(r+\tau')dr-\int_s^t a_i(r)dr\Big|^2\Big]^{r/2}ds.
\end{aligned}$$

Using the almost periodicity of a_i and $\phi_{ij,1}$, we get

$$\begin{aligned}
\tilde{J}_1 &\leq C_r2^{r-1}\sum_{j=1}^{n}\|\phi_{ij,1}\|_\infty^r\Big(\int_{-\infty}^{t}e^{-2\underline{a_i}(t-s)-2\theta\varepsilon(t-s)}\varepsilon^2(t-s)^2ds\Big)^r\\
&\leq C_r2^{r-1}\sum_{j=1}^{n}\|\phi_{ij,1}\|_\infty^r\varepsilon^r\Big(\int_{-\infty}^{t}e^{-2\underline{a_i}(t-s)}(t-s)^2ds\Big)^{r/2}\\
&\leq 2^{r-1}C_r\Big(\frac{1}{2\underline{a_i}^2}+\frac{1}{4\underline{a_i}^3}\Big)^{r/2}\sum_{j=1}^{n}\|\phi_{ij,1}\|_\infty^r\varepsilon^r
\end{aligned}$$

and

$$\begin{aligned}\tilde{J}_2 &\leq 2^{r-1}\mathbb{E}\Big\|\int_{-\infty}^{t} V(t,s)\sum_{j=1}^{n}[\phi_{ij,1}(s+\tau')-\phi_{ij,1}(s)]dB_j(s)\Big\|^r\\ &\leq C_r 2^{r-1}\Big(\int_{-\infty}^{t}(V(t,s))^{\frac{r}{r-2}}ds\Big)^{\frac{r-2}{2}}\int_{-\infty}^{t}(V(t,s))^{\frac{r}{2}}\sum_{j=1}^{n}\mathbb{E}\|\phi_{ij,1}(s+\tau')-\phi_{ij,1}(s)\|^r ds\\ &\leq C_r 2^{r-1}\varepsilon\Big(\frac{r-2}{r\underline{a_i}}\Big)^{\frac{r-2}{2}}\frac{2}{r\underline{a_i}}\sum_{j=1}^{n}\|\phi_{ij,1}\|_\infty^r\end{aligned}$$

$$\mathbb{E}\|W_{i1}(t+\tau')-W_{i1}(t)\|^r \leq C_r 2^{r-1}\Big[\Big(\frac{1}{2\underline{a_i}^2}+\frac{1}{4\underline{a_i}^3}\Big)^{r/2}\varepsilon^{r-1}+\Big(\frac{r-2}{r\underline{a_i}}\Big)^{\frac{r-2}{2}}\frac{2}{r\underline{a_i}}\Big]\sum_{j=1}^{n}\|\phi_{ij,1}\|_\infty^r\varepsilon.$$

Consequently, the function G_{i1} belongs to $\mathbf{AP_T}(\mathbb{R}, L^r(P,\mathbb{K}))$.
Step 3. $G_{i2} \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P,\mathbb{K}),\alpha,\beta)$.
For $r>2$ and $b>0$, we have

$$\begin{aligned}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|G_{i2}(t)\|^r d\alpha(t) \leq& \frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\left\|\int_{\infty}^{t} V(t,s)\sum_{j=1}^{n}\phi_{ij,2}(s)dB_j(s)\right\|^r d\alpha(t)\\ \leq& \frac{C_r}{\beta([-b,b])}\int_{-b}^{b}\Big(\mathbb{E}\int_{\infty}^{t}\sum_{j=1}^{n}\|V(t,s)\phi_{ij,2}(s)\|\,ds\Big)^r d\alpha(t),\end{aligned}$$

using Hölder's inequality, and Fubini's Theorem we get

$$\begin{aligned}&\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|G_{i2}(t)\|^r d\alpha(t)\\ &\leq \frac{C_r}{\beta([-b,b])}\int_{-b}^{b}\Big(\int_{-\infty}^{t}(V(t,s))^{\frac{r}{r-2}}ds\Big)^{\frac{r-2}{2}}\Big(\int_{-\infty}^{t}(V(t,s))^{\frac{r}{2}}\sum_{j=1}^{n}\mathbb{E}\|\phi_{ij,2}(s)\|^r ds\Big)d\alpha(t)\\ &\leq C_r\Big(\frac{r-2}{r\underline{a_i}}\Big)^{\frac{r-2}{2}}\int_0^{+\infty}\frac{e^{-\frac{r}{2}\underline{a_i}s}}{\beta([-b,b])}\int_{-b}^{b}\sum_{j=1}^{n}\mathbb{E}\|\phi_{ij,2}(t-s)\|^r d\alpha(t)ds.\end{aligned}$$

Moreover, we get

$$\left|\frac{e^{-\frac{r}{2}\underline{a_i}s}}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|\phi_{ij,2}(t-s)\|^r d\alpha(t)\right| \leq e^{-\frac{r}{2}\underline{a_i}s}\|\phi_{ij,2}\|_\infty^r.$$

Since $\phi_{ij,2} \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P,H),\alpha,\beta)$ which is translation invariant and from Lebesgue's Dominated Convergence Theorem, we deduce that

$$\lim_{b\to\infty}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|G_{i2}(t)\|^r d\alpha(t) = 0.$$

This completes the proof. □

Lemma 3.4. *Let the assumptions* $(\boldsymbol{A}_1)$–$(\boldsymbol{A}_2)$ *be fulfilled. If* $I_k \in \mathbf{PAP_T}(\mathbb{R}, L^p(P,\mathbb{K}),$ $\alpha,\beta)$ *and if the function* Γ_k *is defined by*

$$\Gamma_k(t) = \sum_{t_k<t} V(t,t_k)I_k \tag{9}$$

for each $t \in \mathbb{R}$. *Then* $\Gamma_k \in \mathbf{PAP_T}(\mathbb{R}, L^r(P,\mathbb{K}),\alpha,\beta)$.

Proof. We have that $I_k \in \mathbf{PAP_T}(\mathbb{R}, L^r(P,\mathbb{K}), \alpha, \beta)$, which can be decomposed as $I_k = I_{1,k} + I_{2,k}$, where $I_{1,k} \in \mathbf{AP_T}(\mathbb{R}, L^r(P,\mathbb{K}))$ and $I_{2,k} \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P,\mathbb{K}), \alpha, \beta)$, then $\Gamma_k(t)$ can be decomposed as

$$\begin{aligned}\Gamma_k(t) &= \sum_{t_k<t} V(t,t_k)I_{1,k} + \sum_{t_k<t} V(t,t_k)I_{2,k} \\ &= \Gamma_{1,k}(t) + \Gamma_{2,k}(t).\end{aligned} \tag{10}$$

Next we need to verify that $\Gamma_{1,k}(t) \in \mathbf{AP_T}(\mathbb{R}, L^r(P,\mathbb{K}))$ and $\Gamma_{2,k}(t) \in \mathbf{PAP^0_T}(\mathbb{R}, L^r(P,\mathbb{K}), \alpha, \beta)$. Thus our proof will be split into the following three steps.

Step 1. $\Gamma_{1,k}(t) \in \mathbf{UPC}(\mathbb{R}, L^r(P,\mathbb{K}))$.

Let $t, t_1 \in \mathbb{R}$, such that $t > t_1$, for any $\varepsilon > 0$, there exist $\xi = \xi(\varepsilon)$ such that $0 < t - t_1 < \xi$, satisfying

$$\|I_k\|^r_\infty \le \Big(\frac{1-e^{-\delta\underline{a_i}}}{2}\Big)^r\varepsilon.$$

Using a substitution technique and Hölder's inequality, we get

$$\begin{aligned}&\mathbb{E}\|\Gamma_{1,k}(t) - \Gamma_{1,k}(t_1)\|^r \\ &\le 2^{r-1}\mathbb{E}\Big\|\sum_{t_k<t} V(t,t_k)I_{1,k}\Big\|^r + 2^{r-1}\mathbb{E}\Big\|\sum_{t_k<t_1} V(t_1,t_k)I_{1,k}\Big\|^r \\ &\le 2^{r-1}\Big(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)}\Big)^r \sup_{k\in\mathbb{N}}\mathbb{E}\|I_{1,k}\|^r + 2^{r-1}\Big(\sum_{t_k<t_1} e^{-\underline{a_i}(t_1-t_k)}\Big)^r \sup_{k\in\mathbb{N}}\mathbb{E}\|I_{1,k}\|^r \\ &\le \Big(\frac{2}{1-e^{-\delta\underline{a_i}}}\Big)^r\|I_{1,k}\|^r_\infty \le \varepsilon.\end{aligned}$$

Step 2. $\Gamma_{1,k}(t) \in \mathbf{AP_T}(\mathbb{R}, L^r(P,\mathbb{K}))$.

Let a number l_ε such that in any interval $[\delta, \delta + l_\varepsilon]$ one finds a number τ', such that

$$\begin{aligned}&\mathbb{E}\|\Gamma_{1,k}(t+\tau') - \Gamma_{1,k}(t)\|^r \le 2^{r-1}\mathbb{E}\Big\|\sum_{t_k<t+\tau'} V(t+\tau',t_k)I_{1,k} + \sum_{t_k<t} V(t,t_k)I_{1,k}\Big\|^r \\ &\le 2^{r-1}\mathbb{E}\Big\|\sum_{t_k<t} V(t+\tau',t_{k+q})I_{1,k+q} - \sum_{t_k<t} V(t+\tau',t_{k+q})I_{1,k}\Big\|^r \\ &\quad + 2^{r-1}\mathbb{E}\Big\|\sum_{t_k<t} V(t+\tau',t_{k+q})I_{1,k} + \sum_{t_k<t} V(t,t_k)I_{1,k}\Big\|^r \\ &= \hat{J}_1 + \hat{J}_2.\end{aligned}$$

For any $\varepsilon > 0$, by Lemma 2.4, there exist relative dense sets of real numbers Ω_ε and integers Q_ε, for every $\tau' \in \Omega_\varepsilon$, there exists at least one number $q \in Q_\varepsilon$ such that $t_k < t \le t_{k+1}$, $|t - t_k| > \varepsilon$, $|t - t_{k-1}| > \varepsilon$, one has $t + \tau' > t_k + \tau' + \varepsilon > t_{k+q}$ and $t_{k+q+1} > t_{k+1} + \tau' - \varepsilon > t + \tau'$ that is $t_{k+q} < t + \tau' < t_{k+q+1}$ such that $|t^q - \tau| < \varepsilon$ and $\mathbb{E}\|I_{1,k+q} - I_{1,k}\| < \varepsilon,\ q \in Q_\varepsilon,\ k \in \mathbb{N}$, then

$$\begin{aligned}\hat{J}_1 &\le 2^{r-1}\mathbb{E}\Big\|\sum_{t_k<t} V(t+\tau',t_{k+q})[I_{1,k+q} - I_{1,k}]\Big\|^r \\ &\le 2^{r-1}\Big(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)}\Big)^r\mathbb{E}\|I_{1,k+q} - I_{1,k}\|^r \\ &\le \Big(\frac{2}{1-e^{-\delta\underline{a_i}}}\Big)^r\varepsilon\end{aligned}$$

and similarly to the proof of Step 2 in Lemma 3.1, we obtain

$$\begin{aligned}\hat{J}_2 &\leq 2^{r-1}\mathbb{E}\Big\|\sum_{t_k<t}[V(t+\tau',t_{k+q})-V(t,t_k)]I_{1,k}\Big\|^r\\ &\leq 2^{r-1}\Big(\sum_{t_k<t}e^{-\underline{a_i}(t-t_k)}\varepsilon(t-t_k)\Big)^r\sup_{k\in\mathbb{N}}\mathbb{E}\|I_{1,k}\|^r\\ &\leq \frac{2^{r-1}\varepsilon^r e^{-r\delta\underline{a_i}}}{(1-e^{-\delta\underline{a_i}})^r}\sup_{k\in\mathbb{N}}\mathbb{E}\|I_{1,k}\|^r.\end{aligned}$$

Consequently, the function $\Gamma_{1,k}$ belongs to $\mathbf{AP_T}(\mathbb{R},L^r(P,\mathbb{K}))$.
Step 3. $\Gamma_{2,k}\in\mathbf{PAP^0_T}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$.
For $b>0$, by Hölder's inequality, we have

$$\begin{aligned}&\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\|\Gamma_{2,k}(t)\|^r d\alpha(t)\\ \leq\quad & 2^{r-1}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\left\|\sum_{t_k<t}V(t,t_k)I_{2,k}\right\|^r d\alpha(t).\end{aligned}$$

Let $k\in\mathbb{N}$, define the function $u(t)$ by $u(t)=V(t,t_k)I_{2,k}$, $t_k<t$. Then,

$$\lim_{t\to\infty}\mathbb{E}\|u(t)\|^r=\lim_{t\to\infty}\|V(t,t_k)I_{2,k}\|^r\leq\lim_{t\to\infty}M^r e^{-r\underline{a_i}(t-t_k)}\sup_{k\in\mathbb{N}}\mathbb{E}\|I_{2,k}\|^r=0.$$

So, $u\in\mathbf{PC^0_T}(\mathbb{R},L^r(P,\mathbb{K}))$. Define $u_j:\mathbb{R}\to L^r(P,\mathbb{K})$ by

$$u_j=V(t,t_{k-j})I_{2,k-j},\ t_k<t\in\mathbb{R}.$$

Therefore, $u_j\in\mathbf{PAP^0_T}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$. Furthermore,

$$\begin{aligned}\mathbb{E}\|u_j(t)\|^r &= \mathbb{E}\|V(t,t_{k-j})I_{2,k-j}\|^r\\ &\leq e^{-r\underline{a_i}(t-t_{k-j})}\sup_{k\in\mathbb{N}}\mathbb{E}\|I_{2,k}\|^r\\ &\leq e^{-r\underline{a_i}(t-t_k)}e^{-r\underline{a_i}\delta j}\sup_{k\in\mathbb{N}}\mathbb{E}\|I_{2,k}\|^r.\end{aligned}$$

Thus, the series $\sum_{j=0}^{\infty}u_j$ is uniformly convergent on $\mathbb{R}$. By Lemma 2.1, we obtain: $\sum_{t_k<t}V(t,t_k)I_{2,k}=\sum_0^{\infty}u_j(t)\in PAP^0_T(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)$, which also means, $\lim_{b\to\infty}\frac{1}{\beta([-b,b])}\int_{-b}^{b}\mathbb{E}\left\|\sum_{t_k<t}V(t,t_k)I_{2,k}\right\|^r d\alpha(t)=0$. This completes the proof. □

Theorem 3.5. *Suppose that $(\boldsymbol{A}_1)$-$(\boldsymbol{A}_3)$ hold. If $M=3^{r-1}\max_i\Big[\frac{1}{\underline{a_i}^r}\max_j b_{ij}\overline{\varphi_j}+C_r\left(\frac{r-2}{\underline{a_i}^r}\right)^{\frac{r-2}{2}}\frac{2}{\underline{a_i}^r}\max_j\overline{\phi_{ij}}+\frac{C_k}{(1-e^{-\delta\underline{a_i}})^r}\Big]<1$, then the system (1) has a unique r-mean (α,β)-pseudo almost periodic mild solution on $\mathbb{R}^n$.*

Proof. Define the operator $F:\mathbf{PAP_T}(\mathbb{R},L^r(P,\mathbb{K}),\alpha,\beta)\cap\mathbf{UPC}(\mathbb{R},L^r(P,\mathbb{K}))\to\mathbf{PC}(\mathbb{R},L^r(P,\mathbb{K}))$

$$\begin{aligned}(Fy)(t)=\quad&\int_{-\infty}^{t}V(t,s)\sum_{j=1}^{n}\Big[b_{ij}(s)\varphi_j(y_j(s))+f_{ij}(s)\Big]ds\\ &+\int_{-\infty}^{t}V(t,s)\sum_{j=1}^{n}\phi_{ij}(s,y_j(s))dB_j(s)+\sum_{t_k<t}V(t,t_k)I_k(y_i(t_k)).\end{aligned}$$

In order to show that system (3.1) − (3.2) has a r-mean (α,β)-pseudo almost periodic mild solution, we only need to prove the operator F has a fixed point in $\mathbf{PAP_T}(\mathbb{R}, L^r(P,\mathbb{K}),\alpha,\beta)\cap\mathbf{UPC}(\mathbb{R}, L^r(P,\mathbb{K}))$ and divide the proof into two steps.

Step 1. F maps bounded sets into bounded sets in $\mathbf{PAP_T}(\mathbb{R}, L^r(P,\mathbb{K}),\alpha,\beta)\cap\mathbf{UPC}(\mathbb{R}, L^r(P,\mathbb{K}))$. Let $p^*>0$ and

$$y\in B_{p^*}=\{y\in\mathbf{PAP_T}(\mathbb{R}, L^r(P,\mathbb{K}),\alpha,\beta)\cap\mathbf{UPC}(\mathbb{R}, L^r(P,\mathbb{K})) : \mathbb{E}\|y\|\|^r\le p^*\}.$$

It suffices to show that there exists $\hat{K}>0$ such that for each $y\in B_{p^*}$ we have $\mathbb{E}\|Fy\|\|^r\le\hat{K}$. Let $y\in B_{p^*}$, $t\in\mathbb{R}$. For $r>2$, we have

$$\begin{aligned}\mathbb{E}\|(Fy)_i(t)\|^r\le&4^{r-1}\mathbb{E}\Big\|\sum_{j=1}^n\int_{-\infty}^t V(t,s)b_{ij}(s)\varphi_j(y_j(s))ds\Big\|^r\\&+4^{r-1}\mathbb{E}\Big\|\int_{-\infty}^t\sum_{j=1}^n V(t,s)f_{ij}(s)ds\Big\|^r\\&+4^{r-1}\mathbb{E}\Big\|\int_{-\infty}^t\sum_{j=1}^n V(t,s)\phi_{ij}(s,y_j(s))dB_j(s)\Big\|^r\\&+4^{r-1}\mathbb{E}\Big\|\sum_{t_k<t}V(t,t_k)I_k(y_i(t_k))\Big\|^r.\end{aligned}\tag{11}$$

Using Itô integral and Hölder's inequality, we get

$$\begin{aligned}\mathbb{E}\|(Fy)_i(t)\|^r\le&4^{r-1}\Big(\int_{-\infty}^t e^{-\underline{a_i}(t-s)}\Big)^{r-1}\Big(\int_{-\infty}^t e^{-\underline{a_i}(t-s)}\sum_{j=1}^n\mathbb{E}\|b_{ij}(s)\|^r\mathbb{E}\|\varphi_j(y_j(s))\|^r ds\Big)\\&+4^{r-1}\Big(\int_{-\infty}^t e^{-\underline{a_i}(t-s)}\Big)^{r-1}\Big(\int_{-\infty}^t e^{-\underline{a_i}(t-s)}\sum_{j=1}^n\mathbb{E}\|f_{ij}(s)\|^r ds\Big)\\&+4^{r-1}C_r\Big(\int_{-\infty}^t e^{-2\underline{a_i}(t-s)}\sum_{j=1}^n\mathbb{E}\|\phi_{ij}(s,y_j(s))\|^2_{L_2^0}ds\Big)^{\frac{r}{2}}\\&+4^{r-1}\Big(\sum_{t_k<t}e^{-\underline{a_i}(t-t_k)}\Big)^{r-1}\Big(\sum_{t_k<t}e^{-\underline{a_i}(t-s)}\mathbb{E}\|I_k(y_i(t_k))\|^r\Big)\\\le&\Big(\frac{4}{\underline{a_i}}\Big)^{r-1}\sum_{j=1}^n\overline{\varphi_j}b_{ij}\Big(\int_{-\infty}^t e^{-\underline{a_i}(t-s)}\mathbb{E}\|y_j(s)\|^r ds\Big)\\&+\Big(\frac{4}{\underline{a_i}}\Big)^{r-1}\sum_{j=1}^n f_{ij}\Big(\int_{-\infty}^t e^{-\underline{a_i}(t-s)}ds\Big)\\&+4^{r-1}C_r\sum_{j=1}^n\overline{\phi_{ij}}\Big(\int_{-\infty}^t e^{-\frac{r}{r-2}\underline{a_i}(t-s)}ds\Big)^{\frac{r-2}{2}}\Big(\int_{-\infty}^t e^{-\frac{r}{2}\underline{a_i}(t-s)}\mathbb{E}\|y_j(s)\|^r ds\Big)\\&+\Big(\frac{4}{1-e^{-\delta\underline{a_i}}}\Big)^{r-1}C_k\Big(\sum_{t_k<t}e^{-\underline{a_i}(t-s)}\mathbb{E}\|y_i(t_k)\|^r\Big).\end{aligned}\tag{12}$$

So

$$\begin{aligned}\mathbb{E}\|(Fy)_i(t)\|^r\quad\le\quad&\frac{4^{r-1}}{\underline{a_i}^r}\max_j\overline{\varphi_j}b_{ij}\|y\|^r_\infty+\frac{4^{r-1}}{\underline{a_i}^r}\sum_{j=1}^n f_{ij}\\&+4^{r-1}C_r\left(\frac{r-2}{\underline{a_i}r}\right)^{\frac{r-2}{2}}\frac{2}{\underline{a_i}r}\max_j\overline{\phi_{ij}}\|y\|^r_\infty+\frac{4^{r-1}}{(1-e^{-\delta\underline{a_i}})^r}C_k\|y_i\|^r_\infty.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\|(Fy)(t)\|^r &\leq \max_i \Big[(4n)^{r-1}\frac{1}{\underline{a_i}^r}\max_j \overline{\varphi_j} b_{ij}\|y\|_\infty^r + (4n)^{r-1}\frac{1}{\underline{a_i}^r}\sum_{j=1}^n f_{ij} \\ &\quad + (4n)^{r-1} C_r\left(\frac{r-2}{\underline{a_i}r}\right)^{\frac{r-2}{2}}\frac{2}{\underline{a_i}r}\max_j \overline{\phi_{ij}}\|y\|_\infty^r + \frac{4^{r-1}}{(1-e^{-\delta \underline{a_i}})^r} C_k\|y\|_\infty^r\Big] \\ &= \hat{K}.\end{aligned}$$

Step 2. For any $x, y \in B_{p^*}$

$$\begin{aligned}\mathbb{E}\|(Fx-Fy)_i(t)\|^r \leq & 3^{r-1}\mathbb{E}\Big\|\sum_{j=1}^n\int_{-\infty}^t V(t,s)b_{ij}(s)[\varphi_j(x_j(s))-\varphi_j(y_j(s))]ds\Big\|^r \\ &+ 3^{r-1}\mathbb{E}\Big\|\int_{-\infty}^t\sum_{j=1}^n V(t,s)[\phi_{ij}(s,x_j(s))-\phi_{ij}(s,y_j(s))]dB_j(s)\Big\|^r \\ &+ 3^{r-1}\mathbb{E}\Big\|\sum_{t_k<t} V(t,t_k)[I_k(x_i(t_k))-I_k(y_i(t_k))]\Big\|^r.\end{aligned} \tag{13}$$

Using Hölder's inequality and (A_3), we get

$$\begin{aligned}&\mathbb{E}\Big\|\sum_{j=1}^n\int_{-\infty}^t V(t,s)b_{ij}(s)[\varphi_j(x_j(s))-\varphi_j(y_j(s))]ds\Big\|^r \\ &\leq \Big(\int_{-\infty}^t e^{-\underline{a_i}(t-s)}ds\Big)^{r-1}\Big(\int_{-\infty}^t e^{-\underline{a_i}(t-s)}\sum_{j=1}^n b_{ij}\mathbb{E}\Big\|\varphi_j(x_j(s))-\varphi_j(y_j(s))\Big\|^r ds\Big) \\ &\leq \frac{1}{\underline{a_i}^r}\max_j b_{ij}\overline{\varphi_j}\sum_{j=1}^n\sup_{s\in\mathbb{R}}\mathbb{E}\|x_j(s)-y_j(s)\|^r \\ &\leq \frac{1}{\underline{a_i}^r}\max_j b_{ij}\overline{\varphi_j}\mathbb{E}\|x-y\|^r\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\Big\|\sum_{t_k<t}V(t,t_k)[I_k(x_i(t_k))-I_k(y_i(t_k))]\Big\|^r \\ &\leq \Big(\sum_{t_k<t}e^{-\underline{a_i}(t-t_k)}\Big)^{r-1}\Big(\sum_{t_k<t}e^{-\underline{a_i}(t-s)}\mathbb{E}\|I_k(x(t_k))-I_k(y(t_k))\|^r\Big) \\ &\leq \frac{1}{(1-e^{-\delta\underline{a_i}})^r}C_k\sup_{s\in\mathbb{R}}\mathbb{E}\|x_i(s)-y_i(s)\|^r.\end{aligned}$$

Using Itô integral, Hölder's inequality and (A_3), we get

$$\begin{aligned}&\mathbb{E}\Big\|\int_{-\infty}^t\sum_{j=1}^n V(t,s)[\phi_{ij}(s,x_j(s))-\phi_{ij}(s,y_j(s))]dB_j(s)\Big\|^r \\ &\leq C_r\left(\frac{r-2}{\underline{a_i}^r}\right)^{\frac{r-2}{2}}\frac{2}{\underline{a_i}^r}\max_j\overline{\phi_{ij}}\sum_{j=1}^n\sup_{s\in\mathbb{R}}\mathbb{E}\|x_j(s)-y_j(s)\|^r \\ &\leq C_r\left(\frac{r-2}{\underline{a_i}^r}\right)^{\frac{r-2}{2}}\frac{2}{\underline{a_i}^r}\max_j\overline{\phi_{ij}}\mathbb{E}\|x-y\|^r.\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{E}\|(Fx-Fy)_i(t)\|^r &\leq 3^{r-1}\Big[\frac{1}{\underline{a_i}^r}\max_j b_{ij}\overline{\varphi_j}+C_r\left(\frac{r-2}{\underline{a_i}^r}\right)^{\frac{r-2}{2}}\frac{2}{\underline{a_i}^r}\max_j\overline{\phi_{ij}}\Big]\mathbb{E}\|x-y\|^r\\
&\quad+\frac{3^{r-1}C_k}{(1-e^{-\delta\underline{a_i}})^r}\sup_{s\in\mathbb{R}}\mathbb{E}\|x_i(s)-y_i(s)\|^r.\\
\mathbb{E}\|(Fx-Fy)(t)\|^r &\leq 3^{r-1}\max_i\Big[\frac{1}{\underline{a_i}^r}\max_j b_{ij}\overline{\varphi_j}+C_r\left(\frac{r-2}{\underline{a_i}^r}\right)^{\frac{r-2}{2}}\frac{2}{\underline{a_i}^r}\max_j\overline{\phi_{ij}}\\
&\quad+\frac{C_k}{(1-e^{-\delta\underline{a_i}})^r}\Big]\mathbb{E}\|x-y\|^r.
\end{aligned}$$

Since by construction $M<1$, we have the strict contraction of operator F and by the Banach contraction mapping principle, F has a unique fixed point such that $Fy=y$. □

4. Exponential stability

Theorem 4.1. *Under hypothesis of Theorem* 3.1. *the r-mean (α,β)-pseudo almost periodic mild solution of the system* (1) *is exponentially stable if*

$$\begin{aligned}
L &= \min_i\underline{a_i}-4^{r-1}\Big[C_r\sum_{j=1}^n\max_j\overline{\phi_{ij}}\left(\frac{r-2}{2(r-1)\underline{a_i}}\right)^{\frac{r-2}{2}}\left(1-e^{-\frac{2(r-1)}{r-2}\underline{a_i}(t-t_0)}\right)^{\frac{r-2}{2}}\\
&\quad+\sum_{j=1}^n\left(\frac{1-e^{-\underline{a_i}(t-t_0)}}{\underline{a_i}}\right)^{r-1}\max_j\overline{b_{ij}}\overline{\varphi_j}\Big]>0,\ \textit{for } r>2.
\end{aligned}$$

Proof. It follows from Theorem 3.1 that system (1) has at least one mild solution $x(t)=(x_1(t),...,x_n(t))^T\in B$ with initial value $\psi(t)=(\psi_1(t),...,\psi_n(t))^T$. Let $y(t)$ be an arbitrary solution of system (1) with initial value $\psi^*(t)=(\psi_1^*(t),...,\psi_n^*(t))^T$. Let $z_i(t)=x_i(t)-y_i(t)$, $u_i(t)=\psi_i(t)-\psi_i^*(t)$, $i=1...n$, then for $r>2$

$$\begin{aligned}
z_i(t)&=\psi_i(t_0)e^{-\int_{t_0}^t a_i(s)ds}+\int_{t_0}^t e^{-\int_{t_0}^t a_i(s)ds}\sum_{j=1}^n b_{ij}(s)\Big[\varphi_j(x_j(s))-\varphi_j(y_j(s))\Big]ds\\
&\quad+\int_{t_0}^t e^{-\int_{t_0}^t a_i(s)ds}\sum_{j=1}^n\Big[\phi_{ij}(s,x_i(s))-\phi_{ij}(s,y_i(s))\Big]dB_j(s)+\sum_{t_k<t}e^{-\int_{t_0}^{t_k}a_i(s)ds}I_k(z_i(t_k)).
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\|z_i(t)\|^r &= \mathbb{E}\|x_i(t)-y_i(t)\|^r\\
&\leq 4^{r-1}e^{-r\underline{a_i}(t-t_0)}\mathbb{E}\|\psi_i(t)-\psi_i^*(t)\|^r\\
&\quad+4^{r-1}\mathbb{E}\Big\|\int_{t_0}^t e^{-\underline{a_i}(t-s)}\sum_{j=1}^n b_{ij}(s)\Big[\varphi_j(x_j(s))-\varphi_j(y_j(s))\Big]ds\Big\|^r\\
&\quad+4^{r-1}\mathbb{E}\Big\|\int_{t_0}^t e^{-\underline{a_i}(t-s)}\sum_{j=1}^n\Big[\phi_{ij}(s,x_j(s))-\phi_{ij}(s,y_j(s))\Big]dB_j(s)\Big\|^r\\
&\quad+4^{r-1}\mathbb{E}\Big\|\sum_{t_k<t}e^{-\underline{a_i}(t-t_k)}\Big[I_k(x_i(t_k))-I_k(y_i(t_k))\Big]\Big\|^r\\
&\leq 4^{r-1}e^{-r\underline{a_i}(t-t_0)}\mathbb{E}\|\psi_i(t)-\psi_i^*(t)\|^r+4^{r-1}\sum_{k=1}^3 U_k.
\end{aligned}$$

Using Hölder inequality, we get

$$\begin{aligned}
U_1 &\leq \mathbb{E}\left(\int_{t_0}^{t} e^{-\underline{a_i}(t-s)} \sum_{j=1}^{n} \|b_{ij}(s)\|\|\varphi_j(x_j(s)) - \varphi_j(y_j(s))\| ds\right)^r \\
&\leq \left(\int_{t_0}^{t} e^{-\underline{a_i}(t_1-s)} ds\right)^{r-1} \int_{t_0}^{t} e^{-\underline{a_i}(t-s)} \sum_{j=1}^{n} \mathbb{E}\|b_{ij}(s)\|^r \mathbb{E}\|\varphi_j(x_j(s)) - \varphi_j(y_j(s))\|^r ds \\
&\leq \Big(\frac{1-e^{-\underline{a_i}(t-t_0)}}{\underline{a_i}}\Big)^{r-1} \max_j \overline{b_{ij}}\,\overline{\varphi_j} \int_{t_0}^{t} e^{-\underline{a_i}(t-s)} \sum_{j=1}^{n} \mathbb{E}\|x_j(s) - y_j(s)\|^r ds, \\
U_3 &= \mathbb{E}\Big\| \sum_{t_k<t} e^{-\underline{a_i}(t-t_k)} \Big[I_k(x_i(t_k)) - I_k(y_i(t_k))\Big] \Big\|^r \\
&\leq \left(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)}\right)^{r-1} \left(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)} \mathbb{E}\|I_k(x_i(t_k)) - I_k(y_i(t_k))\|^r\right) \\
&\leq C_k \left(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)}\right)^{r-1} \left(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)} \mathbb{E}\|x_i(t_k) - y_i(t_k))\|^r\right) \\
&\leq \frac{C_k}{\big(1-e^{-\delta\underline{a_i}}\big)^{r-1}} \left(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)} \mathbb{E}\|x_i(t_k) - y_i(t_k))\|^r\right).
\end{aligned}$$

Using Itô integral and Hölder inequality, we obtain

$$\begin{aligned}
U_2 &= \mathbb{E}\Big\| \int_{t_0}^{t} e^{-\underline{a_i}(t-s)} \sum_{j=1}^{n} \Big[\phi_{ij}(s, x_j(s)) - \phi_{ij}(s, y_j(s))\Big] dB_j(s) \Big\|^r \\
&\leq C_r \mathbb{E}\left(\int_{t_0}^{t} e^{-2\underline{a_i}(t_1-s)} \sum_{j=1}^{n} \|\phi_{ij}(s, x_j(s)) - \phi_{ij}(s, y_j(s))\|^2 ds\right)^{r/2} \\
&\leq C_r \left(\int_{t_0}^{t} e^{-\frac{2(r-1)}{r}\underline{a_i}(t-s)} e^{-\frac{2}{r}\underline{a_i}(t-s)} \sum_{j=1}^{n} \|\phi_{ij}(s, x_j(s)) - \phi_{ij}(s, y_j(s))\|^2 ds\right)^{r/2} \\
&\leq C_r \left(\int_{t_0}^{t} e^{-\frac{2(r-1)}{r-2}\underline{a_i}(t-s)} ds\right)^{\frac{r-2}{2}} \left(\int_{t_0}^{t} e^{-\underline{a_i}(t-s)} \sum_{j=1}^{n} \mathbb{E}\|\phi_{ij}(s, x_j(s)) - \phi_{ij}(s, y_j(s))\|^r ds\right) \\
&\leq C_r \max_j \overline{\phi_{ij}} \left(\frac{r-2}{2(r-1)\underline{a_i}}\right)^{\frac{r-2}{2}} \left(1 - e^{-\frac{2(r-1)}{r-2}\underline{a_i}(t-t_0)}\right)^{\frac{r-2}{2}} \\
&\quad \times \left(\int_{t_0}^{t} e^{-\underline{a_i}(t-s)} \sum_{j=1}^{n} \mathbb{E}\|x_j(s) - y_j(s)\|^r ds\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}\|z_i(t)\|^r \quad \leq \quad & 4^{r-1} e^{-r\underline{a_i}(t-t_0)} \mathbb{E}\|\psi_i(t) - \psi_i^*(t)\|^r \\
& + 4^{r-1} \frac{C_k}{\big(1-e^{-\delta\underline{a_i}}\big)^{r-1}} \left(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)} \mathbb{E}\|x_i(t_k) - y_i(t_k))\|^r\right) \\
& + \Big[C_r \max_j \overline{\phi_{ij}} \left(\frac{r-2}{2(r-1)\underline{a_i}}\right)^{\frac{r-2}{2}} \left(1 - e^{-\frac{2(r-1)}{r-2}\underline{a_i}(t-t_0)}\right)^{\frac{r-2}{2}}
\end{aligned}$$

$$+\Big(\frac{1-e^{-\underline{a_i}(t-t_0)}}{\underline{a_i}}\Big)^{r-1}\max_j \overline{b_{ij}}\,\overline{\varphi_j}\Big]\times\int_{t_0}^{t}e^{-\underline{a_i}(t-s)}\sum_{j=1}^{n}\mathbb{E}\|x_j(s)-y_j(s)\|^r ds.$$

Consequently

$$\begin{aligned}
&\mathbb{E}\|x(t)-y(t)\|^r \le 4^{r-1}\max_i e^{-r\underline{a_i}(t-t_0)}\mathbb{E}\|\psi(t)-\psi^*(t)\|^r\\
&+4^{r-1}\frac{C_k}{\left(1-e^{-\delta\underline{a_i}}\right)^{r-1}}\times\left(\sum_{t_k<t}\max_i e^{-\underline{a_i}(t-t_k)}\mathbb{E}\|x(t_k)-y(t_k))\|^r\right)\\
&+4^{r-1}\Big[C_r\sum_{j=1}^{n}\max_j\overline{\phi_{ij}}\left(\frac{r-2}{2(r-1)\underline{a_i}}\right)^{\frac{r-2}{2}}\left(1-e^{-\frac{2(r-1)}{r-2}\underline{a_i}(t-t_0)}\right)^{\frac{r-2}{2}}\\
&+\sum_{j=1}^{n}\Big(\frac{1-e^{-\underline{a_i}(t-t_0)}}{\underline{a_i}}\Big)^{r-1}\max_j\overline{b_{ij}}\,\overline{\varphi_j}\Big]\times\int_{t_0}^{t}e^{-\underline{a_i}(t-s)}\mathbb{E}\|x(s)-y(s)\|^r ds.
\end{aligned}$$

Let us choose $Z(t)=\mathbb{E}\|x(t)-y(t)\|^r e^{\underline{a_i}t}$, then we have

$$\begin{aligned}
Z(t) \quad\le\quad & 4^{r-1}Z(t_0)+\int_{t_0}^{t}4^{r-1}\Big[C_r\sum_{j=1}^{n}\max_j\overline{\phi_{ij}}\left(\frac{r-2}{2(r-1)\underline{a_i}}\right)^{\frac{r-2}{2}}\left(1-e^{-\frac{2(r-1)}{r-2}\underline{a_i}(t-t_0)}\right)^{\frac{r-2}{2}}\\
&+\sum_{j=1}^{n}\Big(\frac{1-e^{-\underline{a_i}(t-t_0)}}{\underline{a_i}}\Big)^{r-1}\max_j\overline{b_{ij}}\,\overline{\varphi_j}\Big]Z(s)ds\\
&+4^{r-1}\sum_{t_k<t}\max_i\frac{C_k}{\left(1-e^{-\delta\underline{a_i}}\right)^{r-1}}Z(t_k).
\end{aligned}$$

Now by using Gronwall-Bellman's lemma [24]

$$\begin{aligned}
Z(t) \quad\le\quad & 4^{r-1}Z(t_0)4^{r-1}Z(t_0)\prod_{t_k<t}\left(1+4^{r-1}\max_i\frac{C_k}{\left(1-e^{-\delta\underline{a_i}}\right)^{r-1}}\right)\\
&\times exp\Big(4^{r-1}\Big[C_r\sum_{j=1}^{n}\max_j\overline{\phi_{ij}}\left(\frac{r-2}{2(r-1)\underline{a_i}}\right)^{\frac{r-2}{2}}\left(1-e^{-\frac{2(r-1)}{r-2}\underline{a_i}(t-t_0)}\right)^{\frac{r-2}{2}}\\
&+\sum_{j=1}^{n}\Big(\frac{1-e^{-\underline{a_i}(t-t_0)}}{\underline{a_i}}\Big)^{r-1}\max_j\overline{b_{ij}}\,\overline{\varphi_j}\Big](t-t_0)\Big).
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{E}\|x(t)-y(t)\|^r\le 4^{r-1}\|x(t_0)-y(t_0)\|^r\left(1+4^{r-1}\max_i\frac{C_k}{\left(1-e^{-\delta\underline{a_i}}\right)^{r-1}}\right)^{t+1}\\
&\times exp\Big(-\min_i\underline{a_i}+4^{r-1}\Big[C_r\sum_{j=1}^{n}\max_j\overline{\phi_{ij}}\left(\frac{r-2}{2(r-1)\underline{a_i}}\right)^{\frac{r-2}{2}}\left(1-e^{-\frac{2(r-1)}{r-2}\underline{a_i}(t-t_0)}\right)^{\frac{r-2}{2}}\\
&+\sum_{j=1}^{n}\Big(\frac{1-e^{-\underline{a_i}(t-t_0)}}{\underline{a_i}}\Big)^{r-1}\max_j\overline{b_{ij}}\,\overline{\varphi_j}\Big](t-t_0)\Big).
\end{aligned}$$

For $r=2$, by using Hölder inequality, we get

$$
\begin{aligned}
U_1 &\leq \mathbb{E}\left(\int_{t_0}^{t} e^{-\underline{a_i}(t-s)}\sum_{j=1}^{n}\|b_{ij}(s)\|\|\varphi_j(x_j(s))-\varphi_j(y_j(s))\|ds\right)^2\\
&\leq \Big(\frac{1-e^{-\underline{a_i}(t-t_0)}}{\underline{a_i}}\Big)\max_j \overline{b_{ij}}\overline{\varphi_j}\int_{t_0}^{t} e^{-\underline{a_i}(t-s)}\sum_{j=1}^{n}\mathbb{E}\|x_j(s)-y_j(s)\|^2ds,\\
U_3 &= \mathbb{E}\Big\|\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)}\Big[I_k(x_i(t_k))-I_k(y_i(t_k))\Big]\Big\|^2\\
&\leq \left(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)}\right)\left(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)}\mathbb{E}\|I_k(x_i(t_k))-I_k(y_i(t_k))\|^2\right)\\
&\leq \left(\frac{C_k}{1-e^{-\underline{a_i}\delta}}\right)\left(\sum_{t_k<t} e^{-\underline{a_i}(t-t_k)}\mathbb{E}\|x_i(t_k)-y_i(t_k))\|^2\right).
\end{aligned}
$$

Using Itô integral and Hölder inequality, we obtain

$$
\begin{aligned}
U_2 &= \mathbb{E}\Big\|\int_{t_0}^{t} e^{-\underline{a_i}(t-s)}\sum_{j=1}^{n}\Big[\phi_{ij}(s,x_j(s))-\phi_{ij}(s,y_j(s))\Big]dB_j(s)\Big\|^2\\
&\leq 4\max_j\overline{\phi_{ij}}\left(\int_{t_0}^{t} e^{-\underline{a_i}(t-s)}ds\right)\left(\int_{t_0}^{t} e^{-\underline{a_i}(t-s)}\mathbb{E}\|x(s)-y(s)\|^2ds\right)\\
&\leq 4\max_j\overline{\phi_{ij}}\left(\frac{1-e^{-\underline{a_i}(t-t_0)}}{\underline{a_i}}\right)\left(\int_{t_0}^{t} e^{-\underline{a_i}(t-s)}\mathbb{E}\|x(s)-y(s)\|^2ds\right).
\end{aligned}
$$

Therefore, now by using Gronwall-Bellman's lemma [24] we obtain

$$
\begin{aligned}
\mathbb{E}\|x(t)-y(t)\|^2 &\leq 4^{r-1}\|x(t_0)-y(t_0)\|^2\Big(1+4\max_i\frac{C_k}{(1-e^{-\delta\underline{a_i}})}\Big)^{t+1}\\
&\times exp\Big(-\min_i\underline{a_i}+4\max_i\Big(\frac{1-e^{-\underline{a_i}(t-t_0)}}{\underline{a_i}}\Big)\Big[\max_j\overline{b_{ij}}\overline{\varphi_j}+4\sum_{j=1}^{n}\overline{\phi_{ij}}\Big](t-t_0)\Big),
\end{aligned}
$$

which implies that the r-mean (α,β)-pseudo almost periodic mild solution of the system (1) is exponential stable. □

5. Applications

In this section, we illustrate our main results by an example. Consider the two-dimensional impulsive stochastic neural networks

$$
\begin{cases}
dy_i(s)=\Big[-a_i(s)y_i(s)+\sum_{j=1}^{2}b_{ij}(s)\varphi_j(y_j(s))+\sum_{j=1}^{2}f_{ij}(s)\Big]ds\\
\ +\sum_{j=1}^{2}\phi_{ij}(s,y_i(s))dB_j(s),\ \ s\geq 0,\ \ s\neq s_k,\ \ k\in\mathbb{N},\\
y_i(s_k^+)=I_{i,k}(y(s_k)),\ \ s=s_k,\\
y_i(s_0)=\psi_i,\ \ i=1,2
\end{cases}
\tag{14}
$$

where $(B_j(s), s\in\mathbb{R})$ is two-sided standard Weiner process, $\mathcal{H}=L^2([0,\pi])$, $s_k=k+\frac{1}{4}|\sin k-\sin\sqrt{2}k|$, $k\in\mathbb{N}$, this gives $\delta=\inf(s_{k+1}-s_k)=\frac{1}{3}>0$, $\psi_i=(1,1)^T$,

$$
a_1(s)=a_2(s)=8+\sin(2\pi s)+\sin(\sqrt{2}\pi s)\in\mathbf{AP_T}(\mathbb{R},\mathbb{R})\ \Rightarrow\underline{a_i}=6
$$

$$(b_{ij}(s))_{1\le i,j\le 2} = \begin{pmatrix} \frac{\sin(2\pi s)+\sin(\sqrt{2}\pi s)+e^{-|s|}}{20} & \frac{\cos(2\pi s)+\cos(\sqrt{2}\pi s)+e^{-|s|}}{20} \\ \frac{\cos(2\pi s)+\cos(\sqrt{2}\pi s)+\frac{1}{1+s^2}}{20} & \frac{\sin(2\pi s)+\sin(\sqrt{2}\pi s)+e^{-s^2}}{20} \end{pmatrix},$$

$$(f_{ij}(s))_{1\le i,j\le 2} = 20(b_{ij}(s))_{1\le i,j\le 2},$$

$$\varphi_1(s) = \varphi_2(s) = \frac{1}{2}\sin s,$$

$$\phi_{ij}(s, y_i(s)) = \frac{1}{50}\begin{pmatrix} \cos(2\pi s)+\cos(\sqrt{2}\pi s)+e^{-|s|} & \cos(2\pi s)+\cos(\sqrt{2}\pi s)+e^{-|s|} \\ \cos(2\pi s)+\cos(\sqrt{2}\pi s)+e^{-|s|} & \cos(2\pi s)+\cos(\sqrt{2}\pi s)+e^{-|s|} \end{pmatrix}\sin(y_i(s))$$

$$I_1 = I_2 = \frac{1}{40}[\cos(2\pi k)+\cos(\sqrt{2}\pi k)+e^{-k^2}]\sin(y_i),$$

this gives that $b_{ij} = 0,3$, $\overline{\varphi_j} = \frac{\sqrt{2}}{2}$, $f_{ij} = 6$, $\overline{\phi_{ij}} = 0,12$ and $C_k = \frac{3\sqrt{2}}{40}$. It is obvious to verify that the functions $\phi_{ij} \in \mathbf{PAP_T}(\mathbb{R}\times L^2(P,\mathcal{H}), L^2(P, L_0^2), \alpha, \beta)$, b_{ij}, $f_{ij} \in \mathbf{PAP_T}(\mathbb{R}, L^2(P,\mathcal{H}), \alpha, \beta)$, $I_k \in \mathbf{PAP_T}(\mathbb{N}\times L^2(P,\mathcal{H}), L^2(P,\mathcal{H}), \alpha, \beta)$, and $\varphi_j \in \mathbf{PAP_T}(\mathbb{R}, L^2(P, L^2(P,\mathcal{H})), \alpha, \beta)$. So we have $M = 0,695 < 1$, $L = 5,54 > 0$. All conditions from Theorems 3.5, 4.1 are satisfied, then the system 14 has has exactly one r-mean (α,β)-pseudo almost periodic mild solution, which is globally exponentially stable.

Example 5.1. We will choose the two measures α and β in the previous application defined by the following double weights, respectively:

$$\rho_1(t) = e^{\sin(t)},\ t\in\mathbb{R} \ \text{ and } \ \rho_2(t) = \begin{cases} e^t & \text{if } t\le 0, \\ 1 & \text{if } t > 0. \end{cases}$$

We have $\frac{2r}{e} \le \alpha([-r,r]) = \int_{-r}^{r} e^{\sin(t)}dt \le 2er$. Then $\alpha\in\mathcal{M}$ satisfies (H_1). In deed $\sin(\tau+a)\le 2+\sin(a)$ for all $\tau\in\mathbb{R}$ and $a\in A$, which implies that $\alpha(\tau+A)\le e^2\alpha(A)$. From [1], $\beta\in\mathcal{M}$ satisfies (H_1). Since $\limsup_{r\to+\infty}\dfrac{\alpha([-r,r])}{\beta([-r,r])} < \infty$, then (H_2) is true.

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