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The Padovan-Circulant-Hurwitz Sequences

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Abstract. In this paper, we define the Padovan-circulant-Hurwitz sequences of the first, second, third and fourth kinds by using the Hurwitz matrices, which are obtained from the characteristic polynomials of the Padovan-circulant sequences of the first, second, third and fourth kinds. First, we derive relationships between the Padovan-circulant-Hurwitz numbers of the first, second, third and fourth kinds and the generating matrices of these sequences. Then we obtain the miscellaneous properties of these sequences by aid of these matrices.

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1. Introduction

The Padovan sequence is the sequence of the integer $\{P(n)\}$ defined by the initial values $P(0) = P(1) = P(2) = 1$ and the recurrence relation:

$$P(n) = P(n-2) + P(n-3)$$

for all $n \geq 3$.

In [5], Deveci defined the Padovan-circulant sequences of first, second, third and fourth kind as follows, respectively:

$$x_n^1 = x_{n-2}^1 - x_{n-3}^1 - x_{n-4}^1 \text{ for } n \geq 5 \text{ where } x_1^1 = x_2^1 = x_3^1 = 0 \text{ and } x_4^1 = 1,$$

$$x_n^2 = -x_{n-2}^2 + x_{n-4}^2 - x_{n-5}^2 \text{ for } n \geq 6 \text{ where } x_1^2 = x_2^2 = x_3^2 = x_4^2 = 0 \text{ and } x_5^2 = 1,$$

$$x_n^3 = -x_{n-3}^3 - x_{n-4}^3 + x_{n-6}^3 \text{ for } n \geq 7 \text{ where } x_1^3 = x_2^3 = x_3^3 = x_4^3 = x_5^3 = 0 \text{ and } x_6^3 = 1$$

and

$$x_n^4 = x_{n-4}^4 - x_{n-5}^4 - x_{n-6}^4 \text{ for } n \geq 7 \text{ where } x_1^4 = x_2^4 = x_3^4 = x_4^4 = x_5^4 = 0 \text{ and } x_6^4 = 1.$$

Let P be a nth degree real polynomial given by

$$P(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n.$$

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In [11], A. Hurwitz defined the Hurwitz matrix $H_n = [h_{ij}]_{n\times n}$ associated to P was defined as follows:

$$H_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & a_1 & a_3 & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_2 & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & 0 & a_1 & \ddots & \ddots & \ddots & a_n & \vdots & \vdots \\ \vdots & \vdots & a_0 & \ddots & \ddots & \ddots & a_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & \cdots & \cdots & \cdots & a_{n-2} & a_n & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n-4} & a_{n-2} & a_n \end{bmatrix}.$$

Suppose that the $(n+k)$th term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [12], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

$$A = [a_{i,j}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geqslant 0$.

As it is well-known recurrence sequences, circulant matrix and Hurwitz matrix appear in modern research in many fields from mathematics, physics, computer science, architecture to nature and art see, for example; [4, 9, 13, 15, 16, 19, 20, 21, 23, 24, 25].

Recurrence sequences are widely utilized to solve some problems in various scientific fields, or different problems in different scientific disciplines are directly created by taking the structural aspects of these sequences into account. In the literature, many interesting properties and applications of the recurrence sequences relevant to this paper have been studied by many authors; see, for example, [1, 6, 7, 8, 10, 14, 22]. In this paper, we give new sequences which are called Padovan-circulant sequences of first, second, third and fourth kind. Firstly, we define the Padovan-circulant-Hurwitz sequences of first, second, third and fourth kind by using the Hurwitz matrices which are obtained from the characteristic polynomials of the Padovan-circulant sequences of first, second, third and fourth kind and then we derive relationships between the Padovan-circulant-Hurwitz numbers of the first, second, third and fourth kind and the generating matrices of these sequences. Also, we give miscellaneous properties of the Padovan-circulant-Hurwitz sequences of the first, second, third and fourth kind such as the generating function, the Binet formula, the permanental, determinantal and combinatorial representations and the sums by the aid of the generating functions and the generating matrices of the sequences defined.

2. The Main Results

It is easy to see that the characteristic polynomials of the Padovan-circulant sequences of first, second, third and fourth kind are as follows, respectively:

$$f^{(1)}(x) = x^4 - x^2 + x + 1,$$
$$f^{(2)}(x) = x^5 + x^3 - x + 1,$$
$$f^{(3)}(x) = x^6 + x^3 + x^2 - 1$$

and

$$f^{(4)}(x) = x^6 - x^2 + x + 1.$$

Then we can write the following Hurwitz matrices for the polynomials $f^{(1)}(x)$, $f^{(2)}(x)$, $f^{(3)}(x)$ and $f^{(4)}(x)$, respectively:

$$H^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix},$$

$$H^{(2)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$H^{(3)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

and

$$H^{(4)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}.$$

Now we define the Padovan-circulant-Hurwitz sequences of first, second, third and fourth kind by using $H^{(1)}$, $H^{(2)}$, $H^{(3)}$ and $H^{(4)}$ the matrices as follows, respectively:

$$a^1(n+4) = a^1(n+2) - a^1(n+1) + a^1(n) \text{ where } a^1(1) = a^1(2) = a^1(3) = 0 \text{ and } a^1(4) = 1, \quad (1)$$

$$a^2(n+5) = -a^2(n+2) + a^2(n+1) + a^2(n) \text{ where } a^2(1) = a^2(2) = a^2(3) = a^2(4) = 0 \text{ and } a^2(5) = 1, \quad (2)$$

$$a^3(n+6) = -a^3(n+3) + a^3(n+2) + a^3(n) \text{ where } a^3(1) = a^3(2) = a^3(3) = a^3(4) = a^3(5) = 0 \text{ and } a^3(6) = 1 \quad (3)$$

and

$$a^4(n+6) = a^4(n+3) - a^4(n+2) + a^4(n) \text{ where } a^4(1) = a^4(2) = a^4(3) = a^4(4) = a^4(5) = 0 \text{ and } a^4(6) = 1 \quad (4)$$

for $n \geq 1$.

The generating functions of the Padovan-circulant-Hurwitz sequences of first, second, third and fourth kind are then:

$$g^{(1)}(x) = \frac{x^4}{1 - x^2 + x^3 - x^4},$$

$$g^{(2)}(x) = \frac{x^5}{1 + x^3 - x^4 - x^5},$$

$$g^{(3)}(x) = \frac{x^6}{1 + x^3 - x^4 - x^6}$$

and

$$g^{(4)}(x) = \frac{x^6}{1 - x^3 + x^4 - x^6}.$$

By equation (1),(2), (3) and (4), we can write the following companion matrices, respectively:

$$PH^{(1)} = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$PH^{(2)} = \begin{bmatrix} 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$PH^{(3)} = \begin{bmatrix} 0 & 0 & -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$PH^{(4)} = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

and we call the matrices $PH^{(1)}$, $PH^{(2)}$, $PH^{(3)}$ and $PH^{(4)}$ Padovan-circulant-Hurwitz matrices of the first, second, third and fourth kind.

Let $a^k(\alpha)$ be denoted by a^k_α for $k = 1, 2, 3, 4$. By an inductive argument, we may write for $\alpha \geq 3$

$$\left(PH^{(1)}\right)^\alpha = \begin{bmatrix} a^1_{\alpha+4} & a^1_{\alpha+5} & a^1_{\alpha+2} - a^1_{\alpha+3} & a^1_{\alpha+3} \\ a^1_{\alpha+3} & a^1_{\alpha+4} & a^1_{\alpha+1} - a^1_{\alpha+2} & a^1_{\alpha+2} \\ a^1_{\alpha+2} & a^1_{\alpha+3} & a^1_{\alpha} - a^1_{\alpha+1} & a^1_{\alpha+1} \\ a^1_{\alpha+1} & a^1_{\alpha+2} & a^1_{\alpha-1} - a^1_{\alpha} & a^1_{\alpha} \end{bmatrix},$$

$$\left(PH^{(2)}\right)^\alpha = \begin{bmatrix} a^2_{\alpha+5} & a^2_{\alpha+6} & a^2_{\alpha+7} & a^2_{\alpha+3} + a^2_{\alpha+4} & a^2_{\alpha+4} \\ a^2_{\alpha+4} & a^2_{\alpha+5} & a^2_{\alpha+6} & a^2_{\alpha+2} + a^2_{\alpha+3} & a^2_{\alpha+3} \\ a^2_{\alpha+3} & a^2_{\alpha+4} & a^2_{\alpha+5} & a^2_{\alpha+1} + a^2_{\alpha+2} & a^2_{\alpha+2} \\ a^2_{\alpha+2} & a^2_{\alpha+3} & a^2_{\alpha+4} & a^2_{\alpha} + a^2_{\alpha+1} & a^2_{\alpha+1} \\ a^2_{\alpha+1} & a^2_{\alpha+2} & a^2_{\alpha+3} & a^2_{\alpha-1} + a^2_{\alpha} & a^2_{\alpha} \end{bmatrix},$$

$$\left(PH^{(3)}\right)^\alpha = \begin{bmatrix} a^3_{\alpha+6} & a^3_{\alpha+7} & a^3_{\alpha+8} & a^3_{\alpha+3} + a^3_{\alpha+5} & a^3_{\alpha+4} & a^3_{\alpha+5} \\ a^3_{\alpha+5} & a^3_{\alpha+6} & a^3_{\alpha+7} & a^3_{\alpha+2} + a^3_{\alpha+4} & a^3_{\alpha+3} & a^3_{\alpha+4} \\ a^3_{\alpha+4} & a^3_{\alpha+5} & a^3_{\alpha+6} & a^3_{\alpha+1} + a^3_{\alpha+3} & a^3_{\alpha+2} & a^3_{\alpha+3} \\ a^3_{\alpha+3} & a^3_{\alpha+4} & a^3_{\alpha+5} & a^3_{\alpha} + a^3_{\alpha+2} & a^3_{\alpha+1} & a^3_{\alpha+2} \\ a^3_{\alpha+2} & a^3_{\alpha+3} & a^3_{\alpha+4} & a^3_{\alpha-1} + a^3_{\alpha+1} & a^3_{\alpha} & a^3_{\alpha+1} \\ a^3_{\alpha+1} & a^3_{\alpha+2} & a^3_{\alpha+3} & a^3_{\alpha-2} + a^3_{\alpha} & a^3_{\alpha-1} & a^3_{\alpha} \end{bmatrix}$$

and

$$\left(PH^{(4)}\right)^\alpha = \begin{bmatrix} a^4_{\alpha+6} & a^4_{\alpha+7} & a^4_{\alpha+8} & a^4_{\alpha+3} - a^4_{\alpha+5} & a^4_{\alpha+4} & a^4_{\alpha+5} \\ a^4_{\alpha+5} & a^4_{\alpha+6} & a^4_{\alpha+7} & a^4_{\alpha+2} - a^4_{\alpha+4} & a^4_{\alpha+3} & a^4_{\alpha+4} \\ a^4_{\alpha+4} & a^4_{\alpha+5} & a^4_{\alpha+6} & a^4_{\alpha+1} - a^4_{\alpha+3} & a^4_{\alpha+2} & a^4_{\alpha+3} \\ a^4_{\alpha+3} & a^4_{\alpha+4} & a^4_{\alpha+5} & a^4_{\alpha} - a^4_{\alpha+2} & a^4_{\alpha+1} & a^4_{\alpha+2} \\ a^4_{\alpha+2} & a^4_{\alpha+3} & a^4_{\alpha+4} & a^4_{\alpha-1} - a^4_{\alpha+1} & a^4_{\alpha} & a^4_{\alpha+1} \\ a^4_{\alpha+1} & a^4_{\alpha+2} & a^4_{\alpha+3} & a^4_{\alpha-2} - a^4_{\alpha} & a^4_{\alpha-1} & a^4_{\alpha} \end{bmatrix},$$

from which it is clear that

$$\det\left(PH^{(1)}\right)^\alpha = \left(PH^{(3)}\right)^\alpha = \left(PH^{(4)}\right)^\alpha = (-1)^\alpha \, and \det\left(PH^{(2)}\right)^\alpha = 1.$$

It is clear that each of the eingenvalues of the matrices $PH^{(1)}$, $PH^{(2)}$, $PH^{(3)}$ and $PH^{(4)}$ are distinct. Let

$$\left\{\varepsilon^{(1)}_1, \varepsilon^{(1)}_2, \varepsilon^{(1)}_3, \varepsilon^{(1)}_4\right\},$$

$$\left\{\varepsilon^{(2)}_1, \varepsilon^{(2)}_2, \varepsilon^{(2)}_3, \varepsilon^{(2)}_4, \varepsilon^{(2)}_5\right\}$$

and

$$\left\{\varepsilon^{(t)}_1, \varepsilon^{(t)}_2, \varepsilon^{(t)}_3, \varepsilon^{(t)}_4, \varepsilon^{(t)}_5, \varepsilon^{(t)}_6\right\}$$

be the sets of the eingevalues of the matrices $PH^{(1)}$, $PH^{(2)}$ and $PH^{(3)}$ respectively and let V^k be $(k+3)\times(k+3)$ Vandermonde matrix as follows:

$$V^{(k)}=\begin{bmatrix}\left(\varepsilon_1^{(k)}\right)^{k+2} & \left(\varepsilon_2^{(k)}\right)^{k+2} & \cdots & \left(\varepsilon_{k+3}^{(k)}\right)^{k+2}\\ \left(\varepsilon_1^{(k)}\right)^{k+1} & \left(\varepsilon_2^{(k)}\right)^{k+1} & \cdots & \left(\varepsilon_{k+3}^{(k)}\right)^{k+1}\\ \vdots & \vdots & & \vdots\\ \varepsilon_1^{(k)} & \varepsilon_2^{(k)} & & \varepsilon_{k+3}^{(k)}\\ 1 & 1 & \cdots & 1\end{bmatrix}$$

where $k=1,2,3$. Suppose now that

$$W_i^{(k)}=\begin{bmatrix}\left(\varepsilon_1^{(k)}\right)^{\alpha+k+3-i}\\ \left(\varepsilon_2^{(k)}\right)^{\alpha+k+3-i}\\ \vdots\\ \left(\varepsilon_{k+3}^{(k)}\right)^{\alpha+k+3-i}\end{bmatrix}$$

and $V_{i,j}^{(k)}$ is a $(k+3)\times(k+3)$ matrix obtained from $V^{(k)}$ by replacing the jth column of $V^{(k)}$ by $W_i^{(k)}$. This yields the Binet-type formulas for the Padovan-circulant-Hurwitz matrices of the first, second and third kind, as stated in the following theorem.

Theorem 2.1. *Let a_α^k be the αth term of the sequence of the kth kind for $k=1,2,3$. Then*

$$p_{ij}^{(k,\alpha)}=\frac{\det V_{i,j}^{(k)}}{\det V^{(k)}}$$

where $\left(PH^{(k)}\right)^\alpha=\left[p_{ij}^{(k,\alpha)}\right]$ such that $k=1,2,3$.

Proof. Since the eigenvalues of the matrix $PH^{(k)}$ are distinct, the matrix $PH^{(k)}$ is diagonalizable. Let

$$D^{(1)}=diag\left(\varepsilon_1^{(1)},\varepsilon_2^{(1)},\varepsilon_3^{(1)},\varepsilon_4^{(1)}\right),$$

$$D^{(2)}=diag\left(\varepsilon_1^{(2)},\varepsilon_2^{(2)},\varepsilon_3^{(2)},\varepsilon_4^{(2)},\varepsilon_5^{(2)}\right)$$

and

$$D^{(3)}=diag\left(\varepsilon_1^{(3)},\varepsilon_2^{(3)},\varepsilon_3^{(3)},\varepsilon_4^{(3)},\varepsilon_5^{(3)},\varepsilon_6^{(3)}\right),$$

then it is readily seen that $PH^{(k)}V^{(k)}=V^{(k)}D^{(k)}$. Since the matrix $V^{(k)}$ is invertible,

$$\left(V^{(k)}\right)^{-1}PH^{(k)}V^{(k)}=D^{(k)}$$

Thus, the matrix $PH^{(k)}$ is similar to $D^{(k)}$. So we get

$$\left(PH^{(k)}\right)^\alpha V^{(k)}=V^{(k)}\left(D^{(k)}\right)^\alpha$$

for $\alpha \geq 3$. Then we can write the following linear system of equations:

$$\begin{cases} p_{i1}^{(k,\alpha)}\left(\varepsilon_1^{(k)}\right)^{k+2} + p_{i2}^{(k,\alpha)}\left(\varepsilon_1^{(k)}\right)^{k+1} + \cdots + p_{ik+3}^{(k,\alpha)} = \left(\varepsilon_1^{(k)}\right)^{\alpha+k+3-i} \\ p_{i1}^{(k,\alpha)}\left(\varepsilon_2^{(k)}\right)^{k+2} + p_{i2}^{(k,\alpha)}\left(\varepsilon_2^{(k)}\right)^{k+1} + \cdots + p_{ik+3}^{(k,\alpha)} = \left(\varepsilon_2^{(k)}\right)^{\alpha+k+3-i} \\ \vdots \\ p_{i1}^{(k,\alpha)}\left(\varepsilon_{k+3}^{(k)}\right)^{k+2} + p_{i2}^{(k,\alpha)}\left(\varepsilon_{k+3}^{(k)}\right)^{k+1} + \cdots + p_{ik+3}^{(k,\alpha)} = \left(\varepsilon_{k+3}^{(k)}\right)^{\alpha+k+3-i} \end{cases}.$$

So, we obtain that

$$p_{ij}^{(k,\alpha)} = \frac{\det V_{i,j}^{(k)}}{\det V^{(k)}} \text{ for } k = 1,2,3 \text{ and } i,j = 1,2,\ldots,k+3.$$

□

If we choose

$$V^{(4)} = \begin{bmatrix} \left(\varepsilon_1^{(4)}\right)^5 & \left(\varepsilon_2^{(4)}\right)^5 & \cdots & \left(\varepsilon_6^{(4)}\right)^5 \\ \left(\varepsilon_1^{(4)}\right)^4 & \left(\varepsilon_2^{(4)}\right)^4 & \cdots & \left(\varepsilon_6^{(4)}\right)^4 \\ \vdots & \vdots & & \vdots \\ \varepsilon_1^{(4)} & \varepsilon_2^{(4)} & & \varepsilon_6^{(4)} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \text{ and } W_i^{(4)} = \begin{bmatrix} \left(\varepsilon_1^{(4)}\right)^{\alpha+6-i} \\ \left(\varepsilon_2^{(4)}\right)^{\alpha+6-i} \\ \vdots \\ \left(\varepsilon_6^{(4)}\right)^{\alpha+6-i} \end{bmatrix}$$

then we obtain the Binet formula for the Padovan-circulant-Hurwitz sequence of the fourth kind as follows:

$$p_{ij}^{(4,\alpha)} = \frac{\det V_{i,j}^{(4)}}{\det V^{(4)}} \text{ for each } i,j = 1,2,3,4,5,6.$$

Then we can give the Binet formulas for the Padovan-circulant-Hurwitz numbers of the first, second, third and fourth kind by the following Corollary.

Corollary 2.2. *Let a_α^k be the αth term of the Padovan-circulant-Hurwitz numbers of the first, second, third and fourth kind. Then*

(i) For $k = 1,2$

$$a_\alpha^k = \frac{\det V_{k+3,k+3}^{(k)}}{\det V^{(k)}}.$$

(ii) For $k = 3,4$

$$a_\alpha^k = \frac{\det V_{5,5}^{(k)}}{\det V^{(k)}} = \frac{\det V_{6,6}^{(k)}}{\det V^{(k)}}.$$

Now we consider the relationship between the Padovan-circulant-Hurwitz sequences of the first, second, third and fourth kind and the permanent of a certain matrix which is obtained using the Padovan-circulant-Hurwitz matrices of the first, second, third and fourth kind matrix $PH^{(k)}$, where $k = 1,2,3$.

Definition 2.1. A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.

Suppose that $x_1, x_2, \dots, x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0$,$m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [2], Brualdi and Gibson obtained that $per\,(M) = per\,(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Let $K^{(1)}(n) = \left[k_{i,j}^{(1)}\right]$, $K^{(2)}(n) = \left[k_{i,j}^{(2)}\right]$, $K^{(3)}(n) = \left[k_{i,j}^{(3)}\right]$ and $K^{(4)}(n) = \left[k_{i,j}^{(4)}\right]$ be the $n\times n$ super-diagonal matrices, defined by, respectively:

$$k_{i,j}^{(1)} = \begin{cases} 1 & \begin{array}{c}\text{if } i=p \text{ and } j=p+1 \text{ for } 1\le p\le n-1,\\ \text{if } i=p \text{ and } j=p+3 \text{ for } 1\le p\le n-3\\ \text{and}\\ i=p+1 \text{ and } j=p \text{ for } 1\le p\le n-1,\end{array}\\ -1 & \text{if } i=p \text{ and } \; j=p+2 \text{ for } 1\le p\le n-2,\\ 0 & \text{otherwise.}\end{cases}, \text{ for } n\ge 4,$$

$$k_{i,j}^{(2)} = \begin{cases} 1 & \begin{array}{c}\text{if } i=p \text{ and } j=p+3 \text{ for } 1\le p\le n-3,\\ \text{if } i=p \text{ and } j=p+4 \text{ for } 1\le p\le n-4\\ \text{and}\\ i=p+1 \text{ and } j=p \text{ for } 1\le p\le n-1,\end{array}\\ -1 & \text{if } i=p \text{ and } \; j=p+2 \text{ for } 1\le p\le n-2,\\ 0 & \text{otherwise.}\end{cases}, \text{ for } n\ge 5,$$

$$k_{i,j}^{(3)} = \begin{cases} 1 & \begin{array}{c}\text{if } i=p \text{ and } j=p+3 \text{ for } 1\le p\le n-3,\\ \text{if } i=p \text{ and } j=p+5 \text{ for } 1\le p\le n-5\\ \text{and}\\ i=p+1 \text{ and } j=p \text{ for } 1\le p\le n-1,\end{array}\\ -1 & \text{if } i=p \text{ and } \; j=p+2 \text{ for } 1\le p\le n-2,\\ 0 & \text{otherwise.}\end{cases}, \text{ for } n\ge 6$$

and

$$k_{i,j}^{(4)} = \begin{cases} 1 & \begin{array}{c}\text{if } i=p \text{ and } j=p+2 \text{ for } 1\le p\le n-2,\\ \text{if } i=p \text{ and } j=p+5 \text{ for } 1\le p\le n-5\\ \text{and}\\ i=p+1 \text{ and } j=p \text{ for } 1\le p\le n-1,\end{array}\\ -1 & \text{if } i=p \text{ and } \; j=p+3 \text{ for } 1\le p\le n-3,\\ 0 & \text{otherwise.}\end{cases}, \text{ for } n\ge 6.$$

Then we have the following Theorem.

Theorem 2.3. (i) *For* $k = 1,2,3$ *and* $n \geq k+3$,

$$perK^{(k)}(n) = a^{(k)}(n+k+3).$$

(ii) *For* $k=4$ *and* $n \geq 6$,

$$perK^{(4)}(n) = a^{(4)}(n+6).$$

Proof. (i) Let us consider $k=1$ and let the equation hold for $n \geqslant 4$. Then we show that the equation holds for $n+1$. If we expand the $perK^{(1)}(n)$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perK^{(1)}(n+1) = perK^{(1)}(n-1) - perK^{(1)}(n-2) + perK^{(1)}(n-3).$$

Since $perK^{(1)}(n-1)=a^{(1)}(n+3)$, $perK^{(1)}(n-2)=a^{(1)}(n+2)$ and $perK^{(1)}(n-3)=a^{(1)}(n+1)$, we easily obtain that $perK^{(1)}(n+1)=a^{(1)}(n+5)$. So the proof is complete.

The proofs for (ii), (iii) and (iv) are similar to the above and are omitted. □

Let $L^{(1)}(n)=\left[l_{i,j}^{(1)}\right]$, $L^{(2)}(n)=\left[l_{i,j}^{(2)}\right]$, $L^{(3)}(n)=\left[l_{i,j}^{(3)}\right]$ and $L^{(4)}(n)=\left[l_{i,j}^{(4)}\right]$ be the $n\times n$ matrices, defined by, respectively:

$$l_{i,j}^{(1)}=\begin{cases} 1 & \begin{array}{c}\text{if } i=p \text{ and } j=p+1 \text{ for } 1\le p\le n-2,\\ \text{if } i=p \text{ and } j=p+3 \text{ for } 1\le p\le n-3\\ \text{and}\\ i=p+1 \text{ and } j=p \text{ for } 1\le p\le n-1,\end{array}\\ -1 & \text{if } i=p \text{ and } j=p+2 \text{ for } 1\le p\le n-2,\\ 0 & \text{otherwise.}\end{cases}, \text{ for } n\ge 4,$$

$$l_{i,j}^{(2)}=\begin{cases} 1 & \begin{array}{c}\text{if } i=p \text{ and } j=p+3 \text{ for } 1\le p\le n-4,\\ \text{if } i=p \text{ and } j=p+4 \text{ for } 1\le p\le n-4\\ \text{and}\\ i=p+1 \text{ and } j=p \text{ for } 1\le p\le n-1,\end{array}\\ -1 & \text{if } i=p \text{ and } j=p+2 \text{ for } 1\le p\le n-4,\\ 0 & \text{otherwise.}\end{cases}, \text{ for } n\ge 5,$$

$$l_{i,j}^{(3)}=\begin{cases} 1 & \begin{array}{c}\text{if } i=p \text{ and } j=p+3 \text{ for } 1\le p\le n-5,\\ \text{if } i=p \text{ and } j=p+5 \text{ for } 1\le p\le n-5\\ \text{and}\\ i=p+1 \text{ and } j=p \text{ for } 1\le p\le n-1,\end{array}\\ -1 & \text{if } i=p \text{ and } j=p+2 \text{ for } 1\le p\le n-5,\\ 0 & \text{otherwise.}\end{cases}, \text{ for } n\ge 6$$

and

$$l_{i,j}^{(4)}=\begin{cases} 1 & \begin{array}{c}\text{if } i=p \text{ and } j=p+2 \text{ for } 1\le p\le n-5,\\ \text{if } i=p \text{ and } j=p+5 \text{ for } 1\le p\le n-5\\ \text{and}\\ i=p+1 \text{ and } j=p \text{ for } 1\le p\le n-1,\end{array}\\ -1 & \text{if } i=p \text{ and } j=p+3 \text{ for } 1\le p\le n-5,\\ 0 & \text{otherwise.}\end{cases}, \text{ for } n\ge 6.$$

Assume that the $n\times n$ matrices $M^{(k)}(n)=\left[m_{i,j}^{(k)}\right]$ for $k=1,2,3$ and $M^{(4)}(n)=\left[m_{i,j}^{(4)}\right]$ is defined by, respectively:

$$M^{(k)}(n)=\begin{bmatrix} 1 & \cdots & \overset{\overset{(n-k-3)\text{th}}{\downarrow}}{1} & 0 & 0\\ 1 & & & & \\ 0 & & & & \\ 0 & & & L^{(k)}(n-1) & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}, \text{ for } n>k+3$$

and

$$M^{(4)}(n)=\begin{bmatrix} 1 & \cdots & \overset{\overset{(n-6)\,\text{th}}{\downarrow}}{1} & 0 & 0 \\ 1 & & & & \\ 0 & & & & \\ 0 & & L^{(4)}(n-1) & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}, \text{ for } n>6.$$

Then we can give more general results by using other the permanental representations than the above.

Theorem 2.4. *Let $a^{(k)}(n)$ be the nth the Padovan-circulant-Hurwitz sequences of the first, second, third and fourth kind for $k=1,2,3,4$. Then*

(i). For $n \geq k+3$,

$$perL^{(k)}(n)=a^{(k)}(n).$$

(ii). For $n>k+3$,

$$perM^{(k)}(n)=\sum_{i=1}^{m-1} a^{(k)}(i).$$

Proof. (i). Let us consider the matrix $L^{(2)}(n)$ and let the equation hold for $n \geq 5$. Then we show that the equation holds for $n+1$. If we expand $perL^{(2)}(n)$ by the Laplace expansion of permanent according to the first row, then we obtain

$$perL^{(2)}(n+1)=-perL^{(2)}(n-2)+perL^{(2)}(n-3)+perL^{(2)}(n-4).$$

Also, since $perL^{(2)}(n-2)=a^{(2)}(n-2)$, $perL^{(2)}(n-3)=a^{(2)}(n-3)$ and $perL^{(2)}(n-4)=a^{(2)}(n-4)$, it is clear that $perL^{(2)}(n+1)=a^{(2)}(n+1)$.

The proofs for the matrices $L^{(1)}(n)$, $L^{(3)}(n)$ and $L^{(4)}(n)$ are similar.

(ii). If we extend $perM^{(k)}(n)$ with respect to the first row, we write

$$perM^{(k)}(n)=perM^{(k)}(n-1)+perL^{(k)}(n-1)$$

for $k=1,2,3,4$. By induction on n, taking into consideration the results of Theorem 2.3 and part (i) in Theorem 2.4, the conclusion is easily seen. □

Let the notation $A \circ K$ denotes the Hadamard product of A and K. A matrix A is called convertible if there is an $m \times m$ (1, -1)-matrix K such that per $A = \det(A \circ K)$.

Let $n>k+3$ and let R be the $n \times n$ matrix, defined by

$$R=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

It is easy to see that $perK^{(k)}(n)=\det\left(K^{(k)}(n)\circ R\right)$, $perL^{(k)}(n)=\det\left(L^{(k)}(n)\circ R\right)$ and $perM^{(k)}(n)=\det\left(M^{(k)}(n)\circ R\right)$ for $n>k+3$. Then we have the following useful results.

Corollary 2.5. *(i)*

$$\det\left(K^{(k)}(n)\circ R\right)=a^{(k)}(n+k+3)\,,\ \textit{for } k=1,2,3 \textit{ and } n>k+3$$

and

$$\det\left(K^{(4)}(n)\circ R\right)=a^{(4)}(n+6)\,,\ \textit{for } k=4 \textit{ and } n>6.$$

(ii) For $k=1,2,3,4$ and $n>k+3$,

$$\det\left(L^{(k)}(n)\circ R\right)=a^{(k)}(n)$$

and

$$\det\left(M^{(k)}(n)\circ R\right)=\sum_{i=1}^{m-1}a^{(k)}(i)\,.$$

Let $C(c_1,c_2,\dots,c_v)$ be a $v\times v$ companion matrix as follows:

$$C(c_1,c_2,\dots,c_v)=\begin{bmatrix} c_1 & c_2 & \cdots & c_v\\ 1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 1 & 0\end{bmatrix}.$$

See [17, 18] for more information about the companion matrix.

Theorem 2.6. *(Chen and Louck [3]). The (i,j) entry $c_{i,j}^{(n)}(c_1,c_2,\dots,c_v)$ in the matrix $C^n(c_1,c_2,\dots,c_v)$ is given by the following formula:*

$$c_{i,j}^{(n)}(c_1,c_2,\dots,c_v)=\sum_{(t_1,t_2,\dots,t_v)}\frac{t_j+t_{j+1}+\cdots+t_v}{t_1+t_2+\cdots+t_v}\times\binom{t_1+\cdots+t_v}{t_1,\dots,t_v}c_1^{t_1}\cdots c_v^{t_v}\quad(5)$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+vt_v=n-i+j$, $\binom{t_1+\cdots+t_v}{t_1,\dots,t_v}=\frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (5) are defined to be 1 if $n=i-j$.

Then we can give other combinatorial representations than for the Padovan-circulant Hurwitz sequences of the first, second, third and fourth kind by the following Corollary.

Corollary 2.7. *Let $a^{(k)}(\alpha)$ be the αth the Padovan-circulant Hurwitz sequences of the first, second, third and fourth kind.*

(i) For $k=1,2$,

$$a^{(k)}(\alpha)=\sum_{(t_1,t_2\dots,t_{k+3})}\frac{t_{k+3}}{t_1+t_2+\cdots+t_{k+3}}\binom{t_1+\cdots+t_{k+3}}{t_1,\dots,t_{k+3}}(-1)^{t_3}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(k+3)\,t_{k+3}=\alpha$.

(ii) For $k=3,4$,

$$\begin{aligned}a^{(k)}(\alpha)&=\sum_{(t_1,t_2\dots,t_6)}\frac{t_6}{t_1+t_2+\cdots+t_6}\binom{t_1+\cdots+t_6}{t_1,\dots,t_6}(-1)^{t_k}\\&=\sum_{(t_1,t_2\dots,t_6)}\frac{t_5+t_6}{t_1+t_2+\cdots+t_6}\binom{t_1+\cdots+t_6}{t_1,\dots,t_6}(-1)^{t_k}\end{aligned}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + 6t_6 = \alpha$.

Proof. (i) Let us consider $k = 1$. In Theorem 2.6, if we take $i = j = 4$, $c_1 = 0$, $c_2 = 1$, $c_3 = -1$ and $c_4 = 1$, then the proof is immediately seen from $\left(PH^{(1)}\right)^{\alpha}$.

(ii) The proof is similar to the above and is omitted. □

Now we consider the sums of Padovan-circulant Hurwitz numbers of first, second, third and fourth kind.

Let

$$S_\alpha = \sum_{i=1}^{\alpha} a^{(k)}(i)$$

for $i \geq 1$ and $k = 1, 2, 3$. Suppose that $\left(T^{(k)}\right)^{\alpha}$ are the $(k+3) \times (k+3)$ matrices such that

$$T^{(k)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & PH^{(k)} & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

Then it can be shown by induction that:

$$\left(T^{(k)}\right)^{\alpha} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ S_{\alpha+k+2} & & & & & \\ S_{\alpha+k+1} & & & & & \\ \vdots & & & \left(PH^{(k)}\right)^{\alpha} & & \\ S_{\alpha+1} & & & & & \\ S_{\alpha} & & & & & \end{bmatrix}.$$

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