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A Fisher-Type Evolution Equation with Applications in Image Processing

Bogdan Maxim

Abstract. The aim of this paper is to present some applications of the Neumann Laplacian in image processing, along with the necessary mathematical background. We prove weak and strong versions of the maximum principle for weak solutions of elliptic and parabolic problems and apply them to a Fisher K.P.P.-type equation. The original contribution lies in the application of this equation in image processing, where various diffusion-like effects can be achieved. Additionally, a review of the basics of linear and nonlinear PDEs with Neumann boundary conditions is provided, along with updated bibliography and recent qualitative results. There are also some new theoretical results developed in this work.

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1. Introduction and notations

The Fisher equation is primarily used in population dynamics models. However, due to its rich properties, such as global boundedness, it is also well-suited for applications in image processing.

In this research article, we will conduct a thorough study of the following semilinear evolution problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u = \alpha u \cdot \big(r(x) - p(x)u\big), & (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & (t,x) \in (0,T) \times \partial\Omega \\ u(0,x) = u_0(x), & x \in \Omega \end{cases} \tag{1}$$

The problem is set up in a domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Of particular interest will be the case when $N = 2$, especially considering its relevance to applications in image processing.

Simulating PDEs on images can provide valuable insights into how differential operators behave and their specific characteristics.

Engineers commonly refer to Neumann boundary conditions as *natural conditions*, whereas in image processing, they are termed *reflective conditions* and are suitable for applying PDEs to images. These conditions describe an isolated domain where there

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is no flux through the boundary. They are notable because, in formulating weak versions of PDEs, the space of test functions is not constrained by the boundary conditions, unlike in the case of Dirichlet conditions.

This section begins by proving weak versions of the maximum principle for elliptic and parabolic PDEs, which will serve as the main results applied in this work. Complete proofs are provided since they may not be readily available in basic monographs on this subject.

In Section 2, we address linear heat equations with Neumann boundary conditions and lay the groundwork necessary for Section 3, where we study a semilinear parabolic equation with a logistic term. We emphasize the importance of strong forms of the maximum principle, enabling analysis of our solutions in terms of the problem's parameters. Here, we utilize an elegant functional method involving the notion of almost interior points, introduced by J. Glück and M. Weber in [13].

Sections 4 and 5 are devoted to studying the steady-states of the parabolic problem. Subsequently, in Section 6, we present the main result of the paper: the uniform asymptotic stability of the only nontrivial steady state of our main problem.

Finally, we discuss how the gathered results can be applied to modify images and develop an algorithm implemented using Matlab R2023b, capable of transforming one picture into another using diffusion. Additionally, we explore the application of diffusion for two more purposes: causing an image to dissapear or gradually transforming it into a mathematical solution of (1).

There are some new results like Theorem 2.1 **(5)** and Lemma 3.5 **(2),(4),(5),(6),(7)** and many non-obvious details and tricks that fill the existent gaps in the proofs of some well-known results. The hypotheses in which we work are very general: we use Lipschitz domains and weak solutions for PDEs.

Concluding remarks are provided, along with suggestions for further research and improvement.

In the Appendix, we compile some non-standard definitions and results, including proofs for some of them.

Throughout the paper, we will use the notations: $L^p(\Omega)^+ = \{f \in L^p(\Omega) \mid f(x) \geq 0, \text{ for a.a. } x \in \Omega\}$ where $p \in [1, \infty]$ and $H^1(\Omega)^+ = \{f \in H^1(\Omega) \mid f(x) \geq 0, \text{ for a.a. } x \in \Omega\}$.

1.1. The weak minimum principle for elliptic and parabolic problems.

Theorem 1.1 (**Weak Minimum Principle for elliptic problems**). *Fix some open and bounded set $\Omega \subset \mathbb{R}^N$ and let $c \in L^p(\Omega)$ for some p satisfying* $\begin{cases} p > 1 \text{ if } N=2 \\ p \geq \dfrac{N}{2} \text{ if } N > 2 \end{cases}$, *with $c \geq 0$ a.e. on Ω. If $u \in H^1(\Omega)$ satisfies in the **weak** sense the following inequalities:*

$$\begin{cases} -\Delta u + c(x)u \geq 0, & x \in \Omega \\ \dfrac{\partial u}{\partial \nu}(x) \geq 0, & x \in \partial\Omega \end{cases} \quad i.e. \int_\Omega \nabla u(x) \cdot \nabla \phi(x) + c(x)u(x)\phi(x)\, dx \geq 0 \quad (2)$$

for all test functions $\phi \in H^1(\Omega)$ *with* $\phi \geq 0$ *a.e. on* Ω[1] *then exactly one of the following two situations takes place:*
(1) $u \geq 0$ *a.e. on* Ω*;*
(2) u *is a constant strict negative function and* $c \equiv 0$.

Proof. First, let's explain why the integral inequality from the statement is well-defined. Since $\nabla u, \nabla\varphi \in L^2(\Omega;\mathbb{R}^N)$, from *Cauchy inequality* we deduce that $\nabla u\cdot\nabla\varphi \in L^1(\Omega)$. If $N = 2$ from *Sobolev embedding theorem*[2] $u, \phi \in H^1(\Omega) \hookrightarrow L^r(\Omega)$ for each $r \geq 1$. Setting $r = 2p' = \frac{2p}{p-1} > 2$ we obtain first from *Cauchy inequality* that $u\phi \in L^{p'}(\Omega)$ and then from *Hölder inequality* that $cu\phi \in L^1(\Omega)$. If $N > 2$ from *Sobolev embedding theorem* we have that $u, \phi \in H^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$. So $u\phi \in L^{\frac{N}{N-2}}(\Omega)$ from *Cauchy inequality*. Now, since $p \geq \frac{N}{2} \Rightarrow p' \leq \frac{N}{N-2}$. Being in a bounded domain we deduce from *Hölder inequality* that $u\phi \in L^{\frac{N}{N-2}}(\Omega) \subset L^{p'}(\Omega)$ (see [17, page 240]). Applying one more time *Hölder inequality* we will get that $cu\phi \in L^1(\Omega)$.
Now we can start the proof. Write $u = u^+ - u^-$, where $u^+ = \max\{u, 0\} \geq 0$ and $u^- = -\min\{u, 0\} \geq 0$ a.e. on Ω. From [18, Theorem 2.2] we know that $u^+, u^- \in H^1(\Omega)^+$ and moreover $\nabla u^+ = \begin{cases} \nabla u, & \text{a.e. on } \{x \in \Omega \mid u(x) > 0\} \\ 0, & \text{a.e. on } \{x \in \Omega \mid u(x) \leq 0\} \end{cases}$, $\nabla u^- = \begin{cases} \nabla u, & \text{a.e. on } \{x \in \Omega \mid u(x) < 0\} \\ 0, & \text{a.e. on } \{x \in \Omega \mid u(x) \geq 0\} \end{cases}$. So $u^+u^- = 0$ and $\nabla u^+ \cdot \nabla u^- = 0$ a.e. on Ω.
Choose $\phi = u^- \in H^1(\Omega)^+$ as test function. Then (2) becomes:

$$\int_\Omega (\nabla u^+ - \nabla u^-)\cdot\nabla u^- \, dx + \int_\Omega c(u^+ - u^-)\cdot u^- \, dx \geq 0$$
$$\Rightarrow \int_\Omega \nabla u^+\cdot\nabla u^- - |\nabla u^-|^2 \, dx + \int_\Omega c(x)u^+u^- - c(u^-)^2 \, dx \geq 0$$
$$\Rightarrow \int_\Omega |\nabla u^-|^2 \, dx + \int_\Omega c(u^-)^2 \leq 0.$$

Knowing that $c \geq 0$ a.e. on Ω we get that $\nabla u^- = 0$ a.e. on Ω. Taking into account that Ω is a connected domain we deduce that there is a constant $k \in \mathbb{R}$ with $u^- = k$ a.e. on Ω. If $k \leq 0$ then $0 \leq u^- = k \leq 0 \Rightarrow u^- = 0$ and $u = u^+ \geq 0$. If $k > 0$ then $k = u^- = -\min\{u, 0\} = -u \Rightarrow u = -k < 0$ and $\int_\Omega c\,dx = 0$. Therefore $c = 0$ a.e. on Ω as claimed. □

Theorem 1.2 (**Weak Minimum Principle for parabolic problems**). *Let any* $d, T > 0$ *and* $c \in L^\infty((0,T)\times\Omega)$*. If* $u \in C([0,T]; L^2(\Omega)) \cap H^1_{\mathrm{loc}}((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$ *satisfies in the* ***weak*** *sense the inequalities:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u + c(t,x)u \geq 0, & (t,x) \in (0,T)\times\Omega \\ \dfrac{\partial u}{\partial \nu} \geq 0, & (t,x) \in (0,T)\times\partial\Omega \end{cases} \quad \textit{i.e.} \tag{3}$$

[1] This definition is adapted from [33, pages 212-213].
[2] A proof can be found in [1, Theorem 4.51 and Theorem 6.3] or in [22, Theorem 12.17 and Theorem 12.18].

$$\int_{\Omega} \frac{\partial u}{\partial t}(t,\cdot)\phi \, dx + d\int_{\Omega} \nabla u(t,\cdot)\cdot\nabla\phi \, dx \; + \int_{\Omega} c(t,\cdot)u(t,\cdot)\phi \, dx \geq 0 \tag{4}$$

for a.a. $t \in (0,T)$*, for all* $\phi \in H^1(\Omega)$ *with* $\phi \geq 0$ *a.e. on* Ω *and:* $u(0,x) \geq 0$ *for a.a.* $x \in \Omega$ *then* $u(t,x) \geq 0$ *for a.a.* $(t,x) \in (0,T)\times\Omega$*.*

Proof. First consider, if needed, some real constant $M \geq 0$ with $\operatorname*{ess\,inf}_{(0,T)\times\Omega} c + M \geq 0$ and let $v = e^{-Mt}\cdot u$. Then, since $u(t,\cdot) \in H^1(\Omega)$ and $t \mapsto e^{-Mt}$ is a smooth mapping we deduce that $v(t,\cdot) \in H^1(\Omega) \subset L^2(\Omega)$, $\forall\; t \in [0,T]$, from [18, Lemma 1.12 (5), page 9]. Observe that for any $t, t+h \in [0,T]$ we have that $\|v(t+h,\cdot) - v(t,\cdot)\|_{L^2(\Omega)} \leq e^{-Mt}\|u(t+h,\cdot)e^{-Mh} - u(t,\cdot)\|_{L^2(\Omega)} \leq \|u(t+h,\cdot)e^{-Mh} - u(t,\cdot)\|_{L^2(\Omega)} \leq \|u(t+h,\cdot) - u(t,\cdot)\|_{L^2(\Omega)} + |1-e^{-Mh}|\cdot \underbrace{\|u(t+h,\cdot)\|_{L^2(\Omega)}}_{\leq \|u\|_{C((0,T);L^2(\Omega))}} \overset{h\to 0}{\to} 0$, which shows that v is continuous from $[0,T]$ to $L^2(\Omega)$. Since $\|v(t,\cdot)\|_{L^2(\Omega)} = e^{-Mt}\|u(t,\cdot)\|_{L^2(\Omega)} \leq \|u(t,\cdot)\|_{L^2(\Omega)}$ for any $t \in [0,T]$ we deduce that $v \in C([0,T], L^2(\Omega))$. In the same manner $\int_0^T \|v(t,\cdot)\|^2_{H^1(\Omega)} \leq \int_0^T \|u(t,\cdot)\|^2_{H^1(\Omega)} = \|u\|_{L^2((0,T);H^1(\Omega))} < \infty$, so $v \in L^2((0,T);H^1(\Omega))$. Because $u \in H^1([a,b];L^2(\Omega))$ for any $[a,b] \subset (0,T)$, we have in particular that $\frac{\partial u}{\partial t} \in L^2((a,b);L^2(\Omega))$ which means by definition that the two Bochner integrals verify: $\int_a^b u(t,\cdot)\psi'(t)\,dt = -\int_a^b \frac{\partial u}{\partial t}(t,\cdot)\psi(t)\,dt$, $\forall\;\psi \in C_c^\infty([a,b])$. By checking the same definition of the weak derivative we obtain immediately that $v \in H^1([a,b];L^2(\Omega))$ and $\frac{\partial v}{\partial t} = e^{-Mt}\left(-Mu + \frac{\partial u}{\partial t}\right)$. From the *AM-GM* inequality one gets:

$$\begin{aligned}\|v\|_{H^1([a,b];L^2(\Omega))} &= \left(\int_a^b \|v(t,\cdot)\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial t}(t,\cdot)\right\|^2_{L^2(\Omega)}\right)^{1/2} \\ &\leq \sqrt{2M^2+2}\cdot\|u\|_{H^1([a,b];L^2(\Omega))} < \infty.\end{aligned}$$

This proves that $v \in H^1_{\text{loc}}((0,T);L^2(\Omega))$. Notice that v satifies in the weak sense the inequalities:

$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta v + (c(t,x)+M)v \geq 0, & (t,x) \in (0,T)\times\Omega \\ \dfrac{\partial v}{\partial \nu} \geq 0, & (t,x) \in (0,T)\times\partial\Omega \end{cases} \tag{5}$$

and $v(0,x) = u(0,x) \geq 0$ a.e. on Ω. If we show that $v(t,x) = \underbrace{e^{-Mt}}_{>0} u(t,x) \geq 0$ a.e. on $(0,T)\times\Omega$ then we are done.

For an arbitrary $t \in (0,T)$ we know that $v(t,\cdot) \in H^1(\Omega)$ and we may write $v(t,\cdot) = v^+(t,\cdot) - v^-(t,\cdot)$, where $v^+(t,\cdot) = \max\{v(t,\cdot),0\}$ and $v^-(t,\cdot) = -\min\{v(t,\cdot),0\}$ lie both in $H^1(\Omega)$ and are a.e. positive on Ω. By choosing $\phi = v^-(t,\cdot) \in H^1(\Omega)$ as test function we obtain that:

$$\int_\Omega \frac{\partial v}{\partial t} v^-(t,\cdot)\,dx + d\int_\Omega \nabla v(t,\cdot)\cdot\nabla v^-(t,\cdot)\,dx + \int_\Omega (c(t,\cdot)+M)v(t,\cdot)v^-(t,\cdot)\,dx \ge 0$$

$$\Rightarrow -\int_\Omega \frac{\partial v}{\partial t} v^-(t,\cdot)\,dx + d\int_\Omega |\nabla v^-(t,\cdot)|^2\,dx + \int_\Omega \underbrace{(c(t,\cdot)+M)}_{\ge 0}(v^-)^2\,dx \le 0,$$

where we have used that $\nabla v^+(t,\cdot)\cdot\nabla v^-(t,\cdot)=0$ and $v^+(t,\cdot)v^-(t,\cdot)=0$ a.e. on Ω. We deduce that $\int_\Omega -\frac{\partial v}{\partial t}(t,\cdot)v^-(t,\cdot)\,dx \le 0$ for a.a. $t\in(0,T)$.

We know that for any $0<a<b<T$, $v\in H^1([a,b];L^2(\Omega))$. So, from [19, Corollary 5.18] we have that $v^+,\ v^-\in H^1([a,b],L^2(\Omega))$ and moreover:

$$\frac{\partial v^+}{\partial t}(t,\cdot) = \begin{cases} \frac{\partial v}{\partial t}(t,\cdot), \text{ a.e. on } \{x\in\Omega \mid v(t,x)>0\} \\ 0, \text{ a.e. on } \{x\in\Omega \mid v(t,x)\le 0\} \end{cases}$$

$$\frac{\partial v^-}{\partial t}(t,\cdot) = \begin{cases} -\frac{\partial v}{\partial t}(t,\cdot), \text{ a.e. on } \{x\in\Omega \mid v(t,x)<0\} \\ 0, \text{ a.e. on } \{x\in\Omega \mid v(t,x)\ge 0\} \end{cases}.$$

Thenceforth: $0 \ge \int_\Omega -\frac{\partial v}{\partial t}(t,\cdot)v^-(t,\cdot)\,dx = \int_\Omega -\frac{\partial v^+}{\partial t}(t,\cdot)v^-(t,\cdot) + \frac{\partial v^-}{\partial t}(t,\cdot)v^-(t,\cdot)\,dx$ $= \int_\Omega \frac{\partial v^-}{\partial t}(t,\cdot)v^-(t,\cdot)\,dx$ for a.a. $t\in(0,T)$. As $v^-\in H^1([a,b];L^2(\Omega))$ we infer that

$$\infty > \|v^-\|^2_{H^1([a,b];L^2(\Omega))} = \int_a^b\int_\Omega (v^-)^2 + \left(\frac{\partial v^-}{\partial t}\right)^2 dx\,dt \overset{\text{AM-GM}}{\ge} 2\int_a^b\int_\Omega \left|\frac{\partial v^-}{\partial t}v^-\right| dx\,dt.$$

Now from Lemma 10.1 we deduce that $\int_a^b \frac{\partial v^-}{\partial t}\cdot v^-\,dt = \frac{1}{2}\big(v^-(b)^2 - v^-(a)^2\big)$. Applying *Fubini-Tonelli theorem*[3] we finally reach to:

$$0 \ge \int_a^b\int_\Omega \frac{\partial v^-}{\partial t}\cdot v^-\,dx\,dt = \int_\Omega\int_a^b \frac{\partial v^-}{\partial t}\cdot v^-\,dt\,dx = \frac{1}{2}\int_\Omega v^-(b,x)^2 - v^-(a,x)^2\,dx. \quad (6)$$

Define the function $h:[0,T]\to\mathbb{R}$ by $h(t)=\int_\Omega v^-(t,x)^2\,dx$, $\forall t\in[0,T]$. Thus for any $0<a<b<T$ we have that $h(a)\ge h(b)$, i.e. h is decreasing on $(0,T)$. Note that, because $v(0,\cdot)\ge 0$ a.e. on Ω, we get that $v^-(0,\cdot)=0$ a.e. on Ω and therefore $h(0)=0$. Now it's time to use the fact that $v\in C([0,T];L^2(\Omega))$. It is easy to check that $v^+,v^-\in C([0,T];L^2(\Omega))$[4]. In particular h is continuous at $t_0=0$. This is because $\lim_{t\to 0^+} v^-(t,\cdot)=v^-(0,\cdot)=0$ in $L^2(\Omega)$, which means that $\lim_{t\to 0^+} h(t) = \lim_{t\to 0^+}\int_\Omega v^-(t,x)^2\,dx = 0 = h(0)$. Let any $t\in(0,T)$ and consider a sequence $(t_n)_{n\ge 1}\subset(0,T]$ that converges to 0. There is some index $n_t\ge 1$ such that

[3] See [10, Theorem 4.4 and 4.5, page 91].

[4] Just observe that $\|v(t+h,\cdot)-v(t,\cdot)\|^2_{L^2(\Omega)} = \|v^+(t+h,\cdot)-v^+(t,\cdot)\|^2_{L^2(\Omega)} + \|v^-(t+h,\cdot)-v^-(t,\cdot)\|^2_{L^2(\Omega)} + 2\int_\Omega \underbrace{v^+(t+h,\cdot)v^-(t,\cdot)+v^-(t+h,\cdot)v^+(t,\cdot)}_{\ge 0}\,dx$ and $\|v^+(t,\cdot)\|_{L^2(\Omega)}, \|v^-(t,\cdot)\|_{L^2(\Omega)} \le \|v(t,\cdot)\|_{L^2(\Omega)}$, for all $t\in[0,T]$.

$t > t_n$ for each $n \geq n_t$. Therefore $0 \leq h(t) \leq h(t_n)$, $\forall\ n \geq n_t$. Passing to the limit gives us that $0 \leq h(t) \leq \lim_{n\to\infty} h(t_n) = h(0) = 0 \Rightarrow h(t) = 0$. In conclusion $v^-(t,x) = 0$ for a.a. $t \in (0,T)$ and for a.a. $x \in \Omega$. The proof is complete. □

2. Heat equation with Neumann boundary conditions - a review

Let $d, \lambda > 0$ and $f \in L^2((0,T), L^2(\Omega))$, $v_0 \in L^2(\Omega)$ be fixed. Consider the following problem:

$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta v + \lambda v = f(t,x), & (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial v}{\partial \nu} = 0, & (t,x) \in (0,T) \times \partial\Omega \\ v(0,x) = v_0(x), & x \in \Omega \end{cases} \tag{7}$$

Definition 2.1. We say that $v \in C([0,T]; L^2(\Omega)) \cap H^1_{\text{loc}}((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$ is a **weak solution** of (7) if for almost all $t \in (0,T)$ and each $\phi \in H^1(\Omega)$ the following equality holds:

$$\int_\Omega \frac{\partial v}{\partial t}(t,\cdot)\phi\ dx + d\int_\Omega \nabla v(t,\cdot)\cdot\nabla\phi\ dx + \lambda\int_\Omega v(t,\cdot)\phi\ dx = \int_\Omega f\phi\ dx. \tag{8}$$

Let $\big(S_\lambda(t)\big)_{t\geq 0}$ be the λ–translated C_0-semigroup associated to the Neumann Laplacian on the Lipschitz domain Ω, i.e. $S_\lambda(t) : L^2(\Omega) \to L^2(\Omega)$ with $S_\lambda(t)u = v$, where:

$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta v + \lambda v = 0, & (t,x) \in (0,\infty) \times \Omega \\ \dfrac{\partial v}{\partial \nu} = 0, & (t,x) \in (0,\infty) \times \partial\Omega \\ v(0,x) = u(x), & x \in \Omega \end{cases} \text{ in the weak sense.} \tag{9}$$

We will write $S(t) = S_0(t)$ for any $t \geq 0$. Notice that $S_\lambda(t) = e^{-\lambda t}S(t)$, $\forall\ t \geq 0$. The following important theorem takes place:

Theorem 2.1. *For any $T > 0$ problem (7) has a unique weak solution that has the following properties:*

(1) *v is also a **mild solution**[5] of (7), i.e. $v \in C\big([0,T]; L^2(\Omega)\big)$ and $v(t,\cdot) = S_\lambda(t)v_0 + \int_0^t S_\lambda(t-s)f(s,\cdot)\ ds = e^{-\lambda t}S(t)v_0 + \int_0^t e^{-\lambda(t-s)}S(t-s)f(s,\cdot)\ ds$ for any $t \in [0,T]$.*

(2) *If $v_0 \in L^\infty(\Omega)$ and $f \in L^\infty\big((0,T)\times\Omega\big)$ then $v \in L^\infty\big((0,T)\times\Omega\big)$ and:*

$$\min\left\{\operatorname*{ess\,inf}_\Omega v_0, \frac{1}{\lambda}\operatorname*{ess\,inf}_{(0,T)\times\Omega} f\right\} \leq v(t,x) \leq \max\left\{\operatorname*{ess\,sup}_\Omega v_0, \frac{1}{\lambda}\operatorname*{ess\,sup}_{(0,T)\times\Omega} f\right\},\ \textit{a.e. on } (0,T)\times\Omega.$$

In particular $\|v\|_{L^\infty((0,T)\times\Omega)} \leq \max\left\{\|v_0\|_{L^\infty},\ \frac{1}{\lambda}\|f\|_{L^\infty((0,T)\times\Omega)}\right\}.$

[5] See [27, Definition 2.3, page 106].

(3) *If* $f \equiv 0$, *for* $D = \begin{cases} N, \text{ if } N \geq 3 \\ \text{any constant strictly bigger than } 2, \text{ if } N \in \{1,2\} \end{cases}$ *then there is a constant* $C > 0$ *depending on* Ω *and* D, d, λ *such that* $\|v(t,\cdot)\|_{L^\infty(\Omega)} \leq C \cdot e^{\lambda T} \cdot t^{-D/4} \cdot \|v_0\|_{L^2(\Omega)}, \ \forall\, t \in (0,T]$.

(4) *If* $v_0 \in L^2(\Omega)^+ \setminus \{0\}$ *and* $\operatorname*{ess\,inf}_{(0,T)\times\Omega} f \geq 0$ *then* $\operatorname*{ess\,inf}_{\Omega} v(t,\cdot) > 0$ *for any* $t \in (0,T]$.

(5) *If* $v_0 \in L^2(\Omega)^+ \setminus \{0\}$ *and* $\operatorname*{ess\,inf}_{(0,T)\times\Omega} f \geq 0$ *then for any* $t_0 \in (0,T)$ *we have that* $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} v > 0$.[6]

(6) *If* $v_0 \in L^2(\Omega)^+ \setminus \{0\}$ *has the property that* $\operatorname*{ess\,inf}_{\Omega} v_0 > 0$ *and* $\operatorname*{ess\,inf}_{(0,T)\times\Omega} f \geq 0$ *then we have that* $\operatorname*{ess\,inf}_{(0,T)\times\Omega} v > 0$.

Proof. (Sketch) The existence follows via *Galerkin Method* (see [9, Theorem 4.2.1, page 190] and [4, Lemma A.2.7, page 187]) or using *variational methods* (as a reference see [23, Theorem 68, page 67]). Moreover we have that for any $t \in [0,T]$ the following formula holds:

$$v(t,\cdot) = \sum_{n=1}^{\infty} v_n(t) \cdot \varphi_n, \text{ where } v_n(t)$$
$$= e^{-(d\lambda_n+\lambda)t}(v_0,\varphi_n)_{L^2(\Omega)} + \int_0^t e^{-(d\lambda_n+\lambda)(t-s)}(f(s,\cdot),\varphi_n)_{L^2(\Omega)}\, ds, \tag{10}$$

for any $n \geq 1$, where the set $\{\varphi_n \mid n \geq 1\}$ is an orthonormal basis of the Hilbert space $L^2(\Omega)$ formed by the eigenfunctions of the Neumann Laplacian, i.e.

$$\begin{cases} -\Delta\varphi_n = \lambda_n\varphi_n, & x \in \Omega \\ \dfrac{\partial\varphi_n}{\partial\nu} = 0, & x \in \partial\Omega \end{cases},$$

$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$ being the Neumann eigenvalues on Ω counted with their multiplicities.[7]

The uniqueness also follows with ease: let v_1, v_2 be two weak solutions of (7) and denote $v = v_1 - v_2$. For a.a. $s \in [0,T]$ we can choose as a test function $\phi = v(s,\cdot) \in H^1(\Omega)$ and obtain that:

$$\frac{d}{ds}\int_\Omega v^2(s,\cdot)\, dx + d\int_\Omega |\nabla v(s,\cdot)|^2\, dx + \lambda\int_\Omega v^2(s,\cdot)\, dx = 0, \text{ for a.a. } s \in [0,T].$$

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[6]This is the best we can hope since $\operatorname*{ess\,inf}_{(0,T)\times\Omega} v > 0$ is not true in general even for domains with smooth boundary, unless $\operatorname*{ess\,inf}_{\Omega} v_0 > 0$.

[7]Note that if $\Omega = (0,a)\times(0,b)$ is a 2D-rectangle as it is image processing it is known that the spectrum of the Neumann Laplacian is given by $\lambda_0 = 1$ and for each pair $(m,n) \in \mathbb{Z}_+^* \times Z_+^*$ we have that $\lambda_{m,n} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$. The corresponding eigenfunctions are $\phi_0(x,y) = \frac{1}{\sqrt{ab}}$ and $\phi_{m,n}(x,y) = \frac{2}{\sqrt{ab}}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$ for each $(x,y) \in \Omega$. For further details see [14].

Integrating on $[0,t]$ will result in:

$$\int_\Omega v^2(t,\cdot)\ dx + d\int_0^t\int_\Omega |\nabla v(s,\cdot)|^2\ dx + \lambda\int_0^t\int_\Omega v^2(s,\cdot)\ dx = \int_\Omega v^2(0,\cdot)\ dx = 0,$$

for a.a. $t \in [0,T]$. So for a.a. $t \in [0,T]$ we deduce that $v(t,\cdot) = 0$ a.e. on Ω, which is the same as saying that $v \equiv 0$ on $(0,T)\times\Omega$, i.e. $v_1 \equiv v_2$. **(1)** Just use (10) combined with the following formula:[8]

$$S(t)w = \sum_{n=1}^{\infty} e^{-d\lambda_n t}\cdot(w,\varphi_n)_{L^2(\Omega)}\varphi_n,\ \forall\ w\in L^2(\Omega). \tag{11}$$

(2) It follows immediately from the *weak parabolic minimum principle.* **(3)** See the proof of Lemma 10.2 from the Appendix. Note that $v(t,\cdot) = S_\lambda(t)$ for any $t\in[0,T]$. **(4)** Because $f \geq 0$ a.e. on $(0,T)\times\Omega$ we get from the *weak parabolic minimum principle* that $v \geq \tilde{v} := S_\lambda(\cdot)v_0$ a.e. on $(0,T)\times\Omega$. For any fixed $t\in(0,T]$ the bounded linear operator $S_\lambda(t) : L^2(\Omega)\to L^\infty(\Omega)$ (here we have used the ultracontractivity property of the Neumann Laplacian, see Lemma 10.2 from the Appendix) is a positive operator, meaning that for any $w\in L^2(\Omega)^+$ we have that $S_\lambda(t)w \in L^\infty(\Omega)^+$ (it follows from the *weak parabolic minimum principle*). Moreover the constant function $1_\Omega \in L^\infty(\Omega)$ which associates 1 to any $x\in\Omega$ has the property that $S_\lambda(t)1_\Omega = e^{-\lambda t} \Rightarrow \operatorname*{ess\,inf}_\Omega S_\lambda(t)1_\Omega > 0$.[9] Hence $1_\Omega \in L^2(\Omega)^+$ is mapped by $S_\lambda(t)$ into an almost interior point[10] of $L^\infty(\Omega)^+$ and it is an almost interior point of both $L^2(\Omega)^+$ and $L^\infty(\Omega)^+$. So, applying Proposition 10.3 we get that any almost interior point of $L^2(\Omega)^+$ is mapped by $S_\lambda(t)$ into an almost interior point of $L^\infty(\Omega)^+$.
Now, since $v_0\in L^2(\Omega)^+\setminus\{0\}$, we have that $\tilde{v}(t/2,\cdot) = S_\lambda(t/2)v_0$ is an almost interior point of $L^2(\Omega)^+$ because $\big[S_\lambda(t/2)v_0\big](x) = \tilde{v}(t/2,x) > 0$ for a.a. $x\in\Omega$ (see Remark 10.2). [11] Henceforth $\tilde{v}(t,\cdot) = S_\lambda(t)v_0 = S_\lambda(t/2)S_\lambda(t/2)v_0$ is an almost interior point of $L^\infty(\Omega)^+$. Putting all together we conclude that $\operatorname*{ess\,inf}_{x\in\Omega}\ v(t,\cdot) \geq \operatorname*{ess\,inf}_{x\in\Omega}\ \tilde{v}(t,\cdot) > 0$ for any $t\in(0,T]$.
(5) From the *weak parabolic minimum principle* we know that $v \geq S_\lambda(\cdot)v_0 := \tilde{v}$ a.e. on $(0,T)\times\Omega$.
Fix some $t_0\in(0,T)$ and choose some $\delta\in(0,t_0)$. We define the following operator: $\mathcal{S} : L^2(\Omega)\to L^\infty((t_0-\delta,T-\delta)\times\Omega)$ by $\mathcal{S}w := S_\lambda(\cdot)w$ i.e. $[\mathcal{S}w](t,x) = [S_\lambda(t)w](x)$ for $(t,x)\in(t_0-\delta,T-\delta)\times\Omega$.
It is easy to see that $\mathcal{S}$ is a linear operator. Using Lemma 10.2 from the Appendix we deduce that for each $t\in(t_0-\delta,T-\delta)$:

$$\|S_\lambda(t)w\|_{L^\infty(\Omega)} \leq C\cdot t^{-D/4}\cdot\|w\|_{L^2(\Omega)} \leq C\cdot(t_0-\delta)^{-D/4}\cdot\|w\|_{L^2(\Omega)}. \tag{12}$$

Thus $\|\mathcal{S}w\|_{L^\infty((t_0-\delta,T-\delta)\times\Omega)} = \operatorname*{ess\,sup}_{t\in(t_0-\delta,T-\delta)} \|S_\lambda(t)w\|_{L^\infty(\Omega)} \leq C\cdot(t_0-\delta)^{-D/4}\cdot\|w\|_{L^2(\Omega)}$. This shows that $\mathcal{S}$ is a bounded linear operator. From the *weak parabolic minimum principle* we obtain that $\mathcal{S}$ is a positive operator, i.e. if $w\in L^2(\Omega)^+$ then

[8]See Theorem 7.14 from [5].

[9]The Neumann boundary condition was essential here in order to compute $S_\lambda(t)1_\Omega$ because we can consider solutions that do not depend on the spatial variable.

[10]See Definition 10.1 and Remark 10.2 from the appendix.

[11]This follows from the **irreducibility** of the semigroup associated to the Neumann Laplacian. See Definition 2.8 and then Theorem 4.5 in [25].

$\mathcal{S}w \in L^\infty((t_0-\delta, T-\delta)\times\Omega)^+$. Thanks to the Neumann boundary conditions we can compute $[\mathcal{S}1_\Omega](t,x) = e^{-\lambda t} > e^{-\lambda(T-\delta)} > 0$ for $(t,x)\in(t_0-\delta,T-\delta)\times\Omega$. So: $\begin{cases} 1_\Omega \in L^2(\Omega)^+ \text{ is an almost interior point of } L^2(\Omega)^+ \\ \mathcal{S}1_\Omega \text{ is an almost interior point of } L^\infty((t_0-\delta,T-\delta)\times\Omega)^+ \end{cases}$ $\Rightarrow$ $\mathcal{S}$ maps any almost interior point of $L^2(\Omega)^+$ into an almost interior point of $L^\infty((t-\delta,T-\delta)\times\Omega)^+$, from [13, Proposition 2.21]. Now, since $v_0 \in L^2(\Omega)^+\setminus\{0\}$ it follows from **(4)** that $S_\lambda(\delta)v_0 \in L^\infty(\Omega)^+$ is an almost interior point of $L^\infty(\Omega)^+$. In particular $S_\lambda(\delta)v_0$ will be also an almost interior point of $L^2(\Omega)^+$. Thus $\mathcal{S}$ will map $S_\lambda(\delta)v_0$ into an almost interior point of $L^\infty((t_0-\delta,T-\delta)\times\Omega)^+$. All we have to observe at this point is that for each $t\in(t_0,T)$ we have that $t-\delta\in(t_0-\delta,T-\delta)$ and from the semigroup property:

$$[\mathcal{S}(S_\lambda(\delta)v_0)](t-\delta,\cdot) = S_\lambda(t-\delta)\big(S_\lambda(\delta)v_0\big) = S_\lambda(t)v_0 = \tilde{v}(t,\cdot). \tag{13}$$

In conclusion:

$$\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} v \geq \operatorname*{ess\,inf}_{(t_0,T)\times\Omega} \tilde{v} = \operatorname*{ess\,inf}_{(t_0,T)\times\Omega} [\mathcal{S}(S_\lambda(\delta)v_0)](t-\delta,x) = \operatorname*{ess\,inf}_{(t_0-\delta,T-\delta)\times\Omega} \mathcal{S}(S_\lambda(\delta)v_0) > 0. \tag{14}$$

(6) Let $c = \operatorname*{ess\,inf}_{\Omega} v_0 > 0$ and consider the problem

$$\begin{cases} \dfrac{\partial\tilde{v}}{\partial t} - d\Delta\tilde{v} + \lambda\tilde{v} = 0, & (t,x)\in(0,T)\times\Omega \\ \dfrac{\partial\tilde{v}}{\partial\nu} = 0, & (t,x)\in(0,T)\times\Omega \\ \tilde{v}(0,x) = c, & x\in\Omega \end{cases}.$$

We know that $\tilde{v}(t,x) = c\cdot e^{-\lambda t} \geq c\cdot e^{-\lambda T} > 0$ for any $(t,x)\in(0,T)\times\Omega$. If we denote $w = v-\tilde{v}$ we obtain that $\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + \lambda w = f(t,x) \geq 0, & (t,x)\in(0,T)\times\Omega \\ \dfrac{\partial w}{\partial\nu} = 0 \geq 0, & (t,x)\in(0,T)\times\Omega \\ w(0,x) = v_0 - c \geq 0, & x\in\Omega \end{cases}.$
Thus, from the *weak parabolic minimum principle* it follows that $v \geq \tilde{v} \geq c\cdot e^{-\lambda T} > 0$ a.e. on $(0,T)\times\Omega$. So $\operatorname*{ess\,inf}_{(0,T)\times\Omega} v > 0$. The proof of the theorem is now complete. □

3. The parabolic problem

We will study in this section problem (1). We consider the following hypotheses:

(H1) Ω is an open, connected and bounded Lipschitz domain;
(H2) $T > 0$ is the final time under consideration;
(H3) $r, p \in L^\infty(\Omega)$ are some heterogeneous parameters regarding the domain Ω with $0 < \rho := \operatorname*{ess\,inf}_{\Omega} r \leq r(x) \leq 1$ and $r(x) \leq p(x)$ for almost all $x\in\Omega$
(H4) $d > 0$ is the diffusion coefficient;
(H5) $\alpha > 0$ is a strict positive real constant;
(H6) $u_0 \in \mathcal{U} := \big\{w\in L^\infty(\Omega) \mid 0\leq w\leq 1, \text{ a.e. on } \Omega\big\}$;

Definition 3.1. We say that $u \in C([0,T];L^2(\Omega)) \cap H^1_{\mathrm{loc}}((0,T);L^2(\Omega)) \cap L^2((0,T);H^1(\Omega)) \cap L^\infty((0,T);L^\infty(\Omega))$ is a **weak solution** of (1) if for any $\phi \in H^1(\Omega)$ the following identity takes place:

$$\int_\Omega \frac{\partial u}{\partial t}(t,\cdot)\phi\, dx + d\int_\Omega \nabla u(t,\cdot)\cdot\nabla\phi\, dx = \int_\Omega \alpha u(t,\cdot)\big(r - pu(t,\cdot)\big)\phi\, dx. \tag{15}$$

Theorem 3.1 (**Uniqueness**). *If u_1 and u_2 are both weak solutions of (1) then $u_1 \equiv u_2$.*

Proof. Let $v = u_1 - u_2$. Then $c = \alpha(p(u_1+u_2)-r) \in L^\infty((0,T)\times\Omega)$, because $u_1, u_2 \in L^\infty((0,T)\times\Omega)$ and:
$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta v + c(t,x)v = 0 \begin{cases}\geq 0\\ \leq 0\end{cases}, & (t,x) \in (0,T)\times\Omega\\ \dfrac{\partial v}{\partial \nu} = 0 \begin{cases}\geq 0\\ \leq 0\end{cases}, & (t,x) \in (0,T)\times\Omega\,.\\ v(0,x) = 0 \begin{cases}\geq 0\\ \leq 0\end{cases}, & x \in \Omega \end{cases}$$
From the *weak parabolic minimum principle* we get that $v \geq 0$ and $v \leq 0$ a.e. on $(0,T)\times\Omega$, and hence $v \equiv 0$ as needed. □

In the following three results, we establish some lower and upper bounds for the solution of (1).

Proposition 3.2 (**Global boundedness**). *If u is a weak solution of problem (1) then $0 \leq u \leq 1$ a.e. on $(0,T)\times\Omega$.*

Proof. Notice that
$$\begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u + c(t,x)u = 0 \geq 0, & (t,x) \in (0,T)\times\Omega\\ \dfrac{\partial u}{\partial \nu} = 0 \geq 0, & (t,x) \in (0,T)\times\partial\Omega\\ u(0,x) = u_0(x) \geq 0, & x \in \Omega \end{cases}$$
where $c = \alpha(pu - r) \in L^\infty((0,T)\times\Omega)$. From the *weak parabolic minimum principle* we obtain that $u \geq 0$ a.e. on $(0,T)\times\Omega$. Similarly, if we denote $w = 1-u$, we arrive at:
$$\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + \underbrace{\alpha(2p-r)}_{\in L^\infty} w = \alpha\underbrace{(p-r)}_{\geq 0} + \alpha p w^2 \geq 0, & (t,x) \in (0,T)\times\Omega\\ \dfrac{\partial w}{\partial \nu} = 0 \geq 0, & (t,x) \in (0,T)\times\partial\Omega\,.\\ w(0,x) = 1 - u_0(x) \geq 0, & x \in \Omega \end{cases}$$
Using one more time the *weak parabolic minimum principle* we get that $w \geq 0$ a.e. on $(0,T)\times\Omega$. □

Proposition 3.3 (**Thresholds**). *Consider that u is a weak solution of (1).*

(1) *If $u_0 \leq \overline{U} := \left\|\dfrac{r}{p}\right\|_{L^\infty(\Omega)}$ a.e. on Ω then $u \leq \overline{U}$ a.e. on $(0,T)\times\Omega$.*

(2) *If $u_0 \geq \underline{U} := \operatorname{ess\,inf}_{\Omega} \dfrac{r}{p}$ a.e. on Ω then $u \geq \underline{U}$ a.e. on $(0,T)\times\Omega$.*

Proof. **(1)** Let $w = \overline{U} - u$. Proceeding as above we get that:

$$\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + \underbrace{\alpha(2p\overline{U} - r)}_{\in L^\infty(\Omega)} w = \alpha \overline{U} p \underbrace{\left(\overline{U} - \dfrac{r}{p}\right)}_{\geq 0} + \alpha p w^2 \geq 0, & (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial w}{\partial \nu} = 0 \geq 0, & (t,x) \in (0,T) \times \partial\Omega \\ w(0,x) = \overline{U} - u_0(x) \geq 0, & x \in \Omega \end{cases}.$$

From the *weak parabolic minimum principle* we get that $w \geq 0$ a.e. on $(0,T) \times \Omega$.
(2) Set $w = u - \underline{U} \in L^\infty((0,T) \times \Omega)$. Then:

$$\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + \underbrace{\alpha(2p\underline{U} - r + pw)}_{\in L^\infty(\Omega)} w = \alpha \underline{U} p \underbrace{\left(\dfrac{r}{p} - \underline{U}\right)}_{\geq 0} \geq 0, & (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial w}{\partial \nu} = 0 \geq 0, & (t,x) \in (0,T) \times \partial\Omega \\ w(0,x) = u_0(x) - \underline{U} \geq 0, & x \in \Omega \end{cases}.$$

From the same *weak parabolic minimum principle* we conclude that $w \geq 0$ a.e. on $(0,T) \times \Omega$. ☐

Remark 3.1. Let $\tilde{r}$, $\tilde{p} > 0$ and $c \geq 0$ be any three constants.[12] Consider the problem:

$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta v = \alpha v \cdot \left(\tilde{r} - \tilde{p} v\right), & (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial v}{\partial \nu} = 0, & (t,x) \in (0,T) \times \partial\Omega \\ v(0,x) = c \geq 0, & x \in \Omega \end{cases} \tag{16}$$

It can be easily checked that $v(t) = \dfrac{c\tilde{r}}{e^{-\alpha \tilde{r} t}(\tilde{r} - c\tilde{p}) + c\tilde{p}}$, $t \in [0,T]$ is a solution of (16), that we shall denote by $v(\tilde{r}, \tilde{p}, c)$, with the property that:

$$\begin{cases} c \leq v(t) \leq \dfrac{\tilde{r}}{\tilde{p}}, \text{ if } c \leq \dfrac{\tilde{r}}{\tilde{p}} \\ c \geq v(t) \geq \dfrac{\tilde{r}}{\tilde{p}}, \text{ if } c \geq \dfrac{\tilde{r}}{\tilde{p}} \end{cases}, \ \forall\, t \in [0,T]$$

and moreover:

$$\begin{cases} c < v(t) < \dfrac{c\tilde{r}}{e^{-\alpha \tilde{r} T}(\tilde{r} - c\tilde{p}) + c\tilde{p}} < \dfrac{\tilde{r}}{\tilde{p}}, \text{ if } c < \dfrac{\tilde{r}}{\tilde{p}} \\ c > v(t) > \dfrac{c\tilde{r}}{-e^{-\alpha \tilde{r} T}(c\tilde{p} - \tilde{r}) + c\tilde{p}} > \dfrac{\tilde{r}}{\tilde{p}}, \text{ if } c > \dfrac{\tilde{r}}{\tilde{p}} \end{cases}, \text{ for all } t \in (0,T]. \tag{17}$$

Proposition 3.4 (**Barrier functions**). *Consider u to be a solution of the problem* (1). *The following properties take place:*

(1) $v(\rho, \|p\|_{L^\infty(\Omega)}, \operatorname*{ess\,inf}_\Omega u_0) \leq u \leq v(\|r\|_{L^\infty(\Omega)}, \operatorname*{ess\,inf}_\Omega p, \|u_0\|_{L^\infty(\Omega)})$ *a.e. on* $(0,T) \times \Omega$.

[12]The same procedure of solving Bernoulli's equation can be applied if we take $\tilde{r} = \tilde{r}(t)$ and $\tilde{p} = \tilde{p}(t)$ (depending only on $t \in [0,T]$).

(2) *If* $\|u_0\|_{L^\infty(\Omega)} \le \overline{U}$ *then:* $u \le v\left(\|r\|_{L^\infty(\Omega)}, \dfrac{\|r\|_{L^\infty(\Omega)}}{\overline{U}}, \|u_0\|_{L^\infty(\Omega)}\right)$ *a.e. on* $(0,T)\times\Omega$.

(3) *If* $\operatorname*{ess\,inf}_{\Omega} u_0 \ge \underline{U}$ *then:* $u \ge v\left(\|p\|_{L^\infty(\Omega)}\underline{U}, \|p\|_{L^\infty(\Omega)}, \operatorname*{ess\,inf}_{\Omega} u_0\right)$ *a.e. on* $(0,T)\times\Omega$.

Proof. (Sketch) **(1)** First note that if $w = u - v(\tilde{r}, \tilde{p}, c)$ then:

$$\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + \underbrace{\alpha\big[p(u+v) - r\big]}_{\in L^\infty(\Omega)} \cdot w = \underbrace{\alpha v}_{\ge 0}\big[r - \tilde{r} + v(\tilde{p} - p)\big], & (t,x)\in(0,T)\times\Omega \\ \dfrac{\partial w}{\partial \nu} = 0, & (t,x)\in(0,T)\times\partial\Omega \\ w(0,x) = u_0 - c, & x\in\Omega \end{cases}. \tag{18}$$

If $\begin{cases} \tilde{r} = \rho \\ \tilde{p} = \|p\|_{L^\infty(\Omega)} \\ c = \operatorname*{ess\,inf}_{\Omega} u_0 \end{cases}$, then

$$\begin{cases} r - \tilde{r} + v(\tilde{p} - p) = \underbrace{r-\rho}_{\ge 0} + v\underbrace{\left(\|p\|_{L^\infty(\Omega)} - p\right)}_{\ge 0} \ge 0, & (t,x)\in(0,T)\times\Omega \\ u_0 - c = u_0 - \operatorname*{ess\,inf}_{\Omega} u_0 \ge 0, & x\in\Omega \end{cases}.$$

Hence, from the *weak parabolic minimum principle* we obtain that

$$u \ge v\left(\rho, \|p\|_{L^\infty(\Omega)}, \operatorname*{ess\,inf}_{\Omega} u_0\right) \quad \text{a.e. on} \quad (0,T)\times\Omega.$$

(3) If $\begin{cases} \tilde{r} = \|p\|_{L^\infty(\Omega)}\underline{U} \\ \tilde{p} = \|p\|_{L^\infty(\Omega)} \\ c = \operatorname*{ess\,inf}_{\Omega} u_0 \ge \dfrac{\tilde{r}}{\tilde{p}} \end{cases}$, then from Remark 3.1 we know that $v \ge \underline{U}$ a.e. on $(0,T)\times\Omega$ and:

$$\begin{cases} r - \tilde{r} + v(\tilde{p} - p) = r - \|p\|_{L^\infty(\Omega)}\underline{U} + v\underbrace{\left(\|p\|_{L^\infty(\Omega)} - p\right)}_{\ge 0} \ge p\left(\dfrac{r}{p} - \underline{U}\right) \ge 0 \\ u_0 - c = u_0 - \operatorname*{ess\,inf}_{\Omega} u_0 \ge 0 \end{cases}.$$

From the *weak parabolic minimum principle* we get that

$$u \ge v\left(\|p\|_{L^\infty(\Omega)}\underline{U}, \|p\|_{L^\infty(\Omega)}, \operatorname*{ess\,inf}_{\Omega} u_0\right) \quad \text{a.e. on} \quad (0,T)\times\Omega.$$

Similarly, the other two inequalities can be proved and they are left to the reader. □

Fix some $\lambda \ge 2\alpha\|p\|_{L^\infty(\Omega)}$ and define the function $f : \Omega\times[0,1] \to \mathbb{R}$, $f(x,u) = \alpha u(r(x) - p(x)u) + \lambda u$, for any $(x,u)\in\Omega\times[0,1]$. Notice that for any fixed $x\in\Omega$ we have that $\dfrac{\partial f}{\partial u} = \alpha r(x) + \lambda - 2\alpha p(x)u \ge \alpha\rho > 0$ for all $u\in[0,1]$. For any $u \in \mathcal{V} := \big\{w\in L^\infty((0,T)\times\Omega) \,|\, 0\le w(t,x)\le 1, \text{ for a.a. } (t,x)\in(0,T)\times\Omega\big\}$ let v be the unique weak solution (see Theorem 2.1) of the problem:

$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta v + \lambda v = f(x, u(t,x)), & (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial v}{\partial \nu} = 0, & (t,x) \in (0,T) \times \partial\Omega \\ v(0,x) = u_0(x), & x \in \Omega \end{cases} \tag{19}$$

Since f is increasing in the second argument, and $u \in \mathcal{V}$, we immediately get that $0 \le f(x,0) \le f(x,u(t,x)) \le f(x,1) = \lambda - \alpha(p-r)$ for a.a. $(t,x) \in (0,T) \times \Omega$. Applying Theorem 2.1 **(2)** we obtain that for a.a. $(t,x) \in (0,T) \times \Omega$:

$$0 \le \min\left\{\operatorname*{ess\,inf}_{\Omega} u_0,\ \frac{1}{\lambda} \cdot 0\right\} \le v(t,x) \le \max\left\{\operatorname*{ess\,sup}_{\Omega} u_0,\ 1 - \frac{\alpha(p-r)}{\lambda}\right\} \le 1. \tag{20}$$

This allows us to define the nonlinear operator $F : \mathcal{V} \to \mathcal{V}$ given by $F(u) = v$. Next, we give some of the properties of F.

Lemma 3.5. *The nonlinear operator F has the following properties:*

(1) *F is a monotone operator;*

(2) *If $u_1, u_2 \in \mathcal{V}$ with $u_1 \le u_2$ a.e. on $(0,T) \times \Omega$ and $u_1 \not\equiv u_2$ then for each $t_0 \in (t^*, T)$, where $t^* = \sup\left\{\tilde{t} \in (0,T) \mid u_1(t,x) = u_2(t,x),\ \text{for a.a. } (t,x) \in (0,\tilde{t}) \times \Omega\right\}$ we have that: $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} F(u_2) - F(u_1) > 0$.*

(3) *F is Lipschitz continuous with respect to the norms of $L^2((0,T); L^2(\Omega))$ and $L^\infty((0,T); L^\infty(\Omega))$;*

(4) *If $u_0 \le \overline{U}$ a.e. on Ω and $u \le \overline{U}$ a.e. on $(0,T) \times \Omega$ then $F(u) \le \overline{U}$ a.e. on $(0,T) \times \Omega$ and if in addition at least one of the following two conditions holds:*
$$\begin{cases} u_0 \not\equiv \overline{U} \\ \frac{r}{p} \text{ is not constant} \end{cases} \quad \text{then for each } t_0 \in (0,T) \text{ we have } \operatorname*{ess\,inf}_{(t_0,T)\times\Omega} \overline{U} - F(u) > 0.$$
Additionally, in case $u_0 \equiv \overline{U} \equiv \dfrac{r}{p}$, if $u \not\equiv \overline{U}$, then for each $t_0 \in (t^*, T)$ where $t^* = \sup\left\{\tilde{t} \in (0,T) \mid u(t,x) = \overline{U} \text{ for a.a. } (t,x) \in (0,\tilde{t}) \times \Omega\right\}$, we have that $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} \overline{U} - F(u) > 0$.

(5) *If $u_0 \ge \underline{U}$ a.e. on Ω and $u \ge \underline{U}$ a.e. on $(0,T) \times \Omega$ then $F(u) \ge \underline{U}$ a.e. on $(0,T) \times \Omega$ and if in addition at least one of the following two conditions holds:*
$$\begin{cases} u_0 \not\equiv \underline{U} \\ \frac{r}{p} \text{ is not constant} \end{cases} \quad \text{then for each } t_0 \in (0,T) \text{ we have } \operatorname*{ess\,inf}_{(t_0,T)\times\Omega} F(u) - \underline{U} > 0.$$
Moreover, in case $u_0 \equiv \underline{U} \equiv \dfrac{r}{p}$, if $u \not\equiv \underline{U}$, then for each $t_0 \in (t^*, T)$ where $t^* = \sup\left\{\tilde{t} \in (0,T) \mid u(t,x) = \underline{U} \text{ for a.a. } (t,x) \in (0,\tilde{t}) \times \Omega\right\}$, we have that $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} F(u) - \underline{U} > 0$.

(6) *If $u_0 \not\equiv 0$ then for any $t_0 \in (0,T)$ we have that $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} F(u) > 0$.*
In case $u_0 \equiv 0$, if $u \not\equiv 0$, then for any $t_0 \in (t^*, T)$ where $t^* = \sup\left\{\tilde{t} \in (0,T) \mid u(t,x) = 0 \text{ for a.a. } (t,x) \in (0,\tilde{t}) \times \Omega\right\}$, we have that $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} F(u) > 0$.

(7) *If at least one of the two following three conditions is true:* $\begin{cases} u_0 \not\equiv 1 \\ p \not\equiv r \end{cases}$ *then for any* $t_0 \in (0,T)$ *we have that* $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} 1 - F(u) > 0$.
In case $u_0 \equiv 1$ *and* $r \equiv p$, *if* $u \not\equiv 1$, *then for any* $t_0 \in (t^*, T)$ *where* $t^* = \sup\big\{\tilde{t} \in (0,T) \mid u(t,x) = 1$ *for a.a.* $(t,x) \in (0,\tilde{t}) \times \Omega\big\}$, *we have that* $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} 1 - F(u) > 0$.

Proof. **(1)** Let $u_1, u_2 \in \mathcal{V}$ with $u_1 \le u_2$ a.e. on $(0,T)\times\Omega$. Set $v_1 = F(u_1)$, $v_2 = F(u_2)$ and $v = v_2 - v_1$. Then, from the monotony of f we get that:

$$\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta v = f(x, u_2(t,x)) - f(x, u_1(t,x)) \ge 0, & (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial v}{\partial \nu} = 0 \ge 0, & (t,x) \in (0,T) \times \partial\Omega \cdot \\ v(0,x) = 0, & x \in \Omega \end{cases} \tag{21}$$

We may conclude, from the *weak parabolic minimum principle* that $v \ge 0$ a.e. on $(0,T) \times \Omega$.
(2) Consider in addition now that $u_1 \not\equiv u_2$. It is obvious to remark that $t^* < T$. Let's denote $A = \big\{\tilde{t} \in (0,T) \mid u_1(t,x) = u_2(t,x),$ for a.a. $(t,x) \in (0,\tilde{t}) \times \Omega\big\}$. If $t^* \in A$ then $u_1(t,x) = u_2(t,x)$, for a.a. $(t,x) \in (0,t^*) \times \Omega$. If $t^* \notin A$ then there is a strictly increasing sequence $(t_n)_{n\ge1} \subset A$ that tends to t^*. Let $B = \big\{(t,x) \in (0,t^*) \times \Omega \mid u_1(t,x) \ne u_2(t,x)\big\}$ and for each $n \ge 1$, $B_n = \big\{(t,x) \in (0,t_n) \times \Omega \mid u_1(t,x) \ne u_2(t,x)\big\}\big\}$. So $\mathcal{L}^{N+1}(B_n) = 0$, $\forall\ n \ge 1$, because $(t_n)_{n\ge1} \subset A$. Moreover $B_n \subset B$ for $n \ge 1$ and: $\mathcal{L}^{N+1}(B) = \mathcal{L}^{N+1}(B) - \mathcal{L}^{N+1}(B_n) = \mathcal{L}^{N+1}\big(B \setminus B_n\big)$. Now, since $B \setminus B_n \subset (t_n, t^*) \times \Omega$, we deduce that: $\mathcal{L}^{N+1}\big(B \setminus B_n\big) \le \mathcal{L}^{N+1}\big((t_n,t^*) \times \Omega\big) = \mathcal{L}^1\big((t_n,t^*)\big) \cdot \mathcal{L}^N(\Omega) = (t^* - t_n) \cdot \mathcal{L}^N(\Omega) \overset{n\to\infty}{\longrightarrow} 0$, Ω being a bounded set from $\mathbb{R}^N$. Thus $\mathcal{L}^{N+1}(B) = 0$ which means that $t^* \in A$.
Consider some $t_0 \in (t^*, T)$. We introduce the set $C = \big\{t \in (t^*, t_0) \mid u_1(t,\cdot) \not\equiv u_2(t,\cdot)\big\}$. We want to show that C has a strictly positive measure. Suppose that $\mathcal{L}^1(C) = 0$. This means that for a.a. $t \in (t^*, t_0)$ we have that $u_1(t,x) = u_2(t,x)$ for a.a. $x \in \Omega$. If $D = \big\{(t,x) \in (t^*, t_0) \times \Omega \mid u_1(t,x) \ne u_2(t,x)\big\} \subset (t^*, t_0) \times \Omega$ and for each $t \in (t^*, t)$ we set $D_t = \big\{x \in \Omega \mid u_1(t,x) \ne u_2(t,x)\big\}$. Thus for a.a. $t \in (t^*, t_0)$ we have that $\mathcal{L}^N(D_t) = 0$. Now from *Tonelli's theorem* we get that:

$$\begin{aligned} \mathcal{L}^{N+1}(D) &= \int_{(t^*,t_0)\times\Omega} 1_D(t,x)\ d(t,x) = \int_{(t^*,t_0)} \left(\int_\Omega 1_{D_t}(x)\ dx\right)\ dt \\ &= \int_{(t^*,t_0)} \mathcal{L}^N(D_t)\ dt = 0. \end{aligned} \tag{22}$$

This shows that $u_1(t,x) = u_2(t,x)$ for a.a. $(t,x) \in (t^*, t_0) \times \Omega$. But, since $t^* \in A$ we also have that $u_1(t,x) = u_2(t,x)$ for a.a. $(t,x) \in (0,t^*) \times \Omega$. We can conclude that $u_1(t,x) = u_2(t,x)$ for a.a. $(t,x) \in (0,t_0) \times \Omega$. Since $t_0 > t^*$ we have reached a contradiction to the definition of t^*. So $\mathcal{L}^1(C) > 0$, and in any interval (t^*, t_0) we have a subset of strictly positive measure for which $u_1(t,\cdot) \not\equiv u_2(t,\cdot)$.
Therefore we may choose some $\delta > 0$ with $t_0 - \delta \in (t^*, t_0)$ and $u_1(t_0 - \delta) \not\equiv u_2(t_0 - \delta, \cdot)$.

Going further we have that

$$f(x,u_2(t,x)) - f(x,u_1(t,x)) = (u_2-u_1)\cdot\big[\alpha r + \underbrace{\lambda - \alpha p(u_1+u_2)}_{\geq 0}\big] \geq \alpha\rho(u_2-u_1) \not\equiv 0$$

a.e. on $(0,T)\times\Omega$. This shows in particular that $v(t_0-\delta,\cdot)\not\equiv 0$, thence $v(t_0-\delta,\cdot)\in L^\infty(\Omega)^+\setminus\{0\}\subset L^2(\Omega)^+\setminus\{0\}$. Applying a translated version of Theorem 2.1 **(5)** with $t_0-\delta$ as initial value, for $t_0>t_0-\delta$ we deduce that: $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} v > 0$.

(3) Let $u_1,u_2\in\mathcal{V}$, $v_1=F(u_1)$, $v_2=F(u_2)$ and $v=v_2-v_1$. From Theorem 2.1 **(2)**, since $v(0,\cdot)\equiv 0$, we have that:

$$\begin{aligned}\|F(u_2)-F(u_1)\|_{L^\infty((0,T)\times\Omega)} = \|v\|_{L^\infty((0,T)\times\Omega)} &\leq \frac{1}{\lambda}\|f(x,u_2)-f(x,u_1)\|_{L^\infty((0,T)\times\Omega)}\\ &\leq \left(1+\frac{\alpha\|r\|_{L^\infty(\Omega)}}{\lambda}\right)\|u_2-u_1\|_{L^\infty((0,T)\times\Omega)}.\end{aligned}$$

For the continuity with respect to $L^2((0,T)\times\Omega)$–norm, we know from Definition 2.1 that for a.a. $t\in(0,T)$, choosing the test function $\phi=v(t,\cdot)\in H^1(\Omega)$ and integrating on $[0,t]$, the following inequality holds:

$$\begin{aligned}\frac{1}{2}\int_\Omega v^2(t,x)\,dx + \lambda\int_0^t\int_\Omega v^2(s,x)\,dx\,ds &\leq \int_0^t\int_\Omega \big(f(x,u_2)-f(x,u_1)\big)v(s,x)\,dx\,ds\\ &\text{(Cauchy inequality)}\\ &\leq (\alpha\|r\|_{L^\infty(\Omega)}+\lambda)\cdot\|u_2-u_1\|_{L^2((0,t)\times\Omega)}\cdot\|v\|_{L^2((0,t)\times\Omega)} \qquad (23)\end{aligned}$$

We define the real functions $g,h:[0,T]\to\mathbb{R}$, $g(t)=(\alpha\|r\|_{L^\infty(\Omega)}+\lambda)\cdot\|u_2-u_1\|_{L^2((0,t)\times\Omega)}$ and $h(t)=\|v\|_{L^2((0,t)\times\Omega)}$. We have that $g(0)=h(0)$, both g and h are increasing and continuous. Moreover $[h^2(t)]'=\int_\Omega v^2(t,x)\,dx$ for $t\in[0,T]$. Let $t_0=\max\big\{t\in[0,T]\mid h(s)=0 \text{ for } s\in[0,t]\big\}$. Clearly for $t\in(t_0,T]$ (if any) we will have that $h(t)>0$. The inequality (23) rewrites as:

$$\frac{1}{2}\frac{d}{dt}\big[h^2(t)\big]+\lambda h^2(t)\leq g(t)\cdot h(t) \;\Rightarrow\; h'(t)+\lambda h(t)\leq g(t), \forall\, t\in(t_0,T]. \qquad (24)$$

So $(e^{\lambda\tau}h(\tau))'\leq g(\tau)e^{\lambda\tau}$ for $\tau\in(t_0,T]$. Integrating on $[t_0+\varepsilon,t]\subset[0,T]$ leads us to: $e^{\lambda t}h(t)-e^{\lambda(t_0+\varepsilon)}h(t_0+\varepsilon)\leq\int_{t_0+\varepsilon}^t g(\tau)e^{\lambda\tau}\,d\tau\leq\int_0^t g(\tau)e^{\lambda\tau}\,d\tau\leq g(t)\int_0^t e^{\lambda\tau}=\frac{1}{\lambda}g(t)\cdot(e^{\lambda t}-1)$. Making $\varepsilon\to 0^+$ and using the continuity of h will give us that:

$$h(t)\leq\frac{1}{\lambda}g(t)\cdot(1-e^{-\lambda t})\leq\frac{1}{\lambda}g(t)=\left(1+\frac{\alpha\|r\|_{L^\infty(\Omega)}}{\lambda}\right)\|u_2-u_1\|_{L^2((0,t)\times\Omega)}, \qquad (25)$$

for any $t\in(t_0,T]$. But since $h(t)=0$ for $t\in[0,t_0]$ the same inequality holds for any $t\in[0,T]$. Setting $t=T$ will give us the desired continuity of F.

(4) Just put $w = \overline{U} - F(u)$ and observe that:

$$\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + \lambda w = \lambda \overline{U} - f(x, u(t,x)) \\ \qquad\qquad\qquad\qquad \geq \lambda \overline{U} - f(x, \overline{U}) = \alpha \overline{U} p \left(\overline{U} - \dfrac{r}{p} \right) \geq 0, \text{ on } (0,T) \times \Omega \\ \dfrac{\partial w}{\partial \nu} = 0, \text{ on } (0,T) \times \partial\Omega \\ w(0,x) = \overline{U} - u_0(x) \geq 0, \text{ on } \Omega \end{cases} . \quad (26)$$

From the *weak parabolic minimum principle* we get that $F(u) \leq \overline{U}$ a.e. on $(0,T) \times \Omega$. Notice that $\lambda \overline{U} - f(x,u) = f(x, \overline{U}) - f(x,u) + \alpha \overline{U} p \left(\overline{U} - \frac{r}{p} \right) \geq \alpha \rho (\overline{U} - u) + \alpha \overline{U} p \left(\overline{U} - \frac{r}{p} \right)$. Fix any $T > t_0 > \delta > 0$. If any of the three conditions is satisfied then we will have that $w(\delta, \cdot) \not\equiv 0$, and thus $w(\delta, \cdot) \in L^2(\Omega)^+ \setminus \{0\}$. From Theorem 2.1 **(5)** it follows that $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} w > 0$.

Now, if $u_0 \equiv \overline{U} \equiv \frac{r}{p}$ then

$$\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + \lambda w = \lambda \overline{U} - f(x, u(t,x)) = \underbrace{(\lambda - \alpha u p)}_{\geq \alpha \|p\|_{L^\infty(\Omega)} > 0} (\overline{U} - u), \text{ on } (0,T) \times \Omega \\ \dfrac{\partial w}{\partial \nu} = 0, \text{ on } (0,T) \times \partial\Omega \\ w(0,x) = 0, \text{ on } \Omega \end{cases} .$$

With the same measure-theoretic approach that we have used in **(2)** we can prove that there is $\delta > 0$ with $t_0 - \delta \in (t^*, t_0)$ such that $w(t_0 - \delta, \cdot) \not\equiv 0$. Taking this as an initial value and using Theorem 2.1 **(5)** for $t_0 > t_0 - \delta$ it follows that $\operatorname*{ess\,inf}_{(t_0,T)\times\Omega} w > 0$.

(5) The proof is similar to **(4)**.

(6) Note that for $v = F(u)$ then $\begin{cases} \dfrac{\partial v}{\partial t} - d\Delta v + \lambda v \geq \alpha \rho u, & (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial v}{\partial \nu} = 0, & (t,x) \in (0,T) \times \Omega \\ v(0,x) = u_0(x), & x \in \Omega \end{cases}$. The conclusion follows as in **(4)**.

(7) Just observe that if $w = 1 - F(u)$ then

$$\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + \lambda w \geq \alpha \rho (1 - u) + \alpha (p - r), & (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial w}{\partial \nu} = 0, & (t,x) \in (0,T) \times \Omega \\ w(0,x) = 1 - u_0(x), & x \in \Omega \end{cases} .$$

We can proceed as in **(4)** to conclude. □

Now we have all we need in order to prove the main result of this section:

Theorem 3.6. *For each $T > 0$ problem (1) has a unique weak solution $u \in \mathcal{V}$.*

Proof. **(Existence)** Consider the following recurrence:[13]

$$\begin{cases} u_{n+1} = F(u_n), \ n \geq 1 \\ u_1 \in \mathcal{V} \end{cases} . \tag{27}$$

Choosing $u_1 \equiv 1 \in \mathcal{V}$ we have that $F(1) \leq 1$ a.e. on $(0,T) \times \Omega$. Using the monotony of F we obtain by an immediate induction that $1 = u_1 \geq u_2 \geq \cdots \geq u_n \geq \cdots \geq 0$. a.e. on $(0,T) \times \Omega$. For a.a. $(t,x) \in (0,T) \times \Omega$ the sequence of real numbers $\big(u_n(t,x)\big)_{n\geq 1}$ is bounded between 0 and 1 and decreasing. Hence it is convergent to some number $u(t,x) \in [0,1]$. Thus $u_n \to u$ pointwise a.e. on $(0,T) \times \Omega$. Since for each $n \geq 1$ one has that: $|u_n| = u_n \leq 1 \in L^2((0,T) \times \Omega)$, we deduce from *Lebesgue dominated convergence theorem* that $u \in L^2((0,T) \times \Omega) \simeq L^2((0,T); L^2(\Omega))$ and $u_n \to u$ in $L^2((0,T) \times \Omega)$. Now using the continuity of F (see Lemma 3.5 **(3)**) we deduce that $u_{n+1} = F(u_n) \to F(u)$ in $L^2((0,T) \times \Omega)$. But $u_{n+1} \to u$ in $L^2((0,T) \times \Omega)$. We conclude that $u = F(u)$. Regarding $f(t, u(t,x))$ as a function of (t,x) we may apply Theorem 2.1 to conclude that u is a weak solution of the problem (7) and thence $u \in C([0,T]; L^2(\Omega)) \cap H^1_{\text{loc}}((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega))$. Being also bounded, we conclude that u is a weak solution of (1).The uniqueness was proved before in Theorem 3.1. □

4. Poisson-type problems with Neumann boundary conditions - a review

For Ω being a Lipschitz, open, bounded and connected domain from $\mathbb{R}^N$, any $\lambda > 0$ and any $f \in L^2(\Omega)$ we consider the following problem:

$$\begin{cases} -d\Delta V(x) + \lambda V = f(x), & x \in \Omega \\ \dfrac{\partial V}{\partial \nu}(x) = 0, & x \in \partial\Omega \end{cases} \tag{28}$$

Definition 4.1. We call $V \in H^1(\Omega)$ a **weak** solution of (28) if $a(V,\phi) = b(\phi)$, $\forall\ \phi \in H^1(\Omega)$, where:

$$a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R},\ a(V,\phi) = d \int_\Omega \nabla V \cdot \nabla \phi \ dx + \lambda \int_\Omega V \phi \ dx,\ \forall\ V, \phi \in H^1(\Omega)$$

$$b : H^1(\Omega) \to \mathbb{R},\ b(\phi) = \int_\Omega f \phi \ dx,\ \forall\ \phi \in H^1(\Omega) \tag{29}$$

We have the following result:

Theorem 4.1. *Problem (28) has a unique weak solution $V \in H^1(\Omega)$ and moreover V satisfies the following properties:*

(1) $\|V\|_{L^2(\Omega)} \leq \dfrac{1}{\lambda} \|f\|_{L^2(\Omega)}$.

(2) $\|V\|_{H^1(\Omega)} \leq \dfrac{1}{\min\{d, \lambda\}} \cdot \|f\|_{L^2(\Omega)}$.

[13] See [30, Section 10.6.1].

(3) *$V \in L^\infty(\Omega)$ and if $f \in L^\infty(\Omega)$ then:* $\frac{1}{\lambda}\operatorname*{ess\,inf}_\Omega f \le V(x) \le \frac{1}{\lambda}\operatorname*{ess\,sup}_\Omega f$ *for a.a. $x \in \Omega$. In particular* $\|V\|_{L^\infty(\Omega)} \le \frac{1}{\lambda}\|f\|_{L^\infty(\Omega)}$.

(4) *$\Delta V \in L^2(\Omega)$ and* $\|\Delta V\|_{L^2(\Omega)} \le \frac{1}{d}\cdot\|f\|_{L^2(\Omega)}$.

(5) *There is some $\delta \in (0,1)$ with $V \in C^\delta(\overline{\Omega})$, i.e. V is Hölder continuous with exponent δ up to the boundary.*

(6) *If Ω is an open, bounded and convex set then: $V \in H^2(\Omega)$ and there is a constant $C > 0$ depending only on Ω and λ such that:* $\|V\|_{H^2(\Omega)} \le C\cdot\|f\|_{L^2(\Omega)}$.

(7) *If $f \ge 0$ a.e. on Ω then $V \ge 0$ a.e. on Ω and there is a constant C depending only on Ω and λ such that:* $\int_\Omega V\,dx \le C\cdot\operatorname*{ess\,inf}_\Omega V$. *In particular $V \equiv 0$ or* $\operatorname*{ess\,inf}_\Omega V > 0$.

Proof. (Sketch) It is easy to remark that a is a bilinear form and from *Cauchy inequality* $|a(V,\phi)| \le \max\{d,\lambda\}\cdot\|V\|_{H^1(\Omega)}\cdot\|\phi\|_{H^1(\Omega)}$, for any $V,\phi \in H^1(\Omega)$[14] which shows that a is continuous. Moreover $a(\phi,\phi) \ge \min\{d,\lambda\}\cdot\|\phi\|^2_{H^1(\Omega)}$, $\forall\ \phi \in H^1(\Omega)$ i.e. a is coercive. Since from *Cauchy inequality* $|b(\phi)| \le \|f\|_{L^2(\Omega)}\cdot\|\phi\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}\cdot\|\phi\|_{H^1(\Omega)}$, $\forall\ \phi \in H^1(\Omega)$ we get that b is a continuous linear functional. Applying the *Lax-Milgram theorem*[15] we get that there is a unique $V \in H^1(\Omega)$ such that $a(V,\phi) = b(\phi)$, $\forall\ \phi \in H^1(\Omega)$.

(1) and **(2)**: Choosing $\phi = V$ in (29) we get that

$$\begin{cases}\lambda\|V\|^2_{L^2(\Omega)} \\ \min\{d,\lambda\}\|V\|^2_{H^1(\Omega)}\end{cases} \le d\int_\Omega |\nabla V|^2\,dx + \lambda\int_\Omega V^2\,dx$$
$$= \int_\Omega fV\,dx \le \begin{cases}\|f\|_{L^2(\Omega)}\cdot\|V\|_{L^2(\Omega)} \\ \|f\|_{L^2(\Omega)}\cdot\|V\|_{H^1(\Omega)}\end{cases}.$$

(3) The fact that $V \in L^\infty(\Omega)$ for each $f \in L^2(\Omega)$ follows from [33, Theorem 4]. Assume now that $f \in L^\infty(\Omega)$. Setting $c = \frac{1}{\lambda}\operatorname*{ess\,inf}_\Omega f$ we have that $-d\Delta(V-c) + \lambda(V-c) = f - \lambda c = f - \operatorname*{ess\,inf}_\Omega f \ge 0$ on Ω and $\frac{\partial(V-c)}{\partial\nu} = 0$, on $\partial\Omega$. Since $\lambda > 0$ we get from the weak minimum principle that $V \ge c$. The other inequality follows in an analogue fashion.

(4) From $a(V,\phi) = b(\phi)$, $\forall\ \phi \in H^1(\Omega)$ using *Green's identity*[16] we get that

$$\int_\Omega V\Delta\phi\,dx = -\int_\Omega \nabla V\cdot\nabla\phi\,dx = \frac{1}{d}\int_\Omega(\lambda V - f)\cdot\phi\,dx$$

for any $\phi \in H^1(\Omega)$ which means in particular that $\exists\ \Delta V = \frac{\lambda V - f}{d} \in L^2(\Omega)$. So we have that $-d\Delta V = f - \lambda V$ as functions from $L^2(\Omega)$. Thus $\|\Delta V\|^2_{L^2(\Omega)} = -\frac{1}{d}\int_\Omega f\Delta V\ dx + \frac{\lambda}{d}\int_\Omega V\Delta V\ dx \stackrel{\text{Green}}{=} \frac{1}{d^2}\int_\Omega f\cdot(f-\lambda V)\ dx - \frac{\lambda}{d}\lambda\int_\Omega|\nabla V|^2\ dx = \frac{1}{d^2}\|f\|^2_{L^2(\Omega)} - \frac{\lambda}{d^2}\int_\Omega fV\ dx - \frac{\lambda}{d}\lambda\int_\Omega|\nabla V|^2\ dx \le \frac{1}{d^2}\|f\|^2_{L^2(\Omega)}$. Here we have used the fact that $\int_\Omega fV\ dx = b(V) = a(V,V) \ge 0$.

[14] In fact $\|a\| = \max\{d,\lambda\}$.

[15] For a proof, see [10, Corollary 5.8, page 140].

[16] For more details see [7, Proposition 7.6.1, pages 323-325].

(5) This is a well-known regularity result that can be obtained by the classical De Giorgi–Nash–Moser theory[17]. The proof can be found here: [33, Theorem 4] or with complete details in [24, Theorem 3.1.5, page 50].
(6) This result can be found in the monograph [15, Theorem 3.2.1.3, page 149]. The idea is to use the well-known H^2-regularity results for domains of class C^2 by approximating such domains from the inside and from the outside with C^2 domains.
(7) This is a version of the **strong minimum principle** for weak solutions satisfying Neumann boundary conditions given by G. Lieberman in [21, Lemma 2.1]. For further details see [34, Theorems 8.18 and 8.19] and [33, Theorem 4]. □

5. The elliptic problem

In this section we will study the steady-states associated to problem (1).

$$\begin{cases} -d\Delta U = \alpha U \cdot \big(r(x) - p(x)U\big), & x \in \Omega \\ \dfrac{\partial U}{\partial \nu} = 0, & x \in \Omega \\ U(x) \geq 0, & x \in \Omega \end{cases} \tag{30}$$

Definition 5.1. We say that $U \in H^1(\Omega)$ is a **weak solution** of (30) if $U \geq 0$ a.e. on Ω and $d\displaystyle\int_\Omega \nabla U \cdot \nabla\phi \, dx = \int_\Omega \alpha U \cdot \big(r(x) - p(x)U\big)\phi \, dx$ for any $\phi \in H^1(\Omega)$.
We say that $U \in H^1(\Omega)$ is a **weak subsolution** (or **weak supersolution**) of (30) if $U \geq 0$ a.e. on Ω and $d\displaystyle\int_\Omega \nabla U \cdot \nabla\phi \, dx - \int_\Omega \alpha U \cdot \big(r(x) - p(x)U\big)\phi \, dx \leq$ (or $\geq$) 0 for any $\phi \in H^1(\Omega)$ with $\phi \geq 0$ a.e. on Ω.

Remark 5.1. It is straightforward to see that $U \equiv 0$ is a weak solution of (30). This will be called from now on the *trivial solution*. Moreover it is an unstable solution. Indeed, if we take some $\varepsilon < \dfrac{\rho}{\|p\|_{L^\infty(\Omega)}}$, then for any $\delta > 0$ if we take $u_0 \equiv \delta$ we have from Proposition 3.4 **(1)** that: $u(t,x) \geq \dfrac{\rho\delta}{\exp(-\alpha\rho t)\cdot(\rho - \delta\|p\|_{L^\infty(\Omega)}) + \delta\|p\|_{L^\infty(\Omega)}}$ for a.a. $(t,x) \in (0,\infty)\times\Omega$. But as $t \longrightarrow \infty$ the right hand side tends to $\dfrac{\rho}{\|p\|_{L^\infty(\Omega)}}$. So no matter how small is the norm of the initial data $\|u_0\|_{L^\infty(\Omega)}$ we have that $\|u\|_{L^\infty((0,\infty)\times\Omega)} > \varepsilon$. This shows that the trivial solution $U \equiv 0$ is **unstable.**[18]

Remark 5.2. Problem (30) cannot have weak solutions with $U \leq 0$ a.e. on Ω and $U \not\equiv 0$. Indeed: suppose that $U \leq 0$ a.e. on Ω. Then: $-d\Delta U = \alpha \underbrace{U}_{\leq 0} \cdot \underbrace{(r - pU)}_{\geq r \geq \rho} \leq \alpha\rho U$. Thus: $0 = d\int_\Omega \nabla U \cdot \nabla 1 \, dx \overset{\text{Green}}{=} -d\int_\Omega 1\cdot\Delta U \, dx \leq \underbrace{\alpha\rho}_{>0} \int_\Omega U \, dx$. So $0 = \int_\Omega 0 \, dx \geq \int_\Omega U \, dx \geq 0 \;\Rightarrow\; \int_\Omega \underbrace{U}_{\leq 0} \, dx = 0 \;\Rightarrow\; U = 0$ a.e. on Ω, which is impossible.

[17] For further details we recommend the monograph [16, Chapters 3 and 4].
[18] For two definitions of stability see [28, Page 9] and [29, Page 126].

Remark 5.3. Problem (30) can have weak sign-changing (nodal) solutions. For example consider the problem $\begin{cases} -U'' = U \cdot \left(0.4 + \frac{1}{3}\cos(x) - U\right), \ x \geq 0 \\ U(0) = 0.5, \ U'(0) = 0 \end{cases}$. As we can notice in fig. 1 we might have $U \geq \frac{r}{p}$ on a subset of Ω having strict positive measure even though $U \leq \overline{U} = \left\| \frac{r}{p} \right\|_{L^\infty(\Omega)}$ a.e. on Ω. In order to have Neumann boundary conditions we consider the problem on the x-axis interval between points A and B.

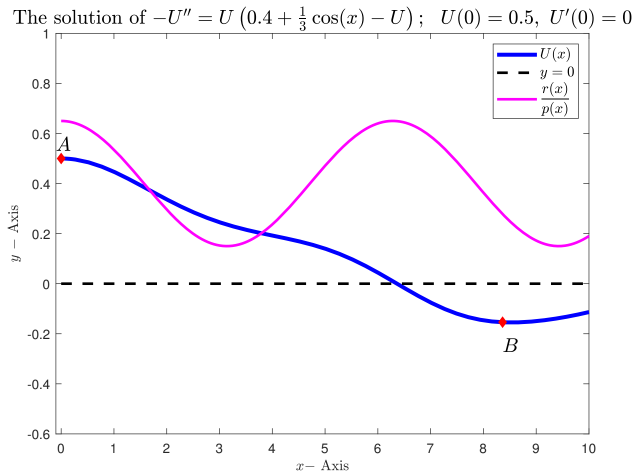

FIGURE 1. A one-dimesional counterexample

Proposition 5.1. *If $U \in H^1(\Omega)$ is a weak solution of (30) then $U \leq \overline{U} = \left\| \frac{r}{p} \right\|_{L^\infty(\Omega)} \leq 1$ a.e. on Ω.*

Proof. Set $W = \overline{U} - U$. We have that:

$$\begin{aligned} -d\Delta W = d\Delta U &= -\alpha U(r - pU) = -\alpha(\overline{U} - W)(r - p\overline{U} + pW) \\ &= \alpha\overline{U}(p\overline{U} - r) - \alpha(2p\overline{U} - r)W + \alpha pW^2 \Rightarrow \end{aligned}$$

$$-d\Delta W + \underbrace{\alpha(2p\overline{U} - r)}_{\geq 0} W = \underbrace{\alpha\overline{U}(p\overline{U} - r)}_{\geq 0} + \underbrace{\alpha pW^2}_{\geq 0} \geq 0$$

We have used that $2p\overline{U} - r \geq p\overline{U} - r = p\left(\overline{U} - \frac{r}{p}\right) = p\left(\left\| \frac{r}{p} \right\|_{L^\infty(\Omega)} - \frac{r}{p}\right) \geq 0$ a.e. on Ω. Since $\frac{\partial W}{\partial \nu} = -\frac{\partial U}{\partial \nu} = 0 \geq 0$ on $\partial\Omega$ we deduce from the *weak minimum principle* that $W \geq 0$ a.e. on Ω which means that $U \leq \overline{U}$ a.e. on Ω. $\square$

Theorem 5.2. *Problem* (30) *has a unique nontrivial weak solution* $U \in H^1(\Omega)$*. Moreover* $\operatorname*{ess\,inf}_{\Omega} U > 0$.

Proof. Let $\lambda \geq 2\alpha\|p\|_{L^\infty(\Omega)}$ be a constant. Define the function $f : \Omega \times [0,1] \to \mathbb{R}$, $f(x,U) = \alpha U(r(x) - p(x)U) + \lambda U$, for any $(x,U) \in \Omega \times [0,1]$. Notice that for any fixed $x \in \Omega$ we have that $\dfrac{\partial f}{\partial U} = \alpha r(x) + \lambda - 2\alpha p(x)U \geq \alpha\rho > 0$ for all $U \in [0,1]$. Thus $f(x,\cdot)$ is a strictly increasing function for any $x \in \Omega$. Moreover for any $U \in \mathcal{U} = \{W \in L^\infty(\Omega) \mid 0 \leq W \leq 1\}$ we have that $0 \leq f(x,0) \leq f(x,U(x)) \leq f(x,1) = \lambda - \alpha(p(x) - r(x)) \leq \lambda$ for a.a. $x \in \Omega$ which means that $\mathcal{U} \ni U \mapsto f(\cdot,U) := \alpha U(r - pU) + \lambda U \in L^\infty(\Omega)^+ \subset L^2(\Omega)$ (because Ω is bounded). Let us denote by V the unique solution (see Theorem 4.1) of the following elliptic problem:

$$\begin{cases} -d\Delta V + \lambda V = f(x,U(x)), & x \in \Omega \\ \dfrac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega \end{cases} \tag{31}$$

Since $-d\Delta V + \lambda V = f(x,U(x)) \geq 0$ on Ω and $\dfrac{\partial V}{\partial \nu} = 0 \geq 0$ on $\partial\Omega$ we deduce from the *weak minimum principle* that $V \geq 0$ a.e. on Ω. In a similar manner: $-d\Delta(1-V) + \lambda(1-V) = \lambda - (-d\Delta V + \lambda V) = \lambda - f(x,U(x)) \geq 0$ on Ω and $\dfrac{\partial(1-V)}{\partial\nu} = -\dfrac{\partial V}{\partial \nu} = 0 \geq 0$ on $\partial\Omega$. Hence from the *weak minimum principle* that $1 \geq V$ a.e on Ω. This proves that $V \in \mathcal{U}$.
Now we can define the following mapping $S : \mathcal{U} \to \mathcal{U}$ given by: $S(U) = V$. Next, we will point out some of the properties of S.
(Fact 1): S is a monotone operator. Let $U_1, U_2 \in \mathcal{U}$ with $U_1 \leq U_2$ a.e. on Ω and $S(U_1) = V_1$, $S(U_2) = V_2$. Consider $V = V_2 - V_1$ Then:

$$\begin{aligned} -d\Delta V + \lambda V = f(x,U_2(x)) - f(x,U_1(x)) &= \alpha \underbrace{(U_2 - U_1)}_{\geq 0} \left(r + \frac{\lambda}{\alpha} - p(U_1 + U_2) \right) \\ &\geq \alpha(U_2 - U_1)\left(\rho + 2\|p\|_{L^\infty(\Omega)} - 2p\right) \geq \alpha\rho(U_2 - U_1) \geq 0 \end{aligned}$$

Having also that $\dfrac{\partial V}{\partial \nu} = 0 \geq 0$ on $\partial\Omega$ we get from the *weak minimum principle* that $V \geq 0$ a.e. on Ω, i.e. $V_1 = S(U_1) \leq V_2 = S(U_2)$ a.e. on Ω.
(Fact 2): S is a strongly monotone operator. We have to show here that for any $U_1, U_2 \in \mathcal{U}$ with $U_2 \geq U_1$ and $U_1 \not\equiv U_2$ then $\operatorname*{ess\,inf}_{\Omega} S(U_2) - S(U_1) > 0$. Setting $V = S(U_2) - S(U_1)$, as we have proved above we get that $-d\Delta V + \lambda V \geq \alpha\rho(U_2 - U_1) \geq 0$ on Ω and $\dfrac{\partial V}{\partial \nu} = 0$ on $\partial\Omega$. Applying Theorem 4.1 **(7)** we obtain that there is a constant $C(\Omega,\lambda)$ with $\int_\Omega V\, dx \leq C(\Omega,\lambda) \cdot \operatorname*{ess\,inf}_{\Omega} V$. Suppose that $\operatorname*{ess\,inf}_{\Omega} V = 0$. Then $\int_\Omega V\, dx = 0$ and since $V \geq 0$ a.e. on Ω it follows that $V \equiv 0$. But this will imply that $0 = -d\Delta V + \lambda V \geq \alpha\rho(U_2 - U_1) \geq 0$, i.e. $U_1 = U_2$ a.e. on Ω which is false.

(Fact 3): S is Lipschitz-continuous with respect to the norms of $L^2(\Omega)$ and $L^\infty(\Omega)$. Let any $U_1, U_2 \in \mathcal{U}$. Then:

$$\begin{cases} -d\Delta V + \lambda V = f(x, U_2(x)) - f(x, U_1(x)), & x \in \Omega \\ \dfrac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega \end{cases},$$

where $V = S(U_2) - S(U_1)$. Using Theorem 4.1 **(1)** we obtain that:

$$\begin{aligned} \|S(U_2) - S(U_1)\|_{L^2(\Omega)} &\le \frac{1}{\lambda}\|f(\cdot, U_2) - f(\cdot, U_1)\|_{L^2(\Omega)} \\ &= \frac{1}{\lambda}\|(U_2 - U_1)(\alpha r + \lambda - \alpha p(U_1 + U_2))\|_{L^2(\Omega)} \\ &\le \frac{1}{\lambda}\|U_2 - U_1\|_{L^2(\Omega)} \cdot \|\underbrace{\alpha r + \lambda - \alpha p(U_1 + U_2)}_{\ge \alpha\rho > 0}\|_{L^\infty(\Omega)} \\ &\le \frac{1}{\lambda}\|U_2 - U_1\|_{L^2(\Omega)} \cdot (\alpha\|r\|_{L^\infty(\Omega)} + \lambda) \\ &= \left(1 + \frac{\alpha\|r\|_{L^\infty(\Omega)}}{\lambda}\right)\|U_2 - U_1\|_{L^2(\Omega)} \end{aligned}$$

From Theorem 4.1 **(3)** we deduce that the same relations as above hold by replacing $L^2(\Omega)$ with $L^\infty(\Omega)$. So: $\|S(U_2) - S(U_1)\|_{L^\infty(\Omega)} \le \left(1 + \frac{\alpha\|r\|_{L^\infty(\Omega)}}{\lambda}\right)\|U_2 - U_1\|_{L^\infty(\Omega)}$.

(Fact 4) If $U \le \overline{U} = \left\|\frac{r}{p}\right\|_{L^\infty(\Omega)}$ a.e. on Ω then $S(U) \le \overline{U}$ a.e. on Ω and if in addition $U \not\equiv \overline{U}$ or $\frac{r}{p}$ is not constant then $\operatorname{ess\,inf}_\Omega \overline{U} - S(U) > 0$.[19]

Indeed if $V = S(U)$ then $-d\Delta\left(\overline{U} - V\right) + \lambda\left(\overline{U} - V\right) = \lambda\overline{U} - f(x, U(x)) = \lambda\overline{U} - f(x, \overline{U}) + f(x, \overline{U}) - f(x, U(x)) = \lambda\overline{U} - \alpha\overline{U}(r - p\overline{U}) - \lambda\overline{U} + (\overline{U} - U)(\alpha r + \lambda - \alpha p(\overline{U} + U)) \ge \alpha p\overline{U}\left(\overline{U} - \frac{r}{p}\right) + \alpha\rho(\overline{U} - U) \ge 0$ on Ω and $\dfrac{\partial(\overline{U} - V)}{\partial\nu} = 0 \ge 0$ on $\partial\Omega$. Thus from the *weak minimum principle* we get that $V \le \overline{U}$ a.e. on Ω. Moreover, if $U \not\equiv \overline{U}$ or $\frac{r}{p}$ is not constant, then from Theorem 4.1 **(7)**, we obtain that $\operatorname{ess\,inf}_\Omega \overline{U} - V > 0$.

(Fact 5) If $U \ge \underline{U} = \operatorname{ess\,inf}_\Omega \frac{r}{p} \ge \frac{\rho}{\|p\|_{L^\infty(\Omega)}} > 0$ a.e. on Ω then $S(U) \ge \underline{U}$ a.e. on Ω and if in addition $U \not\equiv \underline{U}$ or $\frac{r}{p}$ is not constant then $\operatorname{ess\,inf}_\Omega S(U) - \underline{U} > 0$.[20]

Set $V = S(U)$. Then $-\Delta(V - \underline{U}) + \lambda(V - \underline{U}) = f(x, U(x)) - \lambda\underline{U} = f(x, U(x)) - f(x, \underline{U}) + f(x, \underline{U}) - \lambda\underline{U} = (U - \underline{U})(\alpha r + \lambda - \alpha p(U + \underline{U})) + \alpha\underline{U}(r - p\underline{U}) + \lambda\underline{U} - \lambda\underline{U} \ge \alpha\rho(U - \underline{U}) + \alpha\underline{U}p\left(\frac{r}{p} - \underline{U}\right) \ge 0$ on Ω and $\dfrac{\partial(V - \underline{U})}{\partial\nu} = 0 \ge 0$. Using again the *weak minimum principle* we get that $V \ge \underline{U}$ a.e. on Ω. Moreover, if $U \not\equiv \underline{U}$ or $\frac{r}{p}$ is not constant, then from Theorem 4.1 **(7)**, we obtain that $\operatorname{ess\,inf}_\Omega V - \underline{U} > 0$.

(Fact 6) If $U \ge 0$ and $U \not\equiv 0$ then $\operatorname{ess\,inf}_\Omega S(U) > 0$.

[19] In particular, from $\overline{U} \le \overline{U}$ a.e. on Ω we have that $S(\overline{U}) \le \overline{U}$.

[20] In particular, from $\underline{U} \ge \underline{U}$ a.e. on Ω we have that $S(\underline{U}) \ge \underline{U}$.

Notice that $-d\Delta V+\lambda V=f(x,U(x))\geq\alpha\rho U\geq 0$ and $\not\equiv 0$. So from Theorem 4.1 **(7)** we obtain the above statement.

(Fact 7) If $U\not\equiv 1$ **or** $r\not\equiv p$ **then** $\operatorname*{ess\,inf}_{\Omega} 1-S(U)>0$**.**

Observe that $-d\Delta(1-V)+\lambda(1-V)=\lambda-f(x,U(x))=\lambda-f(x,1)+f(x,1)-f(x,U(x))\geq\alpha(p-r)+\alpha\rho(1-U)\geq 0$ and $\not\equiv 0$. Thence from Theorem 4.1 **(7)** we get the conclusion.

We have all ingredients to start the proof. It is obvious that $0<\underline{U}\leq\dfrac{r}{p}\leq\overline{U}\leq 1$ a.e. on Ω and the equality $\underline{U}=\overline{U}$ holds iff $\dfrac{r}{p}$ is a constant function a.e. on Ω (not necessarily equal to 1).

(Existence) We introduce the following recurrence:

$$\begin{cases}U_{n+1}=S(U_n), & n\geq 0\\ U_0\in\mathcal{U}\end{cases}. \tag{32}$$

If we choose $U_0\equiv 1$ then, from $\mathcal{U}\ni S(1)\leq 1$ a.e. on Ω we get inductively using the monotony of S that $U_0=1\geq S(1)=U_1\geq S(S(1))=U_2\geq\cdots\geq U_n\geq\ldots$. Now, since $1\geq\operatorname*{ess\,inf}_{\Omega}\frac{r}{p}=\underline{U}$, using **Fact 5** we easily see inductively that $U_n\geq\underline{U}>0$ a.e. on Ω for any $n\geq 0$. For a.a $x\in\Omega$ the sequence $\big(U_n(x)\big)_{n\geq 0}$ is decreasing and bounded below, hence convergent to some $U(x)\in[\underline{U},1]$. Moreover, because $U_n(x)\leq 1$ for any $n\geq 0$ and a.a. $x\in\Omega$, we deduce from *Lebesgue dominated convergece theorem*, taking into account that $1\in L^2(\Omega)$ (Ω is bounded), that $U_n\to U$ in $L^2(\Omega)$. But, from **Fact 2** we know that S is continuous with respect to the norm $\|\cdot\|_{L^2(\Omega)}$ which means that $U_{n+1}=S(U_n)\to S(U)$ in $L^2(\Omega)$. Since $U_{n+1}\to U$ in $L^2(\Omega)$ we get that $S(U)=U$, i.e. U is a weak nontrivial solution of (30) and $\operatorname*{ess\,inf}_{\Omega}\ U\geq\underline{U}>0$. The existence part is now completed.

(Uniqueness) Suppose that $\tilde{U}\in H^1(\Omega)$, $\tilde{U}\geq 0$ a.e. on Ω, $\tilde{U}\not\equiv 0$ and $\tilde{U}\not\equiv U$ is another nontrivial solution of (30). From the definition 5.1, knowing that $U,\tilde{U}\in H^1(\Omega)$ we obtain that:

$$\begin{cases}\displaystyle\int_\Omega\nabla U\cdot\nabla\tilde{U}\ dx=\alpha\int_\Omega U\tilde{U}(r-pU)\ dx\\ \displaystyle\int_\Omega\nabla\tilde{U}\cdot\nabla U\ dx=\alpha\int_\Omega\tilde{U}U(r-p\tilde{U})\ dx\end{cases}\Rightarrow\int_\Omega pU\tilde{U}\cdot(U-\tilde{U})\ dx=0. \tag{33}$$

From Proposition 5.1 we have that $1\geq\tilde{U}$ a.e. on Ω, so $\tilde{U}\in\mathcal{U}$. Thus, from the monotony of S we infer that $U_1=S(1)\geq S(\tilde{U})=\tilde{U}$ and inductively $U_n\geq\tilde{U}$ a.e. on Ω for each $n\geq 0$. Passing to the limit (pointwise) we get that $U\geq\tilde{U}$ a.e. on Ω.

Henceforth relation (33) gives us that $pU\tilde{U}(U-\tilde{U})=0$ a.e. on Ω. But $p\geq r\geq\rho>0$ and $U\geq\underline{U}>0$ and this means that $\tilde{U}(U-\tilde{U})=0$. Since $\tilde{U}\not\equiv 0$ and $\tilde{U}\not\equiv U$ we have that $\tilde{U}(x)=\begin{cases}0, & x\in\omega\\ U(x), & x\in\Omega\setminus\omega\end{cases}$ for some subset $\omega\subset\Omega$. Because $\tilde{U}\in H^1(\Omega)$ we get that $\omega=\tilde{U}^{-1}(0)$ which is a measurable set with strict positive measure. We mention that $\Omega\setminus\omega$ has strict positive measure too. Again, from $\tilde{U}\in H^1(\Omega)$ and $\underline{U}\in H^1(\Omega)$ we have that $\min\{\tilde{U},\underline{U}\}=\underline{U}\cdot\chi_{\Omega\setminus\omega}\in H^1(\Omega)\ \Rightarrow\ \chi_{\Omega\setminus\omega}\in H^1(\Omega)$ (see [18, Remarks 2.3, page 27]). So $\nabla\chi_{\Omega\setminus\omega}=0$ a.e. on Ω. Using now that Ω is connected we deduce that $\chi_{\Omega\setminus\omega}$ must be a constant a.e. on Ω which is clearly false. The uniqueness is completely proved. □

6. Asymptotic behaviour of u

In this section we further assume that Ω is convex. Let $U \in H^1(\Omega)$ be the unique nontrivial solution of (30). We introduce the following linear eigenvalue problem:

$$\begin{cases} -d\Delta\Psi - \alpha r\Psi + 2\alpha pU\Psi = \lambda\Psi, & x \in \Omega \\ \dfrac{\partial\Psi}{\partial\nu} = 0, & x \in \partial\Omega \end{cases} \tag{34}$$

For any $\zeta > \alpha\|2pU - r\|_{L^\infty(\Omega)}$ and any $f \in L^2(\Omega)$ we can show, as in Theorem 4.1, that the problem $\begin{cases} -d\Delta\psi + (\zeta - \alpha r + 2\alpha pU)\psi = f(x), & x \in \Omega \\ \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega \end{cases}$ has a unique solution $\psi_f \in H^2(\Omega)$ and there is a constant $c > 0$ depending only on Ω and ζ such that $\|\psi_f\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}$. From *Rellich-Kondrachov theorem*[21] we know that $H^2(\Omega) \overset{c}{\hookrightarrow} C(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)$. So there is a constant $c_1 > 0$ depending only on Ω such that $\|\phi\|_{C(\overline{\Omega})} \leq c_1 \cdot \|\phi\|_{H^2(\Omega)}$, $\forall\ \phi \in H^2(\Omega)$. These facts allows us to define the linear operator $\mathcal{T} : L^\infty(\Omega) \to L^\infty(\Omega)$, $\mathcal{T}f := \psi_f$, $\forall\ f \in L^\infty(\Omega)$. We introduce $\mathcal{K} := L^\infty(\Omega)^+ = \left\{f \in L^\infty(\Omega) \mid f \geq 0, \text{ a.e. on } \Omega\right\}$ which is a total order cone on $L^\infty(\Omega)$ with $\text{int}(\mathcal{K}) = \left\{f \in L^\infty(\Omega) \mid \operatorname*{ess\,inf}_{\Omega} f > 0\right\} \neq \emptyset$.

Lemma 6.1. *The linear operator $\mathcal{T}$ has the following properties:*
(1) *$\mathcal{T}$ is bounded;*
(2) *$\mathcal{T}$ is a compact operator;*
(3) *$\mathcal{T}$ is strongly positive;*[22]

Proof. **(1)** For any $f \in L^\infty(\Omega)$ we have that: $\|\mathcal{T}f\|_{L^\infty(\Omega)} = \|\mathcal{T}f\|_{C(\overline{\Omega})} \leq c_1\|\mathcal{T}f\|_{H^2(\Omega)}$ $\leq c_1c\|f\|_{L^2(\Omega)} \leq c_1c\sqrt{\mathcal{L}^N(\Omega)} \cdot \|f\|_{L^\infty(\Omega)}$. Thus $\mathcal{T}$ is bounded.
(2) If $(f_n)_{n\geq 1}$ is a bounded sequence from $L^\infty(\Omega)$, then $(\mathcal{T}f_n)_{n\geq 1}$ is a bounded sequence from $H^2(\Omega)$ since $\|\mathcal{T}f_n\|_{H^2(\Omega)} \leq c\|f_n\|_{L^2(\Omega)} \leq c\sqrt{\mathcal{L}^N(\Omega)} \cdot \|f\|_{L^\infty(\Omega)}$, $\forall\ n \geq 1$.
Because the embedding $H^2(\Omega) \overset{c}{\hookrightarrow} C(\overline{\Omega})$ is compact, we know that the identity operator $I : H^2(\Omega) \to C(\overline{\Omega})$ is compact. So the $H^2(\Omega)$-bounded sequence $(\mathcal{T}f_n)_{n\geq 1}$ is mapped by I into a sequence (itself) that has a convergent subsequence to an element of $C(\overline{\Omega}) \subset L^\infty(\Omega)$. Thus $\mathcal{T}$ is compact.
(3) The positivity of $\mathcal{T}$ follows from the *weak minimum principle* in a standard manner. The strong positivity follows from [21, Lemma 2.1]. □

Now we are in position to apply *Krein-Rutman's theorem* for $\mathcal{T}$ and deduce that $r(\mathcal{T}) > 0$ is a simple eigenvalue of $\mathcal{T}$ with some eigenfunction $\Psi_1 \in L^\infty(\Omega)$ with $\operatorname*{ess\,inf}_{\Omega} \Psi_1 > 0$ and $\|\Psi_1\|_{L^2(\Omega)} = 1$. Define the principal eigenvalue of the problem (34)

[21] For the proof, see [1, Theorem 6.3, page 168] and [30, Theorem 7.97, page 491].
[22] For the definition see the statement of Theorem 10.4 from the appendix.

by $\lambda_1 := \dfrac{1}{r(\mathcal{T})} - \zeta$. It is easy to check that $\Psi = \Psi_1$ and $\lambda = \lambda_1$ is a solution to the eigenvalue problem (34). Moreover from the *Rayleigh-Ritz* variational caracterization of the principal eigenvalue[23] we have that:

$$\lambda_1 = \min_{\psi \in H^1(\Omega)\setminus\{0\}} \frac{d \displaystyle\int_\Omega |\nabla \psi|^2 \, dx + \alpha \int_\Omega (2pU - r)\psi^2 \, dx}{\displaystyle\int_\Omega \psi^2(x) \, dx} \tag{35}$$

Remark 6.1. Taking $\psi = 1_\Omega$ in (35) we get that $\lambda_1 \leq \alpha \displaystyle\int_\Omega 2pU - r \, dx$. Moreover, since the first term in the numerator is positive, we easily obtain that $\lambda_1 \geq \alpha \cdot \operatorname{ess\,inf}_\Omega 2pU - r$. So if $\overline{U} < 2\underline{U}$ then $\lambda_1 > 0$.

Remark 6.2. For the trivial steady-state $U \equiv 0$ we get that $\lambda_1 \leq -\alpha \displaystyle\int_\Omega r(x) \, dx \leq -\alpha\rho\mathcal{L}^N(\Omega) < 0$. This shows one more time that the trivial steady-state is unstable.

Theorem 6.2. *Let u be the solution of (1). If $\lambda_1 > 0$, then for any $u_0 \in \mathcal{U}$ with $u_0 \not\equiv 0$ we have that for $\sigma := \dfrac{1}{\operatorname{ess\,inf}_\Omega \Psi_1}$ and $t_0 := \dfrac{2}{\lambda_1} \ln\left(\dfrac{2\alpha\sigma\|\Psi_1\|_{L^\infty(\Omega)}}{\lambda_1}\right)$ the following inequality holds:*

$$\|u(t,\cdot) - U\|_{L^\infty(\Omega)} \leq \sigma \exp\left(-\frac{\lambda_1 t}{2}\right) \cdot \|\Psi_1\|_{L^\infty(\Omega)}, \ \forall \ t > t_0. \tag{36}$$

In particular $\lim_{t\to\infty} \|u(t,\cdot) - U\|_{L^\infty(\Omega)} = 0$.[24]

Remark 6.3. Note the crucial fact that the convergence speed (i.e., the right-hand side of the inequality) does not depend on the choice of u_0, as neither σ, λ_1, Ψ_1, nor t_0 depend on u_0.

Proof. Define for $(t,x) \in (0,\infty) \times \Omega$: $\overline{u}(t,x) = U(x) + \sigma \exp(-\lambda_1 t) \cdot \Psi_1(x)$ and $\underline{u}(t,x) = U(x) - \sigma \exp\left(-\frac{\lambda_1 t}{2}\right) \cdot \Psi_1(x)$. By direct computation we get that:

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} - d\Delta\overline{u} - \alpha\overline{u}(r - p\overline{u}) = \alpha p \sigma^2 \exp(-2\lambda_1 t)\Psi_1^2 \geq 0 \\ \dfrac{\partial \overline{u}}{\partial \nu} = 0, \ (t,x) \in (0,\infty) \times \Omega \\ \overline{u}(0,x) = U(x) + \sigma\Psi_1(x) \geq U(x) + 1, \ x \in \Omega \end{cases} . \tag{37}$$

For any $T > 0$, setting $w = \overline{u} - u$ it will follow for $c = \alpha(p(u+\overline{u}) - r) \in L^\infty((0,T)\times\Omega)$ that:

$$\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + cw = \alpha p \sigma^2 \exp(-2\lambda_1 t)\Psi_1^2 \geq 0, \ (t,x) \in (0,T) \times \Omega \\ \dfrac{\partial w}{\partial \nu} = 0 \geq 0, \ (t,x) \in (0,\infty) \times \Omega \\ w(0,x) = U(x) + \sigma\Psi_1(x) - u_0(x) \geq U(x) + 1 - u_0(x) \geq 0, \ x \in \Omega \end{cases} . \tag{38}$$

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[23]See [30, Theorem 7.76, page 412].
[24]The idea is taken from [26, Theorem 6.3, page 209].

From the *weak parabolic minimum principle* we deduce that $w \geq 0$ a.e. on $(0,T)\times\Omega$ for any $T>0$. Thus $\overline{u} \geq u$ a.e. on $(0,\infty)\times\Omega$.
Similarly one can get that:

$$\begin{cases} \dfrac{\partial \underline{u}}{\partial t} - d\Delta \underline{u} - \alpha \underline{u}(r - p\underline{u}) = -\frac{1}{2}\sigma \exp(-\lambda_1 t)\Psi_1 \cdot (\lambda_1 - 2\alpha\sigma \exp(-\lambda_1 t/2)\Psi_1) \leq 0 \\ \dfrac{\partial \underline{u}}{\partial \nu} = 0, \ (t,x) \in (0,\infty)\times\Omega \\ \underline{u}(0,x) = U(x) - \sigma\Psi_1(x), \ x \in \Omega \end{cases} \tag{39}$$

if $\lambda_1 \geq 2\alpha\sigma\exp(-\lambda_1 t/2)\Psi_1 \Leftrightarrow t \geq \dfrac{2}{\lambda_1}\ln\left(\dfrac{2\alpha\sigma\Psi_1}{\lambda_1}\right)$. This inequality is true for a.a. $(t,x) \in (t_0,\infty)\times\Omega$. So for any $T > t_0$, denoting $w = u - \underline{u}$ we obtain for $c = \alpha(u-\underline{u})(p(u+\underline{u}) - r) \in L^\infty((0,T)\times\Omega)$ that:

$$\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + cw = \frac{1}{2}\sigma\exp(-\lambda_1 t)\Psi_1 \cdot (\lambda_1 - 2\alpha\sigma\exp(-\lambda_1 t/2)\Psi_1) \geq 0 \\ \dfrac{\partial w}{\partial \nu} = 0 \geq 0, \ (t,x) \in (t_0,T)\times\Omega \\ w(0,x) = u_0(x) - U(x) + \sigma\Psi_1(x) \geq 1 + u_0(x) - U(x) \geq 0, \ x \in \Omega \end{cases} . \tag{40}$$

From the *weak parabolic minimum principle* we deduce that $w \geq 0$ a.e. on $(t_0,T)\times\Omega$ for any $T > t_0$. Thus $u \geq \underline{u}$ a.e. on $(t_0,\infty)\times\Omega$. Therefore we may write:

$$U + \sigma e^{-\lambda_1 t/2}\Psi_1 \geq U + \sigma e^{-\lambda_1 t}\Psi_1 \geq u(t,\cdot) \geq U - \sigma e^{-\lambda_1 t/2}\Psi_1. \tag{41}$$

This is the same as $\|u(t,\cdot) - U\|_{L^\infty(\Omega)} \leq \sigma e^{-\lambda_1 t/2}\|\Psi_1\|_{L^\infty(\Omega)}$ for $t > t_0$. □

7. Applications in digital image processing

In the rest of the paper we consider Ω to be a 2D rectangle, i.e. $\Omega = (0,a)\times(0,b)$. It is well known that PDE's and Calculus of Variations have many applications in image processing (such as segmentation, restoration, etc.) – as a good introduction in this subject we recommend [8]. Other good sources are the book [31] and the free video lectures available online at `https://www.coursera.org/learn/image-processing` both provided by professor Guillermo Sapiro.

7.1. Deforming an image in an other given image. Let's say that we have two grayscale images with the same sizes: the **initial** one which is represented by $u_0 : \Omega \to [0,1]$ and the **final** one that is given by $U : \Omega \to [0,1]$. We associate 0 to black and 1 to white. We shall also modify U such that it will not contain pure black pixels. From Theorem 6.2 we expect that $u(t,\cdot) \approx U$ for large enough t.
We want U to be the unique nontrivial solution of the problem (30), i.e.:

$$\begin{cases} -d\Delta U = \alpha U(r - pU), & x \in \Omega \\ \dfrac{\partial U}{\partial \nu} = 0, & x \in \partial\Omega \end{cases} . \tag{42}$$

Note that the Neumann boundary condition is well-suited for our situation, since the last two layers of pixels are pretty much the same for each picture in the normal

direction. From (42) we get that $p = \frac{r}{U} + \frac{d\Delta U}{\alpha U^2}$. We can choose α big enough such that $p \geq r$ on Ω. In this model r is taken as a mathematical function which creates some particular effect in the way in which u_0 is transformed into U via diffusion.
Equation (1) was discretized using an explicit finite difference scheme with 5 nodes with respect to the time variable and using the 9-points formula for the Laplacian.[25] We took into account the Neumann boundary condition to determine the values for the last layer of pixels. All the M-files and mp4 files (videos exported from Matlab to see how diffusion acts) are freely available at: https://github.com/MaxDBog/Neumann-Laplacian-on-images. The main idea of the algorithm used is described below.

Algorithm 1 The conceptual algorithm

1: Read the two images *Initial* and *Final* with the same resolution as matrices
2: Convert *Initial* and *Final* into grayscale images if necessary
3: Divide by 255 in order to obtain two matrices *u0* and U with all entries from $[0, 1]$
4: Modify the final image so that it is taken apart from pure black pixels. For example: U may be replaced by $0.8 \cdot U + 0.2$
5: Setting dx (the distance between two consecutive pixels), the timestep dt, the number of time iterations Nt and the meshgrid of pixels.
6: Define the effect function $r \geq \rho > 0$ that we want to try
7: Compute $p = \frac{r}{U} + \frac{d\Delta U}{\alpha U^2}$ as a matrix. In practice it is better to set $p = \frac{r}{U}$.
8: Initialization – define the constants of the system α, d, the initial data $u(:, :, 1) = u_0$ and the initial source $f(:, :, 1) = \alpha u_0(r - pu_0)$
9: **for** $k = 1 : Nt - 1$ **do**
10: $u(:, :, k + 1) =$ formula in terms of $u(:, :, k)$
11: $f(:, :, k + 1) = \alpha u(:, :, k)(r - pu(:, :, k))$
12: **end for**
13: Show the diffused images $u(:, :, k)$ for some values of k or even make a mp4 video. As k grows $u(:, :, k) \approx U$
14: Compute $\text{PSNR}(u(:, :, k), U)$ and $\text{PSNRgrad}(u(:, :, k), U)$

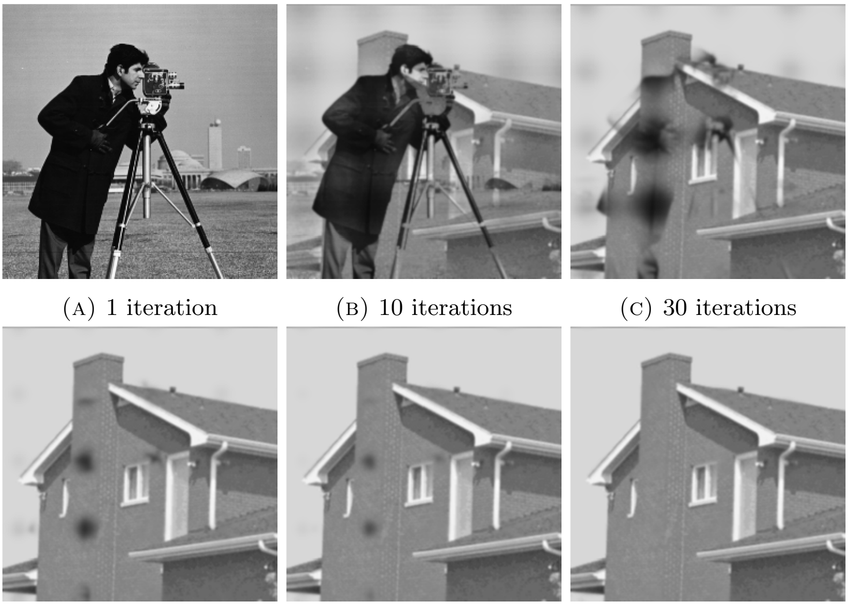
For definitions and further details about the noise estimator, PSNR and PSNRgrad we refer to [2]. This method can be easily adapted for RGB images by applying the same steps to each color channel—red, green, and blue—and then combining the results into a single three-dimensional matrix.
The simulations were performed on Matlab R2023b. The parameters I have used in these simulations are $d = 0.1$, $\alpha = 5$. Now let's see some diffusion effects in just 200 iterations:

(a) **Sinusoidal effect** for $r(x, y) = 2.5 - \sin\left(\frac{x}{10}\right) - \cos\left(\frac{y}{10}\right)$. We get $\text{PSNR}(U, u(:, :, 200)) = 30.53$. See Figure 3 and the video *image_to_image_sinusoidal_effect.mp4*.
(b) **Vertical effect** for $r(x, y) = 1.75 - \sin\left(\frac{x}{5}\right)$. We get $\text{PSNR}(U, u(:, :, 200)) = 30.33$. See Figure 4 and the video *image_to_image_vertical_effect.mp4*.

[25] See [11, Section 4.1] and [32].

(A) Initial image: cameraman (B) Final image: house

FIGURE 2. The two original images



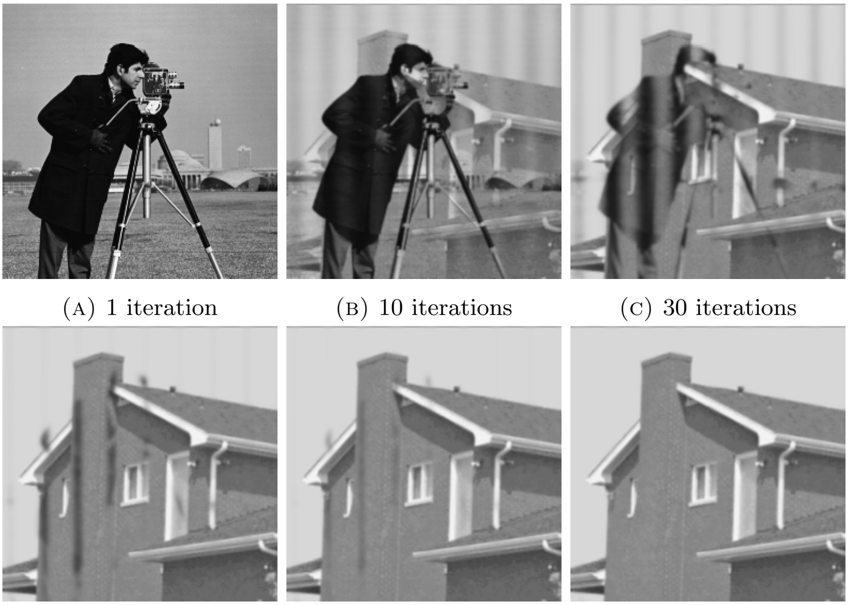
(A) 1 iteration (B) 10 iterations (C) 30 iterations

(D) 70 iterations (E) 100 iterations (F) 200 iterations

FIGURE 3. Cameraman transformed into a house - Sinusoidal effect

(c) **Horizontal effect** for $r(x, y) = 1.75 - \sin\left(\frac{y}{5}\right)$. We get $\text{PSNR}(U, u(:, :, 200)) = 30.15$. See Figure 5 and the video *image_to_image_horizontal_effect.mp4*.

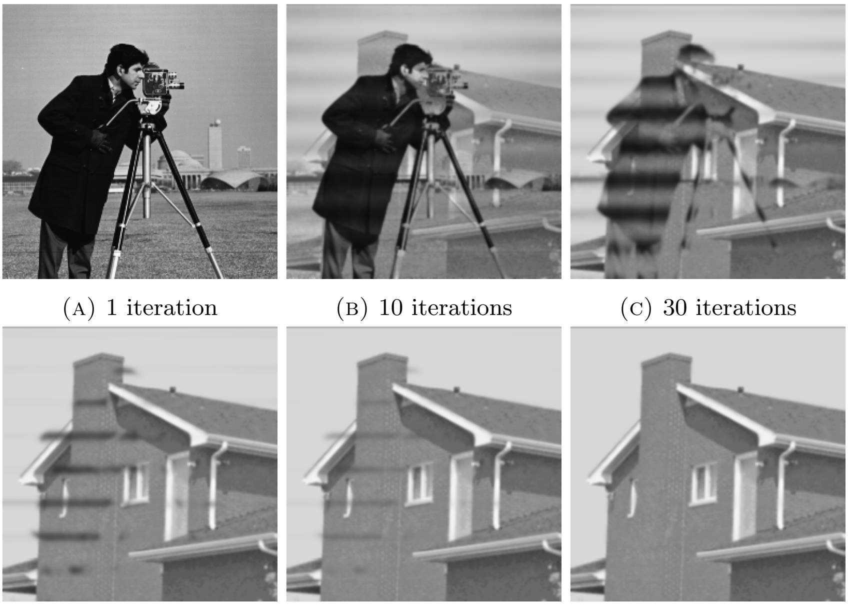
(d) **Diagonal effect** for $r(x, y) = 1.75 + \sin\left(\frac{x + y}{5}\right)$ We get $\text{PSNR}(U, u(:, :, 200)) = 30.38$. See Figure 6 and the video *image_to_image_diagonal_effect.mp4*.



(A) 1 iteration (B) 10 iterations (C) 30 iterations

(D) 70 iterations (E) 100 iterations (F) 200 iterations

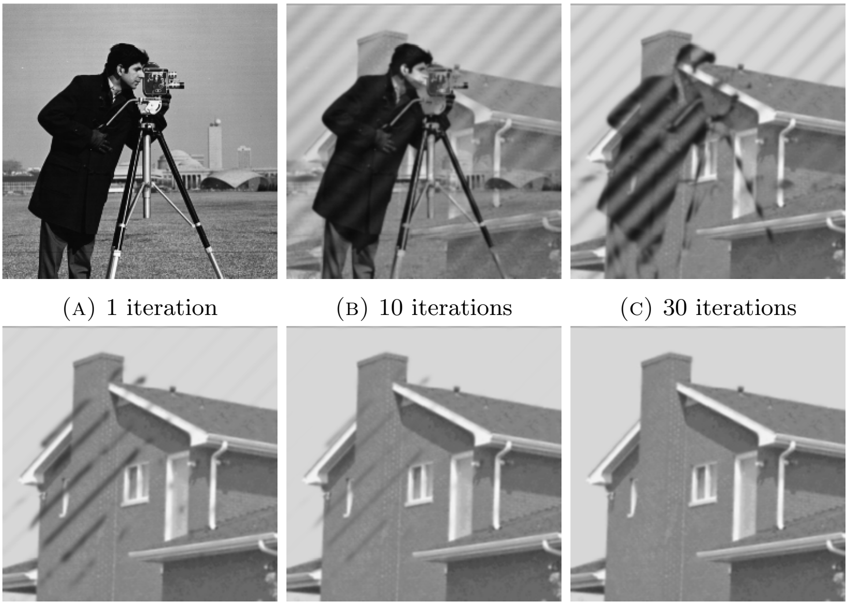
FIGURE 4. Cameraman transformed into a house - Vertical effect



(A) 1 iteration (B) 10 iterations (C) 30 iterations

(D) 70 iterations (E) 100 iterations (F) 200 iterations

FIGURE 5. Cameraman transformed into a house - Horizontal effect



(A) 1 iteration (B) 10 iterations (C) 30 iterations

(D) 70 iterations (E) 100 iterations (F) 200 iterations

FIGURE 6. Cameraman transformed into a house - Diagonal effect

(e) **Dotted effect** for $r(x, y) = 1.5 - \sin\left(\frac{x}{5}\right) \cdot \cos\left(\frac{y}{5}\right)$. We get PSNR$(U, u(:, :, 200)) = 30.08$. See Figure 7 and the video *image_to_image_dotted_effect.mp4*.
(f) **Fancy effect** for $r(x, y) = 1.5 - \sin(x^2 + y^2)$. We get PSNR$(U, u(:, :, 200)) = 30.14$. See Figure 8 and the video *image_to_image_fancy_effect.mp4*.

Typical values for the PSNR in image compression are between 30 and 40, so we have good results, but not excellent.

7.2. Making images to dissapear with Neumann diffusion. We have some initial image u_0 that will converge toward the final image $U \equiv 1$. So we may choose the effect r as we want and put $p \equiv r = \frac{r}{U} + \frac{d\Delta U}{\alpha U^2}$. Choosing $r(x, y) = 3 - \sin\left(\frac{x}{10}\right) - \cos\left(\frac{y}{10}\right)$ we get the resuts illustrated in Figure 9. To watch a video with this simulation go here *disappearing_image.mp4*.
Computing the PSNR between U (complete white) and $u(:, :, 200)$ we obtain 96.98. So our image trully disappeared. Also PSNRgrad is equal to 93.37.

7.3. Deforming grayscale images with Neumann diffusion. Rather than defining p in terms of an image, we can express it as a mathematical function. Next, we will observe how the two images used above converge toward the same mathematical solution U of (30).
For $d = 0.3$, $\alpha = 0.5$, $r(x, y) = 1 - 0.4 \cdot \sin(x/20) - 0.4 \cdot \cos(y/20)$ and $p(x, y) = 2$ we get the results presented in Figures 10 and 11. You can watch the video files associated to these figures at *neumann_diffusion_cameraman.mp4* and *neumann_diffusion_house.mp4*.

(A) 1 iteration

(B) 10 iterations

(C) 30 iterations

(D) 70 iterations

(E) 100 iterations

(F) 200 iterations

FIGURE 7. Cameraman transformed into a house - Dotted effect

(A) 1 iteration

(B) 10 iterations

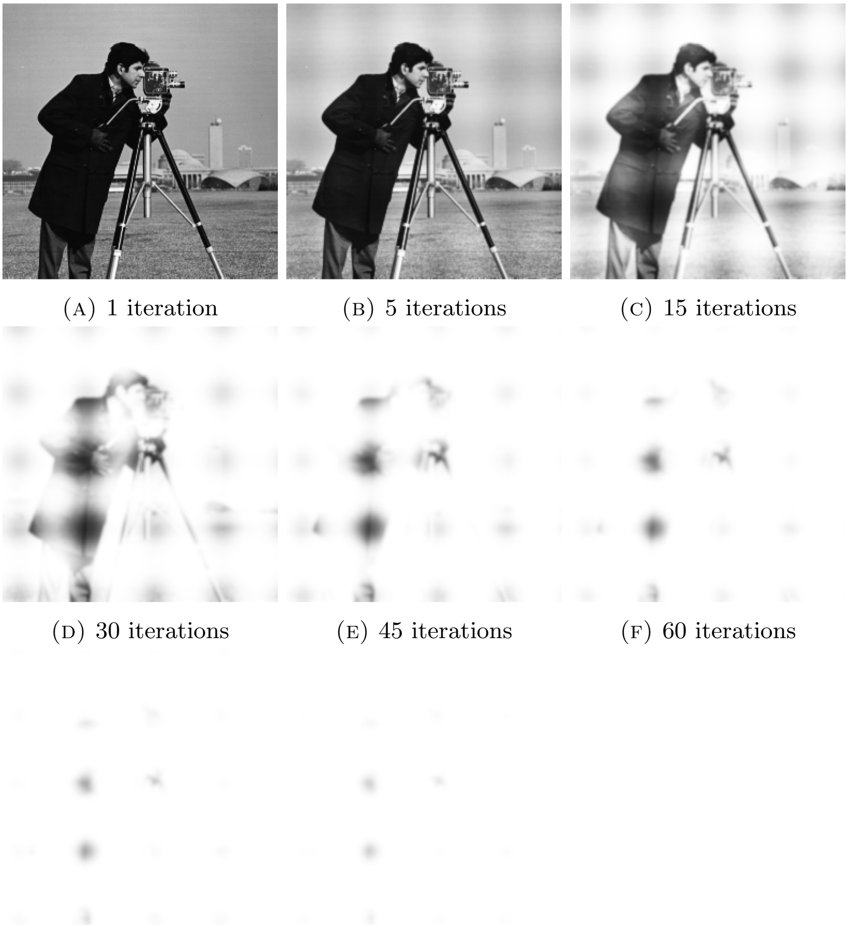
(C) 30 iterations

(D) 70 iterations

(E) 100 iterations

(F) 200 iterations

FIGURE 8. Cameraman transformed into a house - Fancy effect



(A) 1 iteration (B) 5 iterations (C) 15 iterations

(D) 30 iterations (E) 45 iterations (F) 60 iterations

(G) 75 iterations (H) 90 iterations (I) 200 iterations

FIGURE 9. The disappearance of the cameraman

The PSNR between the images obtained after 5000 iterations is equal to 140.39. So they both end up to the same thing.

8. Conclusion

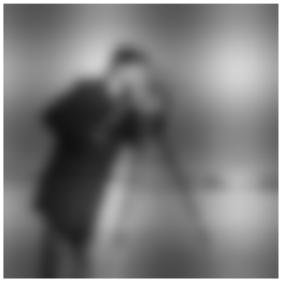
Further investigations have to be made in order to improve the numerical algorithm. Also I believe that the asymptotic stability holds in the case when Ω is not assumed to be convex.

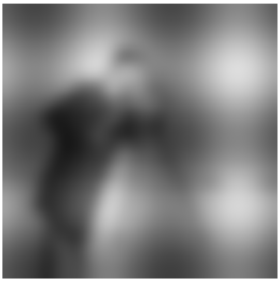
9. Acknowledgement

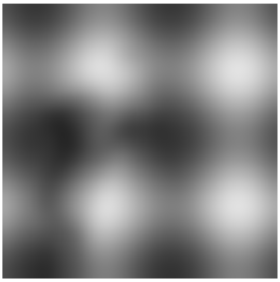
Firstly, I would like to express my gratitude to my supervisor Prof. Vicenţiu Rădulescu whose insights and expertise greatly assisted this research. Secondly I must thank


(A) 1 iteration


(B) 80 iterations


(C) 200 iterations


(D) 300 iterations


(E) 500 iterations

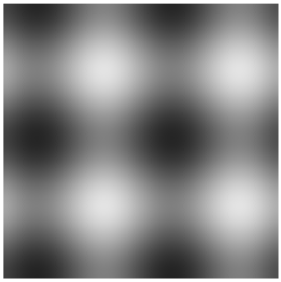
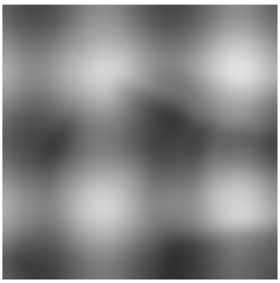

(F) 5000 iterations

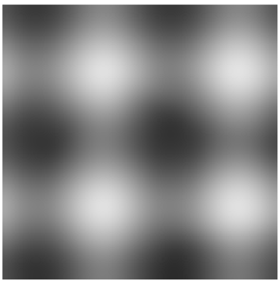
FIGURE 10. The diffusion of the cameraman


(A) 1 iteration


(B) 80 iterations


(C) 200 iterations


(D) 300 iterations


(E) 500 iterations

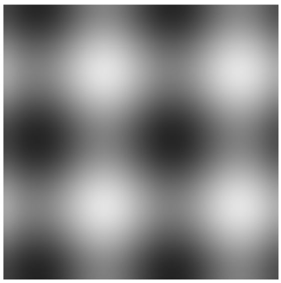

(F) 5000 iterations

FIGURE 11. The diffusion of the house

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10. Appendix

The following result is an adapted version of Corollary 8.10 from [10, page 215], taking into account that in Theorem 3.15 from [19] it is shown that: $H^1([a,b];L^2(\Omega)) \hookrightarrow L^\infty([a,b];L^2(\Omega))$ and in [19, Corollary 3.12] that $C^\infty([a,b];L^2(\Omega))$ is dense in $H^1([a,b];L^2(\Omega))$. See also [3, Proposition V.2.4.7, page 304] or [23, Theorem 35, page 21].

Lemma 10.1. *If $u, v \in H^1([a,b],L^2(\Omega))$, where $a < b$ are real numbers, then $\dfrac{\partial(u\cdot v)}{\partial t} \in L^2([a,b];L^1(\Omega))$ and:*

$$\frac{\partial(uv)}{\partial t} = \frac{\partial u}{\partial t}v + u\frac{\partial v}{\partial t}. \tag{43}$$

Furthermore, the formula for integration by parts holds:

$$\int_{t_0}^{t_1} \frac{\partial u}{\partial t}v\ dt = u(t_1,\cdot)v(t_1,\cdot) - u(t_0,\cdot)v(t_0,\cdot) - \int_{t_0}^{t_1} u\frac{\partial v}{\partial t}\ dt,\ \forall\ a \le t_0 < t_1 \le b. \tag{44}$$

Lemma 10.2 (**Ultracontractivity of the Neumann Laplacian on Lipschitz domains**). *Let any $T, \lambda > 0$ and $\big(S(t)\big)_{t\in[0,T]}$ be the C_0-semigroup associated to the Neumann Laplacian on the Lipschitz domain Ω. Then for each $u \in L^2(\Omega)$ and any $t \in (0,T]$ we have that $S(t)u \in L^\infty(\Omega)$ and moreover there is a constant $C > 0$ depending only on Ω and D, d, λ such that:*

$$\|S(t)u\|_{L^\infty(\Omega)} \le C\cdot e^{\lambda T}\cdot t^{-\frac{D}{4}}\cdot \|u\|_{L^2(\Omega)}. \tag{45}$$

Here $D = N$ if $N \ge 3$ and $D > 2$ is any constant we want in case $N \in \{1,2\}$.

Proof. From the *Sobolev embedding theorem* we know that for Lipschitz domains like Ω we have that $H^1(\Omega) \hookrightarrow L^{\frac{2D}{D-2}}(\Omega)$. Now we will use [25, Theorem 6.4, page 158] for the unbounded linear operator $-d\Delta_N + \lambda I$ associated with the form $a : H^1(\Omega)\times H^1(\Omega) \to \mathbb{R},\ a(u,v) = d\int_\Omega \nabla u\cdot\nabla v\ dx + \lambda\int_\Omega uv\ dx,\ \forall\ u,v \in H^1(\Omega)$. So there is some $C > 0$ depending on Ω and D, d, λ such that for any $u \in L^2(\Omega)$ we have that $\|S_\lambda(t)u\|_{L^\infty(\Omega)} = \|w(t,\cdot)\|_{L^\infty(\Omega)} \le C\cdot t^{-\frac{D}{4}}\cdot\|u\|_{L^2(\Omega)},\ \forall\ t \in (0,T]$, where:

$$\begin{cases} \dfrac{\partial w}{\partial t} - d\Delta w + \lambda w = 0, & (t,x) \in (0,T)\times\Omega \\ \dfrac{\partial w}{\partial \nu} = 0, & (t,x) \in (0,T)\times\partial\Omega \\ w(0,x) = u(x), & x \in \Omega \end{cases}$$

in the weak sense. Now just simply observe that $w(t,\cdot) = e^{-\lambda t}v(t,\cdot)$ where $v(t,\cdot) = S(t)u,\ \forall\ t \in [0,T]$. So $e^{-\lambda T}\|v(t,\cdot)\|_{L^\infty(\Omega)} \le \|e^{-\lambda t}v(t,\cdot)\|_{L^\infty(\Omega)} = \|w(t,\cdot)\|_{L^\infty(\Omega)} \le C\cdot t^{-\frac{D}{4}}\cdot\|u\|_{L^2(\Omega)}$ for any $t \in (0,T]$ and we are done. ∎

Remark 10.1. It is straightforward to demonstrate that $\|S(t)u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$ for any $u \in L^2(\Omega)$ (just take $\phi = v(t, \cdot)$ in the weak formulation of the problem) and $\|S(t)u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ for each $u \in L^\infty(\Omega)$ (follows from the *weak parabolic minimum principle*).

Definition 10.1. We say that $f \in L^2(\Omega)$ with $f \geq 0$ is an **almost interior point of** $L^2(\Omega)^+$ if for every $\phi \in L^2(\Omega)^+ \setminus \{0\}$ we have that $\int_\Omega f(x)\phi(x)\, dx > 0$.
Similarly we say that $f \in L^\infty(\Omega)$ with $f \geq 0$ is an **almost interior point of** $L^\infty(\Omega)^+$ if for every $\varphi \in (L^\infty(\Omega)^*)^+ \setminus \{0\}$ we have that $\varphi(f) > 0$.

Remark 10.2. • Ω being a bounded domain $f \in L^2(\Omega)^+$ is an almost interior point of $L^2(\Omega)^+$ iff $f(x) > 0$ for almost all $x \in \Omega$.
• $f \in L^\infty(\Omega)^+$ is an almost interior point of $L^\infty(\Omega)^+$ iff of there is some $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for almost all $x \in \Omega$. This is a very important distinction between the two cases.[26]

The following proposition is the key ingredient in proving strong positivity of the solution for parabolic problems. It is taken from [13, Proposition 2.21]. Here is its statement:

Proposition 10.3. *Let X and Y be two ordered Banach spaces and $T : X \to Y$ a **bounded, linear** and **positive** operator between them. Then the following two statements are equivalent:*
(1) *There is some $x \in X^+$ such that Tx is an almost interior point of Y^+.*
(2) *T maps all almost interior points of X^+ to almost interior points of Y^+.*

Theorem 10.4 (Krein-Rutman). *Let X be a Banach space and $K \subset X$ a total order cone with $\operatorname{int}(K) \neq \emptyset$. For any **compact** linear operator $\mathcal{T} : X \to X$ that is **strongly positive**, i.e. $\mathcal{T}x \in \operatorname{int}(K)$ for any $x \in K \setminus \{0_X\}$, the following statements are true:*[27]
(1) *The spectral radius of $\mathcal{T}$ is strictly positive, i.e. $r(\mathcal{T}) > 0$.*
(2) *$r(\mathcal{T})$ is a simple eigenvalue of $\mathcal{T}$ that has an eigenvector $v \in \operatorname{int}(K)$.*

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[26] See Corollary 2.8 in [13] and the statement before it.
[27] We recommend on this subject the excellent work [12].

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(Bogdan Maxim) Department of Mathematics, University of Craiova, 13 A.I. Cuza Street, Craiova, 200585, Romania
E-mail address: maxim.bogdan.n6h@student.ucv.ro