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Refinement of convergence rates for tail probabilities

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Abstract. Let $X_1, X_2, ...$ be i.i.d. random variables, and put $S_n = X_1 + ... + X_n$. We find necessary and sufficient moment conditions for $\int_\varepsilon^\infty f(x^q)dx < \infty$, $\varepsilon > \delta$, where $\delta \geq 0$ and $q > 0$, and $f(x) = \sum_n a_n P(|S_n| > xb_n)$ with $a_n > 0$ and b_n is either $n^{1/p}$, $0 < p < 2$, $\sqrt{n \log n}$ or $\sqrt{n \log \log n}$. The series $f(x)$ we deal with are classical series studied by Hsu and Robbins, Erdős, Spitzer, Baum and Katz, Davis, Lai, Gut, etc.

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1. Introduction

Let $X_1, X_2, ...$ be i.i.d. random variables with $P(X \neq 0) > 0$ and $EX = 0$, and consider the random walk $S_0 = 0$, $S_n = X_1 + ... + X_n$, $n \geq 1$. The main purpose of this communication is to find necessary and sufficient moment conditions for $\int_\varepsilon^\infty f(x^q)dx < \infty$, $\varepsilon > \delta$, where $\delta \geq 0$ and $q > 0$, and $f(x) = \sum_n a_n P(|S_n| > xb_n)$ with $a_n > 0$ and b_n is either $n^{1/p}$, $0 < p < 2$, $\sqrt{n \log n}$ or $\sqrt{n \log \log n}$. The results offer refined rates of convergence for the tail probabilities $P(|S_n| > xb_n)$ concerning either the law of large numbers, moderate deviations or the law of the iterated logarithm. The series $f(x)$ we deal with are classical series studied by Hsu and Robbins (1947), Erdős (1949,1950), Spitzer (1956), Baum and Katz (1965), Davis (1968), Lai (1974), Gut (1980), etc.

For $x > 0$, define $A_n = \{|S_n| > xn\}$, $n \geq 1$, consider the random series $N_x = \sum_{n \geq 1} I(A_n) =$ the number of exits of S_n beyond the boundary $\pm xn$, and set $D_x = \{(n, y) : |y| \leq xn\}$. With this notation, we may rephrase the Kolmogorov strong law of large numbers, and the complete convergence theorem as follows.

STRONG LLN (A. N. KOLMOGOROV (1930)): $E|X| < \infty \Longleftrightarrow N_x < \infty$ *a.s.*, $x > 0 \Longleftrightarrow$ *whatever* $x > 0$, $S_n \in D_x$ *a.s. for all but finitely many* n.

COMPLETE CONVERGENCE THEOREM (P. L. HSU and H. ROBBINS (1947); P. ERDŐS (1949,1950)): $EX^2 < \infty \Longleftrightarrow f(x) := EN_x = \sum_{n \geq 1} P(|S_n| > xn) < \infty$ *for any* $x > 0$.

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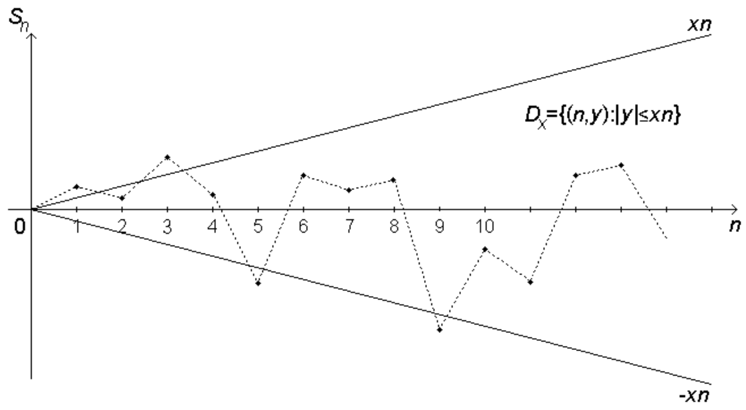

FIGURE 1. Random walk S_n with boundary $\pm xn$

We see that f is a nonincreasing function. Moreover, since $P(X \neq 0) > 0$, $\lim_{x \searrow 0} f(x) = \sum_{n\geq 1} P(S_n \neq 0) = \infty$ by the Borel-Cantelli lemma. The exact asymptotics of $f(x)$ as $x \searrow 0$ was established by Heyde.

C. C. HEYDE (1975): *$EX^2 < \infty \Longrightarrow x^2 EN_x \to EX^2$ i.e. $f(x) \sim x^{-2}EX^2)$ as $x \searrow 0$.*

This means that $\int_0^\infty f(x)dx = \infty$. What about $I_\delta := \int_\delta^\infty f(x)dx$ for some $\delta > 0$? The impetus to study the convergence of I_δ comes from the theory of branching processes, more precisely from the following theorem.

K. B. ATHREYA (1988): *Let $Z_0 = i$, $(Z_n)_{n\geq 1}$ be a critical Galton-Watson process such that $EZ_1^2 < \infty$, and put $M_n = \max_{0\leq k\leq n} Z_k$. Then $EM_n/\log n \to i$ as $n \to \infty$.*

Athreya considered a random walk $(S_n)_{n\geq 0}$ generated by the random variable X = the number of offspring produced by a single parent particle, with offspring distribution $(p_j)_{j\geq 0}$ such that $\sum_{j\geq 1} jp_j = 1$ and $\sum_{j\geq 1} j^2 p_j < \infty$. He proved that

$$\sum_{n\geq 1} \int\limits_{\left\{\frac{|S_n|}{n} > \delta\right\}} \frac{|S_n|}{n} dP < \infty \text{ for any } \delta > 0.$$

This statement is in fact a result of the form $I_\delta < \infty$. Inspection of Athreya's proof made possible the next strengthening of the complete convergence theorem under full generality.

A. SPĂTARU (1990): $EX^2 < \infty \Longleftrightarrow I_\delta = \int_\delta^\infty f(x)dx = \int_\delta^\infty \left(\sum_{n\geq 1} P(|S_n| > xn)\right) dx < \infty,\ \delta > 0$.

Actually, in accordance with the Hsu-Robbins-Erdős theorem, this means that

$$\sum_{n\geq 1} P(|S_n| > xn) < \infty,\ x > 0 \Longleftrightarrow EX^2 < \infty \Longleftrightarrow \sum_{k\geq 1}\sum_{n\geq 1} P(|S_n| > kn) < \infty.$$

An even more general result has been recently obtained by Li and Spătaru.

D. LI and A. SPĂTARU (2005): *For* $q > 0$, *we have*

$$\int_\delta^\infty f(x^q)dx < \infty,\ \delta > 0 \Longleftrightarrow \begin{cases} E|X|^{1/q} < \infty & \text{if } q < 1/2 \\ E[X^2 \log^+ |X|] < \infty & \text{if } q = 1/2 \\ EX^2 < \infty & \text{if } q > 1/2 \end{cases} .$$

Other important boundaries such as $\pm xn^{1/p},\ 0 < p < 2,\ \pm x\sqrt{n \log n},\ \pm x\sqrt{n \log \log n}$ will be considered in what follows.

2. The boundary $\pm xn^{1/p},\ 0 < p < 2,$ (Large deviations)

For $x > 0$ and $0 < p < 2$, define $A_n = \{|S_n| > xn^{1/p}\},\ n \geq 1$, and put $N_x = \sum_{n\geq 1} I(A_n)$ = the number of exits of S_n beyond the boundary $\pm xn^{1/p}$. For $r \geq 1$, consider also the random series $M_x = \sum_{n\geq 1} n^{r-2} I(A_n),\ x > 0$. Set $D_x = \{(n, y) : |y| \leq xn^{1/p}\},\ x > 0$. With this notation, the Marcinkiewicz-Zygmund strong law of large numbers, and the Hsu-Robbins/Erdős/Spitzer/Baum-Katz theorem may be rephrased as follows.

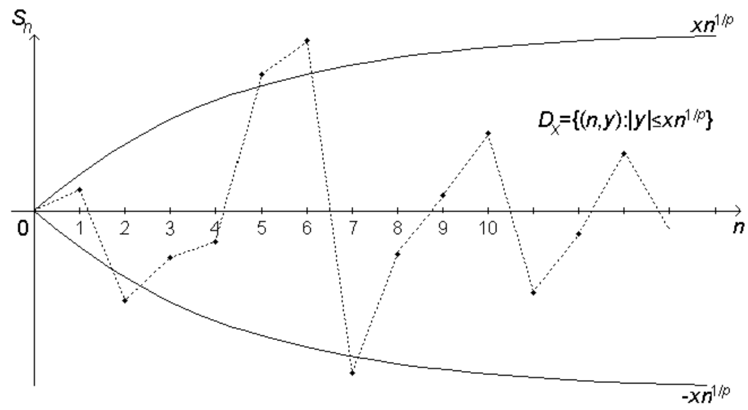

FIGURE 2. Random walk S_n with boundary $\pm xn^{1/p},\ 0 < p < 2$

STRONG LLN (J. MARCINKIEWICZ and A. ZYGMUND (1937)): $E|X|^p < \infty \Longleftrightarrow N_x < \infty$ *a.s.*, $x > 0 \Longleftrightarrow$ *whatever* $x > 0$, $S_n \in D_x$ *a.s. for all but finitely many* n.

THE HSU-ROBBINS/ERDŐS/SPITZER/BAUM-KATZ THEOREM: $E|X|^{pr} < \infty \Longleftrightarrow f(x) := EM_x = \sum_{n\geq 1} n^{r-2} P(|S_n| > xn^{1/p}) < \infty$ *for any* $x > 0$.

(For $p = 1$ and $r = 2$ this is the complete convergence theorem due to Hsu and Robbins, and Erdős. For $p = r = 1$ the result was proved by Spitzer (1956). In the general form stated here the theorem was obtained by Baum and Katz (1965).)

f is nonincreasing. Since X is not degenerated at 0, $\lim\limits_{x\searrow 0} f(x) = \lim\limits_{x\searrow 0} \sum_{n\geq 1} n^{r-2} P(|S_n| > xn^{1/p}) = \sum_{n\geq 1} n^{r-2} P(S_n \neq 0) \geq \sum_{n\geq 1} \frac{1}{n} P(S_n \neq 0) = \infty$ by the translation invariance theorem (see, e.g., Loève (1977), p. 398). The behaviour of $f(x)$ as $x \searrow 0$ was also investigated.

A. GUT and A. SPĂTARU (2000): *If* $1 \le p < 2$ *and* $EX^2 < \infty$, *then* $\sum_{n\ge 1} \frac{1}{n} P(|S_n| > xn^{1/p}) \sim \log(\frac{1}{x}) \frac{2p}{2-p}$ *as* $x \searrow 0$.

The next strengthening of the H-R/E/S/B-K theorem has been obtained.

D. LI and A. SPĂTARU (2005): *For* $q > 0$, *we have*

$$\int_\delta^\infty f(x^q)dx < \infty,\ \delta > 0 \Longleftrightarrow \begin{cases} E|X|^{1/q} < \infty & \text{if} \quad q < 1/pr \\ E[|X|^{pr} \log^+ |X|] < \infty & \text{if} \quad q = 1/pr \\ E|X|^{pr} < \infty & \text{if} \quad q > 1/pr \end{cases}.$$

3. The boundary $\pm x\sqrt{n \log n}$ (Moderate deviations)

For $x > 0$ and $r > 1$, define $M_x = \sum_{n\ge 1} n^{r-2} I\{|S_n| > x\sqrt{n \log n}\}$ = the number of exits of S_n over the boundary $\pm x\sqrt{n \log n}$ with the "weights" n^{r-2}. The next theorem involving moderate deviations is due to Lai.

T. L. LAI (1974): *Put* $f(x) = EM_x = \sum_{n\ge 1} n^{r-2} P(|S_n| > x\sqrt{n \log n})$, $x > 0$.
(i) *If* $E[|X|^{2r}(\log^+ |X|)^{-r}] < \infty$ *and* $EX^2 = \sigma^2$, *then* $f(x) < \infty$, $x > \sigma\sqrt{2r-2}$.
(ii) *If* $f(x) < \infty$ *for some* $x > 0$, *then* $E[|X|^{2r}(\log^+ |X|)^{-r}] < \infty$.

The following strengthening of Lai's theorem was proved.

D. LI and A. SPĂTARU (2005): *Let* $q > 0$.
(i) $\begin{cases} E|X|^{1/q} < \infty & \text{if} \quad q \le 1/2r \\ E[|X|^{2r}(\log^+ |X|)^{-r}] < \infty & \text{if} \quad q > 1/2r \end{cases}$ *and* $EX^2 = \sigma^2$ *imply* $\int_\delta^\infty f(x^q)dx < \infty$, $\delta > (\sigma\sqrt{2r-2})^{1/q}$.
(ii) $\int_\delta^\infty f(x^q)dx < \infty$ *for some* $\delta > 0$ *implies* $\begin{cases} E|X|^{1/q} < \infty & \text{if} \quad q \le 1/2r \\ E[|X|^{2r}(\log^+ |X|)^{-r}] < \infty & \text{if} \quad q > 1/2r \end{cases}$.

Consider now the series $f(x) = \sum_{n\ge 1} \frac{1}{n} P(|S_n| > x\sqrt{n \log n})$, $x > 0$, corresponding to the limiting case $r = 1$ above. There are no simple necessary and sufficient moment conditions for the convergence of this series, in which connection see Spătaru (2000). Nevertheless the next strengthening is possible.

D. LI and A. SPĂTARU (2005): *Let* $q > 0$.

(i) $\begin{cases} E|X|^{1/q} < \infty & \text{if} \quad q < 1/2 \\ E[X^2 \log^+ \log^+ |X|] < \infty & \text{if} \quad q = 1/2 \\ E[X^2 (\log^+ \log^+ |X|)/ \log^+ |X|] < \infty & \text{if} \quad q > 1/2 \end{cases}$ *implies* $\int_\delta^\infty f(x^q)dx < \infty$, $\delta > 0$.

(ii) $\int_\delta^\infty f(x^q)dx < \infty$ *for some* $\delta > 0$ *implies* $\begin{cases} E|X|^{1/q} < \infty & \text{if} \quad q < 1/2 \\ E[X^2 \log^+ \log^+ |X|] < \infty & \text{if} \quad q = 1/2 \\ E[X^2/ \log^+ |X|] < \infty & \text{if} \quad q > 1/2 \end{cases}$.

Notice this result exhibits a slight distance between the necessary and sufficient conditions when $q > 1/2$. It should also point out that it led to the proof of the next general statement.

D. LI and A. SPĂTARU (2005): *Let* $\{b_n;\ n \ge 1\}$ *be a sequence of positive non-decreasing numbers such that* $b_n \to \infty$ *and* $1 < c \le b_{2^n}/b_{2^{n-1}} \le C < \infty$, $n \ge 1$. *Then*

$$\frac{S_n}{b_n} \to 0 \ a.s. \quad \Longleftrightarrow \sum_{n\ge 1} \frac{1}{n} P(|S_n| > xb_n) < \infty,\ x > 0.$$

This result has been known in some special cases. Thus, for $0 < p < 2$, $S_n/n^{1/p} \longrightarrow 0$ a.s. $\iff E|X|^p < \infty \iff \sum_{n\geq 1} \frac{1}{n} P(|S_n| > xn^{1/p}) < \infty,\ x > 0$. The former equivalence here is the Marcinkiewicz-Zygmund strong law of large numbers, while the latter one is the Baum-Katz theorem.

4. The boundary $\pm x\sqrt{n \log\log n}$ (Law of the iterated logarithm)

For $x > 0$, define $M_x = \sum_{n\geq e} \frac{1}{n} I\{|S_n| > x\sqrt{n\log\log n}\}$ = the number of exits of S_n over the boundary $\pm x\sqrt{n\log\log n}$ with the "weights" $\frac{1}{n}$. Set $f(x) = EM_x = \sum_{n\geq e} \frac{1}{n} P(|S_n| > x\sqrt{n\log\log n})$, $x > 0$. Part (i) of the following theorem, related to the law of the iterated logarithm, was proved by Davis, and part (ii) is due to Gut.

J. A. DAVIS (1968); A. GUT (1980):
(i) *If* $EX^2 = \sigma^2 < \infty$, *then* $f(x) < \infty$, $x > \sigma\sqrt{2}$.
(ii) *If* $f(x) < \infty$ *for some* $x > 0$, *then* $EX^2 < \infty$.

The next strengthening of the above results has been obtained.

D. LI and A. SPĂTARU (2005): *Let* $q > 0$.

(i) $\begin{cases} E|X|^{1/q} < \infty & \text{if} \quad q < 1/2 \\ E[X^2(\log^+|X|)/\log^+\log^+|X|] < \infty & \text{if} \quad q = 1/2 \\ EX^2 < \infty & \text{if} \quad q > 1/2 \end{cases}$ *and* $EX^2 = \sigma^2$ *imply* $\int_\delta^\infty f(x^q)dx < \infty$, $\delta > (\sigma\sqrt{2})^{1/q}$.

(ii) $\int_\delta^\infty f(x^q)dx < \infty$ *for some* $\delta > 0$ *implies* $\begin{cases} E|X|^{1/q} < \infty & \text{if} \quad q < 1/2 \\ E[X^2(\log^+|X|)/\log^+\log^+|X|] < \infty & \text{if} \quad q = 1/2 \\ EX^2 < \infty & \text{if} \quad q > 1/2 \end{cases}$.

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