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Multiplicity Results for the (p(x),q(x))−Kirchhoff Equations

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Abstract. We study the existence and multiplicity of weak solutions for the following equation involving variable exponents:

$$\begin{cases} -\triangle^{\tilde{k}_p}_{p(x)} u(x) - \triangle^{\tilde{k}_q}_{q(x)} u(x) + |u|^{p(x)-2}u + |u|^{q(x)-2}u = \lambda f(x, u(x)), & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth enough boundary which is subject to Dirichlet boundary condition, λ is a positive real parameter and p is real continuous function on $\bar{\Omega}$. Using a variational method, we would show the existence and multiplicity of the solutions. To this purpose, we would focus on a generalized variable exponent Lebesgue-Sobolev space.

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1. Introduction

In this paper, we study the following problem:

$$\begin{cases} -\triangle^{\tilde{k}_p}_{p(x)} u(x) - \triangle^{\tilde{k}_q}_{q(x)} u(x) + |u|^{p(x)-2}u + |u|^{q(x)-2}u = \lambda f(x, u(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth enough boundary. Let λ be a positive real parameter and p and q be real continuous functions on $\bar{\Omega}$. The nonlocal Kirchhoff type term $\tilde{k}(r, u)$ is $\tilde{k}(r, u) = a_r + b_r \int_\Omega \frac{1}{r(x)} |\nabla u|^{r(x)} dx$, where a_r and b_r are positive constants. Moreover, we consider $k(r, u)$ instead of $\tilde{k}(r, u)$ in the problem (1) such that $k(r, u) := a_r - b_r \int_\Omega \frac{1}{r(x)} |\nabla u|^{r(x)} dx$. Here, it is considered the sum of two such Kirchhoff type operators $-\triangle^{\tilde{k}_p}_{p(x)} u(x) := -\tilde{k}(p, u)\triangle_{p(x)} u$ and $-\triangle^{\tilde{k}_q}_{q(x)} u(x) := -\tilde{k}(q, u)\triangle_{q(x)} u$, in which $1 < q(x) < p(x) < p^*(x)$, where $p^*(x) = \frac{Np(x)}{N - p(x)}$ and $p(x) < N$ for all $x \in \bar{\Omega}$. $\Delta_{p(x)} u := div(|\nabla u|^{p(x)-2}\nabla u)$ denote the $p(x)$-Laplacian operator (for details, see [2, 3, 11]). In [5], authors studied the following $p(x)$-Kirchhoff

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problem, under some suitable superliner conditions:

$$\begin{cases} -\left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\triangle_{p(x)}u = f(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

They introduced a new method to show the boundedness of Cerami sequences. By using the mountain pass Lemma and the symmetric mountain pass Lemma, they proved that (2) has infinitely many weak solutions. In [9], M. K. Hamdani, by using the theory of variable exponent Sobolev spaces, established the existence of nontrivial weak solutions for the following $p(x)$-Kirchhoff problem:

$$\begin{cases} -\left(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) div\left(|\nabla u|^{p(x)-2}\nabla u\right) & \\ = \lambda|u|^{p(x)-2}u + g(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

In [8], the authors studied the existence of weak solutions and strong generalized solutions, using topological tools, for the following problem:

$$\begin{cases} -\triangle^{k_p}_{p(x)}u(x) - \triangle^{k_q}_{q(x)}u(x) = f(x,u(x),\nabla u(x)), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Here, in section 3 we consider the existence of three weak solutions for the problem (1) by using the three critical point theorem. In section 4, we will show the existence of two solutions for the problem (16).

2. Preliminaries

We recall some necessary definitions and propositions concerning the Lebesgue and Sobolev spaces. Let Ω be a bounded domain of $\mathbb{R}^N$. Set

$$C_+(\Omega) := \{s \in C(\bar{\Omega}); s(x) > 1, \forall x \in \bar{\Omega}\}.$$

For any continuous function $s : \Omega \to (1,\infty)$,

$$s^- := \inf_{x\in\Omega} s(x) \textit{ and } s^+ := \sup_{x\in\Omega} s(x).$$

For $s \in C_+(\bar{\Omega})$

$$L_{s(x)}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ is a measurable function} : \int_\Omega |u|^{s(x)}dx < +\infty\}.$$

Endowed with the norm:

$$\|u\|_{s(x)} := \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{s(x)} dx \le 1\right\}.$$

It is well known that $L_{s(x)}(\Omega)$ is a separable reflexive Banach space [1, 4, 15].

The modular of the $L_{s(x)}(\Omega)$ is defined by

$$\sigma_{s(x)}(u) := \int_\Omega |u(x)|^{s(x)}dx.$$

Proposition 2.1. [7] *$(L_{s(x)}(\Omega), \|.\|_{s(x)})$ is separable, uniformly convex, reflexive Banach space and its conjugate space is $(L_{s'(x)}(\Omega), \|.\|_{s'(x)})$, where*

$$\frac{1}{s(x)} + \frac{1}{s'(x)} = 1, \ \forall x \in \Omega.$$

For all $u \in L_{s(x)}(\Omega)$ and $w \in L_{s'(x)}(\Omega)$, we have

$$|\int_{\Omega} uwdx| \leq (\frac{1}{s^-} + \frac{1}{s'^-})\|u\|_{s(x)}\|w\|_{s'(x)} \leq 2\|u\|_{s(x)}\|w\|_{s'(x)}. \tag{5}$$

Proposition 2.2. [10] *Suppose that $u, u_n \in L_{s(x)}(\Omega)$, we have*

$$\|u\|_{s(x)} < 1 \Rightarrow \|u\|^{s^+}_{s(x)} \leq \sigma_{s(x)}(u) \leq \|u\|^{s^-}_{s(x)}.$$

$$\|u\|_{s(x)} > 1 \Rightarrow \|u\|^{s^-}_{s(x)} \leq \sigma_{s(x)}(u) \leq \|u\|^{s^+}_{s(x)}.$$

$$\|u\|_{s(x)} < 1(resp, = 1; > 1) \Leftrightarrow \sigma_{s(x)}(u) < 1(resp, = 1; > 1). \tag{6}$$

$$\|u_n\|_{s(x)} \to 0(resp, \to +\infty) \Leftrightarrow \sigma_{s(x)}(u_n) \to 0(resp, \to +\infty).$$

$$\lim_{n\to\infty} \|u_n - u\|_{s(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \sigma_{s(x)}(u_n - u) = 0. \tag{7}$$

The Sobolev space $W^{1,s(x)}(\Omega)$ is defined by

$$W^{1,s(x)}(\Omega) := \left\{u \in L_{s(x)}(\Omega) : \ |\nabla u| \in L_{s(x)}(\Omega)\right\}.$$

It is separable and reflexive Banach spaces with norm:

$$\|u\|_{1,s(x)} = \|u\|_{s(x)} + \|\nabla u\|_{s(x)}.$$

On $W^{1,s(x)}_0(\Omega)$, we may consider the following equivalent norm $\|u\|_{s(x)} = \|\nabla u\|_{s(x)}$, where $W^{1,s(x)}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the following norm:

$$\|u\| = \inf\left\{\mu > 0 : \ \int_{\Omega}\left(\left|\frac{\nabla u(x)}{\mu}\right|^{s(x)}\right) dx \leq 1\right\}.$$

It is well known that

$$W^{1,s(x)}_0(\Omega) := \left\{u; \ u\Big|_{\partial\Omega} = 0, \ u \in L^{s(x)}(\Omega), \ |\nabla u| \in L^{s(x)}(\Omega)\right\}.$$

For more details, we refer to [2, 11].

Proposition 2.3. *(Sobolev Embedding [7]) For $s, s' \in C_+(\bar{\Omega})$ and $1 < s'(x) < s^*(x)$ for all $x \in \bar{\Omega}$, there is a continuous compact embedding*

$$W^{1,s(x)}_0(\Omega) \hookrightarrow L_{s'(x)}(\Omega).$$

Therefore, there is a constant $c'_s > 0$ such that

$$\|u\|_{s'(x)} \leq c'_s\|u\|.$$

Proposition 2.4. *(Poincare Inequality [11]) There is a constant $c > 0$ such that*

$$\|u\|_{s(x)} \leq C\|\nabla u\|_{s(x)}, \tag{8}$$

for all $u \in W^{1,s(x)}_0(\Omega)$.

Remark 2.1. From Proposition 2.4, $\|\nabla u\|_{s(x)}$ and $\|u\|_{1,s(x)}$ are equivalent norm on $W^{1,s(x)}_0(\Omega)$.

Remark 2.2. [15] If $N < s^- \leq s(x)$ for any $x \in \bar{\Omega}$, by Theorem 2.2 in [7] and Remark 1 in [13], we deduce that $W_0^{1,s(x)}(\Omega)$ is continuously embedded in $W_0^{1,s^-}(\Omega)$. Since $N < s^-$, it follows that $W_0^{1,s(x)}(\Omega)$ is compactly embedded in $C(\bar{\Omega})$. Defining $\|u\|_\infty = sup_{x\in\bar{\Omega}}|u(x)|$, we find that there exists a positive constant $c_5 > 0$ such that

$$\|u\|_\infty \leq c_5\|u\|,$$

for all $u \in W_0^{1,s(x)}(\Omega)$.

Remark 2.3. [6] $A : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by $A(u) := \int_\Omega \frac{1}{s(x)}|\nabla u|^{s(x)}dx$ is convex. The derivative $A' : W_0^{1,s(x)}(\Omega) \to (W_0^{1,s(x)}(\Omega))'$ is strictly monotone, bounded continuous and of (S_+) type, i.e., if $u_n \rightharpoonup u$ as $n \to \infty$ and $\limsup_{n\to\infty}\langle A'(u_n), u_n - u\rangle \leq 0$ implies $u_n \to u$.

3. Main results

Definition 3.1. $u \in W_0^{1,p(x)}(\Omega)$ is called a *weak solution* for (1) if

$$\begin{aligned}&\langle -\triangle_{p(x)}^{\tilde{k}_p}u, w\rangle + \langle -\triangle_{q(x)}^{\tilde{k}_q}u, w\rangle + \int_\Omega |u|^{p(x)-2}uwdx + \int_\Omega |u|^{q(x)-2}uwdx\\&-\lambda\int_\Omega f(x,u)wdx = 0,\end{aligned}$$

for all $w \in W_0^{1,p(x)}(\Omega)$. Hence,

$$\langle -\triangle_{p(x)}^{\tilde{k}_p}u, u\rangle = \left(a_p + b_p\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)}dx$$

and

$$\langle -\triangle_{q(x)}^{\tilde{k}_q}u, u\rangle = \left(a_q + b_q\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\int_\Omega |\nabla u|^{q(x)}dx.$$

Let $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a caratheodory function and

$$F(x,t) := \int_0^t f(x,s)ds. \tag{9}$$

For $u \in W_0^{1,p(x)}(\Omega)$, define $\Psi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by $\Psi(u) := -\int_\Omega F(u, u(x))dx$. Then $\Psi \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and has compact derivative such that

$$\langle \Psi'(u), w\rangle := -\int_\Omega f(x,u(x))w(x)dx, \text{ for all } u, w \in W_0^{1,p(x)}(\Omega).$$

The energy functional associated to problem (1) can be obtained by

$$\begin{aligned}J(u) :=\ & a_p\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \frac{b_p}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 + a_q\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\\&+\frac{b_q}{2}\Big(\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\Big)^2 + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx\\&+\int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx - \lambda\int_\Omega F(x,u)dx,\end{aligned}$$

for all $u \in W_0^{1,p(x)}(\Omega)$. It is well defined, C^1 functional and for all $u, w \in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned}\langle J'(u), w\rangle = &\left(a_p + b_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla w dx\\ &+\left(a_q + b_q \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\int_\Omega |\nabla u|^{q(x)-2}\nabla u\nabla w dx\\ &+\int_\Omega |u|^{p(x)-2}uwdx + \int_\Omega |u|^{q(x)-2}uwdx - \lambda\int_\Omega f(x,u)wdx.\end{aligned}$$

Therefore, critical points of this energy functional are week solutions for the problem (1). As usual we consider $\Omega \subset \mathbb{R}^N (N > 3)$ a bounded domain with smooth boundary and $p \in C_+(\Omega)$ such that

$$1 < \alpha^- \le \alpha(x) \le \alpha^+ < q^- \le q(x) \le q^+ < p^- \le p(x) \le p^+ < 2q^- < 2p^- < p^*(x) \tag{10}$$

and $p(x) < N$ for any $x \in \bar\Omega$. We consider the following conditions:

(B_1) $f : \bar\Omega \times \mathbb{R} \to \mathbb{R}$ a continuous function such that $|f(x,t)| \le a_1 + a_2|t|^{\alpha(x)-1}$, for all $t \in \mathbb{R}$ and for all $x \in \bar\Omega$, where a_1, a_2 are two positive constants and $\alpha \in C(\bar\Omega)$ such that $1 < \alpha(x) < p^*(x)$.

(B_2) (i) $f(x,t) < 0$ for all $(x,t) \in \Omega \times \mathbb{R}$, and $|t| \in (0,1)$,
(ii) $f(x,t) \ge k > 0$, when $|t| \in (t_0,\infty), t_0 > 1$.

(B_3) $\lim_{s\to 0} \dfrac{f(x,s)}{|s|^{p(x)-2}s} = 0$.

Theorem 3.1. [14] *Let X be a separable reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a continuous Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X', $\Psi : X \to \mathbb{R}$ is a continuous Gateaux differentiable functional whose Gateaux derivative and is compact. Suppose that the following assertions hold:*

(i) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \pm\infty$, *for all* $\lambda > 0$,

(ii) *There exist* $e \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that* $\Phi(u_0) < e < \Phi(u_1)$,

(iii) $\inf_{u\in\Phi^{-1}(-\infty,e]} \Psi(u) > \dfrac{(\Phi(u_1) - e)\Psi(u_0) + (e - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}$.

Then there exist an open interval $\Lambda \subset (0,+\infty)$ and a positive real number γ such that $\Phi'(u) + \lambda\Psi'(u) = 0$ admits at least three weak solutions in X whose norms are less than γ, for all $\lambda \in \Lambda$.

Theorem 3.2. *If (10), B_1 and B_2 hold then there exist an open interval $\Lambda \subset (0,+\infty)$ and a positive real number γ such that for any $\lambda \in \Lambda$, (1) has at least three solutions in $W_0^{1,p(x)}(\Omega)$ whose norms are less than γ.*

Proposition 3.3. [8] *Let $\Phi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ be a functional defined by*

$$\begin{aligned}\Phi(u) := &a_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \frac{b_p}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 + a_q\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\\ &+\frac{b_q}{2}\Big(\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\Big)^2 + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx,\end{aligned}$$

for all $u \in W_0^{1,p(x)}(\Omega)$. Then

(i) $\Phi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ *is sequentially weakly lower semi-continuous, since* $\chi := \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx$ *and* $\sigma_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)}dx$ *are sequentially weakly lower semi-continuous and* $\Phi \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$*. Moreover, its derivative* Φ' *satisfies in:*

$$\begin{aligned}\langle \Phi'(u), w\rangle &= \left(a_p + b_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla w dx \\ &\quad + \left(a_q + b_q \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\int_\Omega |\nabla u|^{q(x)-2}\nabla u \nabla w dx \\ &\quad + \int_\Omega |u|^{p(x)-2}uwdx + \int_\Omega |u|^{q(x)-2}uwdx.\end{aligned}$$

for all $u, w \in W_0^{1,p(x)}(\Omega)$.

(ii) $\Phi' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))'$ *is a continuous, bounded and strictly monotone operator.*

(iii) $\Phi' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))'$ *is of* (S_+) *type.*

(iv) $\Phi' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))'$ *is a homeomorphism.*

Proof of Theorem 3.2. To prove this theorem, we first verify the condition (i) of Theorem 3.1

$$\begin{aligned}\Phi(u) &= a_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \frac{b_p}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 + a_q \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx \\ &\quad + \frac{b_q}{2}\left(\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)^2 x + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\ &\geq a_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + a_q \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx \\ &\quad + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\ &\geq \frac{a_p}{p^+}\int_\Omega |\nabla u|^{p(x)}dx + \frac{a_q}{q^+}\int_\Omega |\nabla u|^{q(x)}dx + \frac{1}{p^+}\int_\Omega |u|^{p(x)}dx + \frac{1}{q^+}\int_\Omega |u|^{q(x)}dx.\end{aligned}$$

Set $C_2 = \min\{\frac{a_p}{p^+}, \frac{1}{p^+}\}$ and $C_3 = \min\{\frac{a_q}{q^+}, \frac{1}{q^+}\}$. If $\sigma_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)}dx$ and $\sigma_{p(x)}(u) > 1$, by proposition 2.4, proposition 2.2 and (10)

$$\Phi(u) \geq C_2\|u\|^{p^-} + C_3\|u\|^{q^-}. \tag{11}$$

On the other hand,

$$\Psi(u) = -\int_\Omega F(x, u(x))dx.$$

By the compact embedding $W_0^{1,s(x)}(\Omega) \hookrightarrow L^1(\Omega)$ and $W_0^{1,s(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$, there exist $c_1, c_\alpha > 0$, such that

$$-\Psi(u) = a_1 \int_\Omega |u(x)|dx + \frac{a_2}{\alpha^-}\int_\Omega |u(x)|^{\alpha(x)}dx \leq a_1 c_1\|u\| + \frac{a_2}{\alpha^-}c_\alpha\|u\|^{\alpha^+}.$$

If $\sigma_{p(x)}(u) > 1$, by Proposition 2.2, Proposition 2.3 and (10)

$$\Psi(u) \geq -a_1 c_1 \|u\| - \frac{a_2}{\alpha^-} c_\alpha \|u\|^{\alpha^+}. \tag{12}$$

By (11), (12) and for any $\lambda > 0$

$$\Phi(u) + \lambda\Psi(u) \geq C_2\|u\|^{p^-} + C_3\|u\|^{q^-} - \lambda(a_1 c_1\|u\| + \frac{a_2}{\alpha^-} c_\alpha \|u\|^{\alpha^+}).$$

According to (10), $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$, for all $\lambda > 0$ and (i) of Theorem 3.1 is verified. Due to $\dfrac{\partial F(x,t)}{\partial t} = f(x,t)$ and (B_2), it is easy to see that $F(x,t)$ is increasing for $t \in (t_0, \infty)$, $t_0 > 1$ respect to $x \in \Omega$ and decreasing in $(0,1)$ respect to $x \in \Omega$. Since $F(x,t) \geq kt$ uniformly for x, so $F(x,t) \to +\infty$ as $t \to +\infty$. Now, from (B_2) we can choose $\delta > 1$ such that $F(x,t) > 0$ for all $t > \delta$, $x \in \Omega$. Then using (B_2)

$$F(x,t) > 0 = F(x,0) \geq F(x,\omega), \text{ for all } t > \delta, \omega \in (0,1). \tag{13}$$

Let m, n be two real numbers such that $0 < m < \min\{1, c_5\}$, where c_5 is given in Remark 3.1 and $n > \delta (n > 1)$ satisfies $n^{p^-}|\Omega| > 1$. It follows from (13) that

$$\int_\Omega \sup_{0\leq t\leq m} F(x,t)dx \leq 0 < \int_\Omega F(x,n)dx.$$

Choosing $k < n^{q^-}|\Omega| < n^{p^-}|\Omega|$, $0 < e < \dfrac{k}{p^+}$, $u_0(x) = 0$ and $u_1(x) = n$ such that $n > 1$, then $\Phi(u_0) = \Psi(u_0) = 0$ and

$$\Phi(u_1) = \int_\Omega \frac{1}{p(x)} n^{p(x)} dx + \int_\Omega \frac{1}{q(x)} n^{q(x)} dx \geq \frac{n^{p^-}}{p^+}|\Omega| > e.$$

Thus, $\Phi(u_0) < e < \Phi(u_1)$. Then (ii) of Theorem 3.1 is verified.

On the other hand, by (B_2), (10), $n > 1$,

$$\begin{aligned} -\frac{(\Phi(u_1) - e)\Psi(u_0) + (e - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} &= -e\frac{\Psi(u_1)}{\Phi(u_1)} \\ = -e\frac{-\int_\Omega F(x,n)dx}{\int_\Omega \frac{1}{p(x)} n^{p(x)} dx + \int_\Omega \frac{1}{q(x)} n^{q(x)} dx} &> 0. \end{aligned} \tag{14}$$

Let $u \in W_0^{1,p(x)}(\Omega)$ such that $\Phi(u) \leq e$ and $e < C_2$. From (11) and Proposition 2.2,

$$C_2\|u\|^{p^-} < C_2\|u\|^{p^-} + C_3\|u\|^{q^-} \leq \Phi(u) \leq e,$$

so $\|u\| \leq (\dfrac{e}{C_2})^{\frac{1}{p^-}} < 1$. From (13)

$$-\inf_{u\in\Phi^{-1}(-\infty,e]} \Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,e]} -\Psi(u) \leq \int_\Omega \sup_{0\leq t\leq m} F(x,t)dx \leq 0. \tag{15}$$

Then (14) and (15) imply that

$$-\inf_{u\in\Phi^{-1}(-\infty,e]} \Psi(u) < -\frac{(\Phi(u_1) - e)\Psi(u_0) + (e - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)},$$

and

$$\inf_{u\in\Phi^{-1}(-\infty,e]}\Psi(u) > \frac{(\Phi(u_1)-e)\Psi(u_0)+(e-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}.$$

This completes the proof. □

4. Case of negative Kirchhoff term

In this section, we study the following problem:

$$\begin{cases} -\triangle^{k_p}_{p(x)}u(x) - \triangle^{k_q}_{q(x)}u(x) + |u|^{p(x)-2}u + |u|^{q(x)-2}u = \lambda f(x,u(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth enough boundary. Let λ be a positive real parameter and p and q be real continuous functions on $\bar{\Omega}$. The negative Kirchhoff term is $k(r,u) = a_r - b_r \int_\Omega \frac{1}{r(x)}|\nabla u|^{r(x)}dx$, such that a_r and b_r are positive constants related to r. Here, we have the sum of two Kirchhoff type operators $-\triangle^{k_p}_{p(x)}u(x) := -k(p,u)\triangle_{p(x)}u$ and $-\triangle^{k_q}_{q(x)}u(x) := -k(q,u)\triangle_{q(x)}u$, such that $1 < q(x) < p(x) < p^*(x)$,

Definition 4.1. $u \in W_0^{1,p(x)}(\Omega)$ is called a *weak solution* for (16) if

$$\langle -\triangle^{k_p}_{p(x)}u, w\rangle + \langle -\triangle^{k_q}_{q(x)}u, w\rangle + \int_\Omega |u|^{p(x)-2}uw dx + \int_\Omega |u|^{q(x)-2}uw dx$$
$$-\lambda \int_\Omega f(x,u)w dx = 0,$$

for all $w \in W_0^{1,p(x)}(\Omega)$, such that

$$\langle -\triangle^{k_p}_{p(x)}u, u\rangle = \left(a_p - b_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)}dx$$

and

$$\langle -\triangle^{k_q}_{q(x)}u, u\rangle = \left(a_q - b_q \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\int_\Omega |\nabla u|^{q(x)}dx.$$

Then energy functional associated to problem (16) can be obtained by

$$\begin{aligned} I(u) := {} & a_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b_p}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 \\ & + a_q \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx - \frac{b_q}{2}\Big(\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\Big)^2 \\ & + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx - \lambda \int_\Omega F(x,u)dx, \end{aligned}$$

for all $u \in W_0^{1,p(x)}(\Omega)$. It is well defined, C^1 functional and for all $u, w \in W_0^{1,p(x)}(\Omega)$,

$$\langle I'(u), w\rangle = \left(a_p - b_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla w dx + \left(a_q - b_q \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\int_\Omega |\nabla u|^{q(x)-2}\nabla u \nabla w dx + \int_\Omega |u|^{p(x)-2}uwdx + \int_\Omega |u|^{q(x)-2}uwdx - \lambda \int_\Omega f(x,u)wdx.$$

Therefore, critical points of this energy functional are week solutions for the problem (16).

Proposition 4.1. [8] *Let us define the functional* $\tau : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ *by*

$$\tau(u) := a_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b_p}{2}(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)^2 + a_q \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx - \frac{b_q}{2}(\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx)^2 + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx,$$

for all $u \in W_0^{1,p(x)}(\Omega)$. *Then*

(i) $\tau \in C^1\left(W_0^{1,p(x)}(\Omega), \mathbb{R}\right)$.

(ii) $\tau : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ *is a continuously Gateaux differentiable functional. Moreover, the derivative operator* τ' *is:*

$$\langle \tau'(u), w\rangle = \left(a_p - b_p \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla w dx + \left(a_q - b_q \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\int_\Omega |\nabla u|^{q(x)-2}\nabla u \nabla w dx + \int_\Omega |u|^{p(x)-2}uwdx + \int_\Omega |u|^{q(x)-2}uwdx.$$

for all $u, w \in W_0^{1,p(x)}(\Omega)$.

(iii) *The mapping* $\tau' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))'$ *is of* (S_+) *type.*

Definition 4.2. [6] The functional I satisfies in the Palais-Smale condition at the level c, $(PS)_c$, if any sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ satisfying

$$I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ as } n \to \infty,$$

has a convergence subsequence.

Theorem 4.2. [12] *Let* X *be a real Banach space,* $\tau, \psi : X \to \mathbb{R}$ *be two continuously Gateaux differentiable functional such that* $\tau(0) = \psi(0) = 0$. *Fix* $r > 0$ *and assume that, for any* $\lambda \in \left]0, \frac{r}{sup_{u\in\tau^{-1}(]-\infty,r[)}\psi(u)}\right[$, *the functional* $I_\lambda := \tau - \lambda\psi$ *satisfies* $(PS)_c$ *condition and it is unbounded from below. Then, for each* $\lambda \in \left]0, \frac{r}{sup_{u\in\tau^{-1}(]-\infty,r[)}\psi(u)}\right[$, I_λ *admits at least two distinct critical points.*

We obtain the existence of two weak solutions for the problem (16) by applying Theorem 4.2 in case $r = 1$.

Theorem 4.3. *Let f satisfies (B_1), F be in (9), and there exist $\beta > p^+$ and $r > 0$ such that*

$$0 < \beta F(x,t) \leq t f(x,t), \tag{17}$$

for each $x \in \Omega$ and for $|t| \geq r$. Then, for $\lambda \in]0, \lambda^*[$, the problem (16) admits two weak solutions, where

$$\lambda^* := \frac{1}{a_1 c_1 \left(\dfrac{p^+}{(a_p - b_p) + (a_q - b_q)}\right)^{\dfrac{1}{p^+}} + \dfrac{a_2}{\alpha^-}[c_\alpha]^\alpha \left(\dfrac{p^+}{(a_p - b_p) + (a_q - b_q)}\right)^{\dfrac{\alpha^+}{p^+}}},$$

where c_1 and c_α denote respectively the constants of the embeddings $W_0^{1,s(x)}(\Omega) \hookrightarrow L^1(\Omega)$ and $W_0^{1,s(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$.

Proof. Suppose that τ is defined by Proposition 4.2 and $\psi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ is $\psi(u) := \int_\Omega F(x,u)dx$. We prove the following steps:

Step 1: We prove that $I := \tau - \lambda\psi$ satisfies $(PS)_c$ condition. Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be a $(PS)_c$ sequence. First we prove that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Assume by contradiction, passing eventually to a subsequence, $\|u_n\| \to +\infty$ as $n \to +\infty$. We choose θ, such that $\max\{\frac{p^+}{2(p^-)^2}, \frac{q^+}{2(q^-)^2}\} < \theta < \min\{\frac{1}{p^+}, \frac{1}{q^+}\}$. By Definition 4.2 for large enough n

$$\begin{aligned}
C + \|u_n\| &\geq I(u_n) - \theta\langle I'(u_n), u_n\rangle \\
&= a_p \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx - \frac{b_p}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)^2 \\
&\quad + a_q \int_\Omega \frac{1}{q(x)}|\nabla u_n|^{q(x)}\,dx - \frac{b_q}{2}\left(\int_\Omega \frac{1}{q(x)}|\nabla u_n|^{q(x)}\,dx\right)^2 \\
&\quad + \int_\Omega \frac{1}{p(x)}|u_n|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}\,dx - \lambda\int_\Omega F(x,u_n)\,dx \\
&\quad - \theta\left(a_p - b_p\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx \\
&\quad - \theta\left(a_q - b_q\int_\Omega \frac{1}{q(x)}|\nabla u_n|^{q(x)}\,dx\right)\int_\Omega |\nabla u_n|^{q(x)}\,dx \\
&\quad - \theta\int_\Omega |u_n|^{p(x)}\,dx - \theta\int_\Omega |u_n|^{q(x)}\,dx + \theta\lambda\int_\Omega f(x,u_n)u_n\,dx \\
&\geq \frac{a_p}{p^+}\int_\Omega |\nabla u_n|^{p(x)}\,dx - \frac{b_p}{2(p^-)^2}\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^2 \\
&\quad + \frac{a_q}{q^+}\int_\Omega |\nabla u_n|^{q(x)}\,dx - \frac{b_q}{2(q^-)^2}\left(\int_\Omega |\nabla u_n|^{q(x)}\,dx\right)^2 \\
&\quad + \frac{1}{p^+}\int_\Omega |u_n|^{p(x)}\,dx + \frac{1}{q^+}\int_\Omega |u_n|^{q(x)}\,dx - \lambda\int_\Omega F(x,u_n)\,dx \\
&\quad - \theta a_p\int_\Omega |\nabla u_n|^{p(x)}\,dx + \frac{\theta b_p}{p^+}\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^2
\end{aligned}$$

$$
\begin{aligned}
&- \theta a_q \int_\Omega |\nabla u_n|^{q(x)}\, dx + \frac{\theta b_q}{q^+}\left(\int_\Omega |\nabla u_n|^{q(x)}\, dx\right)^2 \\
&- \theta \int_\Omega |u_n|^{p(x)}\, dx - \theta \int_\Omega |u_n|^{q(x)}\, dx + \theta\lambda \int_\Omega f(x,u_n)u_n\, dx \\
= {}& a_p\left(\frac{1}{p^+} - \theta\right)\int_\Omega |\nabla u_n|^{p(x)}\, dx + a_q\left(\frac{1}{q^+} - \theta\right)\int_\Omega |\nabla u_n|^{q(x)}\, dx \\
&+ b_p\left(\frac{\theta}{p^+} - \frac{1}{2(p^-)^2}\right)\left(\int_\Omega |\nabla u_n|^{p(x)}\, dx\right)^2 + b_q\left(\frac{\theta}{q^+} - \frac{1}{2(q^-)^2}\right)\left(\int_\Omega |\nabla u_n|^{q(x)}\, dx\right)^2 \\
&+ \left(\frac{1}{p^+} - \theta\right)\int_\Omega |u_n|^{p(x)}\, dx + \left(\frac{1}{q^+} - \theta\right)\int_\Omega |u_n|^{q(x)}\, dx \\
&- \lambda\left[\int_\Omega F(x,u_n)\, dx - \theta\int_\Omega f(x,u_n)u_n\, dx\right].
\end{aligned}
$$

So we have

$$
\begin{aligned}
C + \|u_n\| \geq {}& a_p\left(\frac{1}{p^+} - \theta\right)\int_\Omega |\nabla u_n|^{p(x)}\, dx \\
&+ b_p\left(\frac{\theta}{p^+} - \frac{1}{2(p^-)^2}\right)\left(\int_\Omega |\nabla u_n|^{p(x)}\, dx\right)^2 \\
&+ \left(\frac{1}{p^+} - \theta\right)\int_\Omega |u_n|^{p(x)}\, dx - \lambda c_6|\Omega|,
\end{aligned}
$$

where $|\Omega| = \int_\Omega dx$. Thus, the last inequality together with (8) imply that

$$
\begin{aligned}
C + \|u_n\| \geq {}& (a_p + 1)\left(\frac{1}{p^+} - \theta\right)\|u_n\|^{p^-} \\
&+ b_p\left(\frac{\theta}{p^+} - \frac{1}{2(p^-)^2}\right)\left(\frac{1}{p^+} - \theta\right)\|u_n\|^{2p^-} - \lambda c_6|\Omega|.
\end{aligned}
$$

By (10), it is a contradiction if $\|u_n\| \to \infty$ as $n \to \infty$. Thus $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Then, we prove that $\{u_n\}$ has a convergent subsequence in $W_0^{1,p(x)}(\Omega)$. It follows from Proposition 2.3 and reflexivity of $W_0^{1,p(x)}(\Omega)$, we may assume that

$$
u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega),\ u_n \to u \text{ in } L^{s(x)}(\Omega),\ u_n(x) \to u(x),\ a.e.\ in\ \Omega, \tag{18}
$$

where $1 \leq s(x) < p^*(x)$.
By the virtue of conditions (B_1) and (B_3), one has for any $\epsilon \in (0,1)$ there exists $c_\epsilon > 0$ such that

$$
|f(x,u_n)\, dx| \leq \epsilon|u_n|^{p(x)-1} + c_\epsilon|u_n|^{\alpha(x)-1}. \tag{19}
$$

From Proposition 2.1 and (19),

$$
\begin{aligned}
\left|\int_\Omega f(x,u_n)(u_n - u)\, dx\right| &\leq \int_\Omega \left(\epsilon|u_n|^{p(x)-1}|u_n - u| + c_\epsilon|u_n|^{\alpha(x)-1}|u_n - u|\right) dx \\
&\leq \epsilon||u_n|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}}|u_n - u|_{p(x)} + c_\epsilon||u_n|^{\alpha(x)-1}|_{\frac{\alpha(x)}{\alpha(x)-1}}|u_n - u|_{\alpha(x)}.
\end{aligned}
$$

Since $\{u_n\}$ converges strongly to u in $L^{\alpha(x)}(\Omega)$, that is $|u_n - u|_{\alpha(x)} \to 0$ as $n \to +\infty$ and similarly, $|u_n - u|_{p(x)} \to 0$ as $n \to +\infty$, so

$$\int_\Omega f(x, u_n)(u_n - u)\ dx \to 0,\ \text{ as } n \to +\infty. \tag{20}$$

Similarly, by (5),

$$\int_\Omega |u_n|^{p(x)-2} u_n(u_n - u)\ dx \to 0,\ \text{ as } n \to +\infty \tag{21}$$

and

$$\int_\Omega |u_n|^{q(x)-2} u_n(u_n - u)\ dx \to 0,\ \text{ as } n \to +\infty. \tag{22}$$

From Definition 4.2,

$$\langle I'(u_n), u_n - u\rangle \to 0.$$

Thus,

$$\begin{aligned}\langle I'(u_n), u_n - u\rangle = &\left(a_p - b_p \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx\right) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla(u_n - u) dx \\ &+ \left(a_q - b_q \int_\Omega \frac{1}{q(x)} |\nabla u_n|^{q(x)} dx\right) \int_\Omega |\nabla u_n|^{q(x)-2} \nabla u_n \nabla(u_n - u) dx \\ &+ \int_\Omega |u_n|^{p(x)-2} u_n(u_n - u) dx + \int_\Omega |u_n|^{q(x)-2} u_n(u_n - u) dx \\ &- \lambda \int_\Omega f(x, u_n)(u_n - u) dx \to 0.\end{aligned}$$

From (20), (21) and (22),

$$\begin{aligned}&\left(a_p - b_p \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx\right) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla(u_n - u) dx \\ &+ \left(a_q - b_q \int_\Omega \frac{1}{q(x)} |\nabla u_n|^{q(x)} dx\right) \int_\Omega |\nabla u_n|^{q(x)-2} \nabla u_n \nabla(u_n - u) dx \to 0.\end{aligned} \tag{23}$$

Then by (23) and Proposition 2.3, the sequence $\{u_n\}$ converges strongly to u in $W_0^{1,p(x)}(\Omega)$. Therefore, I satisfies the $(PS)_c$ condition.

Step 2: I is unbounded from below.

First we show there exists $M \in \mathbb{R}^+$ such that for each $x \in \Omega$ and $|t| > M$

$$F(x, t) \geq K|t|^\beta. \tag{24}$$

(17) implies that

$$0 < \beta F(x, \varrho t) \leq \varrho t f(x, \varrho t),\ \text{ for all } \varrho > 0.$$

Let $m(x) := \min_{|\varrho|=M} F(x, \varrho)$ and $g_t(z) := F(x, zt)$ for all $z > 0$. Then

$$0 < \beta g_t(z) = \beta F(x, zt) \leq zt f(x, zt) = z g_t'(z),$$

for all $z > \dfrac{M}{|t|}$ and

$$\int_{\frac{M}{|t|}}^1 \frac{g_t'(z)}{g_t(z)} dz \geq \int_{\frac{M}{|t|}}^1 \frac{\beta}{z} dz.$$

Then

$$\ln\left(\frac{g_t(1)}{g_t(\frac{M}{|t|})}\right) \leq \ln\left(\frac{|t|^\beta}{M^\beta}\right).$$

Therefore,

$$F(x,t) = g_t(1) > F\left(x, \frac{M}{|t|}t\right)\frac{|t|^\beta}{M^\beta} \geq m(x)\frac{|t|^\beta}{M^\beta} \geq K|t|^\beta,$$

so (24) is established. Fix $v \in W_0^{1,p(x)}(\Omega) - \{0\}$. For each $t > 1$ we have

$$\begin{aligned}
I(tv) &= a_p\int_\Omega \frac{1}{p(x)}|t\nabla v|^{p(x)}dx - \frac{b_p}{2}(\int_\Omega \frac{1}{p(x)}|t\nabla v|^{p(x)}dx)^2 \\
&\quad + a_q\int_\Omega \frac{1}{q(x)}|t\nabla v|^{q(x)}dx - \frac{b_q}{2}(\int_\Omega \frac{1}{q(x)}|t\nabla v|^{q(x)}dx)^2 \\
&\quad + \int_\Omega \frac{1}{p(x)}|tv|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|tv|^{q(x)}dx - \lambda\int_\Omega F(x,tv)dx \\
&\leq a_pt^{p^+}\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx - t^{2p^-}\frac{b_p}{2}(\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}dx)^2 \\
&\quad + a_qt^{q^+}\int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx - t^{2q^-}\frac{b_q}{2}(\int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}dx)^2 \\
&\quad + t^{p^+}\int_\Omega \frac{1}{p(x)}|v|^{p(x)}dx + t^{q^+}\int_\Omega \frac{1}{q(x)}|v|^{q(x)}dx - \lambda Kt^\beta\int_\Omega |v|^\beta dx - C.
\end{aligned}$$

From (10) and since $\beta > p^+$ if $t \to +\infty$, then $I \to -\infty$ and is unbounded from below. Fix $\lambda \in]0, \lambda^*[$. For each $u \in \tau^{-1}(]-\infty, 1[)$ such that $\|u\| < 1$, using Proposition 2.2 and Proposition 2.4, we have

$$\begin{aligned}
\tau(u) &= a_p\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b_p}{2}(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)^2 \\
&\quad + a_q\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx - \frac{b_q}{2}(\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx)^2 \\
&\quad + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\
&\geq \frac{a_p}{p^+}\int_\Omega |\nabla u_n|^{p(x)}\,dx - \frac{b_p}{2(p^-)^2}\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^2 \\
&\quad + \frac{a_q}{q^+}\int_\Omega |\nabla u_n|^{q(x)}\,dx - \frac{b_q}{2(q^-)^2}\left(\int_\Omega |\nabla u_n|^{q(x)}\,dx\right)^2 \\
&\quad + \frac{1}{p^+}\int_\Omega |u_n|^{p(x)}\,dx + \frac{1}{q^+}\int_\Omega |u_n|^{q(x)}\,dx \\
&\geq \frac{a_p}{p^+}\|u\|^{p^+} - \frac{b_p}{2(p^-)^2}\|u\|^{2p^-} + \frac{a_q}{q^+}\|u\|^{q^+} - \frac{b_q}{2(q^-)^2}\|u\|^{2q^-} \\
&\quad + \frac{1}{p^+}\|u\|^{p^+} + \frac{1}{q^+}\|u\|^{q^+}
\end{aligned}$$

$$\geq \frac{a_p}{p^+}\|u\|^{p^+} - \frac{b_p}{2(p^-)^2}\|u\|^{2p^-} + \frac{a_q}{q^+}\|u\|^{q^+} - \frac{b_q}{2(q^-)^2}\|u\|^{2q^-}.$$

Since $\{\frac{p^+}{2(p^-)^2}, \frac{q^+}{2(q^-)^2} < \frac{1}{p^+}$ and by (10),

$$\tau(u) \geq \frac{a_p}{p^+}\|u\|^{p^+} - \frac{b_p}{p^+}\|u\|^{p^+} + \frac{a_q}{p^+}\|u\|^{p^+} - \frac{b_q}{p^+}\|u\|^{p^+},$$

so,

$$\|u\| \leq \left(\frac{p^+}{(a_p - b_p) + (a_q - b_q)}\tau(u)\right)^{\frac{1}{p^+}} \leq \left(\frac{p^+}{(a_p - b_p) + (a_q - b_q)}\right)^{\frac{1}{p^+}}. \tag{25}$$

By Proposition 2.2 and Proposition 2.3,

$$\int_\Omega |u|^{\alpha(x)}dx = \sigma_{\alpha(x)}(u) \leq \left[\|u\|_{\alpha(x)}\right]^\alpha \leq [c_\alpha\|u\|]^\alpha, \tag{26}$$

for $u \in W_0^{1,p(x)}(\Omega)$. By the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$, there exist $c_1, c_\alpha > 0$ and by (B_1), (17), (25) and (26)

$$\begin{aligned}
\psi(u) &= \int_\Omega F(x,u)dx \\
&\leq a_1\int_\Omega |u|dx + \frac{a_2}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx \\
&\leq a_1c_1\|u\| + \frac{a_2}{\alpha^-}[c_\alpha\|u\|]^\alpha \\
&\leq a_1c_1\left(\frac{p^+}{(a_p - b_p) + (a_q - b_q)}\right)^{\frac{1}{p^+}} + \frac{a_2}{\alpha^-}[c_\alpha]^\alpha\left(\frac{p^+}{(a_p - b_p) + (a_q - b_q)}\right)^{\frac{\alpha^+}{p^+}} \\
&= \frac{1}{\lambda^*} < \frac{1}{\lambda}.
\end{aligned}$$

Therefore, $\lambda < \dfrac{1}{sup_{u\in\tau^{-1}(]-\infty,r[)}\psi(u)}$. Thus by Theorem 4.2, Problem (16) admits at least two weak solutions. □

5. Conclusion

In this article, we proved the existence of two and three weak solutions with different conditions for the Dirichlet boundary value problem (1) and (16), involving variable exponents by using variational method.

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