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Lacunary Statistical Convergence Sequences Defined by D-Orlicz Function

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Abstract. In this article we have introduced the notion of lacunary statistical convergence sequences defined by D-Orlicz function. In this article we have defined some sequence spaces and studied some geometric, algebraic properties of these sequence spaces like D-module, D-balanced set, D-convex set, D-absorbing set.

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1. Introduction and Background

Bi-complex numbers: Tessarine numbers with commutative quaternions were first introduced by Cockle in [13]. In addition, Segre [19] investigated these numbers by identifying them as bi-complex numbers. Subsequently, Price [12] conducted a thorough analysis of derivatives, integrals, holomorphic functions, bi-complex numbers, and their generalizations to higher dimensions. The bi-complex number was defined by Segre [19] as follows:

$$\xi = z_1^{'} + i_2 z_2^{'} = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,$$

where $z_1^{'}, z_2^{'} \in C_1(i_1); x_1, x_2, x_3, x_4 \in C_0$ and i_1, i_2 are two independent units satisfying the relations $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, The set of bi-complex numbers C_2 is defined as:

$$C_2 = \{\xi : \xi = z_1^{'} + i_2 z_2^{'}; z_1^{'}, z_2^{'} \in C_1(i_1)\},$$

where $C_1(i_1) = \{x_1 + i_1 x_2 : x_1, x_2 \in C_0\}$ and C_0 is set of real numbers. C_2 is a vector space over $C_1(i_1)$. Other than 0 and 1, there are two more idempotent elements in C_2 given by $e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$, where $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.
Every bi-complex number $\xi = z_1^{'} + i_2 z_2^{'}$ can be uniquely expressed as the following form

$$\xi = z_1^{'} + i_2 z_2^{'} = (z_1^{'} - i_1 z_2^{'})e_1 + (z_1^{"} + i_1 z_2^{'})e_2 = \mu_1^{'} e_1 + \mu_2^{'} e_2,$$

where $\mu_1^{'} = (z_1^{'} - i_1 z_2^{'})$ and $\mu_2^{'} = (z_1^{'} + i_1 z_2^{'})$.
For $\xi = z_1^{'} + i_2 z_2^{'} \in C_2$, the norm is defined as

$$\|\xi\|_{C_2} = \sqrt{|z_1^{'}|^2 + |z_2^{'}|^2}.$$

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Hyperbolic Numbers: The hyperbolic number is of the form

$$\alpha = x_1 + i_1 i_2 x_2; x_1, x_2 \in C_0.$$

The idempotent representation of any hyperbolic number $\alpha = x_1 + i_1 i_2 x_2$ is

$$\alpha = v_1 e_1 + v_2 e_2,$$

where $v_1 = x_1 + x_2, v_2 = x_2 - x_1$.
The set of hyperbolic numbers is given by

$$D = \{v_1 e_1 + v_2 e_2 : v_1, v_2 \in C_0\}.$$

The set of positive hyperbolic numbers is given by

$$D_+ = \{v_1 e_1 + v_2 e_2 : v_1, v_2 \geq 0\}.$$

Let $\xi \in C_2$, then hyperbolic norm(D- valued) norm on C_2 is given by

$$|\xi|_D = |\mu_1| e_1 + |\mu_2| e_2 \in D_+.$$

If $\xi, \eta \in C_2$, then

$$|\xi + \eta|_D \leq^{'} |\xi_D + |\eta|_D \text{ and } |\xi\eta|_D = |\xi|_D |\eta|_D.$$

Let S be a subset of D. Consider the two sets $D_1 = \{v_1 : v_1 e_1 + v_2 e_2 \in S\}$ and $D_2 = \{v_2 : v_1 e_1 + v_2 e_2 \in S\}$.
Then supremum of the set S is given by

$$\sup_D S = e_1 \sup D_1 + e_2 \sup D_2.$$

Similarly, infimum of the set S is given by

$$\inf_D S = e_1 \inf D_1 + e_2 \inf D_2.$$

The partial order relation on D is given by

$$\alpha \leq^{'} \beta \text{ if and only if } \beta - \alpha \in D_+ \forall \alpha, \beta \in D.$$

Remark 1.1. Denote D_+^*, by the the non negative extended hyperbolic numbers

$$D_+^* = \{\mu_1 e_1 + \mu_2 e_2, \mu_1, \mu_2 > 0\} \cup \{\infty\} \cup \{-\infty\} \cup \{\infty e_1 + \mu_2 e_2\} \cup \{\mu_1 e_1 - \infty e_2\}.$$

Lacunary Sequence: Freedman et al.[5] did the first research on lacunary sequences [5]. They investigated strongly Cesaro summable and strongly lacunary convergent sequences, taken consideration of a general lacunary sequence θ , and they discovered connections among the two types' classes of sequences. Researchers Ercan et al. [4], Gumus[6], Dowari, and Triptahy[2, 3] have all investigated further lacunary sequences. Recently, generalized difference lacunary weak convergence of sequences was investigated by Tamuli and Tripathy [8].

A sequence of positive integers $\theta = \{k_r\}$ is called lacunary if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = k_r / k_{r-1}$. The space of lacunary strongly convergent sequences, denoted as N_θ was introduced by Freedman et al. [5] and is defined as follows:

$$N_\theta = \left\{ x : \lim_{r \to \infty} \frac{1}{h_r} \sum_{i \in I_r} |x_i - L| = 0, \text{ for some } L \right\}.$$

Statistical convergence: Independently, Fast [14] and Schoenberg [18] introduced the idea of statistical convergence. It was also found in Zygmund[1]. Later on, it was analyzed from the point of view of sequence space and connected to summability by a number of researchers, like Fridy[15], Tripathy [20], Salat [17], Tripathy and Nath [23], Tripathy and Sen [21], Bera and Tripathy [10, 11]. The idea depends on a certain density of subsets of $\mathbb{N}$, the set of natural numbers. A subset E of $\mathbb{N}$ is said to have natural density $\delta(E)$, if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k),$$

where χ_E is the characteristic function on E.

A sequence of bi-complex number (ξ_k) is said to be statistically convergent to $\eta \in C_2$ with respect to the Euclidean norm on C_2 if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : \|\xi_k - \eta\|_{C_2} \geq \varepsilon\}) = 0.$$

Orlicz space: An Orlicz function is a function $\mathcal{M}_{C_0} : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $\mathcal{M}_{C_0}(0) = 0, \mathcal{M}_{C_0}(x) > 0$, for $x > 0$ and $\mathcal{M}_{C_0}(x) \to \infty$, as $x \to \infty$.

Lindendstrauss and Tzafriri [7] used the idea of Orlicz function to construct the sequence space

$$\ell_M := \left\{ x \in \omega : \sum_{k=1}^{\infty} \mathcal{M}_{C_0}\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The sequence space $\ell_{M_{C_0}}$ is Banach space with the norm

$$\|x\| := \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} \mathcal{M}_{C_0}\left(\frac{|x_k|}{\rho}\right) < 1 \right\}.$$

2. Definition and Preliminaries

Definition 2.1. Let $(X, \oplus)$ be a commutative group. If the operations $\oplus : X \times X \to X$ and $\odot : D \times X \to X$ satisfy the properties

$$\begin{aligned}
&(\xi\eta) \odot a = \xi \odot (\eta \odot a),\\
&(\xi + \eta) \odot a = (\xi \odot a) \oplus (\eta \odot a),\\
&\xi \odot (a \oplus b) = (\xi \odot a) \oplus (\xi \odot b),\\
&1 \odot a = a, (1 = 1 + 0i_1i_2),
\end{aligned}$$

for every $\xi, \eta \in D$ and $a, b \in X$, then $(X, D, \oplus, \odot, +,)$ is called D-module.

Definition 2.2. Let A be a subset of X. Then A is said to be D-convex if $(1-\lambda)\xi + \lambda\eta \in A$, for all $\xi, \eta \in A$, for all $0 \leq' \lambda \leq' 1$.

Definition 2.3. Let A be a subset of D-module X. Then A is said to be D-absorbing set if for each $\xi \in X$, there exists $\varepsilon >' 0_D$ such that $\beta\xi \in A$, whenever $0_D \leq' \beta \leq' \varepsilon$.

Definition 2.4. Let A be a subset of a D-module X. Then A is called D-balanced set if for any $\xi \in A$ and $\beta \in C_2$ with $\|\beta\|_D \leq' 1$ such that $\beta\xi \in A$.

Definition 2.5. A function $\Upsilon_D : D \to D_+^*$ is called D-valued convex function if for every $\xi, \eta \in D$ with $0 \leq' \alpha \leq' 1$ such that

$$\Upsilon_D(\alpha\xi + (1-\alpha)\eta) \leq' \alpha\Upsilon_D(\xi) + (1-\alpha)\Upsilon_D(\eta).$$

Definition 2.6. A convex function $\Upsilon_D : D_+ \to D_+^*$ is said to be D-Orlicz function if it satisfies the following conditions
(i) $\Upsilon_D(0_D) = 0_D$;
(ii) $\lim_{\xi\to\infty} \Upsilon_D(\xi) = \infty^*$, where we assume that $\infty^* = \mu_1 e_1 + \infty e_2 = \infty e_1 + \mu_2 e_2 = \infty e_1 + \infty e_2$ and $\lim_{\xi\to\infty} \Upsilon_D(\xi)$ must exist along any line in the hyperbolic plane and must be equal.
We denote the BC-Orlicz function by $\mathcal{M}_D$.

Definition 2.7. An BC-Orlicz function $\mathcal{M}_D$ is said to satisfy the Δ_D^2-condition denoted by $\mathcal{M}_D \in \Delta_D^2$ if there exist some hyperbolic constants $K \geq' 0$ and ξ_0(depending upon K) such that

$$\mathcal{M}_D((2e_1 + 2e_2)\xi) \leq' K\mathcal{M}_D(\xi), \forall\ 0 \leq' \xi \leq' \xi_0.$$

Definition 2.8. A function $g_D : C_2 \to D_+^*$ is called D-paranorm if the following conditions are satisfied;
$p_1 : g(\xi) \geq' 0_D$, for all $\xi \in C_2$;
$p_2 : g(-\xi) = g(\xi)$, for all $\xi \in C_2$;
$p_3 : g(\xi + \eta) \leq' g(\xi) + g(\eta)$, for all $\xi, \eta \in C_2$;
$p_4 : \alpha_k \to \alpha, |\xi_k - \xi|_D \to 0_D$, then $|\alpha_k\xi_k - \alpha\xi|_D \to 0_D$.
A D-paranorm g_D for which $g_D(\xi) = 0_D$ implies $\xi = 0_2$ is called total D-paranorm.

Definition 2.9. Let $\theta = \{k_r\}$ be a lacunary sequence, A sequence of bi-complex numbers (ξ_k) is said to be lacunary statistically convergent to $\eta \in C_2$, if for every $\varepsilon >' 0$ such that

$$\delta\left(\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\sum_{k\in I_r} \|\xi_k - \xi\|_D\right| \geq' \varepsilon\right\}\right) = 0.$$

We denote, $stat_\theta - \lim_k \xi_k = \eta$.
We denote the set of all lacunary statistical convergence seqences of bi-complex numbers by $S_{C_2}^\theta$.

Definition 2.10. [22] A sequence of bi-complex numbers $\xi = (\xi_k)$ is said to be almost convergent to $l \in C_2$ if and only if

$$\lim_{p\to\infty} \nu_{kp}(\xi) = l, \text{ uniformly in } k,$$

where $\nu_{kp}(\xi) = \frac{\xi_k + \xi_{k+1} + \ldots + \xi_{k+p-1}}{p}$.

Definition 2.11. Let $\mathcal{M}_D$ be an D-Orlicz function, and (ξ_k) be a sequence of bi-complex numbers, $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup p_k = H$ and $\theta = \{k_r\}$ be a lacunary sequence. Now we define the following sets

$$N_\theta[\xi, p, \mathcal{M}_D, \|\cdot\|_D] = \Bigg\{(\xi_k) \in \omega^* : \lim_{r\to\infty} \frac{1}{h^r} \sum_{k\in I_r} \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k - \eta)\|_D}{\alpha}\right)\right]^{p_k} = 0,$$

$$\text{uniformly in } k, \text{ for hyperbolic number } \alpha >' 0_D \text{ and for some } \eta \in C_2\Bigg\};$$

$$b^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D] = \Bigg\{\xi \in \omega^* : \text{stat}_\theta - \lim_k \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k - \eta)\|_D}{\alpha}\right)\right]^{p_k} = 0_D,$$

$$\text{uniformly in } k, \text{ for some hyperbolic number } \alpha >' 0_D \text{ and for some } \eta \in C_2\Bigg\};$$

$$b_0^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D] = \Bigg\{\xi \in \omega^* : \text{stat}_\theta - \lim_k \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k} = 0_D,$$

$$\text{uniformly in } k, \text{ for hyperbolic number } \alpha >' 0_D\Bigg\};$$

$$b_\infty^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D] = \Bigg\{\xi \in \omega^* : \sup_k \frac{1}{h_r} \sum_{k\in I_r} \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k} <' \infty_D,$$

$$\text{uniformly in } k, \text{ for some hyperbolic number } \alpha >' 0_D\Bigg\}.$$

Lemma 2.1. *Let $\theta = \{k_r^{'}\}$ and $\theta^{'} = \{k_r^{''}\}$ be two lacunary sequences and if*

$$\lim_{r\to\infty} \inf \frac{h_r^{'}}{h_r^{''}} > 0, \tag{1}$$

then $S_{C_2}^{\theta^{'}} \subseteq S_{C_2}^{\theta}$ and if

$$\lim_{r\to\infty} \frac{h_r^{''}}{h_r^{'}} = 1, \tag{2}$$

then $S_{C_2}^{\theta} \subseteq S_{C_2}^{\theta^{'}}$, for each $r \in \mathbb{N}$.

3. Main Result

Throughout this section we consider (p_k) a sequence of positive real numbers.

Theorem 3.1. *The sets $b^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$, $b_0^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ and $b_\infty^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ are linear spaces over $C(i_1)$ or $C(i_2)$.*

Proof. Let $\eta = \{\eta_k\}$, $\xi = \{\xi_k\} \in b_0^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ and $\beta, \gamma \in C(i_1)$. Then there exist two hyperbolic numbers $\alpha_1 >' 0_D$ and $\alpha_2 >' 0_D$ such that

$$\text{stat}_\theta - \lim_k \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha_1}\right)\right]^{p_k} = 0_D$$

and

$$\text{stat}_\theta - \lim_k \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\gamma_k)\|_D}{\alpha_2}\right)\right]^{p_k} = 0_D.$$

Let $\alpha_3 = \max\{2|\beta|\alpha_1, 2|\gamma|\alpha_2\}$.

Since $\mathcal{M}_D$ is non-decreasing and D-convex.
We have

$$\mathrm{stat}_\theta - \lim_k \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}[\gamma\eta_k + \beta\xi_k]\|_D}{\alpha_3}\right)\right]^{p_k} \leq' \mathrm{stat}_\theta - \lim_k \left(\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\eta\eta_k)\|_D}{\alpha_1}\right)\right]^{p_k}\right) \tag{3}$$

$$+ \mathrm{stat}_\theta - \lim_k \left(\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\beta\xi_k)\|_D}{\alpha_2}\right)\right]^{p_k}\right) \to 0_D, \text{ as } r \to \infty.$$

Thus, $\{\gamma\eta+\beta\xi\} \in b_0^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$. Hence $b_0^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ is a linear space. Similarly, we can prove $b^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ and $b_\infty^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ are linear spaces over the field $C(i_1)$ or $C(i_2)$. □

Theorem 3.2. *Let $\mathcal{M}_D^1$ and $\mathcal{M}_D^2$ be two D-valued Orlicz function satisfying Δ_D^2-condition then*

(i) $b^*[\xi, p, \theta, \mathcal{M}_D^1, \|\cdot\|_D] \subset b^*[\xi, p, \theta, \mathcal{M}_D^1.\mathcal{M}_D^2, \|\cdot\|_D]$.

(ii) $b_0^*[\xi, p, \theta, \mathcal{M}_D^1, \|\cdot\|_D] \subset b_0^*[\xi, p, \theta, \mathcal{M}_\mathcal{D}^\infty.\mathcal{M}_\mathcal{D}^\in, \|\cdot\|_D]$.

(iii) $b_\infty^*[\xi, p, \theta, \mathcal{M}, \|\cdot\|_D] \subset b_\infty^*[\xi, p, \theta, \mathcal{M}_D^1.\mathcal{M}_D^2, \|\cdot\|_D]$.

Proof. If $\eta = \{\eta_k\} \in b_0^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$, then we have

$$\mathrm{stat}_\theta - \lim_k \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k} = 0_D \to 0, \text{ as } r \to \infty.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $\mathcal{M}_D^1(t) < \varepsilon$ for $0 \leq t \leq \delta$.
Let $\eta_k = \mathcal{M}_D^2\left(\frac{\|\xi_k\|_D}{\alpha}\right)$, for all $k \in \mathbb{N}$.
Let

$$A = \lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \|\nu_{kp}(\xi_k - \xi)\|_D \geq' \varepsilon\right\}\right|$$

and

$$A' = \lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \|\nu_{kp}(\xi_k - \xi)\|_D \leq' \varepsilon\right\}\right|.$$

We can write

$$stat_\theta - \lim_k \mathcal{M}_D(\eta_k) = stat_{\theta_(k\in A')} - \lim_k \mathcal{M}_D(\eta_k) + stat_{\theta_(k\in A)} - \lim_k \mathcal{M}_D(\eta_k).$$

So we have

$$\begin{aligned} stat_{\theta_(k\in A')} - \lim_k \mathcal{M}_D(\eta_k) &\leq [\mathcal{M}_D^1(e_1+e_2)] stat_{\theta_(k\in A')} - \lim_k \mathcal{M}_D(\eta_k) \\ &\leq [\mathcal{M}_D^1(2e_1+2e_2)] stat_{\theta_(k\in A')} - \lim_k \mathcal{M}_D(\eta_k). \end{aligned}$$

Since $\mathcal{M}_D^1$ satisfies Δ_D^2-condition, we can write

$$\mathcal{M}_D^1 \leq K\frac{\mathcal{M}_D^1(2e_1+2e_2)}{\delta}$$

□

Theorem 3.3. *Let $\theta = \{k_r'\}$ and $\theta' = \{k_r''\}$ be two lacunary sequences and if (1) holds, then*

$$N_{\theta'}[\xi, p, \mathcal{M}_D, \|\cdot\|_D] \subset N_\theta[\xi, p, \mathcal{M}_D, \|\cdot\|_D].$$

Proof. Let us assume that $I_r = (k^{'}_{r-1}, k^{'}_r], J_r = (k^{''}_{r-1}, k^{''}_r]$ with $h^{'}_r = k^{'}_{r-1} - k^{'}_r, h^{''}_r = k^{''}_{r-1} - k^{''}_r$ and $I_r \subset J_r$, for all $r \in \mathbb{N}$ and (1) holds. Let $(\xi_k) \in N_{\theta'}[\xi, p, \mathcal{M}_D, \|\cdot\|_D]$. Then for hyperbolic number $\alpha >^{'} 0_D$, we have

$$\lim_{r\to\infty} \frac{1}{h^{''}_r} \sum_{k\in I^{''}_r} \left[\mathcal{M}_D\left(\frac{\upsilon_{kp}\Big(\|\xi_k - \eta\|_D\Big)}{\alpha}\right)\right]^{p_k} = 0_D, \text{ uniformly in } k.$$

Now,

$$\lim_{r\to\infty} \frac{h^{'}_r}{h^{''}_r}\frac{1}{h^{'}_r} \sum_{k\in I^{'}_r} \left[\mathcal{M}_D\left(\frac{\upsilon_{kp}\Big(\|\xi_k - \eta\|_D\Big)}{\alpha}\right)\right]^{p_k}$$
$$\leq \lim_{r\to\infty} \frac{1}{h^{''}_r} \sum_{k\in I^{''}_r} \left[\mathcal{M}_D\left(\frac{\upsilon_{kp}\Big(\|\xi_k - \eta\|_D\Big)}{\alpha}\right)\right]^{p_k} = 0, \text{ uniformly in } k.$$

Thus, $(\xi_k) \in N_{\theta'}[\xi, p, \mathcal{M}_D, \|\cdot\|_D]$ and hence the theorem. $\square$

Theorem 3.4. *The sequence space $b^*_0[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ is D-paranormed space with the D-paranorm*

$$g(\xi) = \inf\left\{(\alpha)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \leq^{'} 1, \textit{ for hyperbolic number } \alpha >^{'} 0\right\}, \tag{4}$$

where $H = \max\{1, \sup_k p_k\} < \infty$.

Proof. Clearly, $g(\xi) = g(-\xi)$ and $g(\xi) >^{'} 0$, for all $\xi \in b^*_0[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$.
Let $\xi, \eta \in b^*_0[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$. Then

$$\text{stat}_\theta - \lim_k \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k} = 0_D \text{ and}$$

$$\text{stat}_\theta - \lim_k \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\eta_k)\|_D}{\alpha}\right)\right]^{p_k} = 0_D.$$

Let

$$S = \left\{(\alpha)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k+\eta_k)\|_D}{\alpha}\right)\right]^{p_k} \leq^{'} 1, \text{ for hyperbolic number } \alpha >^{'} 0\right\}$$
$$A = \left\{(\alpha_1)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k+\eta_k)\|_D}{\alpha_1}\right)\right]^{p_k} \leq^{'} 1, \text{ for hyperbolic number } \alpha_1 >^{'} 0\right\}$$
$$B = \left\{(\alpha_2)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k+\eta_k)\|_D}{\alpha_2}\right)\right]^{p_k} \leq^{'} 1, \text{ for hyperbolic number } \alpha_2 > 0\right\}.$$

Let $\alpha = (\alpha_1 + \alpha_2) \in S, \alpha_1 = v^{'}_1 e_1 + v^{'}_2 e_2 \in S_1, \alpha_2 = v^{"}_1 e_1 + v^{"}_2 e_2 \in S_2$ and $\alpha = v_1 e_1 + v_2 e_2$.

Now,

$$\begin{aligned}
g(\xi+\eta) =& \inf\left\{(\alpha)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k+\eta_k)\|_D}{\alpha}\right)\right]^{p_k} \leq' 1\right\}\\
=& \inf\{v_1 : \alpha \in S\}e_1 + \inf\{v_2 : \alpha \in S\}e_2\\
=& \inf\{v_1^{'} : \alpha_1 \in S_1\}e_1 + \inf\{v_1^{"} : \alpha_2 \in S_2\}e_1\\
& + \inf\{v_2^{'} : \alpha_1 \in S_1\}e_2 + \inf\{v_2^{"} : \alpha_2 \in S_2\}e_2\\
=& \inf\{v_1^{'} : \alpha_1 \in S_1\}e_1 + \inf\{v_2^{'} : \alpha_1 \in S_1\}e_2\\
& + \inf\{v_1^{"} : \alpha_2 \in S_2\}e_1 + \inf\{v_2^{"} : \alpha_2 \in S_2\}e_2\\
=& \left\{(\alpha_1)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k+\eta_k)\|_D}{\alpha_1}\right)\right]^{p_k} \leq' 1\right\}\\
& + \left\{(\alpha_2)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\eta_k)\|_D}{\alpha_2}\right)\right]^{p_k} \leq' 1\right\}\\
& = g(\xi) + g(\eta).
\end{aligned}$$

Finally, we prove that the scalar multiplication is continuous. Let β be any bi-complex scalar. Then

$$g(\beta\xi) = \inf\left\{(\alpha)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\beta\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \leq' 1, \text{ for hyperbolic number } \alpha >' 0\right\}.$$

Let

$$\begin{aligned}
M &= \left\{(\alpha)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\beta\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \leq' 1\right\}\\
M_1 &= \{(\alpha_1)^{\frac{p_k}{H}} : \alpha_1 e_1 + \alpha_2 e_2 \in M\}\\
M_2 &= \{(\alpha_2)^{\frac{p_k}{H}} : \alpha_1 e_1 + \alpha_2 e_2 \in M\}, \text{ where } \alpha = \alpha_1 e_1 + \alpha_2 e_2.
\end{aligned}$$

Now,

$$\begin{aligned}
g(\beta\xi) =& \inf M = e_1 \inf M_1 + e_2 \inf M_2\\
=& \inf\left\{(\alpha_1)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\beta\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \leq' 1\right\}e_1\\
& + \inf\left\{(\alpha_2)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\beta\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \leq' 1\right\}e_2\\
=& \inf\left\{(\|\beta\|_D P_1)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \leq' 1\right\}e_1\\
& + \inf\left\{(\|\beta\|_D P_2)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \leq' 1\right\}e_2\\
=& (\|\beta\|_D)^{\frac{p_k}{H}}\left(\inf\left\{(P_1)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \leq' 1\right\}e_1\right.\\
& \left. + \inf\left\{(P_2)^{\frac{p_k}{H}} : \sup\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \leq' 1\right\}e_2\right)\\
=& (\|\beta\|_D)^{\frac{p_k}{H}} g(\xi), \qquad (5)
\end{aligned}$$

where, $P_i = \frac{\alpha_i}{\|\beta\|_D}, i = 1, 2.$ □

Theorem 3.5. *The space $b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ is D-convex.*

Proof. Let $\xi, \eta \in b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ and let $0 \leq' \lambda \leq' 1$. Then

$$\sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha_1}\right)\right]^{p_k} <' \infty_D,$$
$$\sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\eta_k)\|_D}{\alpha_2}\right)\right]^{p_k} <' \infty_D. \tag{6}$$

Let $\alpha = \max\{\alpha_1, \alpha_2\}$. Now

$$\begin{aligned}
&\sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\lambda\xi_k + (1-\lambda)\eta_k)\|_D}{\alpha}\right)\right]^{p_k} \\
&\leq' A\Bigg(\sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\lambda\xi_k)\|_D}{\alpha_1}\right)\right]^{p_k} \\
&\quad + \sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}((1-\lambda)\eta_k)\|_D}{\alpha_2}\right)\right]^{p_k}\Bigg) \\
&\leq' A\Bigg(\lambda \sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha_1}\right)\right]^{p_k} \\
&\quad + (1-\lambda) \sup_r \frac{1}{h_r} \sum_{k\in I_r} \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\eta_k)\|_D}{\alpha_2}\right)\right]^{p_k}\Bigg) <' \infty_D.
\end{aligned}$$

Hence, $(\lambda\xi + (1-\lambda)\eta) \in b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ and therefore the space $b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ is D-convex. □

Theorem 3.6. *The spaces $b^*[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D], b^*_0[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ and $b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ are D-submodule of ω^*.*

Proof. As $b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ is a subspace of ω^*. Now, $\forall\tau \in D$ and $\forall\eta \in b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$
we have

$$\begin{aligned}
&\sup_k \frac{1}{h_r} \sum_{k\in I_r} \left(\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k\tau)\|_D}{\alpha}\right)\right]^{p_k}\right)^{\frac{1}{p_k}} \\
&= \sup_k \frac{1}{h_r} \sum_{k\in I_r} \left(\left[\mathcal{M}_D\left(\frac{\|\tau\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k}\right)^{\frac{1}{p_k}} \\
&\leq' \sup_k \frac{1}{h_r} \sum_{k\in I_r} \left(\left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D\|\tau\|_D}{\alpha}\right)\right]^{p_k}\right)^{\frac{1}{p_k}} \\
&= \|\tau\| \sup_k \frac{1}{h_r} \sum_{k\in I_r} \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right] \\
&<' \infty_D.
\end{aligned}$$

Thus, $\forall \tau \in D$ and $\forall \eta \in b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$, we have $\tau\eta \in b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$. Hence $b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$ is a D-submodule of ω^*. Similar procedure can apply for the other cases. $\square$

Theorem 3.7. *The D-paranorm $g(\xi)$ defined in (4) is D-balanced and D-absorbing subset of the submodule $b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$.*

Proof. Let $\xi \in g(\xi)$ and $\beta \in C_2$, with $||\beta\|_D \leq' 1$. We have from (5). Now

$$\begin{aligned}
&\sup \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\beta\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \\
&= \sup \left[\mathcal{M}_D\left(\frac{\|\beta\nu_{kp}(\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \\
&= \sup \left[\mathcal{M}_D\left(\frac{\|\beta\|_D\|\nu_{kp}(\beta\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \\
&= (\|\beta\|_D)^{p_k} \sup \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\beta\xi_k)\|_D}{\alpha}\right)\right]^{p_k} \\
&\leq' \sup \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\beta\xi_k)\|_D}{\alpha}\right)\right]^{p_k} <' 1.
\end{aligned}$$

Thus, $\beta\xi \in g(\xi)$. Hence, $g(\xi)$ is D-balanced subset of $b^*_\infty[\xi, p, \theta, \mathcal{M}_D, \|\cdot\|_D]$. Choosing $k >' 0_D$ such that $\sup \left[\mathcal{M}_D\left(\frac{\|\nu_{kp}(\beta\xi_k)\|_D}{\alpha}\right)\right]^{p_k} <' k^{\frac{p_k}{H}}$. Set $k = \frac{1}{\varepsilon}$. Then for any $0_D \leq' \beta \leq' \varepsilon$, we have

$$\begin{aligned}
g(\beta\xi) &= (\|\beta\|_D)^{\frac{p_k}{H}} g(\xi) \\
&= (\beta)^{\frac{p_k}{H}} g(\xi) \\
&\leq' (\varepsilon)^{\frac{p_k}{H}} g(\xi) \\
&= \left(\frac{1}{k}\right)^{\frac{p_k}{H}} g(\xi) <' 1.
\end{aligned}$$

Thus $g(\xi)$ is D-absorbing subset. $\square$

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