Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 32, 2005, Pages 227–232
ISSN: 1223-6934

Stability for damped oscillators

Cristian Vladimirescu

Abstract. The stability of the null solution to Eq. (1) below is investigated. This paper presents the method detailed in [16] and based on some Bernoulli type differential inequalities. Extensions to the whole real line are also discussed.

2000 Mathematics Subject Classification. 34A40, 34D20.
Key words and phrases. stability, Bernoulli type differential inequalities.

1. Introduction

In this paper we consider the equation

$$x'' + 2f(t)\,x' + x + g(t,x) = 0,\;\; t \in \mathbb{R}_+, \tag{1}$$

where $\mathbb{R}_+ := [0, +\infty)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are two given continuous functions.

It is well known that (1) describes the motion of the damped nonlinear oscillator of one degree of freedom. Many authors studied the stability questions for such kind of equations (see, e.g., [4], [6]-[15], [18]-[20] and the references therein). For the definitions of different concepts of stability and for standard stability theory see, e.g., [5] and [17].

Eq. (1) can be considered a perturbation to the linear equation

$$x'' + 2f(t)\,x' + x = 0,\;\; t \in \mathbb{R}_+. \tag{2}$$

Regarding Eq. (2), the case of large damping has been also considered in [1] wherein the authors proved that the inequality

$$\int_0^t f(s)\,ds \le C_1 + C_2 t^2, \quad \forall t \in \mathbb{R}_+,$$

where C_1, C_2 are constant implies asymptotic stability. The difficult case is the one of small damping. In [2] it is proved that $\int_0^\infty f = \infty$ is necessary and sufficient for the asymptotic stability, provided that f is monotonous; there is no necessary and sufficient integral condition for the general (non-monotonous) case. In [21] a necessary and sufficient condition working both for the case of large damping in terms of the integral of f, was given, but too difficult to check it.

In [3] the asymptotic stability of the null solution to Eq. (1) has been studied by means of a new approach based on the Schauder fixed point Theorem.

In the present paper, some stability results are proved under assumptions more general than those of [3] (see Remark 2.1 in Section 2). Our approach is based on elementary arguments only, involving in particular some Bernoulli type differential inequalities and has been detailed in [16].

Received: October 11, 2004.

As we will see, under our assumptions f can be chosen in a larger class of functions which allows to obtain extensions of our stability results to the whole real line $\mathbb{R}$, obtaining in particular a strong stability result (see Section 4).

2. The main result

The following hypotheses will be required:
(i) $f \in C^1(\mathbb{R}_+)$ and $f(t) > 0$, for all $t > 0$;
(ii) $\int_0^{+\infty} f(t)\,dt = +\infty$;
(iii) there exist $a \geq 0$ and $K \in (0,1)$ such that

$$\left|f'(t) + f^2(t)\right| \leq K f(t), \text{ for all } t \in [a, +\infty); \tag{3}$$

(iv) $g \in C(\mathbb{R}_+ \times \mathbb{R})$ and g is locally Lipschitzian in x;
(v) there exist $M > 0$ and $\alpha > 1$ such that

$$|g(t,x)| \leq M f(t)\,|x|^{\alpha}, \text{ for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{4}$$

An example of functions f and g is (cf. [3])

$$f(t) = \frac{1}{t+1}, \; g(t,x) = f(t) \cdot x^{\alpha}.$$

Indeed, these functions fulfil (i)-(v) with $a \geq 0$, $K \in (0,1)$, $\alpha > 1$ arbitrary, and $M = 1$.

Remark 2.1. *In* [3] *the following additional assumptions are required:* $\lim_{t\to\infty} f(t) = 0$, *the constant* a *in (iii) is fixed to* $a = 0$, *and a more restrictive condition is assumed instead of (iv), namely*

$$|g(t,x) - g(t,y)| \leq L(\delta) f(t)|x-y|, \quad \forall t \geq 0, \; |x|, |y| \leq \delta,$$

with $L(\delta)$ *continuous and increasing.*

Remark 2.2. *Regarding the discussion from Section 1, it is obvious that we are in the case of "small damping" (see, e.g.,* [9], *p. 415)). Indeed, by* (3) *it follows that* f *is uniformly bounded: there exists a* $c > 0$ *such that* $0 < f(t) \leq c$, $\forall t \geq 0$.

The main result of this paper is the following theorem.

Theorem 2.1. *If the assumptions* (i), (iii), (iv), *and* (v) *are fulfilled, then the null solution to Eq.* (1) *is uniformly stable. If in addition* (ii) *holds, then the null solution to Eq.* (1) *is asymptotically stable.*

Remark 2.3. *Under the assumptions* $(i) - (v)$, *we cannot expect to have uniform asymptotic stability for the null solution. Indeed, if* $f(t) = 1/(t+1)$ *and* $g = 0$, *the general solution to Eq.* (1) *is given by*

$$x(t) = (t+1)^{-1}\left(C_1 \cos(t+1) + C_2 \sin(t+1)\right),$$

and so the null solution to the corresponding first order linear differential system in $(x, y = x')$ *is not uniformly asymptotically stable.*

3. Proof of Theorem 2.1

In this Section we will present the sketch of the proof of Theorem 2.1 (the details of the proof can be found in [16]).

As in [3], we write Eq. (1) as the following first order system

$$z' = A(t)\,z + B(t)\,z + F(t,z)\,, \tag{5}$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A(t) = \begin{pmatrix} -f(t) & 1 \\ -1 & -f(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ f'(t) + f^2(t) & 0 \end{pmatrix},$$

$$F(t,z) = \begin{pmatrix} 0 \\ -g(t,x) \end{pmatrix}.$$

Obviously, our stability question reduces to the stability of the null solution $z(t) = 0$ of the system (5). Notice that the fundamental matrix of the linear system

$$z'(t) = A(t)\,z, \tag{6}$$

which is equal to the identity matrix for $t = t_0$, $t_0 \geq 0$, is given by

$$Z(t,t_0) = \exp\left(-\int_{t_0}^{t} f(s)\,ds\right) \cdot \begin{pmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{pmatrix},$$

for all $t \in \mathbb{R}_+$.

Remark 3.1. *It is known that if* (6) *is uniformly asymptotically stable, then for "small" perturbations* F, (5) *is uniformly asymptotically stable; but in our case one gets a difficulty since the system* (6) *is only asymptotically stable and it is not uniformly asymptotically stable.*

Now, if $z := (x,y)^T$ is a vector of $\mathbb{R}^2$ we set $\|z\| := \sqrt{x^2+y^2}$.

We first assume that (i), (iii)-(v) are fulfilled. In order to prove that the null solution of (5) is stable, take some $z_0 \neq 0$ with $\|z_0\|$ small enough and $t_0 \geq 0$ and denote by $z(t,t_0,z_0)$ the unique solution of (5) which is equal to z_0 for $t = t_0$. By our assumptions, we know that $z(t,t_0,z_0)$ is defined on a maximal right interval, say $[t_0, b)$. This solution satisfies the integral equation

$$\begin{aligned} z(t,t_0,z_0) \;\; = \;\; & Z(t,t_0)z_0 + \int_{t_0}^{t} Z(t,t_0)Z(s,t_0)^{-1}\left[B(s)z(s,t_0,z_0) + \right. \\ & \left. + F(s,z(s,t_0,z_0))\right] ds, \end{aligned} \tag{7}$$

for all $t \in [t_0, b)$. In fact we can show that $b = +\infty$. If $a > 0$ and $t_0 < a$ then it follows by (7) (see [3])

$$\begin{aligned} \|z(t,t_0,z_0)\| \;\; \leq \;\; & \|z_0\| e^{-\int_{t_0}^{t} f(s)ds} + \\ & + \int_{t_0}^{t} e^{-\int_s^t f(u)du} \left|f'(s) + f^2(s)\right| \|z(s,t_0,z_0)\| ds + \\ & + M \int_{t_0}^{t} e^{-\int_s^t f(u)du} f(s)\, \|z(s,t_0,z_0)\|^{\alpha} ds, \end{aligned} \tag{8}$$

for all $t \in [t_0, b)$. Suppose, by contradiction, that $b < +\infty$. Moreover, in a first stage, assume that $b \leq a$.

Since $f \in C^1[0, a]$, it follows by (8) that there exists a constant $D > 0$ such that

$$\|z(t, t_0, z_0)\| \leq \|z_0\| + D \int_{t_0}^{t} (\|z(s, t_0, z_0)\| + \|z(s, t_0, z_0)\|^{\alpha})\, ds =: r\,(t)\,,$$

for all $t \in [t_0, b)$,

$$r\,(t_0) = \|z_0\|,\; r\,(t) \geq \|z_0\| > 0,\;\; t \in [t_0, b).$$

By classical estimates and Bernoulli type differential inequalities, if

$$\|z_0\| < \left(e^{D(\alpha-1)a} - 1\right)^{\frac{1}{1-\alpha}} =: \delta_1,$$

one gets

$$r\,(t) \leq \left(\left(1 + \|z_0\|^{1-\alpha}\right) e^{D(1-\alpha)a} - 1\right)^{\frac{1}{1-\alpha}},\;\; (\forall)\;\; t \in [t_0, b). \tag{9}$$

Since $r(t)$ is bounded on $[t_0, b)$, it follows that $z(t, t_0, z_0)$ and $z'(t, t_0, z_0)$ are both bounded on $[t_0, b)$ and therefore $z(t, t_0, z_0)$ can be extended to the right of b. This fact contradicts the maximality of b. Hence, $z(t, t_0, z_0)$ does exist on $[t_0, b)$ with $b > a$. Let us still assume that b is finite, i.e. $a < b < +\infty$. We are going to establish an estimate for $z(t, t_0, z_0)$ on the interval $[a, b)$. This time, our assumption (iii) comes into play. Indeed, starting from (7), where t_0 and z_0 are replaced by a and $z(a, t_0, z_0)$, we get

$$\begin{aligned}\|z\,(t, t_0, z_0)\,\| \;\;&\leq\;\; \|z(a, t_0, z_0)\| e^{-\int_a^t f(s)ds} + \int_a^t e^{-\int_s^t f(u)du}\,[Kf\,(s)\,|x\,(s, t_0, z_0)| \\ &\quad + Mf\,(s)\,|x\,(s, t_0, z_0)|^{\alpha}]\, ds =: v\,(t)\,,\;\; a \leq t < b.\end{aligned}$$

We have used the fact that $z(t, t_0, z_0) = z(t, a, z(a, t_0, z_0)),\; a \leq t < b$. After easy computations and again by Bernoulli type differential inequalities, we find

$$\begin{aligned}v\,(t) \;\;&\leq\;\; \left\{e^{(\alpha-1)(1-K)\int_a^t f(s)ds}\left[\|z(a, t_0, z_0)\|^{1-\alpha} - \frac{M}{1-K}\right] + \right. \\ &\quad \left. + \frac{M}{1-K}\right\}^{\frac{1}{1-\alpha}},\end{aligned} \tag{10}$$

for all $t \in [a, b)$.

So, if

$$\|z(a, t_0, z_0)\| \in \left(0, \left(\frac{1-K}{M}\right)^{\frac{1}{\alpha-1}}\right),$$

then (10) shows that $z(t, t_0, z_0)$ is bounded on $[a, b)$ and hence $b = +\infty$.

If $t_0 \geq a$, then we similarly get

$$v\,(t) \leq \left\{e^{(\alpha-1)(1-K)\int_{t_0}^t f(s)ds}\left[\|z_0\|^{1-\alpha} - \frac{M}{1-K}\right] + \frac{M}{1-K}\right\}^{\frac{1}{1-\alpha}}, \tag{11}$$

for all $t \in [a, b)$.

Again, for

$$\|z_0\| \in \left(0, \left(\frac{1-K}{M}\right)^{\frac{1}{\alpha-1}}\right),$$

$z(t, t_0, z_0)$ exists on $[t_0, +\infty)$ (i.e., $b = +\infty$).

Now, by the estimates (9) (for $t \in [t_0, a]$), (10), (11), where $b = +\infty$, we see that the null solution of (1) is uniformly stable. If in addition (ii) is fulfilled, then by (10), (11) (where $b = +\infty$), it follows that the null solution of (1) is asymptotically stable. The proof of Theorem 2.1 is complete. □

4. Extensions to $\mathbb{R}$

This section contains some remarks concerning the extension of Theorem 2.1 to the whole real line $\mathbb{R}$.

We consider the equation

$$x'' + 2f(t)\,x' + x + g(t,x) = 0,\ t \in \mathbb{R}, \tag{12}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are two given functions, satisfying the following hypotheses:

(i)' $f \in C^1(\mathbb{R})$ and $t \cdot f(t) > 0$, for all $t \in \mathbb{R}$, $t \neq 0$;

(ii)' $\int_{-\infty}^{0} f(t)\,dt = -\infty$ and $\int_{0}^{+\infty} f(t)\,dt = +\infty$;

(iii)' there exist $a \geq 0$ and $K \in (0,1)$ such that

$$\left|f'(t) + f^2(t)\right| \leq K\,|f(t)|\,,\ \text{for all}\ |t| \geq a; \tag{13}$$

(iv)' $g \in C(\mathbb{R}\times\mathbb{R})$ and g is locally Lipschitzian in x;

(v)' there exist $M > 0$ and $\alpha > 1$ such that

$$|g(t,x)| \leq M\,|f(t)| \cdot |x|^{\alpha}\,,\ \text{for all}\ (t,x) \in \mathbb{R}^2.$$

A prototype of functions f and g is $f(t) = 1/t$ for $|t| \geq a$ $(a > 0)$ and extended on the interval $(-a, a)$ in such a way that $f \in C^1(\mathbb{R})$ and $tf(t) > 0$ for all $t \in \mathbb{R}$, $t \neq 0$ (for instance we can choose $f(t) = t(3/a^2 - 2\,|t|\,/a^3)$, $|t| < a$), and $g(t,x) = f(t)\,x^{\alpha}$. Then hypotheses (i)-(v) are fulfilled with $a = 1$, $K \in (0,1)$, $M \geq 1$, $\alpha > 1$.

We remark that through the changes

$$s = -t,\ u(s) = x(-s),\ t \leq 0,$$

Eq. (12) for $t \leq 0$ becomes

$$\frac{d^2u}{ds^2} + 2f^*(s)\,\frac{du}{ds} + u + g^*(s,u) = 0,\ s \in \mathbb{R}_+,$$

where $f^*(s) = -f(-s)$ and $g^*(s,u) = g(-s,u)$.

Therefore, by this remark and Theorem 2.1 we obtain the following result (for the definition of the strong stability, see [8]).

Theorem 4.1. *Suppose that the hypotheses (i)', (iii)'-(v)' are fulfilled. Then the null solution to Eq.* (12) *is strongly stable, i.e. for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every initial data $|x(0)| < \delta$, $|x'(0)| < \delta$, Eq.* (12) *has a unique solution $x(t)$ defined on $\mathbb{R}$, satisfying $|x(t)| < \epsilon$, $|x'(t)| < \epsilon$, $(\forall)$ $t \in \mathbb{R}$. If, in addition, (ii)' holds, then*

$$x(\pm\infty) = x'(\pm\infty) = 0.$$

Remark 4.1. *Our extensions to the whole real line $\mathbb{R}$ are allowed by the fact that the key condition* (13) *is fulfilled away from the origin.*

References

[1] Z. Artstein, E.F. Infante, On the asymptotic stability of oscillators with unbounded damping, *Quarterly of Applied Mechanics*, **34**, 195-199 (1976).

[2] R.J. Ballieu, K. Peiffer, Attractivity of the origin for the equation $\ddot{x} + f(t, x, \dot{x})|\dot{x}|^\alpha \dot{x} + g(x) = 0$, *Journal of Mathematical Analysis and Applications*, **34**, 321-332 (1978).

[3] T.A. Burton, T. Furumochi, A note on stability by Schauder's theorem, *Funkcialaj Ekvacioj*, **44**, 73-82 (2001).

[4] T.A. Burton, J.W. Hooker, On solutions of differential equations tending to zero, *Journal für die Reine und Angewandte Mathematik*, **267**, 151-165 (1974).

[5] C. Corduneanu, *Principles of differential and integral equations*, Boston, Allyn and Bacon, 1971.

[6] L. Hatvani, On the stability of the zero solution of certain second order non-linear differential equations, *Acta Scientiarum Mathematicarum*, **32**, 1-9 (1971).

[7] L. Hatvani, On the asymptotic behaviour of the solutions of $(p(t)x')' + q(t)f(x) = 0$, *Publicationes Mathematicae*, **19**, 225-237 (1972).

[8] L. Hatvani, On the strong stability by Lyapunov's direct method, *Acta Scientiarum Mathematicarum*, **37**, 241-251 (1975).

[9] L. Hatvani, Integral conditions on the asymptotic stability for the damped linear oscillator with small damping, *Proceedings of the American Mathematical Society*, **124**, 415-422 (1996).

[10] L. Hatvani, T. Krisztin, V. Totik, A necessary and sufficient condition for the asymptotic stability of the damped oscillator, *Journal of Differential Equations*, (2)**119**, 209-223 (1995).

[11] L. Hatvani, V. Totik, Asymptotic stability of the equilibrium of the damped oscillator, *Differential Integral Equations*, **6**, 835-848 (1993).

[12] J. Karsai, On the global asymptotic stability of the zero solution of the equation $\ddot{x} + g(t, x, \dot{x})\dot{x} + f(x) = 0$, *Studia Scientiarum Mathematicarum Hungarica*, **19**, 385-393 (1984).

[13] J. Karsai, On the asymptotic stability of the zero solution of certain nonlinear second order differential equations, *Differential Equations: Qualitative Theory* (Szeged, 1984), Colloq. Math. Soc. János Bolyai, **47**, Amsterdam and New York, North-Holland, 495-503, 1987.

[14] V. Kertész, Stability investigation of the differential equation of damped oscillation, *Alkalmazott Matematikai Lapok*, **8**, 323-339 (1982) (Hungarian, English summary).

[15] J.J. Levin, J.A. Nohel, Global asymptotic stability of nonlinear systems of differential equations and applications to reactor dynamics, *Archive for Rational Mechanics and Analysis*, **5**, 194-211 (1960).

[16] Gh. Moroşanu, C. Vladimirescu, Stability for a nonlinear second order ODE, *Funkcialaj Ekvacioj*, **48**, 49-56 (2005).

[17] L.C. Piccinini, G. Stampacchia, G. Vidossich, *Ordinary Differential Equations in* $\mathbb{R}^n$. *Problems and Methods*, Applied Mathematical Sciences, **39**, New York, Springer-Verlag, 1984.

[18] P. Pucci, J. Serrin, Precise damping conditions for global asymptotic stability for nonlinear second order systems, *Acta Mathematica*, **170**, 275-307 (1993).

[19] P. Pucci, J. Serrin, Precise damping conditions for global asymptotic stability for nonlinear second order systems, II, *Journal of Differential Equations*, **113**, 505-534 (1994).

[20] P. Pucci, J. Serrin, Asymptotic stability for intermittently controlled nonlinear oscillators, *SIAM Journal on Mathematical Analysis*, **25**, 815-835 (1994).

[21] R.A. Smith, Asymptotic stability of $x'' + a(t)x' + x = 0$, *Quarterly Journal of Mathematics, Oxford Series*, (2)**12**, 123-126 (1961).

(Cristian Vladimirescu) Department of Mathematics, University of Craiova,
A.I. Cuza Street, No. 13, Craiova RO-200585, Romania
E-mail address: vladimirescucris@yahoo.com