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Interpolation of compact non-linear operators on Banach triples

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Abstract. We investigate the behavior of Lipschitz and compact non-linear operators under K and J real interpolation methods for Banach triples. We begin with the case when one of the triples reduces to a single Banach space, and we prove that the classical Lions-Peetre compactness theorems for linear operators still hold for Lipschitz and compact non-linear operators. We also establish a compactness result when the interpolation operator is considered from a J-space into a K-space.

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1. Introduction

The behaviour of compact non-linear operators under real interpolation have not even been investigated. Cobos [5] proved that the classical Lions-Peetre [8] compactness theorems for linear operators, in the case of Banach couples, are also valid for Lipschitz operators. Later, Bento [3] generalised some of these results.

In this paper we investigate the behavior of Lipschitz and compact non-linear operators in the multidimensional case. In the literature there are many interpolation methods for Banach n-space ($n \geq 3$). We restrict our attention to Sparr K-and J-methods for Banach triples.

2. Preliminaries and notations

Our notation and terminology are standard and we refer to [4], [7] and [10]. For the reader's convenience we give some definitions and results that will be used later.

Let $\overline{A} = (A_0, A_1, A_2)$ be a Banach triple, i.e. A_0, A_1, A_2 are three Banach spaces which are linearly and continuously embedded in a Hausdorff topological vector space $\mathscr{A}$. Then we can form their sum $\sum(\overline{A}) := A_0 + A_1 + A_2$ and their intersection $\Delta(\overline{A}) := A_0 \cap A_1 \cap A_2$. These two spaces become Banach spaces when endowed with the norms $K(1,1,\cdot)$ and $J(1,1,\cdot)$ respectively, where

$$K(t,s,a) = \inf\left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} + s\|a_2\|_{A_2} :\ a = a_0 + a_1 + a_2,\ a_i \in A_i\right\},\ t,s > 0$$

and

$$J(t,s,a) = \max\left(\|a_0\|_{A_0}, t\|a_1\|_{A_1}, s\|a_2\|_{A_2}\right),\ t,s > 0.$$

Let A be an intermediate space with respect to $\overline{A}$, i.e. A is a Banach space for which $\Delta(\overline{A}) \hookrightarrow A \hookrightarrow \sum(\overline{A})$ (continuous inclusions) and let $\overline{\theta} = (\theta_1, \theta_2) \in (0,1)^2$, $\theta_1 + \theta_2 < 1$. Then we say that

(i) A is of class $\mathscr{C}_K(\overline{\theta}, \overline{A})$ if $K(t,s,a) \leq ct^{\theta_1} s^{\theta_2} \|a\|_A$, $a \in A$;
(ii) A is of class $\mathscr{C}_J(\overline{\theta}, \overline{A})$ if $\|a\|_A \leq ct^{-\theta_1} s^{-\theta_2} J(t,s,a)$, $a \in \Delta(\overline{A})$;

(iii) A is class $\mathscr{C}(\overline{\theta},\overline{A})$ if A is of class $\mathscr{C}_K(\overline{\theta},\overline{A})$ and of class $\mathscr{C}_J(\overline{\theta},\overline{A})$.

An important example of space of class $\mathscr{C}_K(\overline{\theta},\overline{A})$ is the real interpolation K-space (or Sparr K-space [10]) $(A_0,A_1,A_2)_{\overline{\theta},p,K} := \overline{A}_{\overline{\theta},p,K}$. We remind that for $1\le p\le\infty$ and $\overline{\theta}=(\theta_1,\theta_2)\in(0,1)^2$, $\theta_1+\theta_2<1$, the space $\overline{A}_{\overline{\theta},p,K}$ consists of all $a\in\sum(\overline{A})$, which have a finite norm:

$$\|a\|_{\overline{\theta},p,K} = \begin{cases} \left(\displaystyle\int_0^\infty\int_0^\infty \left(t^{-\theta_1}s^{-\theta_2}K(t,s,a)\right)^p \frac{dt}{t}\frac{ds}{s}\right)^{1/p} & \text{if } 1\le p<\infty \\ \displaystyle\sup_{\substack{t>0\\ s>0}} \; t^{-\theta_1}s^{-\theta_2}K(t,s,a) & \text{if } p=\infty \end{cases}$$

On the other hand, the real interpolation J-space (or Sparr J-space [10]) $(A_0,A_1,A_2)_{\overline{\theta},p,J} := \overline{A}_{\overline{\theta},p,J}$ is an important example of space of class $\mathscr{C}_J(\overline{\theta},\overline{A})$. We remind that for $1\le p\le\infty$ and $\overline{\theta}=(\theta_1,\theta_2)\in(0,1)^2$, $\theta_1+\theta_2<1$ the space $\overline{A}_{\overline{\theta},p,J}$ consists of all $a\in\sum(\overline{A})$ which can be represented in the form

$$a=\int_0^\infty\int_0^\infty u(t,s)\frac{dt}{t}\frac{ds}{s} \qquad (\text{convergence in } \textstyle\sum(\overline{A})) \tag{1}$$

where $u(t,s)$ is measurable with values in $\Delta(\overline{A})$ and

$$\int_0^\infty\int_0^\infty \left(t^{-\theta_1}s^{-\theta_2}J(t,s,u(t,s))\right)^p \frac{dt}{t}\frac{ds}{s}<\infty. \tag{2}$$

Moreover

$$\|a\|_{\overline{\theta},p,J} = \inf\left(\int_0^\infty\int_0^\infty (t^{-\theta_1}s^{-\theta_2}J(t,s,u(t,s)))^p\frac{dt}{t}\frac{ds}{s}\right)^{1/p}$$

where the infimum is taken over all representations of a such that (1) and (2) hold, defines a norm on $\overline{A}_{\overline{\theta},p,J}$.

In contrast to the case of Banach couples, when the Equivalence Theorem is true (i.e. $(A_0,A_1)_{\theta,p,J}=(A_0,A_1)_{\theta,p,K}:=(A_0,A_1)_{\theta,p}$; $0<\theta<1$, $1\le p\le\infty$) (see [4]), in the case of Banach triples, in general, only the inclusion $\overline{A}_{\overline{\theta},p,J}\hookrightarrow\overline{A}_{\overline{\theta},p,K}$ is valid, but not converse (see [10]). Though, there are many triples for which the converse inclusion is valid (see [1], [6]). We call an LP-triple, that Banach triple $\overline{A}$ for which $\overline{A}_{\overline{\theta},p,J}=\overline{A}_{\overline{\theta},p,K}:=\overline{A}_{\overline{\theta},p}$ holds for every $\overline{\theta}=(\theta_1,\theta_2)\in(0,1)^2$, $\theta_1+\theta_2<1$ and every $1\le p\le\infty$. For a Banach triple $\overline{A}=(A_0,A_1,A_2)$ we have (see [2], [10])

$$\overline{A}_{\overline{\theta},p,J}\hookrightarrow((A_0,A_2)_{\theta_2,p},(A_1,A_2)_{\theta_2,p})_{\lambda,p}\hookrightarrow\overline{A}_{\overline{\theta},p,K} \tag{3}$$

where $\lambda=\dfrac{\theta_1}{1-\theta_2}$.

The class of all continuous maps $T:\sum(\overline{A})\to\sum(\overline{B})$ such that the restriction of T to A_i is a continuous map from A_i into B_i, $(i=0,1,2)$ will be denoted by $\mathscr{C}(\overline{A},\overline{B})$. If $A_0=A_1=A_2=A$ or $B_0=B_1=B_2=B$, then we write $\mathscr{C}(A,\overline{B})$or, respectively $\mathscr{C}(\overline{A},B)$.

Recall that a non-linear operator T is called compact if it is continuous and if it transforms each bounded set into a set whose closure is compact.

It was proved by Cobos in [5] that if $T\in\mathscr{C}(\overline{A},\overline{B})$ and $T:A_i\to B_i$, $(i=0,1)$ are Lipschitz operators, then for every $\theta\in(0,1)$, $p\in[1,\infty)$

$$T:(A_0,A_1)_{\theta,p}\to(B_0,B_1)_{\theta,p} \text{ is also a Lipschitz operator.} \tag{4}$$

Later, Bento [3] proved that if $T \in \mathscr{C}(\overline{A}, \overline{B})$ is an operator such that $T : A_i \to B_i$, $(i = 0, 1)$ are compact Lipschitz operators, then for every $\theta \in (0, 1)$ and $1 \leq p < \infty$

$$T : (A_0, A_1)_{\theta,p} \to (B_0, B_1)_{\theta,p} \text{ is compact} \tag{5}$$

We extend these results to the case of Banach triples.

3. Interpolation of Lipschitz operators

We begin with the case when one of the triples reduces to a single Banach space.

Proposition 3.1. *Let $\overline{A} = (A_0, A_1, A_2)$ be a Banach triple and let B be a Banach space. Assume that T is a non-linear operator*

(i) *If $T \in \mathscr{C}(\overline{A}, B)$ is an operator such that $T : A_i \to B$, $(i = 0, 1, 2)$ are Lipschitz operators and A is an intermediate space with respect to $\overline{A}$ such that $\Delta(\overline{A})$ is dense in A, then $T : A \to B$ is also a Lipschitz operator.*

(ii) *If $T \in \mathscr{C}(B, \overline{A})$ is an operator such that $T : B \to A_i$ $(i = 0, 1, 2)$ are Lipschitz operators and A is an intermediate space with respect to $\overline{A}$, then $T : B \to A$ is also a Lipschitz operator.*

Proof. (i) Let L_i be the Lipschitz constant of T as a mapping from A_i to B $(i = 0, 1, 2)$. Let $x, y \in \Delta(\overline{A})$ and take any decomposition

$$x - y = a_0 + a_1 + a_2, \text{ with } a_i \in A_i \ (i = 0, 1, 2) \ \text{(in fact, } a_i \in \Delta(\overline{A}),\ i = 0, 1, 2).$$

Put $w_i = x - \sum_{k=0}^{i} a_k$, $(i = 0, 1, 2)$. Then

$$\begin{aligned}
\|Tx - Ty\| &\leq \|Tx - Tw_0\|_B + \|Tw_0 - Tw_1\|_B + \|Tw_1 - Ty\|_B \\
&\leq L_0\|a_0\|_{A_0} + L_1\|a_1\|_{A_1} + L_2\|a_2\|_{A_2} \\
&\leq \max(L_0, L_1, L_2)(\|a_0\|_{A_0} + \|a_1\|_{A_1} + \|a_2\|_{A_2})
\end{aligned}$$

Thus, for any $x, y \in \Delta(\overline{A})$ we have

$$\|Tx - Ty\|_B \leq \max(L_0, L_1, L_2)\|x - y\|_{\sum(\overline{A})}.$$

Hence

$$T : (\Delta(\overline{A}),\ \|\cdot\|_{\sum(\overline{A})}) \to B$$

is a Lipschitz operator. Since $A \hookrightarrow \sum(\overline{A})$

$$T : (\Delta(\overline{A}), \|\cdot\|_A) \to B$$

is also a Lipschitz operator. Therefore we can extend T to a Lipschitz operator from A into B because $\Delta(\overline{A})$ is dense in A.

(ii) Let L_i be the Lipschitz constant of T as a mapping from B to A_i $(i = 0, 1, 2)$. For every $x, y \in B$ we have

$$\|Tx - Ty\|_{\Delta(\overline{A})} = \max_{0 \leq i \leq 2}(\|Tx - Ty\|_{A_i}) \leq \max_{0 \leq i \leq 2}(L_i)\|x - y\|_B.$$

Since $\Delta(\overline{A}) \hookrightarrow A$ it follows that $T : B \to A$ is Lipschitz. $\square$

Corollary 3.1. *Let $\overline{A} = (A_0, A_1, A_2)$ be a Banach triple and let B be a Banach space. Assume that T is a non-linear operator*

(i) *If $T \in \mathscr{C}(\overline{A}, B)$ is an operator such that $T : A_i \to B$ $(i = 0, 1, 2)$ are Lipschitz operators, then for every $\overline{\theta} = (\theta_1, \theta_2) \in (0, 1)^2$, $\theta_1 + \theta_2 < 1$ and every $1 \leq p < \infty$, $T : \overline{A}_{\overline{\theta},p,J} \to B$ is also a Lipschitz operator.*

(ii) *If $T \in \mathscr{C}(A, \overline{B})$ is an operator such that $T : B \to A_i$ $(i = 0, 1, 2)$ are Lipschitz operators, then for every $\overline{\theta} = (\theta_1, \theta_2) \in (0,1)^2$, $\theta_1 + \theta_2 < 1$ and every $1 \le p < \infty$, $T : B \to \overline{A}_{\overline{\theta},p,J}$ (or $\overline{A}_{\overline{\theta},p,K}$) is also a Lipschitz operator.*

Next we consider general Banach triples $\overline{A} = (A_0, A_1, A_2)$ and $\overline{B} = (B_0, B_1, B_2)$.

Proposition 3.2. *Let $\overline{\theta} = (\theta_1, \theta_2) \in (0,1)^2$, $\theta_1 + \theta_2 < 1$ and $1 \le p < \infty$, let $\overline{A} = (A_0, A_1, A_2)$ and $\overline{B} = (B_0, B_1, B_2)$ be two Banach triples. Assume that T is a non-linear operator. If $T \in \mathscr{C}(\overline{A}, \overline{B})$ is an operator such that $T : A_i \to B_i$ $(i = 0, 1, 2)$ are Lipschitz operators, then $T : \overline{A}_{\overline{\theta},p,J} \to \overline{B}_{\overline{\theta},p,K}$ is also a Lipschitz operator.*

Proof. Since $T : A_i \to B_i$ $(i = 0, 1, 2)$ are Lipschitz operators, in view of Cobos result ([5]) we deduce that

$$T : (A_i, A_2)_{\theta_2,p} \to (B_i, B_2)_{\theta_2,p} \quad (i = 0, 1)$$

are Lipschitz operators.

Then

$$T : ((A_0, A_2)_{\theta_2,p}, (A_1, A_2)_{\theta_1,p})_{\lambda,p} \to ((B_0, B_2)_{\theta_2,p}, (B_1, B_2)_{\theta_2,p})_{\lambda,p}$$

where $\lambda = \dfrac{\theta_1}{1-\theta_2}$, is also a Lipschitz operator. Now, by (3) we deduce that

$$T : \overline{A}_{(\theta_1,\theta_2),p,J} \to \overline{B}_{(\theta_1,\theta_2),p,K}$$

is a Lipschitz operator. □

Proposition 3.3. *Let $\overline{\theta} = (\theta_1, \theta_2) \in (0,1)^2$, $\theta_1 + \theta_2 < 1$ and $1 \le p < \infty$,let $\overline{A} = (A_0, A_1, A_2)$ be a Banach triple such that $\Delta(\overline{A})$ is dense in $\overline{A}_{(\theta_1,\theta_2),p,K}$. Assume that T is a non-linear operator and $\overline{B} = (B_0, B_1, B_2)$ is a Banach triple. If $T \in \mathscr{C}(\overline{A}, \overline{B})$ is an operator such that $T : A_i \to B_i$ $(i = 0, 1, 2)$ are Lipschitz operators, then, $T : \overline{A}_{\overline{\theta},p,K} \to \overline{B}_{\overline{\theta},p,K}$ is also a Lipschitz operator.*

Proof. Let L_i be the Lipschitz constant of T as a mapping from A_i to B_i $(i = 0, 1, 2)$. Let $x, y \in \Delta(\overline{A})$ and choose any decomposition

$$x - y = a_0 + a_1 + a_2, \;\; \text{with} \;\; a_i \in A_i \;\; (\text{in fact} \;\; a_i \in \Delta(\overline{A})), \;\; (i = 0, 1, 2).$$

Then

$$Tx - Ty = (Tx - T(x - a_0)) + (T(x - a_0) - T(x - a_0 - a_1)) + (T(x - a_0 - a_1) - Ty)$$

is a decomposition of $Tx - Ty$ in $\sum(\overline{B})$. Therefore

$$K(t, s, Tx - Ty, \overline{B}) \le L_0\|a_0\|_{A_0} + tL_1\|a_1\|_{A_1} + sL_2\|a_2\|_{A_2}$$

and

$$K(t, s, Tx - Ty, \overline{B}) \le L_0 K\left(\frac{L_1}{L_0}t, \frac{L_2}{L_0}s, x - y, \overline{A}\right).$$

Thus, for any $x, y \in \Delta(\overline{A})$

$$\begin{aligned} \|Tx - Ty\|_{\overline{\theta},p,K} &= \left(\int_0^\infty \int_0^\infty \left(t^{-\theta_1}s^{-\theta_2}K(t, s, Tx - Ty, \overline{B})\right)^p \frac{dt}{t}\frac{ds}{s}\right)^{1/p} \\ &\le L_0\left(\int_0^\infty \int_0^\infty \left(t^{-\theta_1}s^{-\theta_2}K\left(\frac{L_1}{L_0}t, \frac{L_2}{L_0}s, x - y, \overline{A}\right)\right)^p \frac{dt}{t}\frac{ds}{s}\right)^{1/p} \\ &\le L_0^{1-\theta_1-\theta_2}L_1^{\theta_1}L_2^{\theta_2}\|x - y\|_{\overline{\theta},p,K}. \end{aligned}$$

Hence

$$T : (\Delta(\overline{A}), \|\cdot\|_{\overline{\theta},p,K}) \to \overline{B}_{\overline{\theta},p,K}$$

is a Lipschitz operator. Therefore $T : \overline{A}_{\overline{\theta},p,K} \to \overline{B}_{\overline{\theta},p,K}$ is also a Lipschitz operator because $\Delta(\overline{A})$ is dense in $\overline{A}_{\overline{\theta},p,K}$ □

4. The compactness results

At first, we derive multidimensional compactness results of Lions-Peetre type.

Theorem 4.1. *Let $\overline{B} = (B_0, B_1, B_2)$ be a Banach triple and let A be a Banach space. Assume that T is a non-linear operator. If $T \in \mathscr{C}(A, \overline{B})$ is an operator such that $T : A \to B_i$ ($i = 0, 1, 2$) are Lipschitz operators, $T : A \to B_0$ is compact and B is a space of class $\mathscr{C}_J(\overline{\theta}, \overline{B})$, then $T : A \to B$ is compact.*

Proof. Let D be any bounded subset of A. We show that $T(D)$ is a precompact subset of B, from which it follows that $T : A \to B$ is compact. Let L_i be the Lipschitz constant of T as a mapping from A to B_i ($i = 0, 1, 2$) and $M = \sup\{\|x\|_A : x \in D\}$. By the assumption on B we have

$$\|b\|_B \leq Ct^{-\theta_1}s^{-\theta_2}J(t, s, b, \overline{B}), \quad b \in \Delta(\overline{B}), \ \ t, s > 0.$$

Hence, given any bounded subset D of A and any $\varepsilon > 0$, we can choose t and s (small enough) so that

$$\max\left(2CL_1Mt^{1-\theta_1}s^{-\theta_2}, 2CL_2Mt^{-\theta_1}s^{1-\theta_2}\right) \leq \varepsilon.$$

Since $T : A \to B_0$ is compact we can find a finite subset $\{v_1, v_2, \dots, v_n\}$ of D such that

$$T(D) \subset \bigcup_{j=1}^{n}\left\{Tv_j + \left\{b \in B_0 : \|b\|_{B_0} \leq \frac{\varepsilon}{Ct^{-\theta_1}s^{-\theta_2}}\right\}\right\}$$

Thus, give any $a \in D$, choosing j so that $\|Ta - Tv_j\|_B \leq \dfrac{\varepsilon}{Ct^{-\theta_1}s^{-\theta_2}}$.
We obtain

$$\begin{aligned}\|Ta - Tv_j\|_B \ \leq\ & \max(Ct^{-\theta_1}s^{-\theta_2}\|Ta - Tv_j\|_{B_0}, Ct^{1-\theta_1}s^{-\theta_2}\|Ta - Tv_j\|_{B_1}, \\ & Ct^{-\theta_1}s^{1-\theta_2}\|Ta - Tv_j\|_{B_2}) \leq \varepsilon.\end{aligned}$$

This proves the compactness of $T : A \to B$. □

Corollary 4.1. *Let $\overline{\theta} = (\theta_1, \theta_2) \in (0, 1)^2$, $\theta_1 + \theta_2 < 1$ and let $1 \leq p < \infty$. Under the same assumption as in Theorem 4.1 we have*

$$T : A \to \overline{B}_{\overline{\theta},p,J} \ \text{ is compact.}$$

Theorem 4.2. *Let $\overline{\theta} = (\theta_1, \theta_2) \in (0, 1)^2$, $\theta_1 + \theta_2 < 1$ and let $1 \leq p < \infty$. Let B be a Banach space. Assume that T is a non-linear operator and that $\overline{A} = (A_0, A_1, A_2)$ is a Banach triple such that $\Delta(\overline{A})$ is dense in $\overline{A}_{\overline{\theta},p,K}$. If $T \in \mathscr{C}(\overline{A}, B)$ is an operator such that $T : A_i \to B$ ($i = 0, 1, 2$) are Lipschitz operators, $T : A_0 \to B$ is compact, then $T : \overline{A}_{\overline{\theta},p,K} \to B$ is compact.*

Proof. Let L_i be the Lipschitz constant of T as a mapping from A_i to B ($i = 0, 1, 2$). Let D any bounded subset of $\overline{A}_{\overline{\theta},p,K}$. Put $M = \sup\{\|a\|_{\overline{A}_{\overline{\theta},p,K}}; a \in D\}$. Since $\Delta(\overline{A})$

is dense in $\overline{A}_{\overline{\theta},p,K}$ we may assume that $D \subset \Delta(\overline{A})$. Since $\overline{A}_{\overline{\theta},p,K}$ is a space of class $\mathscr{C}_K(\overline{\theta}, \overline{A})$ we have

$$t^{-\theta_1} s^{-\theta_2} K(t, s, a, \overline{A}) \leq C\|a\|_{\overline{A}_{\overline{\theta},p,K}} \leq CM, \;\; a \in D, \; t, s > 0.$$

For each t and s, we can decompose any $a \in D$ as $a = a_0 + a_1 + a_2$, with $a_i \in A_i$ $(i = 0, 1, 2)$ and

$$\|a_0\|_{A_0} + t\|a_1\|_{A_1} + s\|a_2\|_{A_2} \leq 2CMt^{\theta_1} s^{\theta_2}.$$

Then , given $\varepsilon > 0$, choosing $t = s$ big enough we can find three subsets D_0, D_1, D_2 of $\Delta(\overline{A})$ such that $D \subset D_0 + D_1 + D_2$, D_0 is bounded in A_0 and

$$\sup\{\|a_1\|_{A_1} : a_1 \in D_1\} \leq \varepsilon/_{3L_1}, \quad \sup\{\|a_2\|_{A_2} : a_2 \in D_2\} \leq \frac{\varepsilon}{3L_2}.$$

Since D_0 is bounded in A_0 we can use the compactness assumption on T to find a finite subset $\{b_1, b_2, \ldots, b_n\}$ of B such that

$$T(D_0) \subset \bigcup_{j=1}^{n} \{b_j + \{b \in B : \; \|b\|_B \leq \varepsilon/_3\}\}$$

Now, for $a \in D$ with $a = a_0 + a_1 + a_2$ and $a_i \in D_i$ (in fact $a_i \in \Delta(\overline{A})$) we have

$$\begin{aligned} \|Ta - b_j\|_B &\leq \|Ta - T(a_0 + a_1)\|_B + \|T(a_0 + a_1) - Ta_0\|_B + \|Ta_0 - b_j\|_B \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \|Ta_0 - b_j\|_B. \end{aligned}$$

Choosing j such that $\|Ta_0 - b_j\|_B \leq \varepsilon/_3$ now it gives $\|Ta - b_j\|_B \leq \varepsilon$. This shows the precompactness of $T(D)$. Thus $T : \overline{A}_{\overline{\theta},p,K} \to B$ is compact. □

Next, we consider general Banach triples $\overline{A} = (A_0, A_1, A_2)$ and $\overline{B} = (B_0, B_1, B_2)$.

Theorem 4.3. *Let $\overline{\theta} = (\theta_1, \theta_2) \in (0, 1)^2$ and $1 \leq p < \infty$, let $\overline{A} = (A_0, A_1, A_2)$ and $\overline{B} = (B_0, B_1, B_2)$ be two Banach triples. If $T \in \mathscr{C}(\overline{A}, \overline{B})$ is an operator such that $T : A_i \to B_i$ $(i = 0, 1, 2)$ are compact Lipschitz operators then*

$$T : \overline{A}_{\overline{\theta},p,J} \to ((B_0, B_2)_{\theta_2,p}, (B_1, B_2)_{\theta_2,p})_{\lambda,p}$$

where $\lambda = \dfrac{\theta_1}{1 - \theta_2}$, is compact Lipschitz operator.

Proof. Since $T : A_i \to B_i$ $(i = 0, 1, 2)$ are compact Lipschitz operators, in view of Bento result ([3], Theorem 5.1) we deduce that

$$T : (A_i, A_j)_{\theta_2,p} \to (B_i, B_2)_{\theta_2,p} \;\; (i = 0, 1)$$

are compact Lipschitz operators.

Then

$$T : ((A_0, A_2)_{\theta_2,p}(A_1, A_2)_{\theta_2,p})_{\lambda,p} \to ((B_0, B_2)_{\theta_2,p}, (B_1, B_2)_{\theta_2,p})_{\lambda,p}$$

is also a compact Lipschitz operator. Now, the result follows from (3). □

Corollary 4.2. *If $\overline{A} = (A_0, A_1, A_2)$ and $\overline{B} = (B_0, B_1, B_2)$ are LP Banach triples, then, under the same assumption as in Theorem 4.3 we have*

$$T : \overline{A}_{\overline{\theta},p} \to \overline{B}_{\overline{\theta},p} \quad \text{is compact Lipschitz operator.}$$

Theorem 4.4. *Let* $\overline{X} = (X_0, X_1)$ *and* $\overline{Y} = (Y_0, Y_1)$ *be two Banach couples. Suppose that* $\overline{A} = (A_0, A_1, A_2)$ *and* $\overline{B} = (B_0, B_1, B_2)$ *are Banach triples such that* A_i *is of class* $\mathscr{C}(\theta_i, \overline{X})$, *with* $0 < \theta_0 < \theta_1 < \theta_2 < 1$ *and* B *is of class* $\mathscr{C}(\psi_i, \overline{Y})$, *with* $0 < \psi_0 < \psi_1 < \psi_2 < 1$. *If* $T \in \mathscr{C}(\overline{X}, \overline{Y})$ *is an operator such that* $T : X_i \to Y_i$, $i = 0, 1$ *are compact Lipschitz operators, then* $T : \overline{A}_{(\alpha,\beta),p} \to \overline{B}_{(\alpha,\beta),p}$ *is compact Lipschitz operator for all values of* $(\alpha, \beta) \in (0,1)^2$, *with* $\alpha + \beta < 1$,

$$(1-\alpha-\beta)\theta_0 + \alpha\theta_1 + \beta\theta_2 = (1-\alpha-\beta)\psi_0 + \alpha\psi_1 + \beta\psi_2 \text{ and } p \in [1, \infty).$$

Proof. Since A_i is of class $\mathscr{C}(\theta_i, \overline{X})$ we have (see [9])

$$(X_0, X_1)_{\eta,p} = \overline{A}_{(\alpha,\beta),p,J} = \overline{A}_{(\alpha,\beta),p,K}$$

where $\eta = (1-\alpha-\beta)\theta_0 + \alpha\theta_1 + \beta\theta_2$. Similarly

$$(Y_0, Y_1)_{\eta,p} = \overline{B}_{(\alpha,\beta),p,J} = \overline{B}_{(\alpha,\beta),p,K}$$

where $\eta = (1-\alpha-\beta)\psi_0 + \alpha\psi_1 + \beta\psi_2$. Now, the result follows from (5). $\square$

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