Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 52(2), 2025, Pages 373–384, DOI: 10.52846/ami.v52i2.1970
ISSN: 1223-6934

On generalized weighted fractional order derivatives and Darboux problem for partial differential equations

Abstract. This paper unveils a novel mathematical construct, namely the weighted fractional derivative, and delves into its comprehensive exploration. By formulating pertinent hypotheses concerning the source term, the manuscript not only verifies the existence, uniqueness, and stability of solutions for Darboux problems but also introduces the transformative aspect of the weighted fractional operator in this context.

2020 *Mathematics Subject Classification.* Primary 34A08; Secondary 47H10.
Key words and phrases. Fractional integral; fractional derivative; Hyers-Ulam stability; partial differential equations.

1. Introduction

Fractional calculus revolutionizes the traditional notions of differentiation and integration by extending them into the realm of non-integer orders. While initially capturing the interest of physicists, this mathematical frontier has since become a focal point for mathematicians and engineers alike. The profound implications of fractional calculus and its application to differential equations of fractional order permeate diverse domains such as electrochemistry, control systems, biology, and viscoelasticity, among others (see [20, 16, 25, 24] for in-depth exploration). Recent years have borne witness to significant strides in both ordinary and partial fractional differential equations, with detailed progress chronicled in [4, 6, 12, 13, 9].

The pioneering contribution of [17] marks a transformative milestone, reshaping conventional perspectives with the integration of Caputo and Riemann-Liouville (RL) weighted operators. These operators, intricately designed with kernels grounded in weighted functions, not only amplify the existing capabilities of the Caputo and RL operators but also embody the essential property of a semi-group. This innovative framework expands the horizons of fractional calculus, introducing a nuanced approach that resonates across diverse academic domains.

Numerous scholarly endeavors have delved into the intricacies of the Darboux problem concerning partial differential equations, offering a rich tapestry of insights. Enthusiastic readers seeking a comprehensive understanding of this topic are encouraged to explore the extensive discussions provided in various publications, including [1, 2, 22, 23, 4, 6]. These works not only scrutinize the Darboux problem from diverse perspectives but also contribute to the broader discourse surrounding this intricate facet of partial differential equations.

This study is dedicated to a thorough examination of the existence, uniqueness, and stability of solutions pertaining to fractional partial differential systems. We

Received June 6, 2024. Accepted August 9, 2025.

specifically concentrate our efforts on systems articulated by the following form:

$$\begin{cases} {}^{C}D_{a^+}^{\upsilon,\varpi}\mathcal{U}(\zeta,\vartheta)=\mathcal{F}(\zeta,\vartheta,\mathcal{U}(\zeta,\vartheta)), & (\zeta,\vartheta)\in J=[a_1,b_1]\times[a_2,b_2], \\ \mathcal{U}(\zeta,a_2)=\varphi(\zeta), & \zeta\in[a_1,b_1], \\ \mathcal{U}(a_1,\vartheta)=\psi(\vartheta), & \vartheta\in[a_2,b_2], \\ \varphi(a_1)=\psi(a_2), \end{cases} \tag{1.1}$$

where $\varpi=(\varpi_1,\varpi_2)$, $\varpi_1(\zeta)\neq 0,\varpi_2(\vartheta)\neq 0$ are continuous, positive and non-decreasing functions on $[a_1,b_1]$ and $[a_2,b_2]$ respectively, $a=(a_1,a_2)\in\mathbb{R}^2$, $\upsilon=(\upsilon_1,\upsilon_2)\in(0,1)^2$, ${}^{C}D_{a^+}^{\upsilon,\varpi}$ is the generalized weighted Caputo fractional derivative of order υ and $\mathcal{F}:J\times\mathbb{R}\to\mathbb{R}$, $\varphi:[a_1,b_1]\to\mathbb{R}$ and $\psi:[a_2,b_2]\to\mathbb{R}$ are given continuous functions.

The subsequent sections of the manuscript unfold as follows: Section 2 provides a comprehensive exposition of diverse definitions and preliminaries crucial for the understanding of the subsequent developments. Moving forward, Section 3 meticulously presents the proofs elucidating the results concerning the existence, uniqueness, and stability aspects.

2. Preliminarily

In this section, we present crucial definitions, lemmas, and propositions essential to underpin our subsequent findings.
Consider $a=(a_1,a_2)\in\mathbb{R}^2$ and $\upsilon=(\upsilon_1,\upsilon_2)$ with υ_1 and υ_2 are positive real numbers and $\varpi_1(\zeta)\neq 0,\varpi_2(\vartheta)\neq 0$ are positive, continuous and non-decreasing functions where

$$\varpi_1^{-1}(\zeta)=\frac{1}{\varpi_1(\zeta)},\quad \text{and}\quad \varpi_2^{-1}(\vartheta)=\frac{1}{\varpi_2(\vartheta)}.$$

Definition 2.1. The weighted RL integral of order υ for $\mathcal{U}(\zeta,\vartheta)\in L^1(J)$ is defined as

$$I_{a^+}^{\upsilon,\varpi}\mathcal{U}(\zeta,\vartheta)=\frac{\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\varpi_1(s)\varpi_2(t)\mathcal{U}(s,t)dtds,$$

Definition 2.2. The weighted RL derivative of order $\upsilon\in(0,1)^2$ for $\mathcal{U}(\zeta,\vartheta)\in L^1(J)$ is defined as

$$\begin{aligned} D_{a^+}^{\upsilon,\varpi}\mathcal{U}(\zeta,\vartheta) &= D^{1,\varpi}I_{a^+}^{1-\upsilon,\varpi}\mathcal{U}(\zeta,\vartheta) \\ &= D^{1,\varpi}\frac{\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)}{\Gamma(1-\upsilon_1)\Gamma(1-\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}\frac{\varpi_1(s)\varpi_2(t)\mathcal{U}(s,t)}{(\zeta-s)^{\upsilon_1}(\vartheta-t)^{\upsilon_2}}dtds \\ &= \frac{\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)}{\Gamma(1-\upsilon_1)\Gamma(1-\upsilon_2)}\frac{\partial^2}{\partial\zeta\partial\vartheta}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}\frac{\varpi_1(s)\varpi_2(t)\mathcal{U}(s,t)}{(\zeta-s)^{\upsilon_1}(\vartheta-t)^{\upsilon_2}}dtds \end{aligned}$$

where

$$D^{1,\varpi}\mathcal{U}(\zeta,\vartheta)=\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\frac{\partial^2}{\partial\zeta\partial\vartheta}\left(\varpi_1(\zeta)\varpi_2(\vartheta)\mathcal{U}(\zeta,\vartheta)\right).$$

Definition 2.3. The weighted Caputo derivative of $\mathcal{U}(\zeta,\vartheta)\in L^1(J)$ of order $\upsilon\in(0,1)^2$ is defined as

$$\begin{aligned}{}^C D_{a^+}^{\upsilon,\varpi}\mathcal{U}(\zeta,\vartheta) &= D_{a^+}^{\upsilon,\varpi}\Big(\mathcal{U}(\zeta,\vartheta)-\varpi_1^{-1}(\zeta)\varpi_1(a_1)\mathcal{U}(a_1,\vartheta)\\ &\quad -\varpi_2^{-1}(\vartheta)\varpi_2(a_2)\mathcal{U}(\zeta,a_2)+\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\varpi_1(a_1)\varpi_2(a_2)\mathcal{U}(a_1,a_2)\Big).\end{aligned}$$

Lemma 2.1. Let $\upsilon=(\upsilon_1,\upsilon_2),\sigma=(\sigma_1,\sigma_2)$ where $\upsilon_1,\upsilon_2,\sigma_1,\sigma_2$ are positive real numbers. If $\mathcal{U}(\zeta,\vartheta)\in C(J)$, then

$$I_{a^+}^{\upsilon,\varpi}I_{a^+}^{\sigma,\varpi}\mathcal{U}(\zeta,\vartheta)=I_{a^+}^{\sigma,\varpi}I_{a^+}^{\upsilon,\varpi}\mathcal{U}(\zeta,\vartheta)=I_{a^+}^{\upsilon+\sigma,\varpi}\mathcal{U}(\zeta,\vartheta).$$

Proof. We have

$$\begin{aligned}I_{a^+}^{\upsilon,\varpi}I_{a^+}^{\sigma,\varpi}\mathcal{U}(\zeta,\vartheta) &= \frac{\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\\ &\quad\times\varpi_1(s)\varpi_2(t)I_{a^+}^{\sigma,\varpi}\mathcal{U}(s,t)dtds\\ &= \frac{\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}\int_{a_1}^{s}\int_{a_2}^{t}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\\ &\quad\times(s-\tau)^{\sigma_1-1}(t-u)^{\sigma_2-1}\varpi_1(\tau)\varpi_2(u)\mathcal{U}(\tau,u)dud\tau dtds.\end{aligned}$$

By using Fubini's theorem, we obtain

$$\begin{aligned}&I_{a^+}^{\upsilon,\varpi}I_{a^+}^{\sigma,\varpi}\mathcal{U}(\zeta,\vartheta)\\ &=\frac{\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}\varpi_1(\tau)\varpi_2(u)\mathcal{U}(\tau,u)\\ &\quad\times\int_{\tau}^{\zeta}\int_{u}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}(s-\tau)^{\sigma_1-1}(t-u)^{\sigma_2-1}dtdsdud\tau\\ &=\frac{\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}\Big(\varpi_1(\tau)\varpi_2(u)\mathcal{U}(\tau,u)\\ &\quad\times\int_{\tau}^{\zeta}(\zeta-s)^{\upsilon_1-1}(s-\tau)^{\sigma_1-1}ds\int_{u}^{\vartheta}(\vartheta-t)^{\upsilon_2-1}(t-u)^{\sigma_2-1}dt\Big)dud\tau.\end{aligned}$$

By using the change of variables $\xi=\frac{s-\tau}{\zeta-\tau}$, $\eta=\frac{t-u}{\vartheta-u}$ and by using the fact that $\int_0^1(1-r)^{\alpha-1}r^{\beta-1}dr=B(\alpha,\beta)$ and $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, we get

$$\begin{aligned}&I_{a^+}^{\upsilon,\varpi}I_{a^+}^{\sigma,\varpi}\mathcal{U}(\zeta,\vartheta)\\ &=\frac{\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}\Big((\zeta-\tau)^{\upsilon_1+\sigma_1-1}(\vartheta-u)^{\upsilon_2+\sigma_2-1}\varpi_1(\tau)\\ &\quad\times\varpi_2(u)\mathcal{U}(\tau,u)\int_0^1\xi^{\sigma_1-1}(1-\xi)^{\upsilon_1-1}d\xi\int_0^1\eta^{\sigma_2-1}(1-\eta)^{\upsilon_2-1}d\eta\Big)dud\tau\\ &=\frac{\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1+\sigma_1)\Gamma(\upsilon_2+\sigma_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-\tau)^{\upsilon_1+\sigma_1-1}(\vartheta-u)^{\upsilon_2+\sigma_2-1}\\ &\quad\times\varpi_1(\tau)\varpi_2(u)\mathcal{U}(\tau,u)dud\tau\\ &=I_{a^+}^{\upsilon+\sigma,\varpi}\mathcal{U}(\zeta,\vartheta).\end{aligned}$$

□

Lemma 2.2. Let $\upsilon=(\upsilon_1,\upsilon_2)\in(0,1)^2$. If $\mathcal{U}(\zeta,\vartheta)\in C(J)$, then we have

$$D_{a^+}^{\upsilon,\varpi}\left(I_{a^+}^{\upsilon,\varpi}\mathcal{U}\right)(\zeta,\vartheta)=\mathcal{U}(\zeta,\vartheta).$$

Proof. From Lemma 2.1 and Definition 2.2, we obtain

$$\begin{aligned} D_{a^+}^{\upsilon,\varpi} I_{a^+}^{\upsilon,\varpi} \mathcal{U}(\zeta,\vartheta) &= D^{1,\varpi}\left(I_{a^+}^{1-\upsilon,\varpi}\ I_{a^+}^{\upsilon,\varpi}\mathcal{U}\right)(\zeta,\vartheta) \\ &= D^{1,\varpi} I_{a^+}^{1,\varpi}\mathcal{U}(\zeta,\vartheta) \\ &= \mathcal{U}(\zeta,\vartheta). \end{aligned}$$

□

Proposition 2.3. Let $\upsilon=(\upsilon_1,\upsilon_2)\in(0,1)^2$ and $\mathcal{U}(\zeta,\vartheta)\in AC^1(J)$. Then, we have

$$\begin{aligned} {}^{C}D_{a^+}^{\upsilon,\varpi}\mathcal{U}(\zeta,\vartheta) &= I_{a^+}^{1-\upsilon,\varpi} D^{1,\varpi}\mathcal{U}(\zeta,\vartheta) \\ &= \frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(1-\upsilon_1)\Gamma(1-\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}\frac{\varpi_1(s)\varpi_2(t)D^{1,\varpi}\mathcal{U}(s,t)}{(\zeta-s)^{\upsilon_1}(\vartheta-t)^{\upsilon_2}}dtds. \end{aligned}$$

Lemma 2.4. If $\mathcal{U}(\zeta,\vartheta)\in AC^1(J)$, then

$$\begin{aligned} &I_{a^+}^{\upsilon,\varpi}\,{}^{C}D_{a^+}^{\upsilon,\varpi}\mathcal{U}(\zeta,\vartheta) \\ &= \Big(\mathcal{U}(\zeta,\vartheta)-\varpi_1^{-1}(\zeta)\varpi_1(a_1)\mathcal{U}(a_1,\vartheta)-\varpi_2^{-1}(\vartheta)\varpi_2(a_2)\mathcal{U}(\zeta,a_2) \\ &\quad +\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\varpi_1(a_1)\varpi_2(a_2)\mathcal{U}(a_1,a_2)\Big). \end{aligned}$$

Definition 2.4. [15] Let $\iota\in\mathbb{N}^*$, $\alpha_j,\beta_j,z,\nu\in\mathbb{C}$, such that $\Re(\alpha_j),\Re(\beta_j)>0$ for $j\in\{1,2,\dots,\iota\}$. The generalized Mittag-Leffler function is defined by

$$\mathbb{E}_{\nu}\left((\alpha_j,\beta_j)_{j=1,\iota};(z)\right)=\sum_{\kappa=0}^{+\infty}\frac{(\nu)_{\kappa}}{\prod_{j=1}^{\iota}\Gamma(\kappa\alpha_j+\beta_j)}\frac{z^{\kappa}}{\kappa!},$$

where

$$(\nu)_{\kappa}=\nu(\nu+1)\dots(\nu+\kappa-1)=\frac{\Gamma(\nu+\kappa)}{\Gamma(\nu)}.$$

In particular, when $\iota=2$ and $\nu=1$, we obtain

$$\mathbb{E}_1\left((\alpha_j,\beta_j)_{j=1,2};(z)\right)=\mathbb{E}\left((\alpha_j,\beta_j)_{j=1,2};(z)\right)=\sum_{\kappa=0}^{+\infty}\frac{z^{\kappa}}{\Gamma(\kappa\alpha_1+\beta_1)\Gamma(\kappa\alpha_2+\beta_2)}.$$

The motivation behind the Lemma below stems from Theorem 1 in [14].

Lemma 2.5. Let $(\upsilon_1,\upsilon_2)\in(0,1)^2$. Suppose that $\mathcal{U}$ and $\mathcal{H}$ are two integrable non-negative functions, and $\mathcal{G}$ is a continuous function on J with ϖ_1 and ϖ_2 are continuous functions on $[a_1,b_1]$ and $[a_2,b_2]$ respectively. Additionally, assume that $\mathcal{G}$, ϖ_1, and ϖ_2 are non-negative and non-decreasing concerning their respective variables. If

$$\begin{aligned} \mathcal{U}(\zeta,\vartheta) \le{}& \mathcal{H}(\zeta,\vartheta)+\mathcal{G}(\zeta,\vartheta)\,\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\int_{a_1^+}^{\zeta}\int_{a_2^+}^{\vartheta}\varpi_1(s)\varpi_2(t) \\ &\times(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\mathcal{U}(s,t)\,dtds, \end{aligned}$$

then

$$\begin{aligned} \mathcal{U}(\zeta,\vartheta) \le{}& \mathcal{H}(\zeta,\vartheta)+\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\int_{a_1^+}^{\zeta}\int_{a_2^+}^{\vartheta}\sum_{k=1}^{\infty}\frac{\left(\mathcal{G}(\zeta,\vartheta)\Gamma(\upsilon_1)\Gamma(\upsilon_2)\right)^k}{\Gamma(k\upsilon_1)\Gamma(k\upsilon_2)} \\ &\times\varpi_1(s)\varpi_2(t)(\zeta-s)^{k\upsilon_1-1}(\vartheta-t)^{k\upsilon_2-1}\mathcal{H}(s,t)\,dtds. \end{aligned}$$

Furthermore, if $\mathcal{H}$ exhibits nondecreasing with respect to each ζ and ϑ, then

$$\mathcal{U}(\zeta,\vartheta) \leq \mathcal{H}(\zeta,\vartheta)\,\mathbb{E}\Big((\upsilon_1,1),(\upsilon_2,1);\mathcal{G}(\zeta,\vartheta)\,\Gamma(\upsilon_1)\,\Gamma(\upsilon_2)\,(\zeta-a_1)^{\upsilon_1}(\vartheta-a_2)^{\upsilon_2}\Big).$$

3. Main results

In this part, we discuss the existence problem of System (1.1). Before, we establish the next lemma.

Lemma 3.1. $\mathcal{U}\in C(J)$ is a solution of (1.1) if and only if

$$\begin{aligned}\mathcal{U}(\zeta,\vartheta) &= \varpi_1^{-1}(\zeta)\varpi_1(a_1)\psi(\vartheta)+\varpi_2^{-1}(\vartheta)\varpi_2(a_2)\varphi(\zeta)\\ &\quad -\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\varpi_1(a_1)\varpi_2(a_2)\varphi(a_1)+I_{a^+}^{\upsilon,\varpi}\mathcal{F}\big(\zeta,\vartheta,\mathcal{U}(\zeta,\vartheta)\big).\end{aligned} \tag{3.1}$$

Proof. Assume that $\mathcal{U}$ satisfies Equation (3.1). Applying ${}^{C}D_{a^+}^{\upsilon,\varpi}$ and from Lemma 2.2, we deduce that $\mathcal{U}$ is a solution of (1.1). Since $I_{a^+}^{\upsilon,\varpi}\mathcal{F}\big(\zeta,\vartheta,\mathcal{U}(\zeta,\vartheta)\big)$ vanishes when $\zeta=a_1$ or $\vartheta=a_2$, then the initial conditions in System (1.1)) are satisfied. Therefore, $\mathcal{U}$ is a solution of System (1.1).

Now, suppose that $\mathcal{U}$ is a solution of System (1.1), and let us consider

$$\begin{aligned} h(\zeta,\vartheta) &= \mathcal{F}(\zeta,\vartheta,\mathcal{U}(\zeta,\vartheta))\,D_{a^+}^{\upsilon,\varpi}\Big(\mathcal{U}(\zeta,\vartheta)-\varpi_1^{-1}(\zeta)\varpi_1(a_1)\mathcal{U}(a_1,\vartheta)\\ &\quad -\varpi_2^{-1}(\vartheta)\varpi_2(a_2)\mathcal{U}(\zeta,a_2)+\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\varpi_1(a_1)\varpi_2(a_2)\mathcal{U}(a_1,a_2)\Big)\\ &\quad D_{a^+}^{1,\varpi}I_{a^+}^{1-\upsilon,\varpi}\Big(\mathcal{U}(\zeta,\vartheta)-\varpi_1^{-1}(\zeta)\varpi_1(a_1)\mathcal{U}(a_1,\vartheta)\\ &\quad -\varpi_2^{-1}(\vartheta)\varpi_2(a_2)\mathcal{U}(\zeta,a_2)+\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\varpi_1(a_1)\varpi_2(a_2)\mathcal{U}(a_1,a_2)\Big).\end{aligned} \tag{3.2}$$

Applying the operator $I_{a^+}^{1,\varpi}$ to (3.2), we get

$$\begin{aligned} I_{a^+}^{1,\varpi}h(\zeta,\vartheta) &= I_{a^+}^{1-\upsilon,\varpi}\Big(\mathcal{U}(\zeta,\vartheta)-\varpi_1^{-1}(\zeta)\varpi_1(a_1)\mathcal{U}(a_1,\vartheta)\\ &\quad -\varpi_2^{-1}(\vartheta)\varpi_2(a_2)\mathcal{U}(\zeta,a_2)+\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\varpi_1(a_1)\varpi_2(a_2)\mathcal{U}(a_1,a_2)\Big).\end{aligned}$$

Applying the operator $D_{a^+}^{1-\upsilon,\varpi}$ to the above equation, we obtain

$$\begin{aligned}&\mathcal{U}(\zeta,\vartheta)-\varpi_1^{-1}(\zeta)\varpi_1(a_1)\mathcal{U}(a_1,\vartheta)-\varpi_2^{-1}(\vartheta)\varpi_2(a_2)\mathcal{U}(\zeta,a_2)\\ &\qquad +\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\varpi_1(a_1)\varpi_2(a_2)\mathcal{U}(a_1,a_2)\\ &\quad = D_{a^+}^{1-\upsilon,\varpi}I_{a^+}^{1,\varpi}h(\zeta,\vartheta)\\ &\quad = D^{1,\varpi}I_{a^+}^{\upsilon,\varpi}I_{a^+}^{1,\varpi}h(\zeta,\vartheta)\\ &\quad = I_{a^+}^{\upsilon,\varpi}h(\zeta,\vartheta),\end{aligned}$$

and the proof is completed. □

In what follows, let us assume that the function $\mathcal{F}$ satisfies the following hypotheses:

($\mathcal{H}_1$) There exist $\mathcal{I}$ and $\mathcal{N}$ in $C(J,\mathbb{R}_+)$ such that

$$|\mathcal{F}(\zeta,\vartheta,\mathcal{U})| \leq \mathcal{I}(\zeta,\vartheta)+\mathcal{N}(x,y)\,|\mathcal{U}|,\quad \forall(\zeta,\vartheta)\in J,\ \mathcal{U}\in\mathbb{R},$$

($\mathcal{H}_2$) There is $L_{\mathcal{F}}>0$ with

$$|\mathcal{F}(\zeta,\vartheta,\mathcal{U})-\mathcal{F}(\zeta,\vartheta,\mathcal{V})| \leq L_{\mathcal{F}}\,|\mathcal{U}-\mathcal{V}|,\quad \forall(\zeta,\vartheta)\in J,\ \mathcal{U},\mathcal{V}\in\mathbb{R}.$$

Theorem 3.2. Assume that $(\mathcal{H}_1)$ hold, then System (1.1) possesses at least one solution.

Proof. Let us define the operator $A : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$\begin{aligned}(\mathcal{A}\mathcal{U})(\zeta,\vartheta) &= \mathcal{T}(\zeta,\vartheta) + \frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)} \int_{a_1}^{\zeta}\int_{a_2}^{\vartheta} (\zeta - s)^{\upsilon_1 - 1} \\ &\quad \times (\vartheta - t)^{\upsilon_2 - 1}\, \varpi_1(s)\varpi_2(t)\mathcal{F}\big(s,t,\mathcal{U}(s,t)\big)dtds,\end{aligned}$$

where the function $\mathcal{T}$ is defined for every $(\zeta,\vartheta) \in J$ as follows:

$$\begin{aligned}\mathcal{T}(\zeta,\vartheta) &= \varpi_1^{-1}(\zeta)\varpi_1(a_1)\psi(\vartheta) + \varpi_2^{-1}(\vartheta)\varpi_2(a_2)\varphi(\zeta) \\ &\quad -\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\varpi_1(a_1)\varpi_2(a_2)\varphi(a_1).\end{aligned} \tag{3.3}$$

Certainly, $\mathcal{A}$ is continuous. We proceed to demonstrate that $\mathcal{A}$ is bounded in $C(J, \mathbb{R})$ into $C(J, \mathbb{R})$. To establish this, we aim to prove that for any $R > 0$, there exists $L > 0$ such that for every $\mathcal{U} \in \mathcal{B}_R = \{\mathcal{U} \in C(J, \mathbb{R}),\ \|\mathcal{U}\|_\infty \leq R\}$, we have $\|\mathcal{A}\mathcal{U}\|_\infty \leq L$. Consider $\mathcal{U} \in \mathcal{B}_R$ and $(\zeta,\vartheta) \in J$; then, we obtain

$$\begin{aligned}|(\mathcal{A}\mathcal{U})(\zeta,\vartheta)| &\leq |\mathcal{T}(\zeta,\vartheta)| + \frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)} \int_{a_1}^{\zeta}\int_{a_2}^{\vartheta} (\zeta - s)^{\upsilon_1 - 1} \\ &\quad \times (\vartheta - t)^{\upsilon_2 - 1}\, \varpi_1(s)\varpi_2(t)\left|\mathcal{F}\big(s,t,\mathcal{U}(s,t)\big)\right| dtds.\end{aligned}$$

From $(\mathcal{H}_1)$, we obtain

$$\begin{aligned}|(\mathcal{A}\mathcal{U})(\zeta,\vartheta)| &\leq |\mathcal{T}(\zeta,\vartheta)| + \frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)} \int_{a_1}^{\zeta}\int_{a_2}^{\vartheta} (\zeta - s)^{\upsilon_1 - 1} \\ &\quad \times (\vartheta - t)^{\upsilon_2 - 1}\, \varpi_1(s)\varpi_2(t)\big(\mathcal{I}(s,t) + \mathcal{N}(s,t)\,|\mathcal{U}(s,t)|\,\big)dtds \\ &\leq \|\mathcal{T}\|_\infty + \frac{\big(\|\mathcal{I}\|_\infty + \|\mathcal{N}\|_\infty R\big)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)} \int_{a_1}^{\zeta}\int_{a_2}^{\vartheta} (\zeta - s)^{\upsilon_1 - 1}(\vartheta - t)^{\upsilon_2 - 1} dtds \\ &\leq \|\mathcal{T}\|_\infty + \frac{\big(\|\mathcal{I}\|_\infty + \|\mathcal{N}\|_\infty R\big)}{\Gamma(\upsilon_1 + 1)\Gamma(\upsilon_2 + 1)} (\zeta - a_1)^{\upsilon_1}(\vartheta - a_2)^{\upsilon_2}.\end{aligned}$$

Then, for any $\mathcal{U} \in \mathcal{B}_R$ there exists

$$L = \|\mathcal{T}\|_\infty + \frac{\big(\|\mathcal{I}\|_\infty + \|\mathcal{N}\|_\infty R\big)}{\Gamma(\upsilon_1 + 1)\Gamma(\upsilon_2 + 1)} (b_1 - a_1)^{\upsilon_1}(b_2 - a_2)^{\upsilon_2}$$

such that $\|\mathcal{A}\mathcal{U}\|_\infty \leq L$.

Next, we establish that $\mathcal{A}$ maps bounded sets in $C(J, \mathbb{R})$ into equicontinuous sets in $C(J, \mathbb{R})$. Consider $(\zeta_1,\vartheta_1), (\zeta_2,\vartheta_2) \in J$ such that $\zeta_1 < \zeta_2$ and $\vartheta_1 < \vartheta_2$, and let

$\mathcal{U} \in \mathcal{B}_R$. From $(\mathcal{H}_1)$, we obtain

$$\begin{aligned}
&|(\mathcal{A}\mathcal{U})(\zeta_2,\vartheta_2)-(\mathcal{A}\mathcal{U})(\zeta_1,\vartheta_1)| \\
&\leq \frac{\left(\|\mathcal{J}\|_\infty+\|\mathcal{N}\|_\infty R\right)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta_1}\int_{a_2}^{\vartheta_1}\Big((\zeta_1-s)^{\upsilon_1-1}(\vartheta_1-t)^{\upsilon_2-1}-(\zeta_2-s)^{\upsilon_1-1}(\vartheta_2-t)^{\upsilon_2-1}\Big)dtds \\
&\quad+\frac{\varpi_2(b_2)\left|\varpi_2^{-1}(\vartheta_2)-\varpi_2^{-1}(\vartheta_1)\right|}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\left(\|\mathcal{J}\|_\infty+\|\mathcal{N}\|_\infty R\right) \\
&\quad\times\int_{a_1}^{\zeta_1}\int_{a_2}^{\vartheta_1}(\zeta_1-s)^{\upsilon_1-1}(\vartheta_1-t)^{\upsilon_2-1}dtds \\
&\quad+\frac{\varpi_1(b_1)\left|\varpi_1^{-1}(\zeta_2)-\varpi_1^{-1}(\zeta_1)\right|}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\left(\|\mathcal{J}\|_\infty+\|\mathcal{N}\|_\infty R\right) \\
&\quad\times\int_{a_1}^{\zeta_1}\int_{a_2}^{\vartheta_1}(\zeta_1-s)^{\upsilon_1-1}(\vartheta_1-t)^{\upsilon_2-1}dtds \\
&\quad+\frac{\left(\|\mathcal{J}\|_\infty+\|\mathcal{N}\|_\infty R\right)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta_1}\int_{\vartheta_1}^{\vartheta_2}(\zeta_2-s)^{\upsilon_1-1}(\vartheta_2-t)^{\upsilon_2-1}dtds \\
&\quad+\frac{\left(\|\mathcal{J}\|_\infty+\|\mathcal{N}\|_\infty R\right)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{\zeta_1}^{\zeta_2}\int_{a_2}^{\vartheta_1}(\zeta_2-s)^{\upsilon_1-1}(\vartheta_2-t)^{\upsilon_2-1}dtds \\
&\quad+\frac{\left(\|\mathcal{J}\|_\infty+\|\mathcal{N}\|_\infty R\right)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{\zeta_1}^{\zeta_2}\int_{\vartheta_1}^{\vartheta_2}(\zeta_2-s)^{\upsilon_1-1}(\vartheta_2-t)^{\upsilon_2-1}dtds+|T(\zeta_2,\vartheta_2)-T(\zeta_1,\vartheta_1)| \\
&\leq \frac{\left(\|\mathcal{J}\|_\infty+\|\mathcal{N}\|_\infty R\right)}{\Gamma(\upsilon_1+1)\Gamma(\upsilon_2+1)}\Big(\Big(\varpi_1(b_1)\left|\varpi_1^{-1}(\zeta_2)-\varpi_1^{-1}(\zeta_1)\right| \\
&\quad+\varpi_2(b_2)\left|\varpi_2^{-1}(\vartheta_2)-\varpi_2^{-1}(\vartheta_1)\right|\Big)(b_1-a_1)^{\upsilon_1}(b_2-a_2)^{\upsilon_2} \\
&\quad+2\Big((\zeta_2-a_1)^{\upsilon_1}(\vartheta_2-\vartheta_1)^{\upsilon_2}+(\zeta_2-\zeta_1)^{\upsilon_1}(\vartheta_2-a_2)^{\upsilon_2}\Big)\Big)+|T(\zeta_2,\vartheta_2)-T(\zeta_1,\vartheta_1)|.
\end{aligned}$$

As $\zeta_1 \to \zeta_2$ and $\vartheta_1 \to \vartheta_2$, the right hand sides of the above inequality tend to zero.

Finally, we prove a priori bounds. Let $\mathcal{U} \in C(J,\mathbb{R})$ such that $\mathcal{U} = \lambda\mathcal{A}(\mathcal{U})$ for some $\lambda \in (0,1)$. Then, for any $(\zeta,\vartheta) \in J$, we have

$$\begin{aligned}
|\mathcal{U}(\zeta,\vartheta)| \leq\ & |\mathcal{T}(\zeta,\vartheta)|+\frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1} \\
&\times\varpi_1(s)\varpi_2(t)\left|\mathcal{F}\left(s,t,\mathcal{U}(s,t)\right)\right|dtds,
\end{aligned}$$

From $(\mathcal{H}_1)$, we have

$$\begin{aligned}
|\mathcal{U}(\zeta,\vartheta)| \leq\ & |\mathcal{T}(\zeta,\vartheta)|+\frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1} \\
&\times\varpi_1(s)\varpi_2(t)\big(\mathcal{J}(s,t)+\mathcal{N}(s,t)\,|\mathcal{U}(s,t)|\big)dtds \\
\leq\ & \|\mathcal{T}\|_\infty+\frac{\|\mathcal{J}\|_\infty}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}dtds \\
&+\frac{\left(\|\mathcal{N}\|_\infty\,\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)\right)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1} \\
&\times\varpi_1(s)\,\varpi_2(t)\,|\mathcal{U}(s,t)|\,dtds
\end{aligned}$$

$$\begin{aligned}
&\leq \|\mathcal{T}\|_{\infty}+\frac{\|\mathcal{I}\|_{\infty}}{\Gamma(\upsilon_1+1)\Gamma(\upsilon_2+1)}(\zeta-a_1)^{\upsilon_1}(\vartheta-a_2)^{\upsilon_2}\\
&+\frac{\left(\|\mathcal{N}\|_{\infty}\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)\right)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\\
&\times\varpi_1(s)\,\varpi_2(t)\,|\mathcal{U}(s,t)|\,dtds.
\end{aligned}$$

In view of Lemma 2.5 we get,

$$\begin{aligned}
|\mathcal{U}(\zeta,\vartheta)| &\leq \left(\|\mathcal{T}\|_{\infty}+\frac{\|\mathcal{I}\|_{\infty}}{\Gamma(\upsilon_1+1)\Gamma(\upsilon_2+1)}(\zeta-a_1)^{\upsilon_1}(\vartheta-a_2)^{\upsilon_2}\right)\\
&\quad\times\mathbb{E}\big((\upsilon_1,1),(\upsilon_2,1);\|\mathcal{N}\|_{\infty}(\zeta-a_1)^{\upsilon_1}(\vartheta-a_2)^{\upsilon_2}\big)\\
&\leq \left(\|\mathcal{T}\|_{\infty}+\frac{\|\mathcal{I}\|_{\infty}}{\Gamma(\upsilon_1+1)\Gamma(\upsilon_2+1)}(b_1-a_1)^{\upsilon_1}(b_2-a_2)^{\upsilon_2}\right)\\
&\quad\times\mathbb{E}\big((\upsilon_1,1),(\upsilon_2,1);\|\mathcal{N}\|_{\infty}(b_1-a_1)^{\upsilon_1}(b_2-a_2)^{\upsilon_2}\big).
\end{aligned}$$

Set

$$\begin{aligned}
\overline{M} &= \left(\|\mathcal{T}\|_{\infty}+\frac{\|\mathcal{I}\|_{\infty}}{\Gamma(\upsilon_1+1)\Gamma(\upsilon_2+1)}(b_1-a_1)^{\upsilon_1}(b_2-a_2)^{\upsilon_2}\right)\\
&\quad\times\mathbb{E}\big((\upsilon_1,1),(\upsilon_2,1);\|\mathcal{N}\|_{\infty}(b_1-a_1)^{\upsilon_1}(b_2-a_2)^{\upsilon_2}\big)
\end{aligned}$$

and let

$$\mathcal{W}=\left\{\mathcal{U}\in C(J,\mathbb{R}),\ \|\mathcal{U}\|_{\infty}<\overline{M}+1\right\},$$

then, one can not find $\mathcal{U}\in\partial\mathcal{W}$ s.t. $\mathcal{U}=\lambda\mathcal{A}(\mathcal{U})$, for any $\lambda\in(0,1)$. It yields from Theorem 2.8 in [5] that $\mathcal{A}$ admits a fixed point and the proof is completed. □

Theorem 3.3. Assume that $(\mathcal{H}_2)$ hold, then System (1.1) possesses a unique solution.

Proof. We know that if (H2) is holds then, (H1) is holds so, the System (1.1) has at least one solution. It remains to prove the uniqueness of solution, to this end we suppose that (1.1) has tow solutions $\mathcal{U}(x,y)$ and $\mathcal{V}(x,y)$ then,

$$\begin{aligned}
|\mathcal{U}(x,y)-\mathcal{V}(x,y)| &\leq \frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\\
&\quad\times\varpi_1(s)\varpi_2(t)\left|\mathcal{F}\big(s,t,\mathcal{U}(s,t)\big)-\mathcal{F}\big(s,t,\mathcal{V}(s,t)\big)\right|dtds\\
&\leq \frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}L_{\mathcal{F}}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\\
&\quad\times\varpi_1(s)\varpi_2(t)\,|\mathcal{U}(s,t)-\mathcal{V}(s,t)|\,dtds
\end{aligned}$$

it follows from Lemma 2.5 that $\mathcal{U}(x,y)=\mathcal{V}(x,y)$ Hence, the System (1.1) has a unique solution. □

4. Ulam stability

In this section, we discuss the Ulam type stability of System (1.1). to define the Ulam-Hyers stability, $\forall\epsilon>0$, and $(\zeta,\vartheta)\in J$, we consider the following inequality

$$\left|{}^{C}D_{a^+}^{\upsilon,\varpi}\mathcal{V}(\zeta,\vartheta)-\mathcal{F}(\zeta,\vartheta,\mathcal{V}(\zeta,\vartheta))\right|<\epsilon. \tag{4.1}$$

Definition 4.1. System (1.1) is said to be Ulam-Hyers stable if there is $C>0$ such that for every $\epsilon>0$, and for all $\mathcal{V}\in C(J,\mathbb{R})$ a solution of the inequality (4.1), there is $\mathcal{U}\in C(J,\mathbb{R})$ a solution of (1.1) such that

$$|\mathcal{V}(\zeta,\vartheta)-\mathcal{U}(\zeta,\vartheta)|\leq C\epsilon,\ \ (\zeta,\vartheta)\in J. \tag{4.2}$$

Remark 4.1. Let $\mathcal{V}$ be a solution of the inequality (4.1) then $\mathcal{V}$ is a solution of the following integral inequality

$$\begin{aligned}\Big|\mathcal{V}(\zeta,\vartheta)-\mathcal{K}(\zeta,\vartheta)-\frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\\ \times\varpi_1(s)\varpi_2(t)\mathcal{F}\big(s,t,\mathcal{V}(s,t)\big)dtds\Big|\\ \leq\epsilon\frac{(\zeta-a_1)^{\upsilon_1}(\vartheta-a_2)^{\upsilon_2}}{\Gamma(\upsilon_1+1)\Gamma(\upsilon_2+1)},\end{aligned}\tag{4.3}$$

where $\mathcal{K}(\zeta,\vartheta)$ is given by

$$\begin{aligned}\mathcal{K}(\zeta,\vartheta)&=\varpi_1^{-1}(\zeta)\varpi_1(a_1)\mathcal{V}(a_1,\vartheta)+\varpi_2^{-1}(\vartheta)\varpi_2(a_2)\mathcal{V}(\zeta,a_2)\\&\quad-\varpi_1^{-1}(\zeta)\varpi_2^{-1}(\vartheta)\varpi_1(a_1)\varpi_2(a_2)\mathcal{V}(a_1,a_2).\end{aligned}$$

Theorem 4.2. Assume that $(\mathcal{H}_2)$ is satisfied. Then System (1.1) is Ulam-Hyers stable.

Proof. Let $\mathcal{V}$ be a solution of (4.1) and $\mathcal{U}$ the unique solution of the following Cauchy problem

$$\begin{cases}{}^{C}D_{a^+}^{\upsilon,\varpi}\mathcal{U}(\zeta,\vartheta)&=\mathcal{F}(\zeta,\vartheta,\mathcal{U}(\zeta,\vartheta)),\quad(\zeta,\vartheta)\in J,\\ \mathcal{U}(\zeta,a_2)&=\mathcal{V}(\zeta,a_2),\ \zeta\in[a_1,b_1],\\ \mathcal{U}(a_1,\vartheta)&=\mathcal{V}(\zeta,a_2),\ \vartheta\in[a_2,b_2],\end{cases}$$

Therefore,

$$\begin{aligned}\mathcal{U}(\zeta,\vartheta)&=\mathcal{K}(\zeta,\vartheta)+\frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\\&\quad\times\varpi_1(s)\varpi_2(t)\mathcal{F}(s,t,\mathcal{U}(s,t))\,dtds\end{aligned}$$

From Remark 4.1 and $(\mathcal{H}_2)$ we have,

$$\begin{aligned}&|\mathcal{V}(\zeta,\vartheta)-\mathcal{U}(\zeta,\vartheta)|\\&\leq\Big|\mathcal{V}(\zeta,\vartheta)-\mathcal{K}(\zeta,\vartheta)-\frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\\&\quad\times\varpi_1(s)\varpi_2(t)\mathcal{F}(s,t,\mathcal{V}(s,t))\,dtds\Big|\\&\quad+\frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\\&\quad\times\varpi_1(s)\varpi_2(t)\left|\mathcal{F}\big(s,t,\mathcal{V}(s,t)\big)-\mathcal{F}\big(s,t,\mathcal{U}(s,t)\big)\right|dtds\\&\leq\epsilon\frac{(\zeta-a_1)^{\upsilon_1}(\vartheta-a_2)^{\upsilon_2}}{\Gamma(\upsilon_1+1)\Gamma(\upsilon_2+1)}+L_f\frac{\varpi_1^{-1}(\zeta)\,\varpi_2^{-1}(\vartheta)}{\Gamma(\upsilon_1)\Gamma(\upsilon_2)}\int_{a_1}^{\zeta}\int_{a_2}^{\vartheta}(\zeta-s)^{\upsilon_1-1}(\vartheta-t)^{\upsilon_2-1}\\&\quad\times\varpi_1(s)\varpi_2(t)\left|\mathcal{V}(s,t)-\mathcal{U}(s,t)\right|dtds.\end{aligned}$$

From Lemma 2.5 we obtain

$$\begin{aligned}
&\left|\mathcal{V}(\zeta,\vartheta)-\mathcal{U}(\zeta,\vartheta)\right| \\
&\leq \epsilon\frac{(\zeta-a_1)^{\upsilon_1}(\vartheta-a_2)^{\upsilon_2}}{\Gamma(\upsilon_1+1)\Gamma(\upsilon_2+1)}\mathbb{E}\Big((\upsilon_1,1),(\upsilon_2,1);L_f(\zeta-a_1)^{\upsilon_1}(\vartheta-a_2)^{\upsilon_2}\Big) \\
&\leq \epsilon\frac{(b_1-a_1)^{\upsilon_1}(b_2-a_2)^{\upsilon_2}}{\Gamma(\upsilon_1+1)\Gamma(\upsilon_2+1)}\mathbb{E}\Big((\upsilon_1,1),(\upsilon_2,1);L_f(b_1-a_1)^{\upsilon_1}(b_2-a_2)^{\upsilon_2}\Big) \\
&:= C\epsilon
\end{aligned}$$

where

$$C=\frac{(b_1-a_1)^{\upsilon_1}(b_2-a_2)^{\upsilon_2}}{\Gamma(\upsilon_1+1)\Gamma(\upsilon_2+1)}\mathbb{E}\left((\upsilon_1,1),(\upsilon_2,1);L_f(b_1-a_1)^{\upsilon_1}(b_2-a_2)^{\upsilon_2}\right).$$

Consequently, System (1.1) is Ulam-Hyers stable and this is complete the proof. □

Example 4.3. Let us consider the following problem

$$\begin{cases}
{}^{C}D_{a^+}^{\upsilon,\varpi}\mathcal{U}(\zeta,\vartheta) = \mathcal{F}(\zeta,\vartheta,\mathcal{U}(\zeta,\vartheta)), & (\zeta,\vartheta)\in J=[0,1]\times[0,1], \\
\mathcal{U}(\zeta,0) = \sinh(\zeta), & \zeta\in[0,1], \\
\mathcal{U}(0,\vartheta) = \exp(\vartheta)-1, & \vartheta\in[0,1],
\end{cases} \tag{4.4}$$

where $a=(0,0)$, $\upsilon=\left(\frac{1}{2},\frac{1}{2}\right)$, $\varpi=(\varpi_1,\varpi_2)$ with $(\varpi_1(\zeta),\varpi_2(\vartheta))=(\exp(\zeta),\exp(\vartheta))$ and $\mathcal{F}(\zeta,\vartheta,\mathcal{U})=\cosh(\zeta+\vartheta)+\arctan(\mathcal{U})$.
For all $\mathcal{U},\mathcal{V}\in\mathbb{R}$ and $(\zeta,\vartheta)\in[0,1]\times[0,1]$, we have

$$\left|\mathcal{F}(\zeta,\vartheta,\mathcal{U})\right|\leq\cosh(\zeta+\vartheta)+|\mathcal{U}|, \tag{4.5}$$

$$\left|\mathcal{F}(\zeta,\vartheta,\mathcal{U})-\mathcal{F}(\zeta,\vartheta,\mathcal{V})\right|\leq|\mathcal{U}-\mathcal{V}|. \tag{4.6}$$

Hence, the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied. As a consequence of Theorem 3.2 and Theorem 4.2 we deduce that the **(IVP)** (4.4) has a unique solution on $[0,1]\times[0,1]$ and it is Ulam-Hyers stable.

5. Conclusion

This paper explores the nuances surrounding a newly formulated mathematical construct known as the weighted fractional derivative. Through a meticulous formulation of hypotheses concerning the source term, the manuscript navigates the complexities of mathematical analysis to ascertain not only the existence but also the uniqueness and stability of solutions for Darboux problems. A distinctive feature of this study is the incorporation of the weighted fractional operator, introducing a nuanced layer that enhances the depth and precision of the analysis. The findings contribute novel insights into fractional calculus, providing a comprehensive understanding of the implications and applications of the weighted fractional derivative in addressing complex mathematical problems.

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