Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 52(1), 2025, Pages 213–227, DOI: 10.52846/ami.v52i1.1977
ISSN: 1223-6934

Bifurcation for Nonlinear Eigenvalue p-Laplacian Problems Involving L^q-norm

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Abstract. In this paper, we study a global bifurcation phenomenon associated with the non-linear problem

$$(E_f) \qquad -\Delta_p u = \lambda \|u\|_q^{p-q} |u|^{q-2} u + f(x, u, \lambda) \text{ in } \Omega,$$

where, the unknown $u \in W_0^{1,p}(\Omega)$. Under some natural hypotheses on the nonlinear perturbation f, we prove that $(\lambda_1, 0)$ is a global bifurcation point of the above problem, where λ_1 stands the first eigenvalue of $(E_{\{f\equiv 0\}})$.

2020 *Mathematics Subject Classification.* Primary 35J15; 35J66; Secondary 47H11; 58J55.
Key words and phrases. p-Laplacian operator; nonlinear eigenvalue problems; topological degree; bifurcation phenomenon.

1. Introduction

The well-known p-Laplacian operator $-\Delta_p u := -\nabla.\left(|\nabla u|^{p-2}\nabla u\right)$ becomes the icon of nonlinear partial differential equations. Motivated by their wide range of applications, eigenvalue problems involving the p-Laplacian operator have attracted increasing attention over the past four decades. This growing interest is driven by their deep connections to various areas of applied sciences, including bifurcation theory, resonance phenomena, fluid dynamics, and quantum mechanics. See [7] and the references therein for more details about motivational physics.

In the present work, we study the bifurcation of the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda \|u\|_q^{p-q} |u|^{q-2} u + f(x, u, \lambda) \text{ in } \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $1 < p < \infty, 1 < q < p^*$, where p^* is defined by

$$p^* := \begin{cases} \infty & \text{if } p \geq N \\ \frac{Np}{N-p} & \text{otherwise.} \end{cases}$$

λ is a real parameter representing eigenvalues. Nonlinearity f stands a function satisfying some conditions to be specified later. The L^q-norm in (1) is justified by the compact embedding of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$ for $1 < q < p^*$, including the case $q = p$.

Received June 12, 2024. December 31, 2024.

Let us mention that for $1 < q < p^*$, Franzina and Lamberti [15] proved that the first eigenvalue, denoted λ_1, of the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda \|u\|_q^{p-q} |u|^{q-2} u \quad \text{in} \quad \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{2}$$

is characterized variationally by the following optimization

$$\lambda_{1:} = min \left\{ \int_\Omega |\nabla u|^p \, dx : u \in W_0^{1,p}(\Omega) \text{ with } \|u\|_q^p = 1 \right\}. \tag{3}$$

The authors also proved that the first eigenvalue λ_1 is simple, and that its corresponding principal eigenfunction u_1 is positive and normalized so that $\|u_1\|_q = 1$. Consequently, u_1 is the unique positive solution to the equation

$$-\Delta_p u = \lambda \|u\|_{p-q}^q |u|^{q-2} u.$$

However, the isolation of the principal eigenvalue was not addressed in their analysis. To overcome this gap and to establish our bifurcation result, we prove in Subsection 2.4 that λ_1 is indeed isolated.

In the special case $p = q$, problem (2) reduces to the classical p-Laplacian eigenvalue problem:

$$-\Delta_p u = \lambda |u|^{p-2} u.$$

Bifurcation phenomena for homogeneous Dirichlet problems in regular domains, under specific assumptions on the nonlinearity f, have been previously investigated—see [1, 6]. These results were later extended by the authors of [11, 12] to encompass arbitrary bounded domains with locally supported, independent weight functions, where bifurcation from the interior was studied. For bifurcation problems formulated on the entire space, we refer the reader to [13].

In this work, we extend classical results corresponding to the case $p = q$ to the more general situation where $p \neq q$, specifically assuming $1 < q < p$. More precisely, we examine a class of nonlinear boundary value problems under relaxed assumptions on the perturbation term f, applicable to arbitrary bounded domains.

It is worth noting that when $1 < p < q$, the problem becomes singular, encompassing a wide variety of nonlinear models. These include, among others, equations arising in the study of chemical catalytic reactions, non-Newtonian fluid dynamics, and heat conduction in electrical conductors whose resistance is temperature-dependent. For a more comprehensive discussion of such models, we refer the reader to [5, 8, 16] and the references therein.

The structure of the paper is as follows. In Section 2, we present the necessary preliminaries and establish a key result concerning the isolation of the principal eigenvalue. In Section 3, we first verify that our operators satisfy the hypotheses of the generalized topological degree theory. We then demonstrate that the topological degree undergoes a discontinuous jump as λ crosses λ_1, leading to the bifurcation result we aim to prove.

2. Hypotheses and preliminary results

2.1. Assumptions. Our assumptions on the nonlinearity f are as follows.

($\mathcal{A}_1$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, f is measurable in $x \in \Omega$ for all $s \in \mathbb{R}$ and continuous in $s \in \mathbb{R}$ a.e. $x \in \Omega$ and we have

$$f(x, s, \lambda) = o(|s|^{p-1}) \tag{4}$$

for s near $s = 0$ uniformly for a.e. with respect to $x \in \Omega$ and uniformly with respect to λ in bounded interval.

($\mathcal{A}_2$) there is $r \in (p, p^*)$ such that

$$\lim_{|s|\to\infty} \frac{|f(x, s, \lambda)|}{|s|^{r-1}} = 0 \tag{5}$$

uniformly for a.e. with respect to $x \in \Omega$ and with respect to λ in bounded interval.

The meaning of ($\mathcal{A}_1$) and ($\mathcal{A}_2$) show that the problem (1) is $(p-1)$-superlinear and $(p-1)$-sublinear at 0 and ∞ respectively.

2.2. Compactness and Variational setting. Let us mention that the Sobolev space $W_0^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}}$$

and the L^p-norm is denoted by $\|u\|_p$. Recall that $W_0^{1,p}(\Omega)$ with $1 < p < \infty$ are reflexive, separable and Banach spaces. Consider the following operators $A = -\Delta_p$, B defined from $W_0^{1,p}(\Omega)$ to its dual space $W^{-1,p'}(\Omega)$ and F from $\mathbb{R} \times W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$, for any $u, v \in W_0^{1,p}(\Omega)$ by

$$\langle A(u), v\rangle := \int_\Omega |\nabla u|^{p-2} \nabla u \nabla u \, dx$$

$$\langle B(u), v\rangle := \|u\|_q^{p-q} \int_\Omega |u|^{q-2} uv \, dx$$

$$\langle F(\lambda, u), v\rangle := \int_\Omega f(x, u, \lambda) v \, dx$$

where $\langle ., . \rangle$ is the usual duality map defined on $W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega)$. When $u = v$ we have

$$\langle Au, u\rangle = \|u\|_{1,p}^p, \quad \text{and} \quad \langle Bu, u\rangle = \|u\|_q^p.$$

Definition 2.1. A pair (λ, u) in $\mathbb{R} \times W_0^{1,p}(\Omega)$ is a weak solution of (1) if and only if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx = \lambda \|u\|_q^{p-q} \int_\Omega |u|^{q-2} uv \, dx + \int_\Omega f(x, u, \lambda) v \, dx \tag{6}$$

holds true for every $v \in W_0^{1,p}(\Omega)$.
The pair $(\lambda, 0)$ is called a trivial solution of (1), for any$\lambda \in \mathbb{R}$.

Here $F(\lambda, \cdot)$ stands the Nemytskii operator generated by f. Observe that (6) is equivalent to

$$Au - \lambda Bu - F(\lambda, u) = 0 \qquad \text{in } W^{-1,p'}(\Omega). \tag{7}$$

To overcome the problem with lack of compactness on the main operator p-Laplacian, we introduce the notion of a class of operators satisfying the called (S^+) property.

Definition 2.2. A map T acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ is said to belong to the class (S^+), if for any sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that

$$\text{If } u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ and } \limsup_{n\to+\infty} \langle Lu_n, u_n - u\rangle \leq 0,$$

then

$$u_n \to u \text{ in } W_0^{1,p}(\Omega).$$

We denote $T \in (S^+)$.

Remark 2.1. The operator A has the following properties:
(i) A is odd, $(p-1)$-homogeneous and strictly monotone, i.e.,

$$\langle Au - Av, u - v\rangle > 0, \quad \text{for all } u \neq v,$$

(ii) $A \in (S+)$. Moreover, A is a homeomorphism.

Lemma 2.1. *The function B is odd, $(p-1)$-homogeneous and compact.*

Proof. It is clear that a map B is odd and $(p-1)$-homogeneous, it suffices to show that it is well defined and then compact. So by applying Hölder's inequality, we have for $u, v \in W_0^{1,p}(\Omega)$

$$\begin{aligned} |\langle Bu, v\rangle| &\leq \|u\|_q^{p-q} \Big(\int_\Omega |u|^{(q-1)q'}\, dx\Big)^{\frac{1}{q'}} \Big(\int_\Omega (|v|^q)\, dx\Big)^{\frac{1}{q}}, \\ &\leq \|u\|_q^{p-q} \|u\|_{q'}^{q-1} \|v\|_q, \end{aligned}$$

where $q' := \frac{q}{q-1}$ is the conjugate of q. Therefore

$$\begin{aligned} |\langle Bu, v\rangle| &\leq C\|u\|_p^{p-q}\|u\|_{q'}^{q-1}\|v\|_p, \\ &\leq CC'\|u\|_{1,p}^{p-q}\|u\|_{1,p}^{q-1}\|v\|_{1,p}. \end{aligned}$$

Then

$$\|\langle Bu\|_* \leq CC'\|u\|_{1,p}^{p-1},$$

where C is obtained by the embedding of $L^p(\Omega)$ in $L^q(\Omega)$ and C' is given by the embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$. Here $\|.\|_*$ stands the dual norm associated with $\|\nabla.\|_p$. For the completely continuity of B, we observe that if $(u_n)_n \subset W_0^{1,p}(\Omega)$ and $u_n \rightharpoonup u$ (converges weakly) in $W_0^{1,p}(\Omega)$. Thanks to the compact Sobolev embedding $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$, we deduce that (u_n) converges strongly to u in $L^q(\Omega)$, so that there exists $g \in L_+^q(\Omega)$ such that

$$|u| \leq g \quad \text{a.e. in } \Omega.$$

Applying the dominated convergence theorem, one may deduce that

$$|u_n|^{q-2}u_n \to |u|^{q-2}u \text{ in } L^{q'}(\Omega).$$

That is,

$$Bu_n \to Bu \text{ in } L^{q'}(\Omega).$$

Recall that the following inclusions

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ and } L^{q'}(\Omega) \hookrightarrow W^{-1,q'}(\Omega)$$

are compact. Consequently

$$Bu_n \to Bu \text{ in } W_0^{1,p}(\Omega)$$

and then the proof of the lemma is achieved. $\square$

2.3. Degree theory. The man tool to prove our bifurcation results is the topological degree. So let us recall here some basic properties of the Leray-Schauder degree (see [3, 4, 19]). Let X be a real separable reflexive Banach space and $T : X \longrightarrow X^*$ be a demi-continuous operator, that is, $u_n \subset X$ a sequence such that

$$u_n \rightharpoonup u \text{ implies that } Tu_n \rightharpoonup Tu$$

and if $T \in (S^+)$, then it is possible to define the degree $Deg[T; D, O]$, where $D \subset X$ is a bounded open set such that $Tu \neq 0$ for any $u \in \partial D$.
A point $u_o \in X$ will be called an isolated critical point of T if

$$\exists\, \epsilon > 0 : \forall u \in B_\epsilon(u_o), Tu \neq 0 \text{ with } u \neq u_o.$$

We define also the index of the isolated critical point u_o, by

$$Ind(T, u_o) = \lim Deg[T; B_\epsilon(u_o), 0].$$

The properties in The following two lemmas which we can find in [9, 10, 13], will be used.

Lemma 2.2. *Assume that T is a potential operator, i.e., for some continuously differentiable functional $\Phi : X \to \mathbb{R}, \Phi'(u) = Tu, u \in X$. Let u_o be a local minimum of ϕ, and an isolated critical point of T. Then*

$$Ind(T, u_0) = 1.$$

Lemma 2.3. *Assume that $\langle Tu, u\rangle > 0$ for all $u \in X, \|u\|_X = R$. Then*

$$Deg[T; B_R(u_o), 0] = 1.$$

Remark 2.2. Note that every continuous map $T :\in W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is demi-continuous and if $T \in (S^+)$ then $(T + K) \in (S^+)$, for any compact operator: $K \in W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$.

Now, it is time to handle to operator F by proving some specific properties.

Lemma 2.4. *The operator $F(\lambda, \cdot)$ satisfies*
(i) *$F(\lambda, \cdot)$ is compact and $F(\lambda, 0) = 0$.*

(ii) *For any $u \in W_0^{1,p}(\Omega)$, we have*

$$\lim_{\|\nabla u\|_p \to 0} \frac{F(\lambda, u)}{\|\nabla u\|_p^{p-1}} = 0 \text{ in } W_0^{-1,p'}(\Omega), \tag{8}$$

uniformly for λ in a bounded subset of $\mathbb{R}$.

Proof. (i) First it is obvious that $F(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}$.
From (4) and (5), one obtains

$$\int_\Omega |f(x, s, \lambda)|^{r'}\, dx \leq C \int_\Omega |u(x)|^r\, dx, \tag{9}$$

where $r \in (p, p^*)$. Such choice of the exponent r ensures that $L^r(\Omega) \hookrightarrow L^{r'(p-1)}(\Omega)$ since $r'(p-1) \leq p'(p-1) = p$, so that the inequality (9) is justified. Then $F(\lambda, u)$ maps from $L^r(\Omega)$ into $L^{r'}(\Omega)$. Moreover, assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, hence

$$u_n \to u \text{ in } L^r(\Omega) \text{ and } F(\lambda, u_n) \to F(\lambda, u) \text{ in } L^{r'}(\Omega).$$

On the other hands, since

$$L^{r'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega),$$

we have

$$F(\lambda, u_n) \to F(\lambda, u) \text{ in } W^{-1,p'}(\Omega).$$

Therefore, the compactness of $F(\lambda, .)$ follows.

(ii) By $(\mathcal{A}_2)$, we claim that $\lim_{\|\nabla u\|_p \to \infty} \frac{F(\lambda,u)}{\|\nabla u\|_p^{p-1}} = 0$, in $L^{r'}(\Omega)$. Indeed, set $v = \frac{u}{\|\nabla u\|_p}$, for $\|\nabla u\|_p$ large enough. Thus

$$\frac{F(\lambda, u)}{\|\nabla u\|_p^{p-1}} = \frac{F(\lambda, u)}{|u|^{p-1}} |v|^{p-1}. \tag{10}$$

From this and Hölder's inequality, we suppose, for some $t > 1$ to be specified, that

$$\left(\int_\Omega \frac{|F(\lambda, u)|^r}{\|\nabla u\|_p^{(p-1)r}} \right)^{r'-1} \leq \left(\int_\Omega \frac{|F(\lambda, u)|^{r't}}{\|\nabla u\|_p^{(p-1)r't}} \right)^{\frac{1}{t}} \cdot \left(\int_\Omega \frac{|u|^{r't'(p-1)}}{\|\nabla u\|_p^{(p-1)r't'}} \right)^{\frac{1}{t}} \tag{11}$$

holds. For this end, t should satisfy

$$\frac{p^*}{p^* - r'(p-1)} < t < \frac{p^*}{r'(r-p)}. \tag{12}$$

This bound is always possible due to the assumption $p < r < p^*$. Now, combining (10) and (11) and regarding to (9), one obtains that

$$\left\| \left| \frac{F(\lambda, u)}{\|\nabla u\|_p^{p-1}} \right|^{r'} \right\|_t^t \leq C^{r't} \int_\Omega |v|^{r't(r-p)} \, dx. \tag{13}$$

(13) and in view that $u \to 0$ in $W_0^{1,p}(\Omega)$, we obtains by compactness due to the bounds (12)

$$\left\| \left| \frac{F(\lambda, u)}{\|\nabla u\|_p^{p-1}} \right|^{r'} \right\|_t \to 0 \text{ as } u_n \to 0 \text{ in } W_0^{1,p}(\Omega). \tag{14}$$

Finally, our setting of v implies that $v \in L^{p^*}(\Omega)$. Moreover, one may find a constant $c > 0$ such that

$$\| |v|^{(p-1)r'} \|_{t'} \leq c$$

by the choice of t satisfying (12). The proof is now archived.

□

Now, we set for any $\lambda \in \mathbb{R}$

$$T_\lambda = A - \lambda B - F(\lambda, .).$$

Then, by (7), Remark 2.1, Lemma 2.1, Remark 2.2, and Lemma 2.4, we can define the topological degree

$$Deg\,[T_\lambda; \mathcal{D}, 0]\,,$$

for any $\lambda \in \mathbb{R}$, such that $T_\lambda u \neq 0$, for any $u \in \partial\mathcal{D}$, where $\mathcal{D}$ is any bounded open set in $W_0^{1,p}(\Omega)$.

2.4. Isolation of the principal eigenvalue. In this subsection, we are concerned to prove the isolation of the first eigenvalue λ_1, of the auxiliary problem (2) related to (1) for $f = 0$. For this, we need some useful results, which we will give later.

First we recall that a pair (λ, u) in $\mathbb{R} \times W_0^{1,p}(\Omega)$ is a weak solution of (2) if and only if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx = \lambda \|u\|_q^{p-q} \int_\Omega |u|^{q-2} uv \, dx. \tag{15}$$

Now we give the following result which is a direct consequence of the Harnack's inequality for nonnegative solutions. We refer to [[14] Theorem 5, 6] and [18].

Lemma 2.5. *Let $u \in W_0^{1,p}(\Omega)$ be a non-negative weak solution of (2) then either $u \equiv 0$ or $u(x) > 0$ for all $x \in \Omega$.*

Lemma 2.6. *Eigenfunctions associated to λ_1 are either positive or negative in Ω.*

Proof. Let u be an eigenfunction associated to λ_1. For $v = |u|$, we have $\|v\|_{1,p} = \|u\|_{1,p}$ and $\|u\|_q^p = 1$, then from the characterization of λ_1 given in (3) it follows that $v = |u|$ is an eigenfunction associated to λ_1, hence by Lemma (2.5), we deduce that $v = |u| > 0$ in Ω, then we conclude that u is either positive or negative in Ω. □

Proposition 2.7. *Let $1 < q < p$ and Ω be a smooth bounded domain in $\mathbb{R}^N$. If λ is an eigenvalue of (2) admitting a positive eigenfunction then $\lambda = \lambda_1$.*

Proof. Our proof is based on the following generalized Picone's inequality (see [2, Proposition 2.9])

$$\Big\langle |\nabla u|^{p-2} \nabla u, \nabla \Big(\frac{v^q}{u^{q-1}} \Big) \Big\rangle \le |\nabla v|^q |\nabla u|^{p-q}, \tag{16}$$

which is hold for every pair of differentiable functions u, v with $u \ge 0$ and $v > 0$. By Young inequality, (16) implies

$$\Big\langle |\nabla u|^{p-2} \nabla u, \nabla \Big(\frac{v^q}{u^{q-1}} \Big) \Big\rangle \le \frac{q}{p} |\nabla v|^p + \frac{p-q}{p} |\nabla u|^p. \tag{17}$$

Let $u_1 \in W^{1,p}(\Omega) \setminus \{0\}$ be the first positive eigenfunction corresponding to λ_1, u is a positive eigenfunction corresponding to λ.
We only need to prove that $\lambda \le \lambda_1$, hence by testing (15) with a nonnegative functions

$$v = \frac{u_1^q}{(\epsilon + u)^{q-1}},$$

and thanks to (16) and (17) we have

$$\begin{aligned} \lambda \|u\|_q^{p-q} \int_\Omega u^{q-1} \frac{u_1^q}{(\epsilon + u)^{q-1}} \, dx &= \int_\Omega |\nabla u|^{p-1} \nabla \Big(\frac{u_1^q}{(\epsilon + u)^{q-1}} \Big) \, dx, \\ &\le \int_\Omega |\nabla u_1|^q |\nabla u|^{p-q} \, dx, \\ &\le \frac{q}{p} \|\nabla u_1\|_p^p + \frac{p-q}{p} \|\nabla u\|_p^p. \end{aligned}$$

By taking the limit as ϵ goes to 0, using the Fatou's Lemma, we get

$$\lambda\|u\|_q^{p-q}\|u_1\|_q^q \leq \frac{q}{p}\|\nabla u_1\|_p^p + \frac{p-q}{p}\|\nabla u\|_p^p, \\ \leq \frac{q}{p}\lambda_1 + \frac{p-q}{p}\lambda. \tag{18}$$

Moreover, we can assume the normalization

$$\int_\Omega |u_1|^q\,dx = \int_\Omega |u|^q\,dx = 1.$$

Then

$$\|\nabla u_1\|_p^p = \lambda_1 \text{ and } \|\nabla u\|_p^p = \lambda,$$

Thus from (16), we obtain

$$\Big(1 - \frac{p-q}{p}\Big)\lambda \leq \frac{q}{p}\lambda_1 \Longrightarrow \lambda \leq \lambda_1.$$

□

In the following lemma we prove an estimate on the measure of $\mathcal{N}$ for an eigenfunction u, where $\mathcal{N}$ is a nodal domain of a function u, that is a closure of a connected component of $\Omega \setminus \{u = 0\}$.

Lemma 2.8. *Let $\mathcal{N}$ be a nodal domain of w, wich is an eigenfunction corresponding to $0 < \lambda \neq \lambda_1$, then w changes sign on $\mathbb{N}$, and the estimate*

$$|\mathcal{N}| \geq \Big(\lambda C^{-1}\Big)^{-\gamma}, \tag{19}$$

holds true for some constant $C = C(p, q, N, \Omega)$,

$$\gamma := \begin{cases} \frac{q(N-1)}{N(p-q)-p+pq} & \text{if } 1 < p < N, \\ \frac{q}{N-q} & \text{if } p = N, \\ \frac{Nq}{pq+N(p-q)} & \text{if } p > N, \end{cases}$$

and, $|B|$ denotes the Lebesgue measure of a subs $B \subset \mathbb{R}^N$.

Proof. By proposition 2.7 we deduce immediately that w must change sign.
To obtain estimate (19) we need to discuss three cases, for this we define on $\mathcal{N}$ the function

$$u(x) := \begin{cases} w(x) & \text{if } x \in \mathcal{N} \\ 0 & \text{if } x \in \Omega \setminus \mathcal{N}. \end{cases}$$

Assume that $w \in W_0^{1,p}(\Omega)$. In what follows, we suppose that $w > 0$; the case $w < 0$ can be treated similarly using the same argument.

The inequality holds for every $v \in W_0^{1,p}(\Omega)$. Now, replacing v by w and applying Hölder's inequality, we obtain:
Case 1: If $1 < p < N$, then

$$\int_{\mathcal{N}} |\nabla w|^p\,dx = \lambda\|w\|_{q,\mathcal{N}}^{p-q}\int_{\mathcal{N}} |w|^q\,dx.$$

Then,

$$\int_{\mathcal{N}} |\nabla w|^p\,dx \le \lambda |\mathcal{N}|^{\frac{p}{q}(1-\frac{q}{p^*})}\Big(\int_{\mathcal{N}} |w|^{p^*}\,dx\Big)^{\frac{p}{p^*}}, \\ \le \lambda |\mathcal{N}|^{p(\frac{1}{q}-\frac{1}{p^*})}\|w\|^p_{p^*,\mathcal{N}}, \tag{20}$$

Now by the injection $W^{1,p}(\mathcal{N}) \hookrightarrow L^{p^*}(\mathcal{N})$, there exists a constant C such that

$$\int_{\mathcal{N}} |\nabla w|^p\,dx = \int_{\Omega} |\nabla u|^p\,dx, \\ \ge C\|u\|^p_{p^*} = C\|w\|^p_{p^*,\mathcal{N}}. \tag{21}$$

Thus

$$\lambda |\mathcal{N}|^{\frac{N(p-q)-p+pq}{q(N-1)}} \ge C.$$

Case 2: If $p = N$, we proceed as in the previous case and obtain:

$$\int_{\mathcal{N}} |\nabla w|^N\,dx \le \lambda |\mathcal{N}|^{\frac{N}{q}-1}\|w\|^N_{N,\mathcal{N}}. \tag{22}$$

Since

$$\int_{\mathcal{N}} |\nabla w|^N\,dx = \int_{\Omega} |\nabla u|^N\,dx, \\ \ge C\|u\|^N_q = C\|w\|^N_{N,\mathcal{N}}, \tag{23}$$

where C is the constant given by the embedding $W^{1,N}(\mathcal{N}) \hookrightarrow L^q(\mathcal{N})$. Hence

$$\lambda |\mathcal{N}|^{\frac{N-q}{q}} \ge C.$$

Case 3: If $p > N$, we have

$$\int_{\mathcal{N}} |\nabla w|^p\,dx + \int_{\mathcal{N}} |w|^p\,dx \le \lambda |\mathcal{N}|^{\frac{p}{q}}\|w\|^p_{\infty,\mathcal{N}}. \tag{24}$$

On the other hand by appealing Morrey's lemma,

$$\Big[\int_{\mathcal{N}} |\nabla w|^p\,dx + \int_{\mathcal{N}} |w|^p\,dx\Big]|\mathcal{N}|^{\frac{p}{N}-1} = \|u\|^p_{1,p}|\mathcal{N}|^{\frac{p}{N}-1}, \\ \ge C\|u\|^p_{\infty,\mathcal{N}} = C\|w\|^p_{\infty,\mathcal{N}}. \tag{25}$$

Thus, we obtain

$$\lambda |\mathcal{N}|^{\frac{p}{N}+\frac{p}{q}-1} \ge C \Longrightarrow \lambda |\mathcal{N}|^{\frac{pq+N(p-q)}{Nq}} \ge C.$$

□

Theorem 2.9. *The principal eigenvalue λ_1 is isolated, that is, there exists $\eta > 0$ such that there is no other eigenvalue of* (2) *in the interval* $(\lambda_1, \lambda_1 + \eta)$.

Proof. Assume by contradiction that λ_1 is not isolated. Then we can say that there exists a sequence of eigenvalue $\{\lambda_k\}$, such that $0 < \lambda_k \searrow \lambda_1$. Let u_k be an eigenfunction corresponding to λ_k, by taking $v = u_k$ in (15), we have

$$0 < \int_{\Omega} |\nabla u_k|^p\,dx = \lambda_k\Big(\int_{\Omega} |u_k|^q\,dx\Big)^{\frac{p}{q}}.$$

Then we can define

$$w_k := \frac{u_k}{\Big(\int_{\Omega} |u_k|^q\,dx\Big)^{\frac{1}{q}}},$$

which is bounded in $W_0^{1,p}(\Omega)$, hence we may extract a subsequence (still denoted w_k) such that $w_k \rightharpoonup w \in W_0^{1,p}(\Omega)$ (weakly) and due to the fact that the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, w_k converges strangly to w in $L^q(\Omega)$. Moreover $\left(\int_\Omega |w|^q \, dx\right)^{\frac{p}{q}} = 1$.
On the other hand

$$0 < \int_\Omega |\nabla w|^p \, dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla w_k|^p \, dx = \lambda_1.$$

Thus, $\int_\Omega |\nabla w|^p \, dx = \lambda_1$ by the characterization of λ_1. Therefore w is an eigenfunction associated to λ_1. Using Lemma 2.6 we can choose $w > 0$ on Ω, and if we denote by Ω_k^- the negative set of w_n. Hence we conclude from the convergence in measure of the sequence w_n towards w that $|\Omega_k^-| \to 0$. This contradicts estimate (19). □

3. Bifurcation result

In this section, we prove the existence of bifurcation solutions to the problem at $\lambda = \lambda_1$. To that end, let us first recall the standard definition of a bifurcation point for a nonlinear continuous map

$$T_\lambda : W_0^{1,p}(\Omega) \longrightarrow W_0^{-1,p'}(\Omega).$$

Definition 3.1. Let $E = \mathbb{R} \times W_0^{1,p}(\Omega)$. A point $(\lambda, 0)$ in E is said to be a bifurcation point of the problem $T_\lambda(u) = 0$, if for any neighborhood of $(\lambda, 0)$ there exists a nontrivial solution of $T_\lambda(u) = 0$.

The following result provides a characterization of the bifurcation points of problem (1).

Proposition 3.1. *If $(\overline{\lambda}, 0) \in E$ is a bifurcation point of (1), then $\overline{\lambda}$ is an eigenvalue of (2).*

Proof. Since $(\overline{\lambda}, 0)$ is a bifurcation point of (1), there is a sequence $\{(\lambda_j, u_j) j \in \mathbb{N}\}$ of nontrivial solutions of the problem (2) such that $\lambda_j \to \overline{\lambda}$ in $\mathbb{R}$ and $u_j \to 0$ in $W_0^{1,p}(\Omega)$, as $j \to +\infty$. By Definition 2.1, pairs (λ_j, u_j) satisfy (6). Thus we have by $(p-1)$-homogeneity

$$Av_j - \lambda_j Bv_j = \frac{F(\lambda_j, u_j)}{\|\nabla u_j\|_p^{p-1}}, \tag{26}$$

where $v_j = \frac{u}{\|\nabla u_j\|_p}$. The sequence (v_j) is bounded in $W_0^{1,p}(\Omega)$. Then there exists a function $v \in W_0^{1,p}(\Omega)$ such that $v_j \rightharpoonup v \in W_0^{1,p}(\Omega)$ and almost everywhere in Ω (up to a subsequence if necessary). Then, from Remark 2.1 and Lemma 2.3, we deduce that

$$Bu_j + \frac{F(\lambda, u_j)}{\|\nabla_j, u_j\|_p^{p-1}} \to \overline{\lambda} Bv \text{ in } W^{-1,p'}(\Omega). \tag{27}$$

Thanks to the continuity of A, it follows from equation (26) that

$$Av_j \to \overline{\lambda} Bv \text{ in } W^{-1,p'}(\Omega).$$

Now, the continuity of A^{-1} implies that $\{v_j\}_{j\geq 1}$ converges to $A^{-1}(\overline{\lambda}Bv)$ in $W_0^{1,p}(\Omega)$, as $j \to +\infty$. We also deduce that $v \neq 0$. The convergence *a.e.* on Ω, yields that v is solution of the equation

$$v = A^{-1}(\overline{\lambda}Bv) \quad \text{in} \quad W_0^{1,p}(\Omega). \tag{28}$$

Finally, in view of equation (28), we conclude that λ is an eigenvalue of problem (2). This completes the proof. □

Remark 3.1. Observe that Proposition 3.1 show that bifurcation points of (1), permit to find new eigenvalues of (2).

Definition 3.2. One says that a subset of the form

$$\mathcal{C} := \{(\lambda, u) \in E : (\lambda, u) \;\text{ is nontrivial solution of }\; (1), \},$$

is to be a continuum of nontrivial solutions of (1), if it is a connected subset of E.

Theorem 3.2. *Assume ($\mathcal{A}_1$) and ($\mathcal{A}_2$). Hence a pair $(\lambda_1, 0)$ is a bifurcation point of (1). Moreover, there is a continuum of nontrivial solutions C of (1) such that $(\lambda_1, 0) \in \overline{\mathcal{C}}$ and $\mathcal{C}$ is either unbounded in $\mathbb{R}\times W_0^{1,p}(\Omega)$ or there is $\overline{\lambda} \neq \lambda_1$, an eigenvalue of (2), with $(\overline{\lambda}, 0) \in \overline{C}$, where $\overline{C}$ stands the closure of C in $\mathbb{R} \times W_0^{1,p}(\Omega)$.*

Proof. We shall employ the classical global bifurcation stated in Theorem 1.3 of [17, Theorem 1.3] and a variation technique of [11, Theorem 3.2] with some appropriate modifications, so that we may conclude that

$$Deg\,[A - \lambda B; B_\epsilon(0), 0]] \,, \tag{29}$$

leaps from -1 to 1 when λ is near to the value of λ_1. This is the key point of the proof. We distinguish two cases:

On the left of λ_1. In view of Lemma 2.3 and The variational characterization (26) of λ_1, we have for λ on the left of λ_1

$$Deg\,[A - \lambda B; B_\epsilon(0), 0] = 1. \tag{30}$$

On the right of λ_1. Since λ_1 is isolated in the spectrum, we can find for each λ near to $\lambda_1, \delta > 0$ such that there is no eigenvalue of (2) in the interval $(\lambda_1, \lambda_1 + \delta)$. To evaluate (27) for on the right of λ_1, we use the following trick. Define an auxiliary function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) = a \begin{cases} 0 & \text{for } t \leq 1 \\ t^3 - 2t^2 + 2t - 1 & \text{for } 1 \leq t \leq 3 \\ 14(t-2) & \text{for } t \geq 3, \end{cases}$$

where $a = \frac{\delta}{7\lambda_1}$ (the graph of ψ is given in Figure 1). The choice of this value of a ensures that ψ is positive and strictly convex in $(1, 3)$ so that the functional

$$\Phi_\lambda(u) := \frac{1}{p}\langle Au, u\rangle - \frac{\lambda}{p}\langle Bu, u\rangle + \psi\left(\frac{1}{p}\langle Au, u\rangle\right). \tag{31}$$

is continuously Fréchet differentiable. Moreover, critical points of Φ_λ correspond to solutions of the associated Euler-Lagrange equation $\Phi;_\lambda = 0$ in $W_0^{1,p}(\Omega)$. That is,

$$Au - \lambda Bu + \psi'\left(\frac{1}{p}\langle Au, u\rangle\right).Au = 0. \tag{32}$$

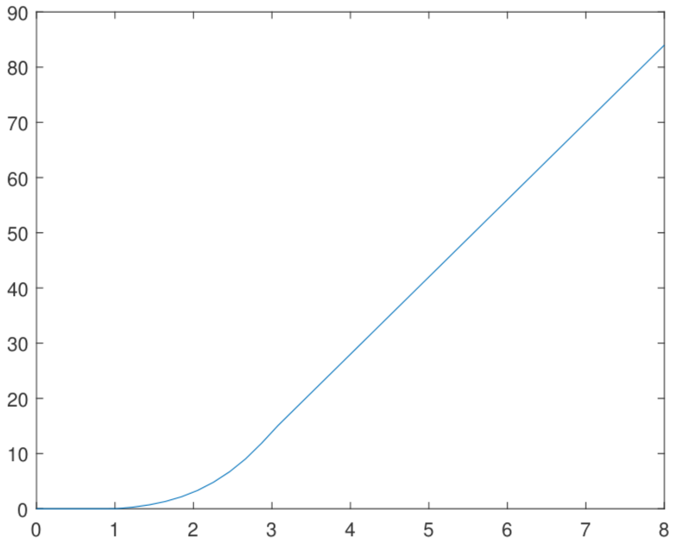

FIGURE 1. The graph of the auxiliary function ψ.

Obviously, 0 is a critical point. Now, we claim that if u^* is a non trivial critical point of Φ_λ, in view of the definition of ψ, we must have $\frac{1}{p}\langle Au^*, u^*\rangle$ in the interval $(1,3)$. Indeed, suppose that $\frac{1}{p}\langle Au^*, u^*\rangle > 3$. In this case, $\psi'\left(\frac{1}{p}\langle Au^*, u^*\rangle\right) = 14a$. Thus

$$\psi'\left(\frac{1}{p}\langle Au^*, u^*\rangle\right) = \frac{2\delta}{\lambda_1}. \tag{33}$$

However, from (10), we deduce that

$$Au^* = \frac{\lambda}{1+\psi'\left(\frac{1}{p}\langle Au^*, u^*\rangle\right)} Bu^*. \tag{34}$$

That means the value $\frac{\lambda}{1+\psi'\left(\frac{1}{p}\langle Au^*, u^*\rangle\right)}$ is an eigenvalue of (2) associated to the eigenfunction u^*. Then we must have

$$\frac{\lambda}{1+\frac{2\delta}{\lambda_1}} \geq \lambda_1.$$

That is, $\lambda \geq \lambda_1 + 2\delta$. This contradicts the fact that $\lambda \in (\lambda_1, \lambda_1+\delta)$, and the claim is proved. Hence we have

$$\frac{1}{p}\langle Au^*, u^*\rangle \in (1,3).$$

Let u_1 be the principal eigenfunction (referred to be positive and normalized). Then regarding to the simplicity of λ_1, u^* is in the one-dimensional eigenspace spanned by u_1. Thus $u^* = \pm\|u^*\|_{1,p}u_1$. Finally, for $\lambda \in (\lambda_1, \lambda_1+\delta)$, we conclude that $-\|u^*\|_{1,p}u_1, 0, \|u^*\|_{1,p}u_1$ are precisely the isolated critical points of Φ_λ.

To complete the proof, we establish the following claims:

- The functional Φ is weakly lower semi-continuous.

- The functional Φ is coercive.

We begin by proving the weak lower semi-continuity of Φ. If $u_n \rightharpoonup u$ weakly, then by Lemma 2.1, we have

$$\langle Bu_n, u_n\rangle \to \langle Bu, u\rangle. \tag{35}$$

Moreover, since the norm is weakly lower semi-continuous, we have

$$\|\nabla u\|_p \le \liminf_{n\to\infty} \|\nabla u_n\|_p. \tag{36}$$

Then, (35), (36) and the fact that ψ is increasing in the interval $(1,\infty)$ lead to

$$\lim_{n\to\infty} \Phi_\lambda(u_n) \ge \Phi_\lambda(u).$$

For coercivity. With respect to the definition of Φ_λ we can write

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{p}\langle Au, u\rangle + \frac{\lambda_1-\lambda}{p}\langle Bu, u\rangle + \frac{\lambda_1}{p}\langle Bu, u\rangle + \psi\left(\frac{1}{p}\langle Au, u\rangle\right).,\\ &\ge \frac{\lambda_1-\lambda}{p}\langle Bu, u\rangle + \psi\left(\frac{1}{p}\langle Au, u\rangle\right).\end{aligned}$$

Hence, by the definition of the function ψ and for $\|\nabla u\|_p$ sufficiently large, we obtain

$$\Phi_\lambda(u) \ge \frac{\lambda_1-\lambda}{p}\langle Bu, u\rangle + \frac{2\delta}{\lambda_1}\left(\frac{1}{p}\langle Au, u\rangle - 2\right),$$

Thus, due to the fact that $\lambda_1 < \lambda$, we have

$$\Phi_\lambda(u) \ge \frac{\lambda_1-\lambda}{p\lambda_1}\|\nabla u\|_p^p + \frac{2\delta}{\lambda_1}\left(\frac{1}{p}\|u\|_{1,p}^p - 2\right).$$

Consequently, we deduce that

$$\Phi_\lambda(u) \ge \frac{1}{p}\Big(\frac{\lambda_1-\lambda}{\lambda_1} + a\Big)\|u\|_{1,p}^p - \frac{4\delta}{\lambda_1}.$$

One readily verifies that

$$\frac{\lambda_1-\lambda}{\lambda_1} + \frac{2\delta}{\lambda_1} > \frac{\lambda_1-\lambda}{\lambda_1} + \frac{\delta}{\lambda_1} > 0. \tag{37}$$

Finally, letting $\lim_{\|\nabla u\|_p\to\infty}$ in (37), we deduce that $\lim_{\|\nabla u\|_p\to\infty}\Phi_\lambda(u) = \infty$.

Now, since, Φ_λ is even and coercive, the three critical points above are precisely two non trivial points at which the minimum of Φ_λ is achieved and 0 is obviously an isolated critical "saddle" point. The last claim is archived.

Thus it follows from Lemma 2.2 that

$$Ind\,[\Phi'_\lambda, \alpha u_1] = Ind\,[\Phi'_\lambda, -\alpha u_1] = 1. \tag{38}$$

Since also

$$\langle \Phi'_\lambda(u), u\rangle > 0,$$

for any $u \in B_R(0)$, with $R > 0$ sufficiently large, one can verify that

$$\langle Au, u\rangle > 3 \text{ and } \langle Au, u\rangle \ge \lambda_1\langle Bu, u\rangle.$$

Then

$$\begin{aligned}\langle \Phi'_\lambda(u), u\rangle &\geq (\lambda_1-\lambda)\langle Bu,u\rangle + \frac{2\delta}{\lambda_1}\langle Au,u\rangle.\\ &\geq \frac{-\delta}{\lambda_1}\langle Au,u\rangle + \frac{2\delta}{\lambda_1}\langle Au,u\rangle\\ &\geq (\frac{2\delta}{\lambda_1}-\frac{\delta}{\lambda_1})\|u\|^p_{1,p} = \frac{\delta}{\lambda_1}\|u\|^p|_{1,p}.\end{aligned}$$

This implies that $\langle \Phi'_\lambda(u), u\rangle \to \infty$, as $\|u\|_{1,p} \to \infty$. We deduce from Lemma 2.3 that

$$Deg[\Phi'_\lambda; B_R(u^*), 0] = 1. \tag{39}$$

The degree being additive and in view of (38) and (39), we conclude that

$$Deg[A-\lambda B; B_\epsilon(u_o), 0] = -1. \tag{40}$$

Also we have

$$\langle Au,u\rangle - \lambda\langle Bu,u\rangle \to 0 \text{ as } \|u\|_{1,p} \to 0.$$

Consequently,

$$Deg[A-\lambda B; B_\epsilon(u^*), 0] = Ind[\Phi'_\lambda, 0], \tag{41}$$

for $\epsilon > 0$ sufficiently small enough. By The invariance principle of the degree we conclude that for any λ near to λ_1 with $\lambda \neq \lambda_1$

$$Deg[T_\lambda; B_\epsilon(u^*), 0] = Deg[A-\lambda B; B_\epsilon(u^*), 0].$$

Finally, (30), (39) and (41) imply, for $\epsilon > 0$ sufficiently small, that

$$Deg[T_\lambda, B_\epsilon(u^*), 0] = 1 \text{ for } \lambda \in (\lambda_1-\delta, \lambda_1)$$

and

$$Deg[T_\lambda, B_\epsilon(u_o), 0] = 1 \text{ for } \lambda \in (\lambda_1, \lambda_1+\delta).$$

Which guaranteed the ”jump” of the degree and the proof is achieved. □

Acknowledgement(s)

The authors would like to express their sincere gratitude to the reviewers for their valuable and constructive comments, which have significantly contributed to improving the final version of this paper.

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