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On an Algebra of Fuzzy m-ary Semihypergroups

Adrian Naço, Anila Peposhi, Kostaq Hila*, Serkan Onar, and Bayram Ali Ersoy

Abstract. In this paper we deals with the fuzzy m-ary semihypergroups, fuzzy hyperideals and homomorphism theorems on m-ary semihypergroups and fuzzy m-ary semihypergroups. We also, introduce and study some classes of fuzzy hyperideals that of pure fuzzy, weakly pure fuzzy hyperideals in m-ary semihypergroups and some properties of them are investigated. We identify those m-ary semihypergroups for which every fuzzy hyperideal is idempotent. We also characterize the m-ary semihypergroups for which every fuzzy hyperideal is weakly pure fuzzy.

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1. Introduction and preliminaries

Hyperstructure theory was introduced in 1934, when F. Marty [15] defined hypergroups based on the notion of hyperoperation, began to analyze their properties and applied them to groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

n-ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. In [6], Davvaz and Vougiouklis introduced the concept of n-ary hypergroups as a generalization of hypergroups in the sense of Marty. Also, we can consider n-ary hypergroups as a nice generalization of n-ary groups. Davvaz and et. al. in [4] considered a class of algebraic hypersystems which represent a generalization of semigroups, hypersemigroups and n-ary semigroups.

The concept of a fuzzy set, introduced by Zadeh in his classic paper [22], provides a natural framework for generalizing some of the notions of classical algebraic structures and of abstract set theory. Fuzzy semigroups have been first considered by Kuroki [12]. After the introduction of the concept of fuzzy sets by Zadeh, several researches conducted the researches on the generalizations of the notions of fuzzy sets with huge

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*Correspoding author.

applications in computer, logics and many branches of pure and applied mathematics. In 1971, Rosenfeld [20] defined the concept of fuzzy group. Since then many papers have been published in the field of fuzzy algebra their employment in fuzzy coding, fuzzy finite state machines and fuzzy languages. Fuzziness enjoys a homely place in the domain of formal languages (see [16], [17]). Recently fuzzy set theory has been well developed in the context of hyperalgebraic structure theory (see [2]). A recent book [1] contains a wealth of applications.

In this paper we deals with the fuzzy m-ary semihypergroups, fuzzy hyperideals and homomorphism theorems on m-ary semihypergroups and fuzzy m-ary semihypergroups. We also, introduce and study some classes of fuzzy hyperideals that of pure fuzzy, weakly pure fuzzy hyperideals in m-ary semihypergroups and some properties of them are investigated. We identify those m-ary semihypergroups for which every fuzzy hyperideal is idempotent. We also characterize the m-ary semihypergroups for which every fuzzy hyperideal is weakly pure fuzzy.

Recall first the basic terms and definitions from the hyperstructure theory.

2. Algebraic hypersystems and m-ary hyperstructures

In this section we recall some known notions on what is meant by an algebraic hypersystem and m-ary hyperstructure.

Let H be a nonempty set and f be a mapping $f : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of all nonempty subsets of H. Then f is called a *binary (algebraic) hyperoperation* on H. In general, a mapping $f : H \times H \times ... \times H \to \mathcal{P}^*(H)$ where H appears m times, is called an *m-ary (algebraic) hyperoperation*, and m is called the *arity* of this hyperoperation. An algebraic system (H, f), where f is an m-ary hyperoperation defined on H, is called an *m-ary hypergroupoid* or an *m-ary hypersystem*. Since we identify the set $\{x\}$ with the element x, any m-ary (binary) groupoid is an m-ary (binary) hypergroupoid. (see [4, 5, 6, 13, 10, 11]). Recently, Davvaz [3] published a book which exhaustively covers all concepts of semihypergroups.

Let f be an m-ary hyperoperation on H and $A_1, A_2, ..., A_m$ subsets of H. We define $f(A_1, A_2, ..., A_m) = \cup\{f(x_1, x_2, ..., x_m) | x_i \in A_i, i = 1, 2, ..., m\}$. We shall use the following abbreviated notation: the sequence $x_i, x_{i+1}, ..., x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty symbol. In this convention, $f(x_1, ..., x_i, y_{i+1}, ..., y_j, z_{j+1}, ..., z_m)$ will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^m)$. In the case when $y_{i+1} = ... = y_j = y$, the last expression will be written in the form $f(x_1^i, \overset{(j-i)}{y}, z_{j+1}^m)$. Similarly, for subsets $A_1, A_2, ..., A_m$ of H we define

$$f(A_1^m) = f(A_1, A_2, ..., A_m) = \cup\{f(x_1^m) | x_i \in A_i, i = 1, ..., m\}.$$

The concept of (i, j)-associative operations was introduced by Thurston [19] and intensively studied by many authors such as Dudek [7] etc. An m-ary hyperoperation f is called $(i, j) - associative$ if

$$f(x_1^{i-1}, f(x_i^{m+i-1}), x_{m+i}^{2m-1}) = f(x_1^{j-1}, f(x_j^{m+j-1}), x_{m+j}^{2m-1}),$$

holds for fixed $1 \leq i < j \leq m$ and all $x_1, x_2, ..., x_{2m-1} \in H$.

Note that (i, k)-associativity follows from (i, j)- and (j, k)-associativities.

If the above condition is satisfied for all $i, j \in \{1, 2, ..., m\}$, then we say that f is *associative*.

By an *algebraic hypersystem* $(H, f_1, f_2, ..., f_m)$ or simply H is meant a set H closed under a collection of m_i-ary hyperoperation f_i and often also satisfying a fixed set of laws, for instant, the associative law. A subset S of H constitues a *subhypersystem* iff S is closed under the same hyperoperations and satisfies the same fixed laws in H.

Let H be an algebraic hypersystem. A *j-hyperideal* $j = 1, 2, ..., m$ relative to the m-ary hyperoperation is defined to be a subhypersystem I_j such that for any $x_1, x_2, ..., x_m \in H$, if $x_j \in I_j$ then $f(x_1, x_2, ..., x_m) \subseteq I_j$. The j-hyperideal relative to f generated by an element $a \in H$ (usually called a principal j-hyperideal) is denoted by $(a)_j = f(H, H, ..., \overset{j}{a}, ..., H) \cup \{a\}$. A subhypersystem I which is a j-hyperideal for each $j = 1, ..., m$ is simply called an *hyperideal*. An m-ary hypergroupoid (H, f) will be called an *m-ary semihypergroup* if and only if f is associative. The element e is called identity element of m-ary semihypergroup (H, f) if $x \in f(\underbrace{e, ..., e}_{i-1}, x, \underbrace{e, ..., e}_{n-i})$ for all $x \in H$ and $1 \leq i \leq m$ [6]. In [4, 5, 8] it is discussed in details the problem of numbers and of existence of neutral elements in m-ary semihypergroups in different situations. In [4], it is proved that: An m-ary hypersemigroup (H, f) has a neutral element if and only if there exists a binary hypersemigroup $(H, \circ)$ such that $f(x_1^m) = x_1 \circ x_2 \circ ... \circ x_m$, for all $x_1^m \in H$. A subset S of m-ary semihypergroup (H, f) is called an m-ary subsemihypergroup of H iff $f(a_1^m) \subseteq H$, for all $a_1, ..., a_m \in S$.

An m-ary semihypergroup H is said to be *regular* [11] if for each $a \in H$ there exist $x_2, x_3, \ldots, x_m; y_1, y_3, \ldots, y_m; \ldots; z_1, z_2, \ldots, z_{m-1} \in H$ such that

$$a \in f(f(a, x_2, \ldots, x_m), f(y_1, a, y_3, \ldots, y_m), \ldots, f(z_1, z_2, \ldots, z_{m-1}, a)).$$

Regular elements (and connection with ideals) in n-ary semigroups were studied by F.M. Sioson [21] and by Dudek and I. Groździńska [9]. An m-ary semihypergroup H is called regular if all of its elements are regular. An m-ary semihypergroup (H, f) is said to be *k-weakly regular* if for each $a \in H, a \in f(f(H, ..., \overset{(k)}{a}, ..., H), ..., f(H, ..., \overset{(k)}{a}, ..., H))$. It is clear that every regular m-ary semihypergroup is k-weakly regular but the converse is not true.

Let (H, f) be an m-ary semihypergroup and ρ be an equivalence relation on H. If A and B are non-empty subsets of H, then $A\overline{\rho}B$ means that for every $a \in A$, there exists $b \in B$ such that $a\rho b$ and for every $b_1 \in B$, there exists $a_1 \in A$ such that $a_1\rho b_1$ and $A\overline{\overline{\rho}}B$ means that for every $a \in A$ and $b \in B$, we have $a\rho b$.

The equivalence relation ρ on an m-ary semihypergroup (H, f) is called k-regular if for all $x_1, ..., x_m \in H$, from $a\rho b$, it follows that

$$f(x_1^{k-2}, a, x_{k+1}^m)\overline{\rho}f(x_1^{k-2}, b, x_{k+1}^m),$$

and is called k-strongly regular if for $x_1, ..., x_m \in H$, $a\rho b$ implies that

$$f(x_1^{k-2}, a, x_{k+1}^m)\overline{\overline{\rho}}f(x_1^{k-2}, b, x_{k+1}^m).$$

ρ is called regular (strongly regular) if it is k-regular (strongly regular) for every $1 \leq k \leq m$ [18].

3. Properties of m-ary semihypergroups

In this section, we investigate some basic properties of m-ary semihypergroups. We introduce the notion of homomorphism between two m-ary semihypergroups and some related characterizations are provided, which will be used in the sequel of the paper. We also introduce the notion of k-pure hyperideals of m-ary semihypergroups and give a characterization of them.

Definition 3.1. Let (H, f) and (H', f') are two m-ary semihypergroups. The mapping $\varphi : H \to H'$ is called a homomorphism if for all $x_1, ..., x_m \in H$, $\varphi(f(x_1, ..., x_m)) = f'(\varphi(x_1), ..., \varphi(x_m))$.

Remark 3.1. Let (H, f) and (H', f') are two m-ary semihypergroups. The mapping $\varphi : H \to H'$ is called an weak homomorphism if for all $x_1, ..., x_m \in H$, $\varphi(f(x_1, ..., x_m)) \subseteq f'(\varphi(x_1), ..., \varphi(x_m))$.

Theorem 3.1. *Let (H, f), (H', f') and (H'', f'') are three m-ary semihypergroups. If mappings $\varphi : H \to H'$ and $\sigma : H' \to H''$ are homomorphisms, then $\sigma \circ \varphi : H \to H''$ is also a homomorphism.*

Proof. Omitted as obvious. □

Definition 3.2. Let $\cong$ be an equivalence relation on the m-ary semihypergroup (H, f) and A_i and B_i be the subsets of H for all $1 \leq i \leq m$. We define $A_i \cong B_i$ iff for all $a_i \in A_i$ there exists $b'_i \in B_i$ such that $a_i \cong b'_i$ and for all $b_i \in B_i$ there exists $a'_i \in A_i$ such that $a'_i \cong b_i$.

An equivalence relation $\cong$ is called a congruence relation on H if the following holds: $\forall a_1, ..., a_m, b_1, ..., b_m \in H$, if $\{a_i\} \cong \{b_i\}$, then $\{f(a_1^m)\} \cong \{f(b_1^m)\}$, where $1 \leq i \leq m$.

Lemma 3.2. *Let (H, f) be an m-ary semihypergroup with identity element and $\cong$ be the congruence relation on H. If $\{x\} \cong \{y\}$, then $\{f(x, a_1^{m-1})\} \cong \{f(y, a_1^{m-1})\}$ for all $x, y, a_1, ..., a_{m-1} \in H$.*

Proof. Let $x, y \in H$ such that $\{x\} \cong \{y\}$ and e be the identity element of H. Then we have $f(x, \underbrace{e, ..., e}_{m-1}) \cong f(y, \underbrace{e, ..., e}_{m-1})$. This implies the followings:

$$\begin{aligned}
f(f(x, \underbrace{e, ..., e}_{m-1}), a_1, \underbrace{e, ..., e}_{m-2}) &\cong f(f(y, \underbrace{e, ..., e}_{m-1}), a_1, \underbrace{e, ..., e}_{m-2}) \\
f(f(x, a_1, \underbrace{e, ..., e}_{m-2}), \underbrace{e, ..., e}_{m-1}) &\cong f(f(y, a_1, \underbrace{e, ..., e}_{m-2}), \underbrace{e, ..., e}_{m-1}) \\
\{f(x, a_1, \underbrace{e, ..., e}_{m-2})\} &\cong \{f(y, a_1, \underbrace{e, ..., e}_{m-2})\}
\end{aligned}$$

This implies the followings:

$$\begin{aligned}
f(f(x, a_1, \underbrace{e, ..., e}_{m-2}), a_2, \underbrace{e, ..., e}_{m-2}) &\cong f(f(y, a_1, \underbrace{e, ..., e}_{m-2}), a_2, \underbrace{e, ..., e}_{m-2}) \\
f(f(x, a_1, a_2, \underbrace{e, ..., e}_{m-3}), \underbrace{e, ..., e}_{m-1}) &\cong f(f(y, a_1, a_2, \underbrace{e, ..., e}_{m-3}), \underbrace{e, ..., e}_{m-1})
\end{aligned}$$

$$\{f(x, a_1, a_2, \underbrace{e, ..., e}_{m-3})\} \quad \cong \quad \{f(y, a_1, a_2, \underbrace{e, ..., e}_{m-3})\}$$

Continuing in this way till a_{m-1}, we finally get $\{f(x, a_1^{m-1})\} \cong \{f(y, a_1^{m-1})\}$. □

Theorem 3.3. *Let (H, f) be an m-ary semihypergroup with identity element and $\cong$ be the congruence relation on H. If $\{a_i\} \cong \{b_i\}$ and $\{x_j\} \cong \{y_j\}$ for all $a_i, b_i, x_j, y_j \in H$ and $i, j \in \{1, ..., m\}$, then for all $k \in \{1, ..., m\}$, we have $\{f(a_1^k, x_{k+1}^m)\} \cong \{f(b_1^k, y_{k+1}^m)\}$.*

Proof. It is similar to the proof of the above lemma. □

Definition 3.3. Let (H, f) be an m-ary semihypergroup and $\cong$ be a congruence on H. Then the quotient of H by $\cong$, written as $[H :\cong]$, is the algebra whose universe is $[H :\cong]$ and whose fundamental operation satisfy

$$f^{[H:\cong]}([x_1]_\cong, ..., [x_m]_\cong) = [f^H(x_1, ..., x_m)]_\cong = \{[a]_\cong : a \in f^H(x_1, ..., x_m)\}$$

where $x_1, ..., x_m \in H$.

Let (H, f) and (H', f') be two m-ary semihypergroups and $\varphi : H \to H'$ be a homomorphism. The relation $\varphi \circ \varphi^{-1}$ is an equivalence ρ on H ($a\rho b$ if and only if $\varphi(a) = \varphi(b)$) known as the kernel of φ. The natural mapping associated with ρ is $\gamma : H \to H/Ker\varphi$, where $\gamma(a) = \rho(a)$. The mapping $\psi : H/\rho \to H'$, where $\psi(\rho(a)) = \varphi(a)$, is then the unique bijection.

Theorem 3.4. *Let (H, f) and (H', f') be two m-ary semihypergroups and $\varphi : H \to H'$ be a homomorphism. Then $\rho = Ker\varphi$ is a congruence and there exists a homomorphism $g : H/\rho \to H'$ such that $Img = Im\varphi$.*

Proof. Clearly, ρ is an equivalence relation. Let we prove that ρ is a congruence. Let us suppose that $a_i \rho a_i'$, for all $1 \le i \le m$. Then we have $\varphi(a_i) = \varphi(a_i')$ for all $1 \le i \le m$. Therefore $\varphi(f(a_1^m)) = f'(\varphi(a_1), ..., \varphi(a_m)) = f'(\varphi(a_1'), ..., \varphi(a_m')) = \varphi(f(a_1', ..., a_m'))$. Consequently, $f(a_1, ..., a_m)\rho f(a_1', ..., a_m')$. Therefore, ρ is a congruence on H.

Let we define $g : H/\rho \to H'$ by $g(\rho(a)) = \varphi(a)$. Then g obviously, is well-defined and one to one. Also g is homomorphism. Indeed:

If $\rho(a_i) \in H/\rho$ for $1 \le i \le m$, then

$$\begin{aligned}
g(f^{H/\rho}(\rho(a_1), ..., \rho(a_m))) &= g(\{\rho(z) | z \in f(\rho(a_1), ..., \rho(a_m))\}) \\
&= \{\varphi(z) | z \in f(\rho(a_1), ..., \rho(a_m))\} \\
&= \varphi\left(\bigcup_{x_1 \in \rho(a_1), ..., x_m \in \rho(a_m)} f(x_1, ..., x_m)\right) \\
&= \bigcup_{x_1 \in \rho(a_1), ..., x_m \in \rho(a_m)} \varphi(f(x_1, ..., x_m)) \\
&= \bigcup_{x_1 \in \rho(a_1), ..., x_m \in \rho(a_m)} f'(\varphi(x_1), ..., \varphi(x_m)) \\
&= f'(\varphi(a_1), ..., \varphi(a_m)) \\
&= f'(g(\rho(a_1), ..., g(\rho(a_m)))
\end{aligned}$$

Therefore g is homomorphism. Clearly, $Img = Im\varphi$. □

Theorem 3.5. [18, Proposition 3.2] *Let (H, f) be an m-ary semihypergroup and $\cong$ be an equivalence relation and strongly regular on H. Then $([H :\cong], f^{[H:\cong]})$ is also an m-ary semihypergroup.*

Definition 3.4. Let (H, f) be an m-ary semihypergroup and $\cong$ be a congruence relation on H. Then the natural map $\sigma_{\cong} : H \to [H :\cong]$ is defined by $\sigma_{\cong}(a_i) = [a_i]_{\cong}$ where $a_i \in H$ for all $1 \leq i \leq m$.

Theorem 3.6. *Let (H, f) be an m-ary semihypergroup and $\cong$ be a congruence relation on H. Then the natural map $\sigma_{\cong} : H \to [H :\cong]$ is an onto homomorphism.*

Proof. For all $a_i \in H$, where $1 \leq i \leq m$, we have:

$$\begin{aligned} \sigma_{\cong}(f^H(a_1, ..., a_m)) &= [f^H(a_1, ..., a_m)]_{\cong} \\ &= f^{[H:\cong]}([a_1]_{\cong}, ..., [a_m]_{\cong}) \\ &= f^{[H:\cong]}(\sigma_{\cong}(a_1), ..., \sigma_{\cong}(a_m)). \quad \square \end{aligned}$$

Theorem 3.7. *Let (H, f) be an m-ary semihypergroup and σ, ρ be two congruence relations on H such that $\rho \subseteq \sigma$. Then*

$$\sigma/\rho = \{(\rho(x), \rho(y)) \in H/\rho \times H/\rho : (x, y) \in \sigma\}$$

is a congruence on H/ρ and $(H/\rho)/(\sigma/\rho) \cong H/\sigma$.

Proof. It can be easily deduced that σ/ρ is an equivalence relation on H/ρ. Let we prove that it is a congruence. Let us suppose that $\rho(a_i)(\sigma/\rho)\rho(b_i)$ for all $1 \leq i \leq m$. Since σ is congruence on H, then $f(a_1^m)\sigma f(b_1^m)$ which implies that $\rho(f(a_1^m))(\sigma/\rho)\rho(f(b_1^m))$. Therefore, σ/ρ is a congruence on H/ρ.

We note that σ/ρ is the kernel of g where $g : H/\rho \to H/\sigma$. Then by Theorem 3.8, it follows that there is an isomorphism $q : (H/\rho)/(\sigma/\rho) \to H/\sigma$ defined by $q((\sigma/\rho)(\rho(a)) = \sigma(a)$ for all $a \in H$. $\square$

In the following we introduce the notion of k-pure hyperideals of m-ary semihypergroups and give a characterization of them.

Definition 3.5. Let (H, f) be an m-ary semihypergroup. A hyperideal A of H is called a *k-pure hyperideal* if for each $x \in A$, there exist elements $x_1, ..., x_{k-1}, x_{k+1}, ..., x_m \in A$ such that $x \in f(x_1, ..., x_{k-1}, x, x_{k+1}, ..., x_m)$.

Proposition 3.8. *Let (H, f) be an m-ary semihypergroup. Let A be a hyperideal of H. Then A is k-pure if and only if for any k-hyperideal B, $B \cap A = f(A, ..., \overset{(k)}{B}, ..., A)$.*

Proof. Suppose A is an k-pure hyperideal of H. For every k-hyperideal B of H, we have always $f(A, ..., \overset{(k)}{B}, ..., A) \subseteq B \cap A$. Let $x \in B \cap A$. Since A is an k-pure hyperideal, there exist $x_1, ..., x_{k-1}, x_{k+1}, ..., x_m \in A$ such that $x \in f(x_1, ..., x_{k-1}, x, x_{k+1}, ..., x_m)$. As $x \in B$ and $x_1, ..., x_{k-1}, x_{k+1}, ..., x_m \in A$, $x \in f(x_1, ..., x_{k-1}, x, x_{k+1}, ..., x_m) \subseteq f(A, ..., \overset{(k)}{B}, ..., A)$. Hence $x \in f(A, ..., \overset{(k)}{B}, ..., A)$. This implies that $B \cap A \subseteq f(A, ..., \overset{(k)}{B}, ..., A)$. Thus $B \cap A = f(A, ..., \overset{(k)}{B}, ..., A)$.

Conversely, assume $B \cap A = f(A, ..., \overset{(k)}{B}, ..., A)$, for any k-hyperideal B of H. We show that A is an k-pure hyperideal. Let $x \in A$ and $B = \{x\} \cup f(H, ..., \overset{(k)}{x}, ..., H)$ be the k-hyperideal of H generated by x. Then we have

$$\begin{aligned}(\{x\} \cup f(H,...,\overset{(k)}{x},...,H)) \cap A &= f(A,...,(\{x\} \cup f(H,...,\overset{(k)}{x},...,H)),...,A)\\ &= f(A,...,x,...,A) \cup f(A,...,f(H,...,\overset{(k)}{x},...,H),...,A)\\ &\subseteq f(A,...,x,...,A) \cup f(A,...,x,...,A)\\ &= f(A,...,x,...,A).\end{aligned}$$

Since $x \in (\{x\} \cup f(H,...,\overset{(k)}{x},...,H)) \cap A$, we have $x \in f(A,...,x,...,A)$. Hence there exist $x_1,...,x_{k-1},x_{k+1},...,x_m \in A$ such that $x \in f(x_1,...,x_{k-1},x,x_{k+1},...,x_m)$. Thus A is k-pure. □

Definition 3.6. Let (H,f) be an m-ary semihypergroup. A hyperideal A of H is called k-*weakly pure* if $A \cap B = f(A,...,B,...,A)$ for all hyperideals B of H.

4. On fuzzy m-ary semihypergroups

In this section, we introduce the notion of fuzzy m-ary semihypergroups and some basic properties of them are obtained.

A mapping $\mu : X \to [0,1]$, where X is an arbitrary non-empty set, is called a *fuzzy subset* in X. A fuzzy subset $\mu : X \to [0,1]$ is called a non-zero fuzzy subset if μ is not the constant map which assumes the value 0. For fuzzy subsets μ and ν of X, $\mu \leq \nu$ means that for all $a \in X, \mu(a) \leq \nu(a)$. For a fuzzy subset μ of X and $t \in [0,1]$, the set $\mu_t = \{x \in X | \mu(x) \geq t\}$ is called the level subset of μ. The characteristic function μ_X of X is a function which gives $\mu_X(a) = 1$ for all $a \in X$.

In [14], the author has obtained the followings:

Definition 4.1. A fuzzy m-ary hyperoperation on H is a map

$$f : \underbrace{H \times H \times .... \times H}_{m} \to F^*(H)$$

which associates a nonzero fuzzy subset $f(a_1^m)$ with any m-uple a_1^m of elements of H, where H is a nonempty set and $F^*(H)$ is the set of all non-zero fuzzy subsets of H.

The couple (H,f) is called a *fuzzy m-ary hypergroupoid*. We say that (H,f) is commutative if for all a_1^m of H and any permutation σ of H_n, we have $f(a_1^m) = f(a_{\sigma(1)}^{\sigma(m)})$.

A fuzzy nullary hyperoperation on H is just an element of $F^*(H)$, i.e. a nonzero fuzzy subset of H.

Definition 4.2. A fuzzy m-ary hypergroupoid (H,f) is called a fuzzy m-ary semihypergroup if for all a_1^{2m-1} of H, we have

$$f(a_1^{i-1}, f(a_i^{m+i-1}), a_{m+i}^{2m-1}) = f(a_1^{j-1}, f(a_j^{m+j-1}), a_{m+j}^{2m-1}),$$

where for any $\mu \in F^*(H)$ and any $r \in H$, we have

$$f(a_1^{i-1}, \mu, a_{m+i}^{2m-1})(r) = \bigvee_{t \in H} (f(a_1^{i-1}, t, a_{m+i}^{2m-1})(r) \wedge \mu(t)).$$

If A is a nonempty subset of H and $x_1^{i-1}, x_{i+1}^m \in H$, then for all $t \in H$ we have:

$$f(x_1^{i-1}, A, x_{i+1}^m)(t) = \bigvee_{a \in A} (f(x_1^{i-1}, a, x_{i+1}^m)(t).$$

If A is a nonempty subset of H, then we denote the characteristic function of A by χ_A. If $A = H$, then for all $t \in H$, we have $\chi_H(t) = 1$.

Let us consider now (H, f) a fuzzy m-ary hypergroupoid. We define the following hyperoperation:

$$\forall a_1^m \in H, \varphi_f(a_1^m) = \{x \in H | f(a_1^m)(x) > 0\}.$$

If (H, φ) is an m-ary hypergroupoid, then we can define the following fuzzy m-ary hyperoperation:

$$\forall a_1^m \in H, f_\varphi(a_1^m) = \chi_{\varphi(a_1^m)}.$$

Then (H, φ_f) is an m-ary hypergroupoid, which is called the associated m-ary hypergroupoid of (H, f). Also, (H, f_φ) is a fuzzy m-ary hypergroupoid, which is called the fuzzy associated m-ary hypergroupoid of (H, φ).

We have the following theorem [14] :

Theorem 4.1. *The following statements hold true:*

(1) *If (H, f) is a fuzzy m-ary semihypergroup, then (H, φ_f) is an m-ary semihypergroup.*
(2) *If (H, φ) is an m-ary semihypergroup, then (H, f_φ) is a fuzzy m-ary semihypergroup.*

According to the above theorem, any m-ary semihypergroup can be seen as a fuzzy m-ary semihypergroup.

Let H_1 and H_2 be two fuzzy m-ary semihypergroups and $h : H_1 \to H_2$ be a map. If μ is a fuzzy subset on H_1, then we define $h(\mu) : H_2 \to [0, 1]$, as follows:

$$(h(\mu))(t) = \bigvee_{r \in h^{-1}(t)} \mu(r), \text{ if } h^{-1}(t) \neq \emptyset,$$

otherwise we consider $(h(\mu))(t) = 0$.

Also, if $h : H_1 \to H_2$ is a map and $a \in H_1$, then $h(\chi_a) = \chi_{h(a)}$.

Definition 4.3. Let (H_1, f_1) and (H_2, f_2) be two fuzzy m-ary semihypergroups and $h : H_1 \to H_2$ be a map. We say that h is a homomorphism of fuzzy m-ary semihypergroups if for all a_1^m of H_1, we have

$$h(f_1(a_1^m)) \leq f_2(h(a_1), ..., h(a_m))$$

The following theorems present connections between fuzzy m-ary semihypergroup homomorphisms and m-ary semihypergroup homomorphisms [14].

Theorem 4.2. *Let (H_1, f_1) and (H_2, f_2) be two fuzzy m-ary semihypergroups and (H_1, φ_{f_1}), (H_2, φ_{f_2}) be the associated m-ary semihypergroups. If $h : H_1 \to H_2$ is a homomorphism of fuzzy m-ary semihypergroups, then h is a homomorphism of the associated m-ary semihypergroups, too.*

Theorem 4.3. *Let (H_1, φ_1) and (H_2, φ_2) be two m-ary semihypergroups and (H_1, f_{φ_1}), (H_2, f_{φ_2}) be the associated fuzzy m-ary semihypergroups. The map $h : H_1 \to H_2$ is a homomophism of m-ary semihypergroups iff it is a homomorphism of the associated fuzzy m-ary semihypergroups.*

Definition 4.4. Let (H, f) be a fuzzy m-ary semihypergroup and (H, φ_f) be the associated m-ary semihypergroup. A nonempty subset S is called a subfuzzy m-ary semihypergroup if for all $s_1^m \in S$, the following conditions hold:

(1) if $s_1^m \in S$ and $f(s_1^m)(x) > 0$, then $x \in S$;

(2) (S, f) is a fuzzy m-ary semihypergroup.

The next two theorems point out on the connections between subfuzzy m-ary semihypergroups and m-ary subsemihypergroups.

Theorem 4.4. *If (S, f) is a subfuzzy m-ary semihypergroup of (H, f), then (S, φ_f) is an m-ary subsemihypergroup of (H, φ_f).*

Theorem 4.5. *(S, φ) is an m-ary subsemihypergroup of (H, φ) if and only if (S, f_φ) is a subfuzzy m-ary semihypergroup of (H, f_φ).*

Let $\mu_1, ..., \mu_m$ be fuzzy subsets of X. With $\min\{\mu_1, ..., \mu_m\}$ and $\sup\{\mu_1, ..., \mu_m\}$ we will mean the following fuzzy subsets of X:

$$\begin{aligned} \min\{\mu_1, ..., \mu_m\}(a) &= \min\{\mu_1(a), ..., \mu_m(a)\} \\ \sup\{\mu_1, ..., \mu_m\}(a) &= \sup\{\mu_1(a), ..., \mu_m(a)\} \end{aligned}$$

for all $a \in X$.

If $\mu_1, ..., \mu_m$ are fuzzy subsets of an m-ary semihypergroup H and x be an element of H, then

$$(\mu_1 \circ ... \circ \mu_m)(x) = \begin{cases} \sup\limits_{x \in f(a_1^m)} \min\{\mu_1(a_1), ..., \mu_m(a_m)\}, & \text{if } x \in f(a_1^m) \\ 0, & \text{otherwise} \end{cases}$$

Lemma 4.6. *For any non-empty subsets $X_1, ..., X_m$ of an m-ary semihypergroup (H, f), we have*
(1) $\mu_{X_1} \circ ... \circ \mu_{X_m} = \mu_{f(X_1,...,X_m)}$
(2) $\min\{\mu_{X_1}, ..., \mu_{X_m}\} = \mu_{X_1 \cap ... \cap X_m}$.

Proof. Proof is straightforward. □

Definition 4.5. Let (H, f) be an m-ary semihypergroup. A fuzzy subset μ of H is called a *fuzzy m-ary subsemihypergroup* of H if $\inf\limits_{t \in f(x_1,...,x_m)} \mu(t) \geq \min\{\mu(x_1), ..., \mu(x_m)\}$ for all $x_1, ..., x_m \in H$.

Definition 4.6. Let (H, f) be an m-ary semihypergroup. A fuzzy subset μ of H is called a *fuzzy k-hyperideal* of H if

$$\inf_{t \in f(x_1,...,x_m)} \mu(t) \geq \mu(x_k) \text{ for all } x_1, ..., x_m \in H.$$

μ is called a *fuzzy hyperideal* of H if μ is a fuzzy k-hyperideal of H for every $k = 1, 2, ..., m$ or equivalently if

$$\inf_{t \in f(x_1^m)} \mu(t) \geq \max\{f(x_1), ..., f(x_m)\} \text{ for all } x_1, ..., x_m \in H.$$

Theorem 4.7. *Let (H, f) be an m-ary semihypergroup. A fuzzy subset μ of H is a fuzzy hyperideal if and only if every non-empty level subset is a hyperideal of H.*

Proof. Let us suppose that μ is a fuzzy hyperideal of H and let μ_t is a level subset of μ. If $x_1, ..., x_m \in \mu_t$ for some $t \in [0, 1]$, then for the definition, we have $\mu(x_1) \geq t, ..., \mu(x_m) \geq t$. Thus, $\min\{\mu(x_1), ..., \mu(x_m)\} \geq t$. From this we get:

$$\inf_{z \in f(x_1^m)} \mu(z) \geq \min\{\mu(x_1), ..., \mu(x_m)\} \geq t.$$

So, we get $\mu(z) \geq t$, for all $z \in f(x_1^m)$. Therefore $f(x_1, ..., x_m) \subseteq \mu_t$.

Conversely, let us suppose that every non-empty level subset μ_t is a hyperideal of H. Let $t_0 = \min\{\mu(x_1), ..., \mu(x_m)\}$ for all $x_1, ..., x_m \in H$. Then we have $\mu(x_1) \geq t_0, ..., \mu(x_m) \geq t_0$. Thus $x_1, ..., x_m \in \mu_{t_0}$. We obtain also that $f(x_1, ..., x_m) \subseteq \mu_{t_0}$. Therefore, $\min\{\mu(x_1), ..., \mu(x_m)\} = t_0 \leq \inf\limits_{z \in f(x_1^m)} \mu(z)$. □

Theorem 4.8. *Let (H, f) be an m-ary semihypergroup and I be a non-empty subset of H. Let μ_I be a fuzzy set defined by:*

$$\mu_I(x) = \begin{cases} s, & \text{if } x \in I \\ t, & \text{otherwise,} \end{cases}$$

where $0 \leq t < s \leq 1$. Then μ_I is a fuzzy k-hyperideal of H if and only if I is a k-hyperideal of H.

Corollary 4.9. *Let (H, f) be an m-ary semihypergroup and μ be a fuzzy set of H such that its upper bound is t_0. Then the followings are equivalent:*
(1) *μ is a fuzzy hyperideal of H*
(2) *Every non-empty level subset of μ is a hyperideal of H.*
(3) *Every level subset μ_t is a hyperideal of H where $t \in [0, t_0]$.*

Theorem 4.10. *Let (H, f) be an m-ary semihypergroup and A a non-empty subset of H. The following statements hold true:*
(1) *A is an m-ary subsemihypergroup of H if and only if μ_A is a fuzzy m-ary subsemihypergroup of H.*
(2) *A is a hyperideal of H if and only if μ_A is a fuzzy hyperideal of H.*

In the following two sections, we introduce and study some classes of fuzzy hyperideals that of pure fuzzy, weakly pure fuzzy hyperideals in m-ary semihypergroups and some properties of them are investigated. We identify those m-ary semihypergroups for which every fuzzy hyperideal is idempotent. We also characterize the m-ary semihypergroups for which every fuzzy hyperideal is weakly pure fuzzy.

5. On pure fuzzy hyperideals in m-ary semihypergroups

Definition 5.1. A fuzzy hyperideal λ of an m-ary semihypergroup (H, f) is called a pure fuzzy hyperideal of H if $\mu \wedge \lambda = \lambda \circ ....\circ \overset{(k)}{\mu} \circ ... \circ \lambda$ for all fuzzy k-hyperideals μ of H.

Proposition 5.1. *Let I be a hyperideal of an m-ary semihypergroup (H, f). Then the following statements are equivalent:*
(1) *I is an k-pure hyperideal in H,*
(2) *The characteristic function of I, denoted by δ_I is pure fuzzy hyperideal of H.*

Proof. (1) ⇒ (2). Let us suppose that I is k-pure in H. Since I is a hyperideal of H, δ_I is obviously a fuzzy hyperideal of H. To prove that δ_I is pure fuzzy, we show that for any fuzzy k-hyperideal γ of H, $\gamma \wedge \delta_I = \underbrace{\delta_I \circ ....\circ \overset{(k)}{\gamma} \circ ... \circ \delta_I}_{m}$. Let $a \in H$. Then

$$\begin{aligned}
(\delta_I \circ ....\circ \overset{(k)}{\gamma} \circ ... \circ \delta_I)(a) &= \bigvee_{a\in f(x_1^m)} \{\delta_I(x_1) \wedge ....\wedge \overset{(k)}{\gamma} (x_k) \wedge ... \wedge \delta_I(x_m)\} \\
&\leq \bigvee_{a\in f(x_1^m)} \{ \bigwedge_{t\in f(x_1^m)} \{\delta_I(t), ..., \overset{(k)}{\gamma} (t), ..., \delta_I(t)\}\} \\
&= \bigvee_{a\in f(x_1^m)} \{\delta_I(a) \wedge ....\wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \delta_I(a)\} \\
&= \delta_I(a) \wedge ....\wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \delta_I(a) \\
&= (\delta_I \wedge ....\wedge \overset{(k)}{\gamma} \wedge ... \wedge \delta_I)(a).
\end{aligned}$$

This implies that $(\delta_I \circ ....\circ \overset{(k)}{\gamma} \circ ...\circ \delta_I)(a) \leq (\gamma \wedge \delta_I)(a)$, so $(\delta_I \circ ....\circ \overset{(k)}{\gamma} \circ ...\circ \delta_I) \leq \gamma \wedge \delta_I$. We have $(\delta_I \wedge ....\wedge \overset{(k)}{\gamma} \wedge ...\wedge \delta_I)(a) = (\delta_I(a) \wedge ....\wedge \overset{(k)}{\gamma} (a) \wedge ...\wedge \delta_I(a)) = 0$ if $a \notin I$. Thus $(\delta_I \wedge ....\wedge \overset{(k)}{\gamma} \wedge ...\wedge \delta_I)(a) = 0 \leq (\delta_I \circ ....\circ \overset{(k)}{\gamma} \circ ...\circ \delta_I)(a)$. Let we consider the case when $a \in I$. Since I is k-pure, then for each $a \in I$, there exist $x_1, ...x_{k-1}, x_{k+1}, ..., x_m \in I$ such that $a \in f(x_1, ..., a, ..., x_m)$. Since $x_1, ...x_{k-1}, x_{k+1}, ..., x_m \in I, \delta_I(x_i) = 1, i = 1, ..., k-1, k+1, ..., m$. Therefore

$$\begin{aligned}
(\delta_I \wedge ....\wedge \overset{(k)}{\gamma} \wedge ... \wedge \delta_I)(a) &= (\delta_I(a) \wedge ....\wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \delta_I(a)) \\
&= (\delta_I(x_1) \wedge ....\wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \delta_I(x_m)) \\
&\leq \bigvee_{a\in f(x_1,...,a,...,x_m)} (\delta_I(x_1) \wedge ....\wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \delta_I(x_m)) \\
&= (\delta_I \circ ....\circ \overset{(k)}{\gamma} \circ ... \circ \delta_I)(a) \\
\delta_I \wedge ....\wedge \overset{(k)}{\gamma} \wedge ... \wedge \delta_I &\leq \delta_I \circ ....\circ \overset{(k)}{\gamma} \circ ... \circ \delta_I.
\end{aligned}$$

Thus $\delta_I \wedge ....\wedge \overset{(k)}{\gamma} \wedge ... \wedge \delta_I = \delta_I \circ ....\circ \overset{(k)}{\gamma} \circ ... \circ \delta_I$. This implies that $\gamma \wedge \delta_I = \delta_I \circ ....\circ \overset{(k)}{\gamma} \circ ... \circ \delta_I$.

(2) $\Rightarrow$ (1). Let us suppose that δ_I is pure fuzzy hyperideal in H. We show that I is k-pure in H. That is for each k-hyperideal J of H, $J \cap I = f(I, ..., \overset{(k)}{J}, ..., I)$. Since J is an k-hyperideal of H, the characteristic function δ_J of J is a fuzzy k-hyperideal of H. Since δ_I is pure fuzzy, we have $\delta_J \wedge \delta_I = \delta_I \circ ...\circ \overset{(k)}{\delta}_J \circ ... \delta_I$. This implies that $\delta_{J\cap I} = \delta_{f(I,...,\overset{(k)}{J},...,I))}$. Hence I is k-pure. □

Proposition 5.2. *Let (H, f) be an m-ary semihypergroup. The following assertions are true:*

(1) *The fuzzy hyperideals φ and μ_H of H, defined respectively as*

$$\varphi(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

and $\mu_H(x) = 1$ for all $x \in H$, are pure fuzzy hyperideals of H.

(2) *If* $\{\lambda_i : i \in I\}$ *is a family of pure fuzzy hyperideals of* H, *then so is* $\bigvee_{i\in I} \lambda_i$.

(3) *If* λ_1 *and* λ_2 *are pure fuzzy hyperideals of* H, *then so is* $\lambda_1 \wedge \lambda_2$.

Proof. (1) We show that for any fuzzy k-hyperideal γ of H we have $\gamma \wedge \varphi = \varphi \circ ... \circ \overset{(k)}{\gamma} \circ ... \circ \varphi$ and $\gamma \wedge \mu_H = \mu_H \circ ... \circ \overset{(k)}{\gamma} \circ ... \circ \mu_H$. For $a \in H$, we have

$$\begin{aligned}
(\varphi \circ ... \circ \overset{(k)}{\gamma} \circ ... \circ \varphi)(a) &= \bigvee_{a\in f(x_1^m)} (\varphi(x_1) \wedge ... \wedge \overset{(k)}{\gamma} (x_k) \wedge ... \wedge \varphi(x_m)) \\
&\le \bigvee_{a\in f(x_1^m)} (\bigwedge_{t\in f(x_1^m)} (\varphi(t), ..., \overset{(k)}{\gamma} (t), ..., \varphi(t))) \\
&= \bigvee_{a\in f(x_1^m)} (\varphi(a) \wedge ... \wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \varphi(a)) \\
&= \varphi(a) \wedge ... \wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \varphi(a) \\
&= (\varphi \wedge ... \wedge \overset{(k)}{\gamma} \wedge ... \wedge \varphi)(a) = (\gamma \wedge \varphi)(a).
\end{aligned}$$

This implies that $(\varphi \circ ... \circ \overset{(k)}{\gamma} \circ ... \circ \varphi) \le \gamma \wedge \varphi$. If $a \neq 0$, then

$$\begin{aligned}
(\gamma \wedge \varphi)(a) &= (\varphi \wedge ... \wedge \overset{(k)}{\gamma} \wedge ... \wedge \varphi)(a) = (\varphi(a) \wedge ... \wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \varphi(a)) \\
&= (0 \wedge ... \wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge 0) = 0 \le (\varphi \circ ... \circ \overset{(k)}{\gamma} \circ ... \circ \varphi)(a) \\
(\gamma \wedge \varphi)(a) &\le (\varphi \circ ... \circ \overset{(k)}{\gamma} \circ ... \circ \varphi)(a).
\end{aligned}$$

If $a = 0$, then

$$\begin{aligned}
(\gamma \wedge \varphi)(0) &= (\gamma(0) \wedge \varphi(0)) \le \bigvee_{0\in f(x_1^m)} (\varphi(x_1) \circ ... \circ \overset{(k)}{\gamma} (x_k) \circ ... \circ \varphi(x_m)) \\
&= (\varphi \circ ... \circ \overset{(k)}{\gamma} \circ ... \circ \varphi)(0) = 0.
\end{aligned}$$

This implies that $(\gamma \wedge \varphi) \le (\varphi \circ ... \circ \overset{(k)}{\gamma} \circ ... \circ \varphi)$. Hence, $\gamma \wedge \varphi = \varphi \circ ... \circ \overset{(k)}{\gamma} \circ ... \circ \varphi$. Let us consider,

$$\begin{aligned}
(\mu_H \circ ... \circ \overset{(k)}{\gamma} \circ ... \circ \mu_H)(a) &= \bigvee_{a\in f(x_1^m)} (\mu_H(x_1) \wedge ... \wedge \overset{(k)}{\gamma} (x_k) \wedge ... \wedge \mu_H(x_m)) \\
&\le \bigvee_{a\in f(x_1^m)} (\bigwedge_{t\in f(x_1^m)} (\mu_H(t), ..., \overset{(k)}{\gamma} (t), ..., \mu_H(t))) \\
&= \bigvee_{a\in f(x_1^m)} (\mu_H(a) \wedge ... \wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \mu_H(a)) \\
&= \mu_H(a) \wedge ... \wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \mu_H(a) \\
&= (\mu_H \wedge ... \wedge \overset{(k)}{\gamma} \wedge ... \wedge \mu_H)(a) = (\gamma \wedge \mu_H)(a).
\end{aligned}$$

This implies that $\mu_H \circ ...\circ \overset{(k)}{\gamma} \circ ... \circ \mu_H \leq \gamma \wedge \mu_H$. Also

$$\begin{aligned}
(\gamma \wedge \mu_H)(a) &= (\mu_H \wedge ...\wedge \overset{(k)}{\gamma} \wedge ... \wedge \mu_H)(a) = (\mu_H(a) \wedge ...\wedge \overset{(k)}{\gamma} (a) \wedge ... \wedge \mu_H(a)) \\
&\leq \bigvee_{a\in f(x_1^m)} (\mu_H(x_1) \wedge ...\wedge \overset{(k)}{\gamma} (x_k) \wedge ... \wedge \mu_H(x_m)) \\
&= (\mu_H \circ ...\circ \overset{(k)}{\gamma} \circ ... \circ \mu_H)(a).
\end{aligned}$$

Hence, $\gamma \wedge \mu_H = \mu_H \circ ...\circ \overset{(k)}{\gamma} \circ ... \circ \mu_H$.

(2) Let $\{\lambda_i : i \in I\}$ is a family of pure fuzzy hyperideals of H. We have to show that $\bigvee\limits_{i\in I} \lambda_i$ is also a pure fuzzy hyperideal of H. That is, we have to show that for any fuzzy k-hyperideal μ of H, $\mu \wedge (\bigvee\limits_{i\in I} \lambda_i) = (\bigvee\limits_{i\in I} \lambda_i) \circ ...\circ \overset{(k)}{\mu} \circ ... \circ (\bigvee\limits_{i\in I} \lambda_i)$. Now for each $a \in H$, we have

$$\begin{aligned}
&\left[(\bigvee_{i\in I} \lambda_i) \circ ...\circ \overset{(k)}{\mu} \circ ... \circ (\bigvee_{i\in I} \lambda_i)\right] \\
&= \bigvee_{a\in f(x_1^m)} \left[(\bigvee_{i\in I} \lambda_i)(x_1) \wedge ...\wedge \overset{(k)}{\mu} (x_k) \wedge ... \wedge (\bigvee_{i\in I} \lambda_i)(x_m)\right] \\
&\leq \bigvee_{a\in f(x_1^m)} \left[\bigwedge_{t\in f(x_1^m)} ((\bigvee_{i\in I} \lambda_i)(t), ..., \overset{(k)}{\mu} (t), ..., (\bigvee_{i\in I} \lambda_i)(t)\right] \\
&= \bigvee_{a\in f(x_1^m)} \left[(\bigvee_{i\in I} \lambda_i)(a) \wedge ...\wedge \overset{(k)}{\mu} (a) \wedge ... \wedge (\bigvee_{i\in I} \lambda_i)(a)\right] \\
&= (\bigvee_{i\in I} \lambda_i)(a) \wedge ...\wedge \overset{(k)}{\mu} (a) \wedge ... \wedge (\bigvee_{i\in I} \lambda_i)(a) \\
&= \left[\mu \wedge (\bigvee_{i\in I} \lambda_i\right] (a).
\end{aligned}$$

This implies that

$$(\bigvee_{i\in I} \lambda_i) \circ ...\circ \overset{(k)}{\mu} \circ ... \circ (\bigvee_{i\in I} \lambda_i) \leq \mu \wedge (\bigvee_{i\in I} \lambda_i).$$

Also,

$$\begin{aligned}
\left[\mu \wedge (\bigvee_{i\in I} \lambda_i)\right] (a) &= \left[(\bigvee_{i\in I} \lambda_i) \wedge ...\wedge \overset{(k)}{\mu} \wedge ... \wedge (\bigvee_{i\in I} \lambda_i)\right] (a) \\
&= \left[(\bigvee_{i\in I} \lambda_i)(a) \wedge ...\wedge \overset{(k)}{\mu} (a) \wedge ... \wedge (\bigvee_{i\in I} \lambda_i)(a)\right]
\end{aligned}$$

$$= \bigvee_{i\in I}\left[\lambda_i(a)\wedge ...\wedge \overset{(k)}{\mu}(a)\wedge ...\wedge \lambda_i(a)\right]$$
$$= \bigvee_{i\in I}\left[(\lambda_i\circ ...\circ \overset{(k)}{\mu}\circ ...\circ \lambda_i)(a)\right].$$

We have,

$$\begin{aligned}(\lambda_i\circ ...\circ \overset{(k)}{\mu}\circ ...\circ \lambda_i)(a) &= \bigvee_{a\in f(x_1^m)}\left[\lambda_i(x_1)\wedge ...\wedge \overset{(k)}{\mu}(x_k)\wedge ...\wedge \lambda_i(x_m)\right]\\ &\leq \bigvee_{a\in f(x_1^m)}\left[(\bigvee_{i\in I}\lambda_i)(x_1)\wedge ...\wedge \overset{(k)}{\mu}(x_k)\wedge ...\wedge (\bigvee_{i\in I}\lambda_i)(x_m)\right]\\ &= \left[(\bigvee_{i\in I}\lambda_i)\circ ...\circ \overset{(k)}{\mu}\circ ...\circ (\bigvee_{i\in I}\lambda_i)\right](a).\end{aligned}$$

This implies that

$$\left[(\bigvee_{i\in I}\lambda_i)\wedge ...\wedge \overset{(k)}{\mu}\wedge ...\wedge (\bigvee_{i\in I}\lambda_i)\right](a)\leq \left[(\bigvee_{i\in I}\lambda_i)\circ ...\circ \overset{(k)}{\mu}\circ ...\circ (\bigvee_{i\in I}\lambda_i)\right](a).$$

This implies that

$$\left[\mu\wedge (\bigvee_{i\in I}\lambda_i)\right]\leq \left[(\bigvee_{i\in I}\lambda_i)\circ ...\circ \overset{(k)}{\mu}\circ ...\circ (\bigvee_{i\in I}\lambda_i)\right].$$

Hence,

$$\left[\mu\wedge (\bigvee_{i\in I}\lambda_i)\right]= \left[(\bigvee_{i\in I}\lambda_i)\circ ...\circ \overset{(k)}{\mu}\circ ...\circ (\bigvee_{i\in I}\lambda_i)\right].$$

(3) Let λ_1 and λ_2 be pure fuzzy hyperideals of H. Then $\mu\wedge\lambda_1 = \lambda_1\circ ...\circ \overset{(k)}{\mu}\circ ...\circ\lambda_1$ and $\mu\wedge\lambda_2 = \lambda_2\circ ...\circ \overset{(k)}{\mu}\circ ...\circ\lambda_2$ for all fuzzy k-hyperideals of H. We have to show that

$$(\lambda_1\wedge\lambda_2)\wedge ...\wedge \overset{(k)}{\mu}\wedge ...\wedge (\lambda_1\wedge\lambda_2) = (\lambda_1\wedge\lambda_2)\circ ...\circ \overset{(k)}{\mu}\circ ...\circ (\lambda_1\wedge\lambda_2).$$

Since λ_2 is a pure fuzzy hyperideal of H. It follows that $\lambda_1\wedge\lambda_2 = \lambda_2\circ ...\circ\lambda_1\circ ...\circ\lambda_2$. Therefore,

$$(\lambda_1\wedge\lambda_2)\circ ...\circ \overset{(k)}{\mu}\circ ...\circ (\lambda_1\wedge\lambda_2) = (\lambda_2\circ ...\circ\lambda_1\circ ...\circ\lambda_2)\circ ...\circ \overset{(k)}{\mu}\circ ...\circ (\lambda_2\circ ...\circ\lambda_1\circ ...\circ\lambda_2).$$

Since $\lambda_2\circ ...\circ\lambda_1\circ ...\circ\lambda_2$ is a fuzzy hyperideal of H, so

$$(\lambda_1\wedge\lambda_2)\circ ...\circ \overset{(k)}{\mu}\circ ...\circ (\lambda_1\wedge\lambda_2) = \mu\wedge (\lambda_2\circ ...\circ\lambda_1\circ ...\circ\lambda_2) = \mu\wedge (\lambda_1\wedge\lambda_2).$$

Thus we have $\mu\wedge (\lambda_1\wedge\lambda_2) = (\lambda_1\wedge\lambda_2)\circ ...\circ \overset{(k)}{\mu}\circ ...\circ (\lambda_1\wedge\lambda_2)$. Hence, $(\lambda_1\wedge\lambda_2)$ is a pure fuzzy hyperideal of H. $\square$

Theorem 5.3. *Let H be an m-ary semihypergroup. The following assertions are equivalent:*

(1) *H is k-weakly regular.*
(2) *Every fuzzy k-hyperideal of H is idempotent.*
(3) $\lambda \wedge \mu = \lambda \circ ... \circ \overset{(k)}{\mu} \circ ... \circ \lambda$ *for every fuzzy k-hyperideal μ and for every fuzzy hyperideal λ of H.*

Proof. (1) $\Rightarrow$ (2). Let δ be a fuzzy k-hyperideal of H. We prove that $\underbrace{\delta \circ ... \circ \delta}_{m} = \delta$. Let $x \in H$. Then

$$(\delta\circ...\circ\delta)(x) = \bigvee_{x\in f(x_1^m)} (\delta(x_1)\wedge...\wedge\delta(x_m)) \leq \bigvee_{x\in f(x_1^m)} (\bigwedge_{t\in f(x_1^m)} (\delta(x_1), ..., \delta(t), ..., \delta(x_m)))$$

where $x_1, ..., x_m \in H$ and H is k-weakly regular. Thus there exist $x_{11}^{1m}, ..., x_{m1}^{mm} \in H$ such that $x \in f(f(x_{11}, ..., x_{1,k-1}, x, x_{1,k+1}, ..., x_{1m}), ..., f(x_{m1}, ..., x, ..., x_{mm}))$. So

$$\delta(x) \leq \bigvee_{x\in f(w_1^m)} (\delta(w_1) \wedge ... \wedge \delta(w_m)) = (\delta \circ ... \circ \delta)(x).$$

Thus $\delta \leq \delta \circ ... \circ \delta$. Hence $\delta = \delta \circ ... \circ \delta$.

(2) $\Rightarrow$ (1). Let $x \in H$. We show that $x \in H, x \in f(f(H, ..., \overset{(k)}{x}, ..., H), ..., f(H, ..., \overset{(k)}{x}, ..., H))$. Let $A = x \cup (f(H, ..., \overset{(k)}{x}, ..., H)$ be the k-hyperideal generated by x. Let δ_A be the characteristic function of A and it is a fuzzy k-hyperideal of H, hence by (2), $\delta_A = \delta_A \circ ... \circ \delta_A = \delta_{f(A,...,A)}$. This implies that $A = f(A, ..., A)$. Since $x \in A$, it follows that $x \in f(A, ..., A) = f(x \cup f(H, ..., \overset{(k)}{x}, ..., H), ..., x \cup f(H, ..., \overset{(k)}{x}, ..., H))$. This implies that

$$\begin{aligned} x &\in f(x \cup f(H, ..., \overset{(k)}{x}, ..., H), ..., x \cup f(H, ..., \overset{(k)}{x}, ..., H)) \\ &\subseteq f(f(H, ..., \overset{(k)}{x}, ..., H), ..., f(H, ..., \overset{(k)}{x}, ..., H)). \end{aligned}$$

This implies that $x \in f(f(H, ..., \overset{(k)}{x}, ..., H), ..., f(H, ..., \overset{(k)}{x}, ..., H))$. Hence H is k-weakly regular.

(1) $\Rightarrow$ (3). Let λ be a fuzzy hyperideal and μ a fuzzy k-hyperideal of H. We show that $\lambda \wedge \mu = \lambda \circ ... \circ \overset{(k)}{\mu} \circ ... \circ \lambda$. Since $\lambda \circ ... \circ \overset{(k)}{\mu} \circ ... \circ \lambda \leq \mu_H \circ ... \mu_H \circ \lambda \leq \lambda$. Also $\lambda \circ ... \circ \overset{(k)}{\mu} \circ ... \circ \lambda \leq \mu \circ \mu_H \circ ... \circ \mu_H \leq \mu$. This implies that $\lambda \circ ... \circ \overset{(k)}{\mu} \circ ... \circ \lambda \leq \mu \wedge \lambda$. Now we show that $\mu \wedge \lambda \leq \lambda \circ ... \circ \overset{(k)}{\mu} \circ ... \circ \lambda$. Let $x \in H$ and since H is k-weakly regular, so there exist $t_1^m, ..., s_1^m \in H$ such that $x \in f(f(t_1, ..., x, ..., t_m), ..., f(s_1, ..., x, ..., s_m))$. Thus

$$\begin{aligned} (\mu \wedge \lambda)(x) &= (\lambda(x) \wedge ... \wedge \mu(x) \wedge ... \wedge \lambda(x)) \\ &\leq \bigwedge_{h_1\in f(t_1,...,x,...,t_m),...,h_m\in f(s_1,...,x,...,s_m)} (\lambda(h_1) \wedge ... \wedge \mu(h_k) \wedge ... \wedge \lambda(h_m)) \\ &\leq \bigvee_{x\in f(x_1^m)} (\lambda(x_1) \wedge ... \wedge \mu(x_k) \wedge ... \wedge \lambda(x_k)) = (\lambda \circ ... \circ \overset{(k)}{\mu} \circ ... \circ \lambda)(x). \end{aligned}$$

This implies that $(\mu \wedge \lambda) \leq (\lambda \circ ... \circ \overset{(k)}{\mu} \circ ... \circ \lambda)$. Thus $(\mu \wedge \lambda) = (\lambda \circ ... \circ \overset{(k)}{\mu} \circ ... \circ \lambda)$. Hence λ is pure fuzzy hyperideal.

(3) $\Rightarrow$ (1). We show that H is k-weakly regular. Let $x \in H$ and let $A = \{x\} \cup f(x, H, ..., H) \cup f(H, x, ..., H) \cup ... \cup f(H, ..., x, ..., H) \cup ... \cup f(H, ..., H, x)$ be the hyperideal generated by x. Let δ_A be the characteristic function of A. Then δ_A is fuzzy hyperideal of H, δ_A is pure fuzzy. Thus by the Proposition 5.2, A is k-pure in H. Since $x \in A$ and A is k-pure in H, therefore there exist $x_1, ..., x_{k-1}, x_{k+1}, .., x_m \in A$ such that $x \in f(x_1, ..., x_{k-1}, x, x_{k+1}, ..., x_m)$. This means that

$$\begin{aligned} x &\in f(A, ..., x, ..., A) \\ &= f(\{x\} \cup f(x, H, ..., H) \cup f(H, x, ..., H) \cup ... \cup f(H, ..., x, ..., H) \\ &\quad \cup ... \cup f(H, ..., H, x), ..., x, ..., \{x\} \cup f(x, H, ..., H) \cup f(H, x, ..., H) \\ &\quad \cup ... \cup f(H, ..., x, ..., H) \cup ... \cup f(H, ..., H, x)) \end{aligned}$$

Routine calculations show that $x \in f(f(H, ..., x, ..., H), ..., f(H, ..., x, ..., H))$. Hence H is k-weakly regular m-ary semihypergroup. □

Theorem 5.4. *Let H be an m-ary semihypergroup. The following assertions are equivalent:*

(1) *H is k-weakly regular.*

(2) *Every fuzzy hyperideal λ of H is pure fuzzy.*

Proof. The proof follows from the Theorem 5.4 and Proposition 5.2 □

6. On weakly pure fuzzy hyperideals in m-ary semihypergroups

Definition 6.1. A hyperideal λ of an m-ary semihypergroup H is called k-weakly pure fuzzy if $\lambda \wedge \mu = \lambda \circ ...\circ \overset{(k)}{\mu} \circ.... \circ \lambda$ for all fuzzy hyperideals μ of H.

Proposition 6.1. *Let H be an m-ary semihypergroup. The following assertions are equivalent:*

(1) *Every fuzzy hyperideal of H is k-weakly pure fuzzy.*

(2) *Every fuzzy hyperideal of H is idempotent.*

Proof. (1) $\Rightarrow$ (2). Let us suppose that every fuzzy hyperideal of H is k-weakly pure fuzzy. Let λ be a fuzzy hyperideal of H. Then for every fuzzy hyperideal μ of H, we have $\lambda \wedge \mu = \lambda \circ ... \circ \mu \circ ... \circ \lambda$. In particular $\lambda = \lambda \wedge \lambda = \lambda \circ ... \circ \lambda$. Hence every fuzzy hyperideal of H is idempotent.

(2) $\Rightarrow$ (1). Let us suppose that every fuzzy hyperideal of H is idempotent. Let λ be a fuzzy hyperideal of H, then for any fuzzy hyperideal μ of H, we always have $\lambda \circ ... \circ \mu \circ ... \circ \lambda \leq \lambda \wedge \mu$. On the other hand, $(\lambda \wedge \mu) = (\lambda \wedge \mu) \circ ... \circ (\lambda \wedge \mu) \leq \lambda \circ ... \circ \mu \circ ... \circ \lambda$. Thus we have $\lambda \wedge \mu = \lambda \circ ... \circ \mu \circ ... \circ \lambda$. Hence λ is k-weakly pure fuzzy. □

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(A. Naço) Department of Mathematical Egineering, Polytechnic University of Tirana, Albania
E-mail address: a.naco@fimif.edu.al

(A. Peposhi) Department of Mathematical Egineering, Polytechnic University of Tirana, Albania
E-mail address: a.peposhi@fimif.edu.al

(K. Hila) Department of Mathematical Egineering, Polytechnic University of Tirana, Albania
E-mail address: kostaq_hila@yahoo.com

(S. Onar) Yildiz Technical University, Department of Mathematical Engineering, Davutpaşa-Istanbu, Turkey
E-mail address: serkan10ar@gmail.com

(B.A. Ersoy) Yildiz Technical University, Department of Mathematics, Davutpaşa-Istanbu, Turkey
E-mail address: ersoya@gmail.com