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Fuzzy associative $\mathscr{I}$-ideals of IS-algebras with t-norms

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Abstract. In this paper we introduce the notion of T-fuzzy associative $\mathscr{I}$-ideals in an IS-algebra, and investigate some of their properties. We discuss the properties of homomorphic image and inverse image of T-fuzzy associative $\mathscr{I}$-ideals in an IS-algebra. Connections between direct product and T-product of fuzzy associative $\mathscr{I}$-ideals induced by t-norms are also studied.

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1. Introduction

The notion of BCK-algebras was first introduced by K. Iseki [9] which is a subclass of BCI-algebras [8] introduced by Y.Imai and K.Iseki. This notion is originated from two different ways: one of the motivation is based on set theory, another motivation is from classical and non-classical propositional calculus. Jun et al.[10] introduced a class of algebras related to BCI-algebras and semigroups, called a *BCI-semigroup*. For the convenience of study, Jun et al. [11] renamed the BCI-semigroup as an *IS-algebra* and studied some of its properties.

The notion of fuzzy sets was first introduced by L. A. Zadeh [13] in 1965. On the other hand, B. Schweizer and A. Sklar [6, 7] introduced the notions of *Triangular norm* (t-norm) and *Triangular conorm* (t-conorm). The triangular norms(t-norms) and the triangular conorms (t-conorms) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators respectively. Thus, the t-norm generalizes (classical) conjunctive(AND) operator and the t-conorm generalizes (classical) disjunctive(OR) operator. In application, the t-norm T and the t-conorm S can be regarded as functions that map the unit square into the unit interval.

Y. B. Jun and K. H. Kim [3] introduced the notion of imaginable fuzzy ideals of BCK-algebras with respect to a t-norm T, and studied some of their properties. M. Akram and K. H. Dar [4] introduced the notion of T-fuzzy ideals in BCI-algebras, and obtained some of their related results. M. Akram [5] has recently introduced the notion of fuzzy ideals in a class of IS-algebras with respect to a t-conorm, and studied some of their properties. In this paper we introduce the notion of T-fuzzy associative $\mathscr{I}$-ideals in an IS-algebra, and investigate some of their properties. We discuss the properties of homomorphic image and inverse image of T-fuzzy associative $\mathscr{I}$-ideals in an IS-algebra. Connections between direct product and T-product of fuzzy associative $\mathscr{I}$-ideals induced by t-norms are also studied.

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2. Preliminaries

In this section we review some elementary aspects that are necessary for this paper. By a BCI-algebra we mean $(X;*,0)$ of type $(2,0)$ satisfying the following conditions:
(1) $((x*y)*(x*z))*(z*y)=0$,
(2) $x*(x*y)*y=0$,
(3) $x*x=0$,
(4) $x*y=0, y*x=0 \Rightarrow x=y$
for all x, y, $z \in X$. We can define a partial ordering relation " $\leq$ " on X by letting $x \leq y$ if and only if $x*y=0$. Any BCI-algebra X has the following properties:
(5) $x*0=x$.
(6) $(x*y)*z=(x*z)*y$.
(7) $x \leq y \Rightarrow x*z \leq y*z$, $z*y \leq z*x$.
Let S be a nonempty subset of a BCI-algebra X, then S is called a *subalgebra* of X if $x*y \in S$, for all $x, y \in S$. A mapping $f : X \to Y$ of BCI-algebras is a *homomorphism* if $f(x*y)=f(x)*f(y)$, for all x, $y \in X$. A nonempty subset A of a BCI-algebra X is called an *ideal* of X if, for all x, $y \in X$, it satisfies: (I1) $0 \in A$, (I2) $x*y,\ y \in A \Rightarrow x \in A$.
By an IS-algebra we mean a nonempty set X with two binary operations "*" and "$\cdot$" and constant 0 satisfying the following axioms:
(a) $I(X)=(X;*,0)$ is a BCI-algebra,
(b) $S(X)=(X;\cdot)$ is a semigroup,
(c) The operation "$\cdot$ "is distributive (on both sides) over the operation "$*$", that is $x.(y*z)=(x.y)*(x.z)$ and $(x*y).z=(x.z)*(y.z)$ for all x, y, $z \in X$.
A nonempty subset I of a semigroup $S(X)=(X;.)$ is said to be *stable* if $xa \in I$ whenever $x \in S(X)$ and $a \in I$.
A nonempty subset I of an IS-algebra X is called a *left (respectively, right) associative $\mathscr{I}$-ideal* of X if
(a1) I is a left(respectively, right)stable subset of $S(X)$,
(a2) for any x, y, $z \in I(X)$, $(x*y)*z \in I$ and $y*z \in I$ imply that $x \in I$.
We now review some fuzzy logical concepts:
A mapping $\mu : X \to [0,1]$, where X is an arbitrary nonempty set, is called a *fuzzy set* in X. For any fuzzy set μ in X and any $s \in [0,1]$, we define the set $U(\mu;s)=\{x \in X \mid \mu(x) \geq s\}$, which is called the *upper level cut* of μ. A *triangular norm* (t-norm T) is a function $T:[0,1]\times[0,1] \to [0,1]$ that satisfies following conditions:
(T1) $T(x,1)=x$,
(T2) $T(x,y)=T(y,x)$,
(T3) $T(x,T(y,z))=T(T(x,y),z)$,
(T4) $T(x,y) \leq T(x,z)$ whenever $y \leq z$
for all x, y, $z \in [0,1]$.
A fuzzy set μ in a BCI-algebra X is called a *fuzzy ideal with respect to a t-norm* of X if, for all x, $y \in X$ it satisfies:
(TF1) $\mu(0) \geq \mu(x)$,
(TF2) $\mu(x) \geq T(\mu(x*y),\mu(y))$.
A fuzzy set μ in a semigroup $S(X) := (X,.)$ is said to be a *fuzzy left(respectively, right) stable with respect to a t-norm* if $\mu(x.y) \geq \mu(y)$(respectively, $\mu(x.y) \geq \mu(x)$) for all x, $y \in X$. Let T be a t-norm. A fuzzy set μ in X is said to satisfy *imaginable property* if $Im(\mu) \subseteq \Delta_T$, where $\Delta_T=\{x \in [0,1] \mid T(x,x)=x\}$.
For a t-norm T, the statement $T(x,y) \leq \min(x,y)$ for all x, $y \in [0,1]$ holds.

3. Fuzzy associative $\mathscr{I}$-ideals with t-norms

Definition 3.1. *A fuzzy set μ in an IS-algebra X is called a fuzzy left (respectively, right) associative $\mathscr{I}$-ideal of X with respect to a t-norm(briefly, T-fuzzy associative $\mathscr{I}$-ideal) if*

(TF3) $\mu(x.y) \geq \mu(y)$,

(TF4) $\mu(x) \geq T(\mu((x*y)*z), \mu(y*z))$

for all x, y, $z \in X$.

In what follows, associative $\mathscr{I}$-ideal shall mean left associative $\mathscr{I}$-ideal.

Definition 3.2. *T-fuzzy associative $\mathscr{I}$-ideals is said to be an imaginable if it satisfies the imaginable property.*

Example 3.1. *Consider an IS-algebra $X = \{0, a, b, c\}$ with the following Cayley tables:*

*	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

We define a fuzzy set μ: $X \to [0,1]$ by $\mu(0) = \mu(a) = t_1, \mu(b) = \mu(c) = t_2$, $t_1 > t_2$ and let $T_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $T_m(x,y) = max(x+y-1, 0)$ which is a t-norm for all x, $y \in [0,1]$. By routine computations, it is easy to check that μ is a T_m-fuzzy associative $\mathscr{I}$-ideal of X, and $Im(\mu) \subseteq \Delta_{T_m}$. Hence μ is an imaginable fuzzy associative $\mathscr{I}$-ideals of X with respect to T_m.

Theorem 3.1. *Let T be a t-norm. Then every imaginable T-fuzzy associative $\mathscr{I}$-ideal of X is a fuzzy associative $\mathscr{I}$-ideal.*

Proof. Let μ be an imaginable T-fuzzy associative $\mathscr{I}$-ideal of X. Then $\mu(x) \geq T(\mu((x*y)*z), \mu(y*z))$ for all $x, y, z \in X$. Since μ is an imaginable, we have

$$\begin{aligned} & min(\mu((x*y)*z), \mu(y*z)) = \\ & T(min(\mu((x*y)*z), \mu(y*z)), min(\mu((x*y)*z), \mu(y*z))) \\ & \leq T(\mu((x*y)*z), \mu(y*z)) \leq min(T(\mu((x*y)*z), \mu(y*z)). \end{aligned}$$

It follows that $\mu(x) \geq T(\mu((x*y)*z), \mu(y*z)) =$ min $(\mu((x*y)*z), \mu(y*z))$, so μ is a fuzzy associative $\mathscr{I}$-ideal of X. □

Converse of Theorem 3.1 may not be true as seen in the following example.

Example 3.2. *Consider an IS-algebra $X = \{0, a, b, c\}$ with the following Cayley tables:*

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

We define a fuzzy set μ: $X \to [0,1]$ by $\mu(0) = 0.8, \mu(a) = 0.7, \mu(b) = \mu(c) = 0.5$. By routine computation, we can easily see that μ is a fuzzy associative $\mathscr{I}$-ideals of X.

Let $\alpha \in (0,1)$and define the binary operation T_α on (0,1) as follows:

$$T_\alpha(x,y) = \begin{cases} min(x,y) & \text{if max(x, y)=1} \\ 0 & \max(x,y) < 1,\ x+y \leq 1+\alpha \\ \alpha & \text{otherwise} \end{cases}$$

for all $x, y \in [0,1]$. Then T_α is a t-norm. Thus $T_\alpha(\mu(0),\mu(0)) = T_\alpha(0.8,0.8) = \alpha \neq \mu(0)$ whenever $\alpha < 0.6$. So $Im(\mu) \subseteq \Delta_{T_\alpha}$ whenever $\alpha < 0.6$. Hence μ is not an imaginable fuzzy associative $\mathscr{I}$-ideals of X with respect to T_α.

Theorem 3.2. *Let T be a t-norm and let μ be a fuzzy set in an IS-algebra X. Then μ is T-fuzzy associative $\mathscr{I}$-ideal of X if and only if each nonempty level subset $U(\mu;\alpha)$ of μ is an associative $\mathscr{I}$-ideal of X.*

Proof. Let μ be a T-fuzzy associative $\mathscr{I}$-ideal of X and let $\alpha \in [0,1]$. Let x, y, $z \in X$ be such that $x \in U(\mu;\alpha)$, $y \in U(\mu;\alpha)$, $z \in U(\mu;\alpha)$. Using (TF3) and (TF4), we have

$$\mu(x.y) \geq \mu(x) \geq \alpha,$$
$$\mu(x) \geq T(\mu((x*y)*z),\mu(y*z)) \geq \alpha,$$

so $x.y \in U(\mu;\alpha)$ and $x \in U(\mu;\alpha)$.
Conversely, assume that every nonempty level subset $U(\mu;\alpha)$ of μ is an associative $\mathscr{I}$-ideal of X. Then it can be easily checked that μ satisfies ($TF3$). If there exist x, y, $z \in X$ such that $\mu(x) >$ T $(\mu((x*y)*z),\mu(y*z))$, then by taking $t_0 := \frac{1}{2}\{\mu(x) + T(\mu((x*y)*z),\mu(y*z))\}$, we have $(x*y)*z \in U(\mu;t_0)$ and $y*z \in U(\mu;t_0)$. Since μ is an associative $\mathscr{I}$-ideal of X, $x \in U(\mu;t_0)$, we have $\mu(x) \geq t_0$, a contradiction. Hence μ is a T-fuzzy associative $\mathscr{I}$-ideal of X. □

Definition 3.3. *Let f be a mapping on X. If v is a fuzzy set in $f(X)$ then the fuzzy set $\mu = v \circ f$ (i.e., $(v \circ f)(x) = v(f(x))$) in X is called the preimage of v under f.*

Theorem 3.3. *An onto homomorphism preimage of a T-fuzzy associative $\mathscr{I}$-ideal of X is a T-fuzzy associative $\mathscr{I}$-ideal.*

Proof. Let $f : X \to Y$ be an onto homomorphism of IS-algebras. If v is a T-fuzzy associative $\mathscr{I}$-ideal of Y and μ is the preimage of v under f, then for any x, y, $z \in X$, we have

$$\begin{aligned} \mu(x.y) &= (v \circ f)(x.y) = v(f(x.y)) = v(f(x).f(y)) \\ &\geq v(f(y)) = (v \circ f)(y) = \mu(y), \end{aligned}$$

$$\begin{aligned} \mu(x) &= (v \circ f)(x) = v(f(x)) \\ &\geq T(v((f(x)*f(y))*f(z)), v(f(y)*f(z))) \\ &= T((v \circ f)((x*y)*z), (v \circ f)(y*z)) \\ &= T(\mu((x*y)*z), \mu(y*z)). \end{aligned}$$

This shows that μ is a fuzzy associative $\mathscr{I}$-ideal of X with respect to a t-norm T. □

Definition 3.4. *Let μ be a fuzzy set in an IS-algebra X and f a mapping defined on X. Then the fuzzy set μ^f in $f(X)$ defined by*

$$\mu^f(y) = \sup_{x \in f^{-1}(y)} \mu(x) \quad \forall y \in f(X)$$

is called the image of μ under f. A fuzzy set μ in X has the sup *property if for any subset $A \subseteq X$, there exists $a_0 \in A$ such that $\mu(a_0) = \sup_{a \in A} \mu(a)$*

Theorem 3.4. *An onto homomorphism image of a fuzzy associative $\mathscr{I}$-ideal with the* sup *property is a fuzzy associative $\mathscr{I}$-ideal .*

Proof. Let $f : X \to Y$ be an epimorphism of X and μ a fuzzy associative $\mathscr{I}$-ideal of X with the sup property. Consider $f(x)$, $f(y)$, $f(z) \in$ f(X). Now, let x_0, y_0, $z_0 \in$ $f^{-1}(f(x))$ be such that

$$\mu(x_0) = \sup_{t\in f^{-1}(f(x))} \mu(t)$$

$$\mu(y_0) = \sup_{t\in f^{-1}(f(y))} \mu(t)$$

and

$$\mu(z_0) = \sup_{t\in f^{-1}(f(z))} \mu(t)$$

respectively. Then we can deduce that

$$\begin{aligned} \mu^f(f(x).f(y)) &= \sup_{t\in f^{-1}(f(x).f(y))} \mu(t) \\ &\geq \mu(y_0) \\ &= \sup_{t\in f^{-1}(f(y))} \mu(t) \\ &= \mu^f(f(y)), \end{aligned}$$

$$\begin{aligned} \mu^f(f(x)) &= \sup_{t\in f^{-1}(f(x))} \mu(t) \\ &\geq min\{\mu((x_0 * y_0) * z_0), \mu(y_0 * z_0)\} \\ &= min\{\sup_{t\in f^{-1}((f(x)*f(y))*f(z))} \mu(t), \sup_{t\in f^{-1}(f(y)*f(z))} \mu(t)\} \\ &= min\{\mu^f((f(x) * f(y)) * f(z)), \mu^f(f(y) * f(z))\}. \end{aligned}$$

Consequently, μ^f is a fuzzy associative $\mathscr{I}$-ideal of Y. □

Theorem 3.5. *Let T be a t-norm. Let μ be a fuzzy associative $\mathscr{I}$-ideal of an IS-algebra X with a t-norm T and having finite image. Then every descending chain of fuzzy associative $\mathscr{I}$-ideal of X with respect to T terminates at finite steps.*

Proof. Suppose that there exists a strictly descending chain $A_0 \supset A_1 \supset A_2 \supset \cdots$ of fuzzy associative $\mathscr{I}$-ideal of X which does not terminate at finite steps. Define an fuzzy set μ in X by

$$\mu(x) := \begin{cases} \frac{n}{n+1} & \text{if } x \in A_n \setminus A_{n+1}, n = 0, 1, 2, \cdots, \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} A_n, \end{cases}$$

where $A_0 = X$. We prove that μ is a fuzzy associative $\mathscr{I}$-ideal of X with respect to T. Clearly $\mu(x.y) \geq \mu(y)$ for all $x, y \in X$. Let $x, y, z \in X$. Assume that $(x*y)*z \in A_n \setminus A_{n+1}$ and $y * z \in A_k \setminus A_{k+1}$ for $n = 0, 1, 2, \cdots; k = 0, 1, 2, \cdots$. Without loss of generality, we may assume that $n \leq k$. Then, it is obvious that $y * z \in A_n$, and so $x \in A_n$ since A_n is a fuzzy associative $\mathscr{I}$-ideal of X. Hence, we have

$$\mu(x) \geq \tfrac{n}{n+1} = T(\mu((x * y) * z), \mu(y * z)).$$

If $(x * y) * z, y * z \in \bigcap_{n=0}^{\infty} A_n$, then $x \in \bigcap_{n=0}^{\infty} A_n$. Thus, we obtain that

$$\mu(x) = 1 = T(\mu((x * y) * z), \mu(y * z).$$

If $(x*y)*z \notin \bigcap_{n=0}^{\infty} A_n$ and $y*z \in \bigcap_{n=0}^{\infty} A_n$, then there exists a $k \in \mathbb{N}$ such that $(x*y)*z \in A_k \setminus A_{k+1}$. It follows that $x \in A_k$ so that

$$\mu(x) \geq \tfrac{k}{k+1} = T(\mu((x*y)*z), \mu(y*z)).$$

Finally, we suppose that $(x*y)*z \in \bigcap_{n=0}^{\infty} A_n$ and $y*z \notin \bigcap_{n=0}^{\infty} A_n$. Then we have $y*z \in A_r \setminus A_{r+1}$, for some $r \in \mathbb{N}$. This leads to $x \in A_r$, and so

$$\mu(x) \leq \tfrac{r}{r+1} = T(\mu((x*y)*z), \mu(y*z)).$$

Consequently, μ is a fuzzy associative $\mathscr{I}$-ideal of X with the t-norm T and μ has infinite number of different values. However, this is clearly a contradiction, and hence the proof is completed. □

Definition 3.5. *Let T be a t-norm and λ and μ be two fuzzy sets in X. Then the T-product of λ and μ is denoted by $[\lambda \cdot \mu]_T$ and defined by $[\lambda \cdot \mu]_T(x) = T(\lambda(x), \mu(x))$, for all $x \in X$.*

Theorem 3.6. *Let λ and μ be two T- fuzzy associative $\mathscr{I}$-ideals of X. If a t-norm T^* dominates T, that is, if $T^*(T(\alpha,\gamma), T(\beta,\delta)) \geq T(T^*(\alpha,\beta), T^*(\gamma,\delta))$ for all α, β, γ, $\delta \in [0,1]$. Then T^*-product $[\lambda \cdot \mu]_T^*$ is a T-fuzzy associative $\mathscr{I}$-ideal of X.*

Proof. For any $x, y \in X$, we have
$[\lambda \cdot \mu]_T^*(x.y) = T^*(\lambda(x.y), \mu(x.y)) \geq T^*(\lambda(y), \mu(y)) = [\lambda \cdot \mu]_T^*(y)$.
Let x, y, $z \in X$. Then

$$\begin{aligned}
[\lambda \cdot \mu]_T^*(x) &= T^*(\lambda(x), \mu(x)) \\
&\geq T^*(T(\lambda((x*y)*z), \lambda(y*z)), T(\mu((x*y)*z), \mu(y*z))) \\
&\geq T(T^*(\lambda((x*y)*z), \mu((x*y)*z)), T^*(\lambda(y*z), \mu(y*z))) \\
&= T([\lambda \cdot \mu]_T^*((x*y)*z), [\lambda \cdot \mu]_T^*(y*z)).
\end{aligned}$$

This proves that $[\lambda \cdot \mu]_T^*$ is a T- fuzzy associative $\mathscr{I}$-ideals of X. □

Theorem 3.7. *Let T be a t-norm. Let X_1 and X_2 be IS-algebras and $X = X_1 \times X_2$ be the direct product of IS-algebras X_1 and X_2. Let λ be a fuzzy associative $\mathscr{I}$-ideal of an IS-algebra X_1 with respect to T and μ be a fuzzy associative $\mathscr{I}$-ideal of an IS-algebra X_2 with respect to T. Then $\nu = \lambda \times \mu$ is a fuzzy associative $\mathscr{I}$-ideal of $X = X_1 \times X_2$ with respect to T defined by*

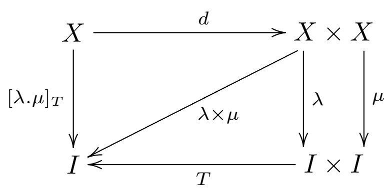
$$\nu(x_1, x_2) = (\lambda \times \mu)(x_1, x_2) = T(\lambda(x_1), \mu(x_2)).$$

Proof. Obviously, $\nu(x.y) \geq \nu(y)$, for all x, $y \in X$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2) \in X_1 \times X_2 = X$. Then

$$\begin{aligned}
&\nu(x) = (\lambda \times \mu)(x) = (\lambda \times \mu)((x_1, x_2)) \\
&= T(\lambda(x_1), \mu(x_2)) \\
&\geq T(T(\lambda((x_1 * y_1) * z_1), \lambda(y_1 * z_1)), (\mu((x_2 * y_2) * z_2), \mu(y_2 * z_2))) \\
&= T(T(\lambda((x_1 * y_1) * z_1), \mu((x_2 * y_2) * z_2)), T(\lambda(y_1 * z_1), \mu(y_2 * z_2))) \\
&= T((\lambda \times \mu)(((x_1 * y_1) * z_1), (x_2 * y_2) * z_2)), (\lambda \times \mu)((y_1 * z_1, y_2 * z_2)) \\
&= T((\lambda \times \mu)(((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)), (\lambda \times \mu)((y_1, y_2) * (z_1, z_2))) \\
&= T((\lambda \times \mu)((x*y)*z), (\lambda \times \mu)(y*z)) \\
&= T(\nu((x*y)*z), \nu(y*z)).
\end{aligned}$$

Hence ν is a fuzzy associative $\mathscr{I}$-ideal of X with respect to T. □

The relationship between T-fuzzy associative $\mathscr{I}$-ideals $\lambda \times \mu$ and $[\lambda \cdot \mu]_T$ can be viewed via the following diagram



where $I = [0, 1]$ and $d : X \to X \times X$ is defined by $d(x) = (x, x)$. It is easy to see that $[\lambda \cdot \mu]_T$ is the preimage of $\lambda \times \mu$ under d.

Definition 3.6. *Let T be a t-norm. If ν is an imaginable fuzzy set S , the strongest T-fuzzy relation on S that is T-fuzzy relation on ν is μ_ν given by $\mu_\nu(x, y) = T(\nu(x), \nu(y))$, for all x, $y \in S$.*

Theorem 3.8. *Let ν be a fuzzy set in an IS-algebra X and μ_ν be the strongest an imaginable T-fuzzy relation on X. Then ν is an imaginable T-fuzzy associative $\mathscr{I}$-ideal of X if and only if μ_ν is an imaginable T-fuzzy associative $\mathscr{I}$-ideal of $X \times X$.*

Proof. Suppose that ν is an imaginable fuzzy associative $\mathscr{I}$-ideal of X. For any $x = (x_1, x_2), y = (y_1, y_2) \in X \times X$, we have

$$\begin{aligned} \mu_\nu(x * y) &= \mu_\nu((x_1, x_2) * (y_1, y_2)) = \mu_\nu((x_1 * y_1, x_2 * y_2)) \\ &= T(\nu(x_1 * y_1), \nu(x_2 * y_2)) \\ &\geq T(\nu(y_1, \nu(y_2))) = \mu_\nu(y)). \end{aligned}$$

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2) \in X \times X$. Then

$$\begin{aligned} \mu_\nu(x) &= \mu_\nu((x_1, x_2)) \\ &= T(\nu(x_1), \nu(x_2)) \\ &\geq T(T(\nu((x_1 * y_1) * z_1), \nu(y_1 * z_1)), T(\nu((x_2 * y_2) * z_2), \nu(y_2 * z_2))) \\ &= T(T(\nu((x_1 * y_1) * z_1), \nu((x_2 * y_2) * z_2), S(\nu(y_1 * z_1), \nu(y_2 * z_2))) \\ &= T(\mu_n((x_1 * y_1) * z_1), (x_2 * y_2) * z_2), \mu_\nu(y_1 * z_1, y_2 * z_2)) \\ &= T(\mu_\nu(((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)), \mu_\nu((y_1, y_2) * (z_1, z_2))) \\ &= T(\mu_\nu((x * y) * z), \mu_\nu(y * z)). \end{aligned}$$

Thus μ_ν is a T-fuzzy associative $\mathscr{I}$-ideals of $X \times X$.
For any $x = (x_1, x_2) \in X \times X$,

$$\begin{aligned} T(\mu_\nu(x), \mu_\nu(x)) &= T(\mu_\nu(x_1, x_2), \mu_\nu(x_1, x_2)) \\ &= T(T(\nu(x_1), \nu(x_2)), T(\nu(x_1), \nu(x_2))) \\ &= T(T(\nu(x_1), \nu(x_1)), T(\nu(x_2), \nu(x_2))) \\ &= T((\nu(x_1), \nu(x_2))) \\ &= \mu_\nu(x). \end{aligned}$$

Hence μ_ν is an imaginable T-fuzzy associative $\mathscr{I}$-ideals of $X \times X$.
The converse is proved similarly. □

References

[1] M. T. Abu Osman, *On some product of fuzzy subgroups*, Fuzzy Sets and Systems **24**(1987) 79-86.
[2] Y. Yu, J. N. Mordeson and S. C. Cheng, *Elements of L-algebra, Lecture Notes in Fuzzy Math. and Computer Sci.*, Creighton Univ., Omaha, Nebraska (USA), 1994.
[3] Y. B. Jun and K. H. Kim, *Imaginable fuzzy ideals of BCK-algebras with respect to a t-norm*, J. Fuzzy Math., **8**(3) (2000) 737-744.
[4] M. Akram and K. H. Dar, *T-fuzzy ideals in BCI-algebras*, IJMMS, **12** (2005) 1899-1907.
[5] M. Akram, *On properties of fuzzy associative $\mathscr{I}$-ideals in IS-algebras with t-conorms*, Inter. Math. Journal.(To appaear).
[6] B. Schweizer, and A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10**(1960) 313-334.
[7] B. Schweizer and A. Sklar, *Associative functions and abstract semigroups*, Publ. Math. Debrecen, **10**(1963) 69-81.
[8] Y. Imai and K. Iseki, *On axiom System of propositional calculi XIV*, Proc., Japonica Acad, **42**(1966) 19-22.
[9] K. Iseki, *An algebra related with a propositional calculus*, Proc. Japan Acad, **42** (1966) 26-29.
[10] Y. B. Jun, S. M. Hong and E. H. Roh, *BCI-semigroups*, Honam Math. J. **15**(1)(1993) 59-64.
[11] Y. B. Jun, X. L. Xin and E. H. Roh, *A class of algebras related to BCI-algebras and semigroups*, Soochow J. Math.,**24**(4)(1998) 309-321.
[12] E. H. Roh, Y. B. Jun and W. H. Shim, *Fuzzy associative $\mathscr{I}$-ideals of IS-algebras*, IJMMS, **24**(11)(2000) 729-735.
[13] L. A. Zadeh, *Fuzzy sets*, Information Control, **8**(1965) 338-353.

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