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A Modified Method with Inertial-type for Solving Fixed Point and Variational Inequalities Problems in Reflexive Banach Spaces

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Abstract. In this paper, using Bregman distance technique, we introduce an inertial type algorithm with self - adaptive step size for approximating a common element of the set of solutions of pseudomonotone variational inequality problem and the set of common fixed point of a finite family of generic generalized Bregman nonspreading mapping in a real reflexive Banach space. Furthermore, we prove a strong convergence theorem of our algorithm without prior knowledge of the Lipschitz constant of the operator under some mild assumptions. We also give a numerical example to illustrate the performance of our algorithm. Our result generalize and improve many existing results in the literature.

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1. Introduction

Let E be a real Banach space and E^* be its dual space. Let C be a nonempty, closed and convex subset of E, and $A : C \to E^*$ be a mapping.

The problem of finding a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \ \forall \ y \in C, \tag{1.1}$$

is called a *variational inequality problem.* The set of solutions of variational inequality problem (1.1) is denoted by $VI(C, A)$. The study of variational inequality problem originates from solving minimization problems involving infinite-dimensional functions and calculus of variation (see, for example, [36] and reference therein). The concept of variational inequality problem was initially introduced by Hartman and Stampacchia [22] as a generalization of boundary value problems in 1966. Such problems are applicable in a wide range of applied sciences and mathematics. Later in 1967 Lions and Stampacchia [32] studied the existence and uniqueness of the solution. Since then, the theory of variational inequality problem has received much attention due to its wide applications in various areas of pure and applied sciences, such as optimal control, image recovery, resource allocations, networking, transportation, signal processing and so on (see, for example, [26, 19, 4] and references therein). The constraints can clearly be expressed as variational inequality problems and (or) as fixed point problems. Consequently, the problem of finding common elements of the set of

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solutions of variational inequality problems and the set of fixed points of nonlinear operators has become an interesting area of research for many researchers working in the area of nonlinear operator theory (see, for example, [33, 34, 23] and the references contained in them).

Many researchers in their quest to find solutions of variational inequality problems have proposed and analyzed various iterative approximation methods (see for example, [25, 14]). A number of results on iterative methods proposed for approximating solutions of variational inequality problems are studied such that the operator A was often considered to be inverse strongly monotone (see, for instance [21, 30] and references therein).

In order to relax the inverse strongly monotone condition imposed on the operator A, Korpelevich [28] proposed the following extragradient method in a finite dimensional Euclidean space $\mathbb{R}^n$:

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda A(y_n)) \ \forall n \geq 0, \end{cases} \tag{1.2}$$

where $\lambda \in (0, \frac{1}{L})$, A is monotone and Lipschitz and P_C is the metric projection onto C. They proved that the sequence $\{x_n\}$ generated by algorithm (1.2) converges weakly to a solution of problem (1.1). However, it is notice that the extragradient method require the computation at each step of the iteration process two projections onto a closed and convex subset C of $\mathbb{R}$. This might affect the efficiency of the extragradient method if the feasible set is not simple enough which might also increase the computational cost.

In order to overcome this drawback, Several modifications of the extragradient method were proposed (see, for example [17, 15, 24, 49, 48] and references therein) for solving variational inequality problem (1.1). In particular, Tseng [49] proposed the following Tseng's extragradient method

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = y_n - \lambda(A(y_n) - A(x_n)) \ \forall n \geq 0, \end{cases} \tag{1.3}$$

where $\lambda \in (0, \frac{1}{L})$, A is monotone and Lipschitz and P_C is the metric projection onto C. They proved that the sequence $\{x_n\}$ generated by algorithm (1.3) converges weakly to a solution of problem (1.1) in a real Hilbert space H. Another modification of the extragradient method was proposed by Censor et al. [17] as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A(x_n)), \\ T_n = \{z \in H : \langle x_n - \lambda A(x_n) - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A(y_n)), \quad \forall \ n \geq 0. \end{cases} \tag{1.4}$$

They modified the extragradient method (1.2) by replacing the second projection onto a closed and convex subset C with a projection onto the half space T_n. Algorithm (1.4) is therefore called subgradient extragradient method. Observe that, the set T_n is a half space, thus algorithm (1.4) is simpler to implement than algorithm (1.2).

They proved that the sequence $\{x_n\}$ generated by algorithm (1.4) converges weakly to a solution of problem (1.1) in a real Hilbert space H under some mild assumptions. Kraikaew and Saejung [29] in order to obtain strong convergence, combined the subgradient extragradient method (1.4) with Halpern method and thus proposed the following iterative algorithm:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A(x_n)), \\ T_n = \{z \in H : \langle x_n - \lambda A(x_n) - y_n, z - y_n\rangle \leq 0\}, \\ z_n = \alpha_n x_0 + (1-\alpha_n)P_{T_n}(x_n - \lambda A(y_n)), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sz_n, \qquad \forall\ n \geq 0, \end{cases} \tag{1.5}$$

where $\beta_n \subset [a,b] \in (0,1),$, and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that the sequence $\{x_n\}$ generated by algorithm (1.5) converges strongly to a point $x^* \in VI(C,A) \cap F(S)$ in a real Hilbert space under some mild assumptions, and $F(S)$ denoted the set of fixed point of relatively nonexpansive mapping S.

In 2018, Chidume et al. [18] proposed the following Krasnoselskii type algorithm in a uniformly smooth, 2 - uniformly convex real Banach space for approximating common element of solutions of a variational inequality problem and common fixed point of a countable family of relatively nonexpansive mappings as:

$$\begin{cases} x_0 = x \in E, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A(x_n)), \\ T_n = \{z \in E : \langle Jx_n - \lambda A(x_n) - Jy_n, z - y_n\rangle \leq 0\}, \\ t_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda A(y_n)), \\ z_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jt_n), \\ x_{n+1} = J^{-1}(\lambda Jx_n + (1-\lambda)J(S_i z_n)), \qquad \forall\ n \geq 0,\ \ i \geq 1, \end{cases} \tag{1.6}$$

where $\lambda \in (0,1)$ such that $\lambda < \frac{\alpha}{K}$, α is a constant, K is Lipschitz constant and $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that the sequence $\{x_n\}$ generated by algorithm (1.6) converges strongly to a point $x^* = \Pi_{\bigcap_{i=1}^{\infty}(S_i)\bigcap VI(C,A)} x_0$ under some mild assumptions.

Bregman [11] introduced an essential and effective technique for designing and analyzing feasibility and optimization algorithms. However, Bregman distance have been studied by many researchers (see, for example [7, 8, 12] and references therein).

Ali et al. [2] introduced a modified inertial subgradient extragradient method for approximating a common element in the set of solutions of variational inequality problem and the set of common fixed point of demigeneralized mapping in a real reflexive Banach space and obtained a strong convergence theorem for the sequence generated by the the following process.

$$\begin{cases} x_0, x_1 \in E, \\ w_n = \nabla f^*(\nabla f(x_n) + \tau_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ y_n = P_C \nabla f^*(\nabla(w_n) - \lambda A(w_n)), \\ T_n = \{z \in E : \langle \nabla f(w_n) - \lambda A(w_n) - \nabla f(y_n), z - y_n \rangle \leq 0\}, \\ z_n = P_{T_n} \nabla f^*(\nabla f(w_n) - \lambda A(y_n)), \\ v_n = \nabla f^*((1-\alpha_n)\nabla f(z_n) + \alpha_n \nabla f(Tz_n)), \\ x_{n+1} = \nabla f^*(\delta_n \nabla f(x_n) + \beta_n \nabla f(v_n) + \gamma_n \nabla f(u)), \quad \forall \; n \geq 0, \end{cases} \tag{1.7}$$

where $\lambda \in (0, \frac{\alpha}{L}), \alpha$ is a constant, L is Lipschitz constant and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$.

Observe that all the methods mentioned above require a prior knowledge of the Lipschitz constant of the operator A as input parameter which is very difficult to estimate when solving some practical problems.

In order to navigate from this setback. Ma [37] introduced a new subgradient extragradient method with a self-adaptive step size for solving monotone variational inequality problems in Banach space without prior knowledge of the Lipschitz constant of the operator. They established a strong convergence result for the problem VIP (1.1) using the following algorithm: Very recently Ali and Ajio [5] proved the strong con-

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Algorithm 1

Initialization: Take $\lambda_0 > 0, x_0 \in E$ be a given starting point, $\mu \in (0,1)$.

(Step 0) Given the current iterate x_n, compute

$$y_n = P_C(Jx_n - \lambda_n A(x_n)),$$

If $x_n = y_n$, then stop: x_n is a solution. Otherwise, go to **Step 2**.

(Step 2) Construct the set

$$T_n = \{w \in E : \langle Jx_n - \lambda_n A(x_n) - Jy_n, w - y_n \rangle \leq 0\},$$

and compute

$$z_n = P_{T_n}(Jx_n - \lambda_n A(y_n)), \qquad x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n),$$

(Step 3) compute

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu(||x_n - y_n||^2 + ||y_n - z_n||^2)}{2\langle A(x_n) - A(y_n), z_n - y_n \rangle}, \lambda_n\} & , if \quad \langle A(x_n) - A(y_n), z_n - y_n \rangle > 0, \\ \lambda_n, & otherwise, \end{cases}$$

Set $n := n+1$ and return to **Step 1**.

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vergence of an inertial type subgradient extragradient algorithm for approximating a common element of the set of solutions of pseudomonotone variational inequality problem and the set of common fixed point of a finite family of generic generalized nonspreading mappings in uniformly smooth and 2 - uniformly convex real Banach spaces, using the following algorithm: Motivated by the above works, in this paper, using Bregman distance technique, we introduce an inertial-type subgradient extragradient algorithm with self adaptive step size for approximating a common element in the set of solutions of pseudomonotone variational inequality problem and the set

Algorithm 2

Initialization: Take $\lambda_1 > 0, \mu, \theta \in (0,1)$. Select initial data $x_0, x_1 \in E$.
Given x_{n-1}, x_n and θ_n for each $n \geq 1$, choose θ_n such that $\theta_n \in [0, \hat{\theta}_n]$.
Iterative steps: Calculate x_{n+1} and λ_{n+1} as follows:

$$\begin{cases} u_n = J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n)), \\ y_n = \Pi_C J^{-1}(Ju_n - \lambda_n A(u_n)), \\ S_n = \{z \in E : \langle Ju_n - \lambda_n A(u_n) - Jy_n, z - y_n \rangle \leq 0\}, \\ z_n = \Pi_{S_n} J^{-1}(Ju_n - \lambda_n A(y_n)), \\ w_n = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n), \\ x_{n+1} = J^{-1}(\beta_n Jz_n + (1 - \beta_n)J(Tw_n)), \quad \forall \ n \geq 1, \end{cases} \tag{1.8}$$

where $T = T_M \circ T_{M-1} \circ, ..., \circ T_1$, $\hat{\theta}_n$ and λ_{n+1} are updated by (3.1) and (3.3) respectively.

$$\hat{\theta}_n = \begin{cases} \min\{\frac{\tau_n}{||x_n - x_{n-1}||}, \theta\}, & if \quad x_n \neq x_{n-1}, \\ \theta, & \text{otherwise} \end{cases} \tag{1.9}$$

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu(||u_n - y_n||^2 + ||y_n - z_n||^2)}{2\langle A(u_n) - A(y_n), z_n - y_n \rangle}, \lambda_n\}, & if \quad \langle A(u_n) - A(y_n), z_n - y_n \rangle > 0, \\ \lambda_n, & \text{otherwise} \end{cases} \tag{1.10}$$

of common fixed point of a finite family of generic generalized Bregman nonspreading mapping in a real reflexive Banach space.

Furthermore, we prove a strong convergence theorem to a solution of the stated problem without prior knowledge of Lipschitz constant of the operator under some mild assumptions. we give a numerical example in order to illustrates the performance of our proposed algorithm. Our result generalize and improve many existing results in the literature.The rest of the paper is structured as follows: In section 2, we give some preliminaries that will be needed in the sequel. In section 3, we present our proposed algorithm and then give its convergence analysis. In section 4, we give some numerical examples in order to illustrate the performance of our proposed algorithm and compare it with some existing ones in the literature.

Finally, we conclude by giving a brief summary of the paper in section 5.

2. Preliminaries

Let E be a real reflexive Banach space with its dual space E^*, C a nonempty, closed and convex subset of E and $f : E \longrightarrow (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. The domain of f is denoted by $dom f := \{x \in E : f(x) < +\infty\}$. Let $x \in int(dom f)$, then

(T1) the subdifferential of f is a function $\partial f : E \longrightarrow E^*$ defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \ \forall \, y \in E\};$$

(T2) the Fenchel conjugate of f is the convex function $f : E \longrightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\};$$

(T3) for any $x \in int(domf)$, and $y \in E$, the right hand derivative of f at x in the direction of y is defined by

$$f^0(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be *Gâteaux* differentiable at x if the limit as $t \longrightarrow 0$ in (2.1) exixts for each y. In this case, the gradient of f at x is the linear function $\nabla f : E \longrightarrow (-\infty, +\infty]$ defined by $\langle \nabla f(x), y\rangle = f^0(x, y)$ for all $y \in E$. The function f is said to be *Gâteaux* differentiable if it is *Gâteaux* differentiable at each $x \in int(domf)$. The function f is said to be Fréchet differentiable at x if the limit as $t \longrightarrow 0$ in (2.1) is attained uniformly in y with $||y|| = 1$. Also, f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $||y|| = 1$. It is well known that if f is *Gâteaux* differentiable (resp. Fréchet) on $int(domf)$, then f is continuous and its *Gâteaux* derivative ∇f is norm to *weak** continuous (resp. norm to norm continuous) on $int(domf)$ (see, for example [6, 10]). A function f on E is said to be strongly coercive (see, for example [51]) if

$$\lim_{||x||\to+\infty} \frac{f(x)}{||x||} = +\infty.$$

Definition 2.1. The function f is said to be:

(1) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;

(2) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $dom\, \partial f$;

(3) Legendre function if it is both essentially smooth and essentially strictly convex.

When the subdifferential of f is single-valued, it coincides with the gradient; that is, $\partial f = \nabla f$ (see, for example [40] and reference therein).

Remark 2.1. If E is a reflexive Banach space and f is a Legendre function, then we have the following results:

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see, for example [8]);

(ii) $(\partial f)^{-1} = \partial f^*$ (see, for example [10]);

(iii) f is Legendre if and only if f^* is Legendre function (see, for example [8]);

(iv) if f is Legendre function, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran $\nabla f = dom\nabla f^* = int(domf^*)$ and ran $\nabla f^* = domf = int(domf)$, where ran represents the range (see, for example [8]).

Definition 2.2. Let $f : E \longrightarrow (-\infty, +\infty]$ be a convex and *Gâteaux* differentiable function. The function $D_f : domf \times int(domf) \longrightarrow (-\infty, +\infty]$ defined by

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y \rangle, \tag{2.2}$$

$for\ all\ x \in domf and\ y \in int(domf)$ is called the Bregman distance with respect to f (see, for more details [11, 16]). It is well known that Bregman distance satisfies the following properties for any $x, w \in domf$ and $y, z \in int(domf)$:

(1) three point identity

$$D_f(z, x) := D_f(z, y) + D_f(y, x) + \langle \nabla f(y) - \nabla f(x), z - y \rangle \tag{2.3}$$

(2) four point identity

$$\begin{aligned} D_f(x, y) + D_f(w, z) - D_f(x, z) - D_f(w, y) \\ = \langle \nabla f(z) - \nabla f(y), x - w \rangle \end{aligned} \tag{2.4}$$

Definition 2.3. A *Gâteaux* differentiable function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined on a reflexive real Banach space E is said to be strongly convex if there exists a constant $\beta > 0$ such that

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \beta ||x - y||^2,\ \forall\ x, y \in domf,$$

equivalently

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\beta}{2} ||x - y||^2,\ \forall\ x, y \in domf.$$

If E is a smooth and strictly convex Banach space, then $f(x) = \frac{1}{2}||x||^2$ is a strongly coercive, bounded, uniformly Fréchet differentiable and strongly convex function with strong convexity constant $\beta \in (0, 1]$ and Fenchel conjugate $f^*(x^*) = \frac{1}{2}||x^*||^2$. It can be easily shown that if f is a strongly convex function with constant $\beta > 0$, then, for all $x \in domf$, and $y \in int(domf)$, (see, [9] for more details),

$$D_f(x, y) \geq \frac{\beta}{2} ||x - y||^2. \tag{2.5}$$

Definition 2.4. Let B and S be the closed unit ball and the unit sphere of a Banach space E defined by $B_r = \{w \in E : ||w|| \leq r\}$ for all $r > 0$ and $S_E = \{x \in E : ||x|| = 1\}$ respectively. Then, the function $f : E \longrightarrow \mathbb{R}$ is said to be uniformly convex on bounded subsets of E (see, for example [51] and reference therein) if $\rho_r : [0, +\infty) \longrightarrow [0, +\infty)$ defined by

$$\rho_r(t) \quad = \quad \inf_{x, y \in B_r, ||x - y|| = t, \alpha \in (0,1)} \frac{\alpha f(x) + (1 - \alpha) f(y) - f(\alpha x + (1 - \alpha) y)}{\alpha(1 - \alpha)}$$

which satisfies $\rho_r(t) > 0$ for all $r, t > 0$. The function ρ_r is called the gauge of uniform convexity of f.

Definition 2.5. Let $T : C \to C$ be a mapping.

(1) A point $x \in C$ is called a fixed point of T if $Tx = x$, where $F(T) := \{x \in C : Tx = x\}$ is the set of fixed point of T.

(2) A point $x \in C$ is said to be asymptotic fixed point of T, if there exists a sequence $\{x_n\} \subseteq C$ such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. We denote the set of all asymptotic fixed point of T by $\hat{F}(T)$.

Ali et al.[3] introduced the following nonlinear mapping in a real reflexive Banach space using Bregman distance.

Definition 2.6. Let $f : E \longrightarrow (-\infty, +\infty]$ be a strongly coercive, Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of $int(domf)$. A mapping $T : C \longrightarrow C$ is called a finite family of generic generalized Bregman nonspreading mapping if there exist real numbers $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ such that for all $x, y \in C$ the following inequalities holds: (i) $(\alpha + \beta + \gamma + \delta) \geq 0$; (ii) $(\alpha + \beta) > 0$; and (iii)

$$\begin{aligned} &\alpha D_f(Tx, Ty) + \beta D_f(x, Ty) + \gamma D_f(Tx, y) + \delta D_f(x, y) \\ &\quad \leq \varepsilon\{D_f(Ty, Tx) - D_f(Ty, x)\} + \xi\{D_f(y, Tx) - D_f(y, x)\}. \end{aligned} \tag{2.6}$$

Remark 2.2. Observe that, (i) if $\alpha + \beta = -\gamma - \delta = 1$, then generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ generalized Bregman nonspreading mapping reduces to generalized Bregman non-spreading; (ii) if $\alpha = 1, \beta = \delta = \xi = 0$ and $\gamma = \varepsilon = -1$ then generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ generalized Bregman nonspreading mapping reduces to Bregman nonspreading mapping; (iii) if E is smooth and the function $f(x) = ||x||^2$, then generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ generalized Bregman nonspreading mapping reduces to generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ generalized nonspreading mapping in the sense of Takahashi [46].

The modulus of total convexity at $x \in int(domf)$ is the function $v_f(x, .) : [0, +\infty) \longrightarrow [0, +\infty)$ defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in domf, ||y - x|| = t\}.$$

The function f is called totally convex at $x \in int(domf)$ if $v_f(x, t)$ is positive for any $t > 0$. This concept was first introduced by [12].

Definition 2.7. Let $C \subseteq int(domf)$ be a nonempty, closed and convex subset of a real Banach space E, where $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex and *Gâteaux* differentiable function. The Bregman projection with respect to f of $x \in int(domf)$ onto C is defined as the unique vector $Proj_C^f(x) \in C$, which satisfies

$$D_f(Proj_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

The normalized duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = ||x||.||x^*||, ||x^*|| = ||x||\}$$

for all $x \in E$.

Remark 2.3. If E is smooth and strictly convex Banach space and $f(x) = ||x||^2$ for all $x \in E$, then we have $\nabla f(x) = 2Jx$ for all $x \in E$ and hence the function

$D_f(x,y) = \phi(x,y)$ and the Bregman projection $Proj_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$ which is defined by

$$\phi(\Pi_C(x),x) = \inf\{\phi(y,x) : y \in C\}.$$

If $E = H$, a real Hilbert space, then the Bregman projection $Proj_C^f(x)$ reduces to the metric projection $P_C(x)$ of H onto C.

Lemma 2.1. *[13, 8] Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \longrightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Let $x \in E$. Then the Bregman projection $Proj_C^f : E \longrightarrow C$ satisfies the following properties:*

(i) $z = Proj_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \ \forall\, y \in C,$

(ii) $D_f(y, proj_C^f(x)) + D_f(proj_C^f(x), x) \leq D_f(y,x), \ \forall\, y \in C$ *and* $x \in E$.

Let $f : E \longrightarrow (-\infty, +\infty]$ be convex, Legendre function Gâteaux differentiable function. Following [1, 16] we make use of the function $V_f : E \times E^* \longrightarrow [0, +\infty)$ defined by

$$V_f(x,x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \ \forall\, x \in E \text{ and } x^* \in E^*. \tag{2.7}$$

Then, the following assertions hold:

(i) *V_f is nonnegative and*

$$V_f(x,x^*) = D_f(x, \nabla f^*(x^*)), \ \forall\, x \in E \text{ and } x^* \in E^*. \tag{2.8}$$

Thus, from (2.7) it is obvious that $D_f(x,y) = V_f(x, \nabla f(y))$ and V_f is convex in the second variable. Therefore for $\lambda \in (0,1)$ and $x, y \in E$, we have

$$D_f(z, \nabla f^*(\lambda \nabla f(x) + (1-\lambda)\nabla f(y))) \quad \leq \quad \lambda D_f(z,x) + (1-\lambda) D_f(z,y) \tag{2.9}$$

Moreover by subdifferential inequality (see, for example [27] and reference therein), we have

$$V_f(x,x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \quad \leq \quad V_f(x, x^* + y^*), \ \forall\, x \in E \text{ and } x^*, y^* \in E^* \tag{2.10}$$

Lemma 2.2. *[39] Let E be a Banach space and $f : E \longrightarrow \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E. Let $r > 0$ be a constant and ρ_r is the gauge of uniform convexity of f. Then for any $x, y \in B_r$ and $\alpha \in (0,1)$,*

$$f(\alpha x + (1-\alpha)y) \quad \leq \quad \alpha f(x) + (1-\alpha) f(y) - \alpha(1-\alpha)\rho_r(||x-y||). \tag{2.11}$$

In addition if f is bounded on bounded subsets and uniformly convex on bounded subsets of E then, for any $x \in E, y^*, z^* \in B_r$ and $\alpha \in (0,1)$,

$$V_f(x, \alpha y^* + (1-\alpha)z^*) \quad \leq \quad \alpha V_f(x,y^*) + (1-\alpha)V_f(x,z^*) - \alpha(1-\alpha)\rho_r^*(||y^* - z^*||).$$

Lemma 2.3. *[42] Let $f : E \longrightarrow (-\infty, +\infty]$ be a uniformly Fréchet differentiable function and bounded on bounded subsets of E. Then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to strong topology of E^*.*
Recalll that the function f is called sequentially consistent [13] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \Longrightarrow \lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{2.12}$$

Lemma 2.4. *[43] Let $f : E \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function. If $x \in E$ and the sequence $\{D_f(x_n, x)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Definition 2.8. Let $A : C \to E^*$ be a mapping. Then A is said to be

(1) monotone if the following inequality hold
$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall\, x, y \in C.$$
(2) pseudomonotone if
$$\langle A(x), y - x\rangle \geq 0 \Rightarrow \langle A(y), y - x\rangle \geq 0, \quad \forall\, x, y \in C.$$
(3) Lipschitz continuous if there exists a constant $L > 0$ such that
$$||Ax - Ay|| \leq L||x - y||, \quad \forall\, x, y \in C.$$
(4) weakly sequentially continuous if for any $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$.

Definition 2.9. [38, 31] Let $A : C \longrightarrow E^*$ be an operator. The Minty variational inequality problem $(MVIP)$ consist of finding a point $x^* \in C$ such that
$$\langle A(x^*), y - x^*\rangle \geq 0,\ \forall\ y \in C, \tag{2.13}$$
The set of solutions of (2.13) is denoted by $M(C, A)$. Some existing results for the $(MVIP)$ have been presented in [31]. Also, the assumption that $M(C, A) \neq \emptyset$ has been used in solving the variational inequality problem $VI(C, A)$ in finite dimensional spaces (see, for example[45]).

Lemma 2.5. *[38] Consider the variational inequality problem $VI(C, A)$. Suppose the mapping $h : [0, 1] \longrightarrow E^*$ defined by $h(t) = A(tx+(1-t)y)$ and $t \in [0, 1]$ is continuous for all $x, y \in C$ (i.e, h is hemicontinuous), then $M(C, A) \subset VI(C, A)$. Moreover, if A is pseudomonotone, then $VI(C, A)$ is closed, convex and $VI(C, A) = M(C, A)$*

Lemma 2.6. *[50] If $\{b_n\}$ is a sequence of nonnegative real numbers satisfying the following inequality:*
$$b_{n+1} \leq (1 - \psi_n)b_n + \psi_n\sigma_n + \gamma_n,\ \ n \geq 0,$$
where (i) $\{\psi_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \psi_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then, $b_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 2.7. *[35] Let $\{b_n\}$ be a sequence of real numbers such that there exists a subsequence $\{b_{n_i}\}$ of $\{b_n\}$ such that $b_{n_i} < b_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \longrightarrow \infty$ and the following properties are satisfied for all $k \in \mathbb{N}$;*
$$b_{m_k} \leq b_{m_k+1} \quad and \quad b_k \leq b_{m_k+1},$$
$$In\ \ fact, \quad m_k = \max\{j \leq k : b_j < b_{j+1}\}.$$

Lemma 2.8. *[3] Let $f : E \longrightarrow (-\infty, +\infty]$ be a strongly coercive, Legendre function which bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of $int(domf)$ and $T_i : C \longrightarrow C$ for $i = 1, 2, ..., M$ be a finite family of generic generalized Bregman nonspreading mapping such that $\Im = \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$. Then T_i is quasi Bregman nonexpansive mapping for each $i = 1, 2, \cdots, M$.*
We give the proof of Lemma 2.8 for the sake of completeness.

Proof. Since $T_i : C \longrightarrow C$ for $i = 1, 2, ..., M$ is a finite family of generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ generalized Bregman nonspreading mapping with $\Im = \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$, then for any $y \in C$, let $p \in \Im$ and replace x with p in equation (3.54) of definition (2.6), we obtain

$$\begin{aligned}\alpha D_f(p, T_i y) + \beta D_f(p, T_i y) + \gamma D_f(p, y) + \delta D_f(p, y) &\leq \varepsilon\{D_f(T_i y, p) - D_f(T_i y, p)\} \\ &\quad +\xi\{D_f(y, p) - D_f(y, p)\}\end{aligned}$$

$$\alpha D_f(p, T_i y) + \beta D_f(p, T_i y) + \gamma D_f(p, y) + \delta D_f(p, y) \leq 0.$$

This implies that

$$(\alpha + \beta)D_f(p, T_i y) + (\gamma + \delta)D_f(p, y) \leq 0.$$

Thus

$$(\alpha + \beta)D_f(p, T_i y) \leq -(\gamma + \delta)D_f(p, y).$$

Using conditions (i) and (ii) of definition (2.6), we have

$$\begin{aligned}D_f(p, T_i y) &\leq -\frac{(\gamma + \delta)}{(\alpha + \beta)} D_f(p, y) \\ &\leq D_f(p, y)\end{aligned}$$

$$D_f(p, T_i y) \leq D_f(p, y). \tag{2.14}$$

□

3. Main Results

In order to obtain strong convergence of our algorithm, we make the following assumptions:

(A1) Let E be a real reflexive Banach space and C be nonempty, closed and convex subset of E.

(A2) The operator $A : E \longrightarrow E^*$ is pseudomonotone, L - Lipschitz continuous and weakly sequentially continuous on E.

(A3) For each $i \in \{1, 2, ..., M\}, \{T_i\}$ be a finite family of generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ - generalized Bregman nonspreading mapping of E into itself such that $\hat{F}(T_i) = F(T_i)$. Assume $\Omega = F(T_M \circ T_{M-1} \circ, ..., \circ T_1) = \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$.

(A4) The solution set $\Gamma = VI(C, A) \bigcap \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$.

(A5) The function $f : E \longrightarrow \mathbb{R}$ satisfies the following:

(1) f is proper, convex and lower semi-continuous;

(2) f is uniformly Fréchet differentiable;

(3) f is strongly convex on E with strong convexity constant $\beta > 0$;

(4) f is a strongly coercive and Legendre function which is bounded on bounded subsets of E.

(A6) Assume that the control sequences satisfy:

(i) $\{\alpha_n\} \subset (0,1)$ satisfies $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) Choose a positive sequence $\{\tau_n\}$ such that $\lim\limits_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0$.

(iii) $\beta_n \in (a,b)$, where $0 < a < b < 1$.

Algorithm 3

Initialization: Take $\lambda_1 > 0, \mu \in (0,\beta)$, $\theta > 0$. Select initial data $x_0, x_1 \in E$ and set $n = 1$.

Step 1 : Given x_{n-1}, x_n and θ_n for each $n \geq 1$, choose θ_n such that $\theta_n \in [0, \hat{\theta}_n]$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\{\frac{\tau_n}{||x_n - x_{n-1}||}, \theta\}, & if \quad x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2: Compute

$$\begin{cases} u_n = \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ y_n = Proj_C^f(\nabla f^*(\nabla f(u_n) - \lambda_n A(u_n))). \end{cases} \tag{3.2}$$

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu(||u_n - y_n||^2 + ||y_n - z_n||^2)}{2\langle A(u_n) - A(y_n), z_n - y_n\rangle}, \lambda_n\}, & if \quad \langle A(u_n) - A(y_n), z_n - y_n\rangle > 0, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{3.3}$$

If $y_n = u_n$, then set $z_n = u_n$ for some $n \geq 1$. Else go to **Step 3**.

Step 3: Construct

$$S_n = \{z \in E : \langle \nabla f(u_n) - \lambda_n A(u_n) - \nabla f(y_n), z - y_n\rangle \leq 0\}$$

and Compute

$$\begin{cases} z_n = Proj_{S_n}^f(\nabla f^*(\nabla f(u_n) - \lambda_n A(y_n))), \\ w_n = \nabla f^*(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n)), \quad \forall \;\; n \geq 1. \end{cases} \tag{3.4}$$

Set $n := n+1$ and return to **Step 1**, where $T = T_M \circ T_{M-1} \circ, ..., \circ T_1$.

In order to prove the strong convergence result of Algorithm 3, we first prove the following lemma which plays an important role in the proof of the main result.

Lemma 3.1. *Suppose that $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{\lambda_n\}$ are sequences generated by Algorithm 3 and assumptions (A1) - (A6) hold, then*

(1) *If $u_n = y_n$ for some $n \geq 1$, then $u_n \in VI(C, A)$.*

(2) *The sequence* $\{\lambda_n\}$ *generated by* (3.3) *is a nonincreasing sequence and* $\lim_{n\to\infty} \lambda_n = \lambda \geq \min\{\frac{\mu}{L}, \lambda_1\}$.

Proof. (1) Suppose that $u_n = y_n$ for some $n \geq 1$. Then from Algorithm 3, we have

$$u_n = Proj_C^f(\nabla f^*(\nabla f(u_n) - \lambda_n A(u_n))).$$

Thus, $u_n \in C$. Using the definition of $\{y_n\}$ in Algorithm 3 and the property of Bregman projection $Proj_C^f$ onto C in Lemma 2.1, we have

$$\langle \nabla f(u_n) - \lambda_n A(u_n) - \nabla f(u_n), u_n - y\rangle \geq 0, \quad \forall\, y \in C.$$

Thus,

$$\langle -\lambda_n A(u_n), u_n - y\rangle = \lambda_n \langle A(u_n), y - u_n\rangle \geq 0, \quad \forall\, y \in C.$$

Since $\lambda_n \geq 0$, we obtain that $\langle A(u_n), y - u_n\rangle \geq 0$. Hence, $u_n \in VI(C, A)$.
(2) It follows from (3.3) that $\lambda_{n+1} \leq \lambda_n$, for all $n \in \mathbb{N}$. Furthermore, Since A is a Lipschitz continuous mapping with positive constant L, in a case where $A(u_n) - A(y_n) \neq 0$, and the sequence $\{\lambda_n\}$ is nonincreasing, we obtain

$$\begin{aligned} \frac{\mu(||u_n - y_n||^2 + ||y_n - z_n||^2)}{2\langle A(u_n) - A(y_n), z_n - y_n\rangle} \geq \frac{2\mu||u_n - y_n||||y_n - z_n||}{2||A(u_n) - A(y_n)||||z_n - y_n||} &\geq \frac{\mu||u_n - y_n||}{L||u_n - y_n||} \\ &= \frac{\mu}{L} \end{aligned} \tag{3.5}$$

Thus $\{\lambda_n\}$ is bounded below by $\min\{\frac{\mu}{L}, \lambda_1\}$, we conclude that

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\{\frac{\mu}{L}, \lambda_1\}.$$

□

Remark 3.1. We have from Definition 3.1 of Algorithm 3 that $\theta_n||x_n - x_{n-1}|| \leq \tau_n$ for each $n \geq 1$, which together with $\lim_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0$ implies

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \leq \lim_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0. \tag{3.6}$$

Lemma 3.2. *Suppose that assumptions* $(A1) - (A6)$ *hold, and the sequences* $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$ *and* $\{x_n\}$, *be generated by Algorithm 3 Then* $\{x_n\}$ *is bounded.*

Proof. First, we show that

$$D_f(u, z_n) \quad \leq \quad D_f(u, u_n) - (1 - \frac{\mu\lambda_n}{\beta\lambda_{n+1}})(D_f(y_n, u_n) + D_f(z_n, y_n)).$$

Let $u \in VI(C, A) \bigcap \bigcap_{i=1}^{M} F(T_i) \neq \emptyset$. Observe that $y_n \in C$, then we have $\langle A(u), y_n - u\rangle \geq 0$, for all $n \in \mathbb{N}$.
Since A is pseudomonotone, we have $\langle A(y_n), y_n - u\rangle \geq 0$, for all $n \in \mathbb{N}$. Then

$$0 \leq \langle A(y_n), y_n - u + z_n - z_n\rangle = \langle A(y_n), y_n - z_n\rangle - \langle A(y_n), u - z_n\rangle$$

which implies that

$$\langle A(y_n), y_n - z_n\rangle \geq \langle A(y_n), u - z_n\rangle, \qquad \forall \; n \in \mathbb{N}. \tag{3.7}$$

From the definition of S_n in Algorithm 3, we have that

$$\langle \nabla f(u_n) - \lambda_n A(u_n) - \nabla f(y_n), z_n - y_n \rangle \leq 0.$$

Thus,

$$\begin{aligned} & \langle \nabla f(u_n) - \lambda_n A(y_n) - \nabla f(y_n), z_n - y_n \rangle \\ = \ & \langle \nabla f(u_n) - \lambda_n A(u_n) - \nabla f(y_n), z_n - y_n \rangle + \lambda_n \langle A(u_n) - A(y_n), z_n - y_n \rangle \\ \leq \ & \lambda_n \langle A(u_n) - A(y_n), z_n - y_n \rangle \end{aligned} \tag{3.8}$$

Applying Lemma 2.1, (2.8), (2.7) and (2.3), we have

$$\begin{aligned} & D_f(u, z_n) \\ = \ & D_f(u, Proj^f_{T_n}(\nabla f^*(\nabla f(u_n) - \lambda_n A(y_n)))) \\ \leq \ & D_f(u, \nabla f^*(\nabla f(u_n) - \lambda_n A(y_n))) - D_f(z_n, \nabla f^*(\nabla f(u_n) - \lambda_n A(y_n))) \\ = \ & V_f(u, \nabla f(u_n) - \lambda_n A(y_n)) - V_f(z_n, \nabla f(u_n) - \lambda_n A(y_n)) \\ = \ & f(u) - \langle \nabla f(u_n) - \lambda_n A(y_n), u \rangle + f^*(\nabla f(u_n) - \lambda_n A(y_n)) \\ & -[f(z_n) - \langle \nabla f(u_n) - \lambda_n A(y_n), z_n \rangle + f^*(\nabla f(u_n) - \lambda_n A(y_n))] \\ = \ & f(u) - \langle \nabla f(u_n) - \lambda_n A(y_n), u \rangle - f(z_n) - \langle \nabla f(u_n) + \lambda_n A(y_n), z_n \rangle \\ = \ & f(u) - \langle \nabla f(u_n), u \rangle + \langle \lambda_n A(y_n), u \rangle - f(z_n) \\ & + \langle \nabla f(u_n), z_n \rangle - \langle \lambda_n A(y_n), z_n \rangle \\ = \ & f(u) - \langle \nabla f(u_n), z_n \rangle + f(u_n) - f(u_n) + \langle \lambda_n A(y_n), u \rangle \\ & - f(z_n) + \langle \nabla f(u_n), z_n \rangle - \langle \lambda_n A(y_n), z_n \rangle \\ = \ & D_f(u, u_n) - [f(z_n) - f(u_n) - \langle \nabla f(u_n), z_n \rangle] \\ & + \langle \lambda_n A(y_n), u \rangle - \langle \lambda_n A(y_n), z_n \rangle \\ = \ & D_f(u, u_n) - D_f(z_n, u_n) + \langle \lambda_n A(y_n), u - z_n \rangle \\ = \ & D_f(u, u_n) - D_f(z_n, u_n) + \lambda_n \langle A(y_n), u - z_n \rangle \\ \leq \ & D_f(u, u_n) - D_f(z_n, u_n) + \lambda_n \langle A(y_n), u - y_n \rangle + \lambda_n \langle A(y_n), y_n - z_n \rangle \end{aligned} \tag{3.9}$$

Using equation (2.3) and (3.8), we obtain

$$\begin{aligned} & D_f(u, z_n) \\ \leq \ & D_f(u, u_n) - D_f(z_n, u_n) + \lambda_n \langle A(y_n), u - y_n \rangle + \lambda_n \langle A(y_n), y_n - z_n \rangle \\ = \ & D_f(u, u_n) - [D_f(z_n, y_n) + D_f(y_n, u_n) + \langle \nabla f(y_n) - \nabla f(u_n), z_n - y_n \rangle] \\ & + \lambda_n \langle A(y_n), y_n - z_n \rangle \\ = \ & D_f(u, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) - \langle \nabla f(y_n) - \nabla f(u_n), z_n - y_n \rangle \\ & + \lambda_n \langle A(y_n), y_n - z_n \rangle \\ \leq \ & D_f(u, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) - \langle \nabla f(y_n) - \lambda_n A(y_n) - \nabla f(u_n), y_n - z_n \rangle \\ = \ & D_f(u, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) + \langle \nabla f(y_n) - \lambda_n A(y_n) - \nabla f(u_n), z_n - y_n \rangle \\ \leq \ & D_f(u, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) + \lambda_n \langle A(u_n) - A(y_n), z_n - y_n \rangle \\ \leq \ & D_f(u, u_n) - D_f(z_n, y_n) - D_f(y_n, u_n) + \frac{\lambda_n \mu}{2\lambda_{n+1}}(||u_n - y_n||^2 + ||z_n - y_n||^2) \end{aligned}$$

$$\begin{aligned}
&\leq D_f(u,u_n) - D_f(z_n,y_n) - D_f(y_n,u_n) + \frac{\lambda_n\mu}{\beta\lambda_{n+1}}(D_f(u_n,y_n) + D_f(z_n,y_n)) \\
&= D_f(u,u_n) - (1 - \frac{\lambda_n\mu}{\beta\lambda_{n+1}})(D_f(u_n,y_n) + D_f(z_n,y_n)) \qquad (3.10)
\end{aligned}$$

Applying Lemma 3.1 (2), we have that $\mu \in (0,\beta)$, $\lim_{n\to\infty}(1 - \frac{\lambda_n\mu}{\beta\lambda_{n+1}}) = 1 - \frac{\mu}{\beta} > 0$. This implies that, there exists a positive integer $N_0 > 0$ such that for all $n > N_0$, $(1 - \frac{\lambda_n\mu}{\beta\lambda_{n+1}}) > 0$. Hence, from (3.10), we have

$$D_f(u,z_n) \leq D_f(u,u_n) \qquad (3.11)$$

Using the definition of $\{u_n\}$ in Algorithm 3, we obtain

$$\begin{aligned}
D_f(u,u_n) &= D_f(u,\nabla f^*((1-\theta_n)\nabla f(x_n) + \theta_n\nabla f(x_{n-1})) \\
&= V_f(u,(1-\theta_n)\nabla f(x_n) + \theta_n\nabla f(x_{n-1})) \\
&= f(u) - \langle(1-\theta_n)\nabla f(x_n) + \theta_n\nabla f(x_{n-1}),u\rangle \\
&\quad + f^*((1-\theta_n)\nabla f(x_n) + \theta_n\nabla f(x_{n-1})) \\
&\leq (1-\theta_n)D_f(u,x_n) + \theta_n D_f(u,x_{n-1}) \qquad (3.12)
\end{aligned}$$

From the definition of $\{w_n\}$ in Algorithm 3, we have

$$\begin{aligned}
D_f(u,w_n) &= D_f(u,\nabla f^*(\alpha_n\nabla f(x_0) + (1-\alpha_n)\nabla f(z_n))) \\
&= V_f(u,\alpha_n\nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)) \\
&= f(u) - \langle\alpha_n\nabla f(x_0) + (1-\alpha_n)\nabla f(z_n),u\rangle \\
&\quad + f^*(\alpha_n\nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)) \\
&\leq \alpha_n D_f(u,x_0) + (1-\alpha_n)D_f(u,z_n) \qquad (3.13)
\end{aligned}$$

Using the definition of $\{x_{n+1}\}$ in Algorithm 3, we have

$$\begin{aligned}
D_f(u,x_{n+1}) &= D_f(u,\nabla f^*(\beta_n\nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n))) \\
&= V_f(u,\beta_n\nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n)) \\
&= f(u) - \langle\beta_n\nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n),u\rangle \\
&\quad + f^*(\beta_n\nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n)) \\
&= f(u) - \beta_n\langle\nabla f(z_n),u\rangle - (1-\beta_n)\langle\nabla f(Tw_n),u\rangle \\
&\quad + \beta_n f^*(\nabla f(z_n)) + (1-\beta_n)f^*(\nabla f(Tw_n))
\end{aligned}$$

$$\begin{aligned}
D_f(u,x_{n+1}) &\leq \beta_n f(u) + (1-\beta_n)f(u) - \beta_n\langle\nabla f(z_n),u\rangle - (1-\beta_n)\langle\nabla f(Tw_n),u\rangle \\
&\quad + \beta_n f^*(\nabla f(z_n)) + (1-\beta_n)f^*(\nabla f(Tw_n)) \\
&\leq \beta_n D_f(u,z_n) + (1-\beta_n)D_f(u,Tw_n) \\
&= \beta_n D_f(u,z_n) + (1-\beta_n)D_f(u,T_M \circ T_{M-1}\circ,\ldots,\circ T_1(w_n)) \\
&\leq \beta_n D_f(u,z_n) + (1-\beta_n)D_f(u,T_{M-1}\circ,\ldots,\circ T_1(w_n)) \\
&\leq \beta_n D_f(u,z_n) + (1-\beta_n)D_f(u,w_n) \qquad (3.14)
\end{aligned}$$

Substituting (3.12) and (3.13) into (3.14), we have

$$\begin{aligned}
D_f(u, x_{n+1}) &\leq \beta_n D_f(u, z_n) + (1-\beta_n)[\alpha_n D_f(u, x_0) + (1-\alpha_n) D_f(u, z_n)] \\
&= \beta_n D_f(u, z_n) + \alpha_n(1-\beta_n) D_f(u, x_0) + (1-\beta_n)(1-\alpha_n) D_f(u, z_n) \\
&= (\beta_n + (1-\beta_n)(1-\alpha_n)) D_f(u, z_n) + \alpha_n(1-\beta_n) D_f(u, x_0) \\
&= (\beta_n - \beta_n + 1 - \alpha_n + \alpha_n\beta_n) D_f(u, z_n) + \alpha_n(1-\beta_n) D_f(u, x_0) \\
&= (1-(1-\beta_n)\alpha_n) D_f(u, z_n) + (1-\beta_n)\alpha_n D_f(u, x_0) \\
&\leq (1-(1-\beta_n)\alpha_n)[(1-\theta_n) D_f(u, x_n) + \theta_n D_f(u, x_{n-1})] \\
&\quad + (1-\beta_n)\alpha_n D_f(u, x_0) \qquad (3.15)
\end{aligned}$$

Thus, we obtain,

$$\begin{aligned}
D_f(u, x_{n+1}) &\leq \max\{D_f(u, x_0), \max\{D_f(u, x_n), D_f(u, x_{n-1})\}\} \\
&\vdots \\
&\leq \max\{D_f(u, x_0), \max\{D_f(u, x_1), D_f(u, x_0)\}\} \qquad (3.16)
\end{aligned}$$

Hence, $\{D_f(u, x_n)\}$ is bounded. Since $\frac{\beta}{2}||x_n - u||^2 \leq D_f(u, x_n)$, we have that $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are also bounded. □

Theorem 3.3. *Suppose that assumptions* $(A1)-(A6)$ *holds, and the sequence* $\{\alpha_n\} \subset (0, 1)$ *satisfy* $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Let* $\{x_n\}$ *be the sequence generated by Algorithm 3. Then* $\{x_n\}$ *converges strongly to a solution*

$$x^* = Proj^f_{VI(C,A)\ \cap\ \cap_{i=1}^M F(T_i)} x_0.$$

Proof. Let $x^* = Proj^f_{VI(C,A)\ \cap\ \cap_{i=1}^M F(T_i)} x_0$. From Lemma 2.1, we have

$$\langle \nabla f(x_0) - \nabla f(x^*), z - x^* \rangle \leq 0, \qquad \forall\ z \in VI(C, A)$$

From Lemma 3.2, we have that, there exists $N_0 \geq 0$, such that for all $n \geq N_0$,

$$D_f(u, z_n) \leq D_f(u, u_n) \ \textit{ and } \ D_f(u, u_n) \leq (1-\theta_n) D_f(u, x_n) + \theta_n D_f(u, x_{n-1}),$$

and the following sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are bounded. Furthermore, we estimate $D_f(u, x_{n+1})$ using inequality (3.10) for every $n \geq N_0$.

$$\begin{aligned}
D_f(u, x_{n+1}) &= D_f(u, \nabla f^*(\beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n))) \\
&= V_f(u, \beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n)) \\
&= \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, Tw_n) \\
&= \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, T_M \circ T_{M-1}\circ, ..., \circ T_1(w_n)) \\
&\leq \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, T_{M-1}\circ, ..., \circ T_1(w_n)) \\
&\leq \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, w_n) \\
&\leq \beta_n D_f(u, z_n) + (1-\beta_n)[\alpha_n D_f(u, x_0) + (1-\alpha_n) D_f(u, z_n)] \\
&= \beta_n D_f(u, z_n) + (1-\beta_n)(1-\alpha_n) D_f(u, z_n) + \alpha_n(1-\beta_n) D_f(u, x_0) \\
&= (\beta_n + (1-\beta_n)(1-\alpha_n)) D_f(u, z_n) + \alpha_n(1-\beta_n) D_f(u, x_0)
\end{aligned}$$

$$
\begin{aligned}
&= (\beta_n + (1-\beta_n)(1-\alpha_n))(D_f(u,u_n) \\
&\quad -(1-\frac{\lambda_n\mu}{\beta\lambda_{n+1}})(D_f(u_n,y_n)+D_f(y_n,z_n))) + \alpha_n(1-\beta_n)D_f(u,x_0) \\
&= (1-(1-\beta_n)\alpha_n)(D_f(u,u_n) - (1-\frac{\lambda_n\mu}{\beta\lambda_{n+1}})(D_f(u_n,y_n)+D_f(y_n,z_n))) \\
&\quad +\alpha_n(1-\beta_n)D_f(u,x_0) \\
&\leq D_f(u,u_n) - (1-\frac{\lambda_n\mu}{\beta\lambda_{n+1}})(D_f(u_n,y_n)+D_f(y_n,z_n)) \\
&\quad +\alpha_n(1-\beta_n)D_f(u,x_0) \\
&\leq (1-\theta_n)D_f(u,x_n) + \theta_n D_f(u,x_{n-1}) \\
&\quad -(1-\frac{\lambda_n\mu}{\beta\lambda_{n+1}})(D_f(u_n,y_n)+D_f(y_n,z_n)) + \alpha_n(1-\beta_n)D_f(u,x_0) \\
&= D_f(u,x_n) - \theta_n D_f(u,x_n) + \theta_n D_f(u,x_{n-1}) \\
&\quad -(1-\frac{\lambda_n\mu}{\beta\lambda_{n+1}})(D_f(u_n,y_n)+D_f(y_n,z_n)) + \alpha_n(1-\beta_n)D_f(u,x_0) \\
&= D_f(u,x_n) + \theta_n(D_f(u,x_{n-1}) - D_f(u,x_n)) \\
&\quad -(1-\frac{\lambda_n\mu}{\beta\lambda_{n+1}})(D_f(u_n,y_n)+D_f(y_n,z_n)) + \alpha_n(1-\beta_n)D_f(u,x_0) \qquad (3.17)
\end{aligned}
$$

The remaining part of the proof will be divided into two cases.
Case I. Suppose that there exists $N_1 \in \mathbb{N}$ $(N_1 \geq N_0)$ such that $\{D_f(u,x_n)\}_{n=N_1}^{\infty}$ is nonincreasing. Since the sequence $\{D_f(u,x_n)\}_{n=1}^{\infty}$ is bounded then it converges for all $n > N_1 \geq N_0$. That is,

$$
\lim_{n\to\infty}(D_f(u,x_n) - D_f(u,x_{n+1})) = \lim_{n\to\infty}(D_f(u,x_{n-1}) - D_f(u,x_n)) = 0 \qquad (3.18)
$$

This implies from (3.17) that

$$
\begin{aligned}
(1-\frac{\lambda_n\mu}{\beta\lambda_{n+1}})(D_f(u_n,y_n)+D_f(y_n,z_n)) &\leq \theta_n(D_f(u,x_{n-1}) - D_f(u,x_n)) \\
&\quad +D_f(u,x_n) - D_f(u,x_{n+1}) + \alpha_n(1-\beta_n)D_f(u,x_0) \qquad (3.19)
\end{aligned}
$$

Using (3.17), (3.18) and the fact that $(1-\frac{\lambda_n\mu}{\beta\lambda_{n+1}}) > 0, (1-\beta_n) > 0$ together with $\lim_{n\to\infty}\alpha_n = 0$, we have from (3.19) that

$$
\begin{aligned}
(1-\frac{\lambda_n\mu}{\beta\lambda_{n+1}})(D_f(u_n,y_n)+D_f(y_n,z_n)) &\leq \theta_n(D_f(u,x_{n-1}) - D_f(u,x_n)) \\
&\quad +D_f(u,x_n) - D_f(u,x_{n+1}) + \alpha_n(1-\beta_n)D_f(u,x_0) \longrightarrow 0, \quad as \ \ n \longrightarrow \infty.
\end{aligned}
$$

Hence,

$$
\lim_{n\to\infty} D_f(u_n,y_n) = \lim_{n\to\infty} D_f(y_n,z_n) = 0 \qquad (3.20)
$$

Thus, from Lemma 2.3, we have that

$$
\lim_{n\to\infty} ||u_n - y_n|| = \lim_{n\to\infty} ||y_n - z_n|| = 0 \qquad (3.21)
$$

Using Lemma 2.2 and the definitions of $\{x_{n+1}\}, \{u_n\}, \{w_n\}$, we obtain

$$\begin{aligned}
D_f(u, x_{n+1}) &= D_f(u, \nabla f^*(\beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n))) \\
&= V_f(u, \beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n)) \\
&\leq \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, Tw_n) \\
&\quad -\beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \\
&= \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, T_M \circ T_{M-1}\circ, ..., \circ T_1(w_n)) \\
&\quad -\beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \\
&\leq \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, T_{M-1}\circ, ..., \circ T_1(w_n)) \\
&\quad -\beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \\
&\leq \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, w_n) \\
&\quad -\beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \\
&\leq \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, \nabla f^*(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n))) \\
&\quad -\beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \\
&\leq \beta_n D_f(u, z_n) + (1-\beta_n)[\alpha_n D_f(u, x_0) + (1-\alpha_n) D_f(u, z_n)] \\
&\quad -\beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \\
&\leq (1-(1-\beta_n)\alpha_n) D_f(u, z_n) + (1-\beta_n)\alpha_n D_f(u, x_0) \\
&\quad -\beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \\
&\leq (1-(1-\beta_n)\alpha_n) D_f(u, u_n) + (1-\beta_n)\alpha_n D_f(u, x_0) \\
&\quad -\beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||)
\end{aligned}$$

$$\begin{aligned}
D_f(u, x_{n+1}) &\leq (1-(1-\beta_n)\alpha_n)[(1-\theta_n) D_f(u, x_n) + \theta_n D_f(u, x_{n-1})] \\
&\quad +(1-\beta_n)\alpha_n D_f(u, x_0) - \beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \\
&= (1-\theta_n) D_f(u, x_n) + \theta_n D_f(u, x_{n-1}) \\
&\quad -(1-\beta_n)\alpha_n[(1-\theta_n) D_f(u, x_n) + \theta_n D_f(u, x_{n-1})] \\
&\quad +(1-\beta_n)\alpha_n D_f(u, x_0) - \beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \\
&= D_f(u, x_n) + \theta_n(D_f(u, x_{n-1}) - D_f(u, x_n)) + (1-\beta_n)\alpha_n D_f(u, x_0) \\
&\quad -(1-\beta_n)\alpha_n[(1-\theta_n) D_f(u, x_n) + \theta_n D_f(u, x_{n-1})] \\
&\quad -\beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \qquad (3.22)
\end{aligned}$$

This implies from (3.22) that

$$\begin{aligned}
0 &\leq \beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \leq D_f(u, x_n) - D_f(u, x_{n+1}) \\
&\quad +\theta_n(D_f(u, x_{n-1}) - D_f(u, x_n)) + (1-\beta_n)\alpha_n D_f(u, x_0) \\
&\quad -(1-\beta_n)\alpha_n[(1-\theta_n) D_f(u, x_n) + \theta_n D_f(u, x_{n-1})] \qquad (3.23)
\end{aligned}$$

From (3.18) and the fact that $(1-\beta_n) > 0$ together with $\lim_{n\to\infty} \alpha_n = 0$, we have from (3.23)

$$\begin{aligned}
&\beta_n(1-\beta_n)\rho_r^*(||\nabla f(z_n) - \nabla f(Tw_n)||) \\
\leq\ & D_f(u, x_n) - D_f(u, x_{n+1}) + \theta_n(D_f(u, x_{n-1}) - D_f(u, x_n)) \\
&-(1-\beta_n)\alpha_n[(1-\theta_n) D_f(u, x_n) + \theta_n D_f(u, x_{n-1})] + (1-\beta_n)\alpha_n D_f(u, x_0) \\
&\longrightarrow 0, \quad as \ \ n \longrightarrow \infty. \qquad (3.24)
\end{aligned}$$

Thus, using the property of ρ_r^* in Lemma 2.2,we have

$$\lim_{n\to\infty} ||\nabla f(z_n) - \nabla f(Tw_n)|| = 0. \tag{3.25}$$

Since f is uniformly Fréchet differentiable, then ∇f^* is uniformly norm to norm continuous on bounded subsets of E^*. Hence, we have from (3.25) that

$$\lim_{n\to\infty} ||z_n - Tw_n|| = 0. \tag{3.26}$$

Using the definition of $\{x_{n+1}\}$ in Algorithm 3, we have

$$\begin{aligned} \nabla f(x_{n+1}) &= \beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n) \\ \nabla f(x_{n+1}) - \nabla f(z_n) &= (\beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n)) - \nabla f(z_n) \end{aligned}$$

$$\begin{aligned} & ||\nabla f(x_{n+1}) - \nabla f(z_n)|| \\ = \;& ||\beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n) - [\beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(z_n)]|| \\ = \;& ||\beta_n \nabla f(z_n) - \beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n) - (1-\beta_n)\nabla f(z_n)|| \\ = \;& ||(1-\beta_n)(\nabla f(Tw_n) - \nabla f(z_n))|| \\ = \;& (1-\beta_n)||\nabla f(Tw_n) - \nabla f(z_n)||. \end{aligned} \tag{3.27}$$

Thus, from (3.25) and the fact that f is uniformly Fréchet differentiable, then ∇f^* is uniformly norm to norm continuous on bounded subsets of E^*, hence, we have from (3.27) that

$$\lim_{n\to\infty} ||x_{n+1} - z_n|| = 0. \tag{3.28}$$

From the definition of $\{w_n\}$ in Algorithm 3, we have

$$\begin{aligned} w_n &= \nabla f^*(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)) \\ \nabla f(w_n) &= \alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n) \\ \nabla f(w_n) - \nabla f(z_n) &= (\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)) - \nabla f(z_n) \end{aligned}$$

$$\begin{aligned} & |\nabla f(w_n) - \nabla f(z_n)|| \\ = \;& ||\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n) - (\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(z_n))|| \\ = \;& ||\alpha_n \nabla f(x_0) - \alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(z_n) - (1-\alpha_n)\nabla f(z_n))|| \\ = \;& \alpha_n ||\nabla f(x_0) - \nabla f(z_n)|| \end{aligned}$$

Now, using the fact that $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$\lim_{n\to\infty} ||\nabla f(w_n) - \nabla f(z_n)|| = 0 \tag{3.29}$$

Since f is uniformly Fréchet differentiable, then ∇f^* is uniformly norm to norm continuous on bounded subsets of E^*, we have from (3.29) that

$$\lim_{n\to\infty} ||w_n - z_n|| = 0 \tag{3.30}$$

Thus from (3.30), (3.26), we get

$$\begin{aligned} ||Tw_n - w_n|| &= ||Tw_n - z_n + z_n - w_n|| \\ &\leq ||Tw_n - z_n|| + ||z_n - w_n|| \longrightarrow 0, \; as\ n \longrightarrow \infty. \end{aligned} \tag{3.31}$$

Hence,

$$\lim_{n\to\infty} ||Tw_n - w_n|| = 0 \tag{3.32}$$

From the definition of $\{u_n\}$ in Algorithm 3 and Remark 3.1, we obtain

$$\begin{aligned} u_n &= \nabla f^*(\nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))) \\ \nabla f(u_n) &= \nabla f(x_n) + \theta_n(\nabla f(x_{n-1}) - \nabla f(x_n)) \\ ||\nabla f(u_n) - \nabla f(x_n)|| &= ||\theta_n(\nabla f(x_{n-1}) - \nabla f(x_n))|| \\ &= \alpha_n.\frac{\theta_n}{\alpha_n}||\nabla f(x_{n-1}) - \nabla f(x_n)|| \longrightarrow 0, \quad as\ \ n \longrightarrow \infty \end{aligned} \tag{3.33}$$

Hence,

$$\lim_{n\to\infty} ||\nabla f(u_n) - \nabla f(x_n)|| = 0 \tag{3.34}$$

Since f is uniformly Fréchet differentiable , then ∇f^* is uniformly norm to norm continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} ||u_n - x_n|| = 0 \tag{3.35}$$

Furthermore, we have from (3.35) and (3.21) that

$$\begin{aligned} ||y_n - x_n|| &= ||y_n - u_n + u_n - x_n|| \\ &\leq ||y_n - u_n|| + ||u_n - x_n|| \longrightarrow 0, \quad as\ \ n \longrightarrow \infty. \end{aligned} \tag{3.36}$$

Thus,

$$\lim_{n\to\infty} ||y_n - x_n|| = 0 \tag{3.37}$$

From (3.37), (3.28) and (3.21), we obtain

$$\begin{aligned} ||x_{n+1} - x_n|| &= ||x_{n+1} - z_n + z_n - y_n + y_n - x_n|| \\ &\leq ||x_{n+1} - z_n|| + ||z_n - y_n|| + ||y_n - x_n|| \longrightarrow 0, \ \ as\ n \longrightarrow \infty \end{aligned} \tag{3.38}$$

Hence,

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0 \tag{3.39}$$

Furthermore, from (3.39), (3.28) and (3.30), we have

$$\begin{aligned} ||x_n - w_n|| &= ||x_n - x_{n+1} + x_{n+1} - z_n + z_n - w_n|| \\ &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - z_n|| + ||z_n - w_n|| \longrightarrow 0, \ as\ n \longrightarrow \infty \end{aligned} \tag{3.40}$$

Thus,

$$\lim_{n\to\infty} ||x_n - w_n|| = 0 \tag{3.41}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup u^*$, which implies that $w_{n_k} \rightharpoonup u^*$ as $k \longrightarrow \infty$. Since $\lim\limits_{k\to\infty} ||Tw_{n_k} - w_{n_k}|| = 0$, it follows that $u^* \in F(T)$.

Next, we show that $u^* \in VI(C, A)$.

We have $\{u_{n_k}\}$ converges weakly to $u^* \in C$ since $||x_{n_k} - u_{n_k}|| \longrightarrow 0\ as\ k \longrightarrow \infty$, then $y_{n_k} \rightharpoonup u^*$ since $||y_{n_k} - u_{n_k}|| \longrightarrow 0$ *as* $k \longrightarrow \infty$. From the definition of $y_{n_k} = Proj_C^f(\nabla f(u_{n_k}) - \lambda_{n_k} A(u_{n_k}))$, we have from Lemma 2.1 that for all $z \in C$,

$$\langle \nabla f(u_{n_k}) - \lambda_{n_k} A(u_{n_k}) - \nabla f(y_{n_k}), z - y_{n_k} \rangle \leq 0.$$

This implies that

$$\langle \nabla f(u_{n_k}) - \nabla f(y_{n_k}), z - y_{n_k} \rangle \leq \lambda_{n_k} \langle A(u_{n_k}), z - y_{n_k} \rangle.$$

Then for all $z \in C$, we have

$$\frac{1}{\lambda_{n_k}} \langle \nabla f(u_{n_k}) - \nabla f(y_{n_k}), z - y_{n_k} \rangle + \langle A(u_{n_k}), y_{n_k} - u_{n_k} \rangle \leq \langle A(u_{n_k}), z - u_{n_k} \rangle \tag{3.42}$$

Fixing $z \in C$ and letting $k \longrightarrow +\infty$ in (3.42) since $||y_{n_k} - u_{n_k}|| \longrightarrow 0$ *as* $k \longrightarrow \infty$ and ∇f is uniformly norm to norm continuous on bounded subsets of E, then $\underset{n\to\infty}{lim} ||\nabla f(y_{n_k}) - \nabla f(u_{n_k})|| = 0$. Now, considering the fact that $\underset{n\to\infty}{lim} \inf \lambda_{n_k} > 0$, we have

$$\liminf_{n\to\infty} \langle A(u_{n_k}), z - u_{n_k} \rangle \geq 0 \tag{3.43}$$

Let $\{\varepsilon_k\}$ be a decreasing nonnegative sequence such that $\underset{n\to\infty}{lim} \varepsilon_k = 0$. For each ε_k, we denote the smallest positive integer N_k such that for all $k \geq N_k$,

$$\langle A(u_{n_k}), z - u_{n_k} \rangle + \varepsilon_k \geq 0 \tag{3.44}$$

Furthermore, as $\{\varepsilon_k\}$ is decreasing, $\{N_k\}$ is increasing. Thus, if there exists a subsequence $\{u_{n_{k_i}}\} \subset \{u_{n_k}\}$, such that for each $i \geq 1, A(u_{n_{k_i}}) \neq 0$, and setting

$$s_{n_{k_i}} = \frac{A(u_{n_{k_i}})}{||A(u_{n_{k_i}})||^2},$$

we have $\langle A(u_{n_{k_i}}), s_{n_{k_i}} \rangle = 1$ for each $i \geq 1$. It follows from (3.44) that for each $i \geq 1$

$$\langle A(u_{n_{k_i}}), z + \varepsilon_k s_{n_{k_i}} - u_{n_{k_i}} \rangle \geq 0. \tag{3.45}$$

Thus, since A is pseudomonotone, we obtain from (3.45)

$$\langle A(z + \varepsilon_k s_{n_{k_i}}), z + \varepsilon_k s_{n_{k_i}} - u_{n_{k_i}} \rangle \geq 0. \tag{3.46}$$

Since $\{u_{n_k}\}$ converges weakly to $u^* \in C$, and A is weakly sequentially continuous on C, we have that $A(u_{n_k})$ converges weakly to $A(u^*)$. If $A(u^*) = 0$, then $u^* \in VI(C, A)$. Suppose that $A(u^*) \neq 0$. Then, by sequentially weakly lower semicontinuity of the norm, we have the following

$$0 < ||A(u^*)|| \leq \liminf_{k\to\infty} ||A(u_{n_k})||.$$

Since $\{u_{n_{k_i}}\} \subset \{u_{n_k}\}$ and $\varepsilon_k \longrightarrow 0$ as $k \longrightarrow \infty$, we obtain

$$0 \leq \limsup_{k\to\infty} ||\varepsilon_k s_{n_k}|| = \limsup_{k\to\infty} (\frac{\varepsilon_k}{||A(u_{n_k})||}) \leq \frac{\limsup_{k\to\infty} \varepsilon_k}{\liminf_{k\to\infty} ||A(u_{n_k})||} \leq \frac{0}{||A(u^*)||} = 0$$

Thus, $\underset{k\to\infty}{lim} \varepsilon_k = 0$ and hence, taking limit as $k \longrightarrow \infty$ in (3.46) we obtain

$$\liminf_{k\to\infty} \langle A(z), z - u_{n_k} \rangle \geq 0.$$

Therefore,

$$\langle A(z), z - u^* \rangle = \underset{k\to\infty}{lim} \langle A(z), z - u_{n_k} \rangle = \liminf_{k\to\infty} \langle A(z), z - u_{n_k} \rangle \geq 0, \quad \forall \ z \in C.$$

Hence,

$$\langle A(z), z - u^* \rangle \geq 0.$$

Thus, it follows from Lemma 2.5 that $u^* \in VI(C, A)$. Furthermore, from (3.32) and (3.41) we have that $u^* \in F(T)$. Hence, $u^* \in \Gamma$.
Next, we show that $\{x_n\}$ converges strongly to a point $x^* = Proj_{\Gamma}^{f} x_0$. Since $\{x_n\}$ is bounded, then, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup u^*$ and

$$\begin{aligned}\limsup_{n\to\infty} \langle \nabla f(x_0) - \nabla f(x^*), x_n - x^* \rangle &= \lim_{k\to\infty} \langle \nabla f(x_0) - \nabla f(x^*), x_{n_k} - x^* \rangle \\ &= \langle \nabla f(x_0) - \nabla f(x^*), u^* - x^*, \rangle \qquad (3.47)\end{aligned}$$

Thus, from Lemma 2.1 and (3.47), we have

$$\limsup_{n\to\infty} \langle \nabla f(x_0) - \nabla f(x^*), x_n - x^* \rangle = \langle \nabla f(x_0) - \nabla f(x^*), u^* - x^*, \rangle \leq 0 \qquad (3.48)$$

Hence, it follows from (3.48) that

$$\limsup_{n\to\infty} \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle \leq 0 \qquad (3.49)$$

Furthermore, from the definition of $D_f(x^*, x_{n+1})$ in Algorithm 3 Lemma 2.8 and inequality (2.10) of Lemma 2.1, we obtain

$$\begin{aligned}
& D_f(u, x_{n+1}) \\
= \; & D_f(u, \nabla f^*(\beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n))) \\
= \; & V_f(u, \beta_n \nabla f(z_n) + (1-\beta_n)\nabla f(Tw_n)) \\
= \; & \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, Tw_n) \\
= \; & \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, T_M \circ T_{M-1}\circ, ..., \circ T_1(w_n)) \\
\leq \; & \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, T_{M-1}\circ, ..., \circ T_1(w_n)) \\
\leq \; & \beta_n D_f(u, z_n) + (1-\beta_n) D_f(u, w_n) \\
\leq \; & \beta_n D_f(x^*, z_n) + (1-\beta_n)[D_f(x^*, \nabla f^*(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)))] \\
\leq \; & \beta_n D_f(x^*, z_n) + (1-\beta_n)[V_f(x^*, \alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n))] \\
\leq \; & \beta_n D_f(x^*, z_n) \\
& + (1-\beta_n)[V_f(x^*, \alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n) - \alpha_n(\nabla f(x_0) - \nabla f(x^*))) \\
& - \langle \nabla f^*(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n))) - x^*, -\alpha_n(\nabla f(x_0) - \nabla f(x^*)) \rangle] \\
= \; & \beta_n D_f(x^*, z_n) + (1-\beta_n)[V_f(x^*, \alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n) - \alpha_n \nabla f(x_0) \\
& + \alpha_n \nabla f(x^*)) + \alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle] \\
= \; & \beta_n D_f(x^*, z_n) + (1-\beta_n)[V_f(x^*, \alpha_n \nabla f(x^*) + (1-\alpha_n)\nabla f(z_n)) \\
& + \alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle]
\end{aligned}$$

$$
\begin{aligned}
& D_f(x^*, x_{n+1}) \\
\leq\ & \beta_n D_f(x^*, z_n) + (1-\beta_n)(D_f(x^*, \nabla f^*[\alpha_n \nabla f(x^*) + (1-\alpha_n)\nabla f(z_n)]) \\
& +\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle) \\
=\ & \beta_n D_f(x^*, z_n) + (1-\beta_n)[\alpha_n D_f(x^*, x^*) + (1-\alpha_n) D_f(x^*, z_n) \\
& +\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle] \\
\leq\ & \beta_n D_f(x^*, z_n) \\
& +(1-\beta_n)[(1-\alpha_n) D_f(x^*, z_n) + \alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle] \\
=\ & \beta_n D_f(x^*, z_n) + (1-\beta_n)(1-\alpha_n) D_f(x^*, z_n) \\
& +(1-\beta_n)\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle \\
=\ & (\beta_n + (1-\beta_n)(1-\alpha_n)) D_f(x^*, z_n) + (1-\beta_n)\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle \\
\leq\ & (\beta_n + (1-\beta_n)(1-\alpha_n)) D_f(x^*, u_n) + (1-\beta_n)\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle \\
=\ & (1-(1-\beta_n)\alpha_n) D_f(x^*, u_n) + (1-\beta_n)\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle \\
=\ & (1-\theta_n) D_f(x^*, x_n) + \theta_n D_f(x^*, x_{n-1}) - (1-\beta_n)\alpha_n[(1-\theta_n) D_f(x^*, x_n) \\
& +\theta_n D_f(x^*, x_{n-1})] + (1-\beta_n)\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle \\
=\ & D_f(x^*, x_n) + \theta_n (D_f(x^*, x_{n-1}) - D_f(x^*, x_n)) - \alpha_n[(1-\theta_n) D_f(x^*, x_n) \\
& +\theta_n D_f(x^*, x_{n-1})] + (1-\beta_n)\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle \\
& +\alpha_n \beta_n [(1-\theta_n) D_f(x^*, x_n) + \theta_n D_f(x^*, x_{n-1})] \\
=\ & D_f(x^*, x_n) + \theta_n (D_f(x^*, x_{n-1}) - D_f(x^*, x_n)) - \alpha_n D_f(x^*, x_n) \\
& +\alpha_n \beta_n D_f(x^*, x_n) - \alpha_n[\theta_n (D_f(x^*, x_{n-1}) - D_f(x^*, x_n))] \\
& +\alpha_n \beta_n [\theta_n (D_f(x^*, x_{n-1}) - D_f(x^*, x_n))] \\
& +(1-\beta_n)\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle \\
=\ & D_f(x^*, x_n) + \theta_n (D_f(x^*, x_{n-1}) - D_f(x^*, x_n)) - \alpha_n (1-\beta_n) D_f(x^*, x_n) \\
& -\alpha_n (1-\beta_n)[\theta_n (D_f(x^*, x_{n-1}) - D_f(x^*, x_n))] \\
& +(1-\beta_n)\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle \\
=\ & (1-(1-\beta_n)\alpha_n) D_f(x^*, x_n) + \theta_n (D_f(x^*, x_{n-1}) - D_f(x^*, x_n)) \\
& -\alpha_n (1-\beta_n)[\theta_n (D_f(x^*, x_{n-1}) - D_f(x^*, x_n))] \\
& +(1-\beta_n)\alpha_n \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle
\end{aligned}
\tag{3.50}
$$

Setting $\psi_n = (1-\beta_n)\alpha_n$, $\sigma_n = \langle \nabla f(x_0) - \nabla f(x^*), w_n - x^* \rangle$ and $\gamma_n = [1 - \alpha_n(1 - \beta_n)]\theta_n(D_f(x^*, x_{n-1}) - D_f(x^*, x_n))$.
Now, applying Lemma 2.6, (3.49), (3.50) and from the fact that $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$\lim_{n\to\infty} D_f(x^*, x_n) = 0 \tag{3.51}$$

Thus, from Lemma 2.3, we have

$$\lim_{n\to\infty} ||x^* - x_n|| = 0 \tag{3.52}$$

Hence, $x_n \longrightarrow x^*$ where $x^* = Proj^f_\Gamma x_0$ and $\Gamma = VI(C, A) \bigcap \bigcap_{i=1}^M F(T_i) \neq \emptyset$.
Case II. Suppose that the sequence $\{D_f(p, x_n)\}_{n=1}^\infty$ is not a nonincreasing sequence. Then, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$D_f(p, x_{n_k}) < D_f(p, x_{n_k+1}), \quad for \ \ all \quad k \in \mathbb{N}.$$

Then, using Lemma 2.7, there exists a nondecreasing sequence $\{m_s\} \subseteq \mathbb{N}$ such that $m_s \longrightarrow \infty$ as $s \longrightarrow \infty$, then,

$$D_f(p, x_{m_s}) \leq D_f(p, x_{m_s+1}) \quad \textit{and} \quad D_f(p, x_s) \leq D_f(p, x_{m_s+1}).$$

Since $\{D_f(p, x_{m_s})\}$ is bounded, then $\underset{s\to\infty}{lim} D_f(p, x_{m_s})$ exist.
Therefore, using the same approach as in case (I), we have the following

$$(i) \underset{s\to\infty}{lim} ||x_{m_s} - w_{m_s}|| = 0, (ii) \underset{s\to\infty}{lim} ||u_{m_s} - y_{m_s}|| = 0,$$
$$(iii) \underset{s\to\infty}{lim} ||z_{m_s} - y_{m_s}|| = 0, (iv) \underset{s\to\infty}{lim} ||x_{m_s+1} - x_{m_s}|| = 0.$$

Now, following the same steps as in the proof of case (I), we obtain

$$\limsup_{s\to\infty} \langle \nabla f(x_0) - \nabla f(x^*), w_{m_s+1} - x^* \rangle = \limsup_{s\to\infty} \langle \nabla f(x_0) - \nabla f(x^*), w_{m_s} - x^* \rangle \leq 0$$

Furthermore, from (3.50) for all $m_s \geq N_0$ and $D_f(x^*, x_{m_s}) \leq D_f(x^*, x_{m_s+1})$, we have

$$\begin{aligned} D_f(x^*, x_{m_s+1}) &\leq (1 - (1 - \beta_{m_s})\alpha_{m_s}) D_f(x^*, x_{m_s}) \\ &\quad - \alpha_{m_s}(1 - \beta_{m_s})[\theta_{m_s}(D_f(x^*, x_{m_s-1}) - D_f(x^*, x_{m_s}))] \\ &\quad + (1 - \beta_{m_s})\alpha_{m_s} \langle \nabla f(x_0) - \nabla f(x^*), w_{m_s} - x^* \rangle \\ &\quad + \theta_{m_s}(D_f(x^*, x_{m_s-1}) - D_f(x^*, x_{m_s})) \\ &\leq (1 - (1 - \beta_{m_s})\alpha_{m_s}) D_f(x^*, x_{m_s+1}) \\ &\quad + (1 - \beta_{m_s})\alpha_{m_s} \langle \nabla f(x_0) - \nabla f(x^*), w_{m_s} - x^* \rangle \\ &\quad - \alpha_{m_s}(1 - \beta_{m_s})[\theta_{m_s}(D_f(x^*, x_{m_s-1}) - D_f(x^*, x_{m_s}))] \\ &\quad + \theta_{m_s}(D_f(x^*, x_{m_s-1}) - D_f(x^*, x_{m_s})) \end{aligned}$$

$$\begin{aligned} (1 - \beta_{m_s})\alpha_{m_s} D_f(x^*, x_{m_s+1}) \leq & [1 - (1 - \beta_{m_s})\alpha_{m_s}]\theta_{m_s}(D_f(x^*, x_{m_s-1}) - D_f(x^*, x_{m_s})) \\ & + (1 - \beta_{m_s})\alpha_{m_s} \langle \nabla f(x_0) - \nabla f(x^*), w_{m_s} - x^* \rangle \\ \leq & \langle \nabla f(x_0) - \nabla f(x^*), w_{m_s} - x^*, \rangle \end{aligned}$$

Since $0 < (1 - \beta_{m_s})\alpha_{m_s} < 1$ for all $s \geq 0$ and $D_f(x^*, x_{m_s}) \leq D_f(x^*, x_{m_s+1})$, we have

$$D_f(x^*, x_{m_s}) \leq D_f(x^*, x_{m_s+1}) \leq \langle \nabla f(x_0) - \nabla f(x^*), w_{m_s} - x^* \rangle.$$

This implies

$$\limsup_{s\to\infty} D_f(x^*, x_{m_s}) \leq \limsup_{s\to\infty} \ \langle \nabla f(x_0) - \nabla f(x^*), w_{m_s} - x^* \rangle \leq 0.$$

Thus,

$$\limsup_{s\to\infty} D_f(x^*, x_{m_s}) = 0,$$

which by Lemma 2.3, we have $\underset{s\to\infty}{lim} ||x^* - x_{m_s}|| = 0$.
However, we know that $D_f(x^*, x_s) \leq D_f(x^*, x_{m_s+1})$ for all $s \in \mathbb{N}$, hence, $\underset{s\to\infty}{lim} D_f(x^*, x_s) = 0$, which by Lemma 2.3, we have

$$\underset{s\to\infty}{lim} ||x^* - x_s|| = 0.$$

Hence, $x_s \longrightarrow x^*$ where $x^* = Proj_\Gamma^f x_0$. $\square$

Corollary 3.4. *Let E be a real reflexive Banach space, $A : E \longrightarrow E^*$ be a monotone and Lipschitz continuous operator, $\{T_i\}_{i=1}^M$ be a finite family of generic generalized Bregman nonspreading mapping. Let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function satisfying assumption (A5). Let $\{u_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{\alpha_n\} \subset (0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ be sequences satisfying assumptions $(A1) - (A6)$ of Algorithm 3. Suppose $\Gamma = VI(C,A) \cap \bigcap_{i=1}^M F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to a solution $x^* = Proj^f_{VI(C,A) \,\cap\, \bigcap_{i=1}^M F(T_i)} x_0$.*

Proof. Observe that in this case the weak sequential continuity of A in assumption (A2) of Algorithm 3 has to be droped since it follows from the monotonicity of A and (3.42) that

$$\begin{aligned} \frac{1}{\lambda_{n_k}}\langle \nabla f(u_{n_k}) - \nabla f(y_{n_k}), z - y_{n_k}\rangle + \langle A(u_{n_k}), y_{n_k} - u_{n_k}\rangle &\leq \langle A(u_{n_k}), z - u_{n_k}\rangle \\ &\leq \langle A(z), z - u_{n_k}\rangle \end{aligned} \tag{3.53}$$

Furthermore, passing limit as $k \longrightarrow \infty$ in inequality (3.53) and applying the fact that $||u_{n_k} - y_{n_k}|| \longrightarrow 0, \;\; as \;\; k \longrightarrow \infty$ and since ∇f is uniformly norm to norm continuous on bounded subsets of E, then $\lim_{n\to\infty} ||\nabla f(y_{n_k}) - \nabla f(u_{n_k})|| = 0$, we obtain

$$\langle A(z), z - u^*\rangle \geq 0, \;\; \forall \;\; z \in C.$$

Hence, it follows from Theorem (3.3) that the sequence $\{x_n\}$ converges strongly to a solution $x^* = Proj^f_{VI(C,A) \,\cap\, \bigcap_{i=1}^M F(T_i)} x_0$. □

Corollary 3.5. *Let $f : E \longrightarrow (-\infty, +\infty]$ be a strongly coercive, Legendre function which bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of $int(dom f)$ and $\{T_i\}_{i=1}^M$ be a finite family of generic generalized Bregman nonspreading mapping. Let $\{u_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$ and $\{\alpha_n\} \subset (0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ be sequences satisfying assumptions $(A1) - (A6)$ of Algorithm 3. Suppose $\Gamma = VI(C,A) \cap \bigcap_{i=1}^M F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to a solution $x^* = Proj^f_{VI(C,A) \,\cap\, \bigcap_{i=1}^M F(T_i)} x_0$.*

Proof. By Remark 2.2 if E is smooth and the function $f(x) = ||x||^2$, then generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ generalized Bregman nonspreading mapping reduces to generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ generalized nonspreading mapping in the sense of Takahashi [46], that is for all $x, y \in C$ the following inequalities holds: (i) $(\alpha + \beta + \gamma + \delta) \geq 0$; (ii) $(\alpha + \beta) > 0$; and (iii)

$$\begin{aligned} \alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) &\leq \varepsilon\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\quad + \xi\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{3.54}$$

Thus by Theorem (3.3), we have that the sequence $\{x_n\}$ converges strongly to a solution $x^* = Proj^f_{VI(C,A) \,\cap\, \bigcap_{i=1}^M F(T_i)} x_0$. □

Corollary 3.6. *Let H be a real Hilbert space, $A : H \longrightarrow H$ be pseudomonotone and Lipschitz continuous operator, and $\{T_i\}_{i=1}^M$ be a finite family of normally generalized hybrid mappings of H into itself. Let $\{u_n\}$, $\{y_n\}$, $\{w_n\}$ and $\{z_n\}$ be sequences generated by Algorithm 3 and $\beta_n \in (a,b)$ where $0 < a < b < 1$, $\{\alpha_n\} \subset (0,1)$ satisfy $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ be sequences satisfying assumptions $(A1)-(A6)$ of Algorithm 3. Suppose $\Gamma = VI(C,A) \bigcap \bigcap_{i=1}^M F(T_i) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to a solution $x^* = Proj^f_{VI(C,A) \bigcap \bigcap_{i=1}^M F(T_i)} x_0$.*

Proof. By Remark 2.2, the generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \xi)$ - generalized nonspreading mappings reduces to normally generalized hybrid mapping in Hilbert space i.e, there exists $\alpha_1, \beta_1, \gamma_1, \delta_1 \in \mathbb{R}$ such that

$$\alpha_1 ||Tx - Ty||^2 + \beta_1, ||x - Ty||^2 + \gamma_1 ||Tx - y||^2 + \delta_1 ||x - y||^2 \leq 0, \quad \forall\, x, y \in C,$$

where $\alpha_1 = \alpha - \epsilon, \beta_1 = \beta + \epsilon, \gamma_1 = \gamma - \zeta$ and $\delta_1 = \delta + \zeta$ satisfying $\alpha_1 + \beta_1 = \alpha + \beta > 0$ and $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 = \alpha + \beta + \gamma + \delta \geq 0$. Thus by Theorem (3.3), we have that the sequence $\{x_n\}$ converges strongly to a solution $x^* = Proj^f_{VI(C,A) \bigcap \bigcap_{i=1}^M F(T_i)} x_0$. □

4. Numerical Example

We give a numerical example to illustrate the performance of the newly introduced Algorithm..

Example 4.1. Let $E = \mathbb{R}^2$ be equipped with inner product $\langle x, y \rangle = x_1 y_1 + x_2 y_2$ and $\|x\|^2 = \sum_{i=1}^2 |x_i|^2 \ \forall\ x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$. Let for each $i = 1, 2$, $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T_i x = \frac{1}{4}(x_1, x_2), \forall x \in \mathbb{R}^2.$$

Then T_i, for each $i = 1, 2$, is generic generalized Bregman nonspreading mapping with $\beta = \gamma = 0 = \xi = \epsilon$, $\alpha = 4$ and $\delta = -4$. Indeed, for any $x, y \in \mathbb{R}^2$ and $\beta, \gamma, \xi, \epsilon, \alpha, \delta$ with above values, we have

(i) $\alpha + \beta + \gamma + \delta = 4 + 0 + 0 - 4 \geq 0$;

(ii) $\alpha + \beta = 4 + 0 > 0$;

(iii)

$$\begin{aligned}
&\alpha D_f(T_i x, T_i y) + \beta D_f(x, T_i y) + \gamma D_f(T_i x, y) + \delta D_f(x, y) \\
&- \epsilon\big\{D_f(T_i x, T_i y) - D_f(T_i x, x)\big\} - \xi\big\{D_f(y, T_i x) - D_f(x, y)\big\} \\
&= \alpha\|T_i x - T_i y\| + \delta\|x - y\| \\
&= \|(x_1 - y_1, x_2 - y_2)\| - 4\|(x_1 - y_1, x_2 - y_2)\| \\
&= \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} - 4\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \leq 0.
\end{aligned}$$

Define $A : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A(x_1, x_2) = 5(x_2, -x_1), \forall x_1, x_2 \in \mathbb{R}.$$

Clearly A is pseudomonotone and 5-Lipschitz continuous. As for $x, y \in \mathbb{R}^2$

$$\begin{aligned}\langle Ax - Ay, x - y\rangle &= 5\big((x_2, -x_1) - (y_2, -y_1)\big)(x_1 - y_1, x_2 - y_2)\\ &= 5(x_2 - y_2, y_1 - x_1)(x_1 - y_1, x_2 - y_2)\\ &= 5(x_2 - y_2)(x_1 - y_1) + 5(y_1 - x_1)(x_2 - y_2) \geq 0\end{aligned}$$

and

$$\begin{aligned}\|Ax - Ay\| &= \|5(x_2, -x_1) - 5(y_2, -y_1)\|\\ &= \|5(x_2 - y_2, -x_1 + y_1)\|\\ &\leq 5\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}\\ &= 5\|(x_1, x_2) - (y_1, y_2)\|\\ &= 5\|x - y\|.\end{aligned}$$

Let $C = \overline{B}_{\mathbb{R}^2}(0, 2) := \{x \in \mathbb{R}^2 : \|x\| \leq 2\}$. Clearly C is nonempty, closed and convex subset of $\mathbb{R}^2$. Hence, we have

$$P_C x = \begin{cases} x, \ \|x\| \leq 2,\\ \frac{2x}{\|x\|}, \ \text{otherwise}.\end{cases}$$

Choose $\mu = \frac{7}{10}$, $\theta_n = \hat{\theta}_n$, $\lambda_n = \frac{2}{5}$, $\beta_n = \frac{n}{3n+2}$ and $\alpha_n = \frac{1}{2n}$. Then Algorithm 3.0 now takes the following form;

$$\begin{cases} u_n = (1 - \hat{\theta}_n)x_n + \hat{\theta}_n x_{n-1},\\ y_n = P_C(u_n - (\frac{2}{5}Au_n)),\\ s_n = \{z \in \mathbb{R}^2 : \langle u_n - \frac{2}{5}Au_n - y_n, z - y_n\rangle \leq 0\},\\ z_n = P_{s_n}(u_n - (\frac{2}{5}Ay_n)),\\ w_n = (\frac{1}{2n})x_0 + (\frac{2n-1}{2n})z_n,\\ x_{n+1} = (\frac{n}{3n+2})z_n + (\frac{2n+2}{3n+2})Tw_n, n \geq 2.\end{cases} \tag{4.1}$$

Let $\{x_n\}$ be a sequence generated by Algorithm (4.1).

Case I. Take $x_1 = (0.5, 0.25)^T$, $x_2 = (1, 0.5)^T$ and $x_0 = (0.001, 0.001)^T$.
Case II. Take $x_1 = (-0.5, -0.25)^T$, $x_2 = (-1, -0.5)^T$ and $x_0 = (0.001, 0.001)^T$.
Case III. Take $x_1 = (0.3, 0.06)^T$, $x_2 = (0.2, 0.9)^T$ and $x_0 = (0.00004, 0.00004)^T$.
Case IV. Take $x_1 = (-1, -0.5)^T$, $x_2 = (-0.5, 0.1)^T$ and $x_0 = (-0.5, -0.9)^T$.

5. Conclusion

This paper introduced a new inertial type subgradient extragradient algorithm with self adaptive step size for approximating common element of the set of solutions of pseudomonotone variational inequality problem and common fixed point of a finite family of generic generalized Bregman nonspreading mapping in a real reflexive Banach space .
Furthermore, we proved a strong convergence theorem of our algorithm without prior knowledge of Lipschitz constant of the operator under some mild assumptions. we present a numerical example in order to illustrates the performance of our proposed algorithm. Our result generalize and improve many existing results in the literature.

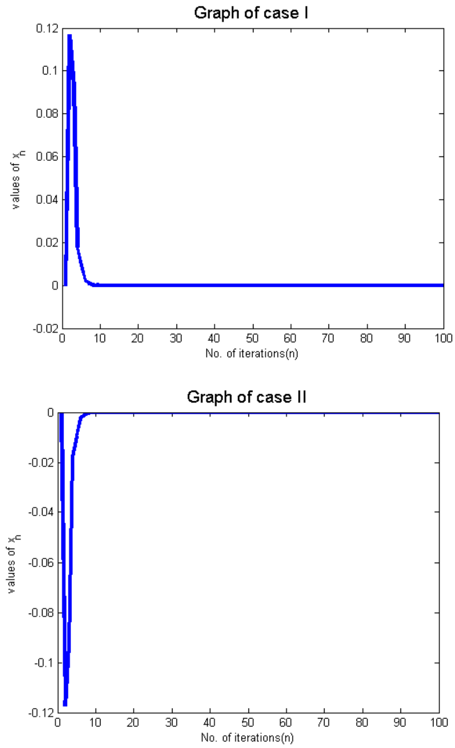

FIGURE 1. The graphs of sequence $\{x_n\}$ generated by Algorithm (4.1) versus number of iterations (Case I and Case II).

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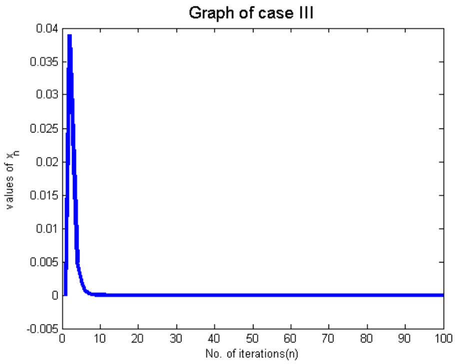

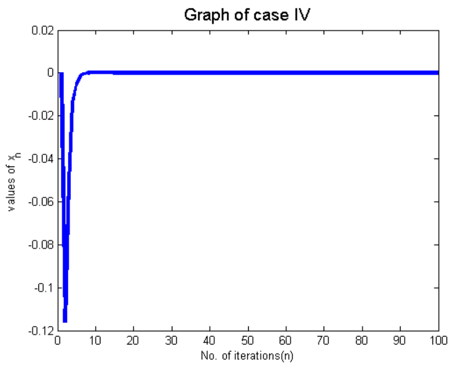

FIGURE 2. The graphs of sequence $\{x_n\}$ generated by Algorithm (4.1) versus number of iterations (Case III and Case IV).

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