Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 52(2), 2025, Pages 416–435, DOI: 10.52846/ami.v52i2.1982
ISSN: 1223-6934

Discussion on a New Tripled System of Hybrid Type of FDEs with $p-$Laplacian Involving $\varphi-$Caputo Derivatives

Hamid Beddani, Moustafa Beddani, and Mohammad Esmael Samei*

Abstract. Our research is about the analysis of a new type of triple system of hybrid differential equations of fractional order with nonlocal integro multi point boundary conditions, whose results can certainly be useful in solving practical problems. We focus on a mathematical operator called the $p-$Laplacian and another type of derivative called the $\varphi-$Caputo derivative. The displayed comes about are gotten by the hybrid Dhage fixed point theorem for a entirety of three operators. A few illustrative illustrations is displayed at the conclusion to appear the pertinence of the gotten comes about. To the leading of our information, this is often the primary time where such issue is considered.

2020 *Mathematics Subject Classification.* Primary 34A38; Secondary 32A65; 26A33.
Key words and phrases. multi point boundary, $p-$Laplacian operator, fractional derivative of $\varphi-$Caputo type, hybrid case theorem.

1. Introduction

Fractional calculus is useful in many research areas, like the ones mentioned in sources [1, 21, 31, 32]. Scientists from various areas of study became interested in fractal order differential equations (DEs) after the mission's exploration [17, 30]. Most of the research has been done using different types of fractional derivatives such as Riemann-Liouville (RL), Grunwald-Letnikov, Atangana-Baleanu, Hadamard, Caputo, and Katugampola [6, 26, 30, 33].

When, we find the fractional derivatives of functions in relation to other functions, they are distinctive than standard fractional derivatives since they include utilizing other uncommon functions, called φ, see [3, 4, 22]. Many articles use the theories of Schauder, Krasnoselskii, Darbo, or Monch to show that there are solutions for nonlinear fractional differential equations (FDEs), however, these theories only work under certain conditions [2, 7, 9, 28, 36]. Certainly, valuable works have been published on the existence and uniqueness (EU) of solutions for FDEs with $p-$Laplacian operators, including Li, Wang, Khan, Chabanea *et al.* studied a nonlinear FDEs with the operator for the EU of solutions [8, 10, 11, 13, 23, 25, 27, 34] and others for hybrid type of the equations [5, 12, 15, 16, 20, 35].

Received July 4, 2024. Accepted August 9, 2025.
* Corresponding Author

Ferraoun *et al.* have been concerned with the hybrid FDEs involving RL-operator the following problem:

$$\begin{cases} \mathcal{D}_0^{\alpha_1}\left[\left(\frac{u_1(\mathrm{t})}{g_1(\mathrm{t},\mathrm{u}(\mathrm{t}))}\right)\right] = f_1\big(\mathrm{t},\mathrm{u}(\mathrm{t})\big) + \mathcal{I}_0^{\delta_1}h_1\big(\mathrm{t},\mathrm{u}(\mathrm{t})\big), \\ \mathcal{D}_0^{\alpha_2}\left[\left(\frac{u_2(\mathrm{t})}{g_2(\mathrm{t},\mathrm{u}(\mathrm{t}))}\right)\right] = f_2\big(\mathrm{t},\mathrm{u}(\mathrm{t})\big) + \mathcal{I}_0^{\delta_2}h_2\big(\mathrm{t},\mathrm{u}(\mathrm{t})\big), \\ \quad\vdots \\ \mathcal{D}_0^{\alpha_n}\left[\left(\frac{u_n(\mathrm{t})}{g_n(\mathrm{t},\mathrm{u}(\mathrm{t}))}\right)\right] = f_n\big(\mathrm{t},\mathrm{u}(\mathrm{t})\big) + \mathcal{I}_0^{\delta_n}h_n\big(\mathrm{t},\mathrm{u}(\mathrm{t})\big), \end{cases}$$

here $\mathrm{u}(\mathrm{t}) = \begin{bmatrix} u_1(\mathrm{t}) & u_2(\mathrm{t}) & \dots & u_n(\mathrm{t}) \end{bmatrix}$, $\mathrm{t} \in L := [0,1]$, with

$$u_i(0) = \theta_i \int_0^{\beta_i} \varphi_i(s)u_i(s)\,\mathrm{d}s, \qquad 0 < \beta_i,$$

$\alpha_i, \delta_i < 1$, $i = \overline{1,n}$, where φ_i are continuous functions on $[0,\beta_i]$ on L, $g_i \in C\big(L \times \mathbb{R}^n, \mathbb{R}^*\big)$, $f_i, h_i \in C\big(L \times \mathbb{R}^n\big)$ [19]. Beddani *et al.* worked on the nonlinear FDEs involving ψ_p Laplacian operator:

$$\begin{cases} \mathcal{D}_{0^+}^{r_{1k};\varphi}\psi_p\Big[\mathcal{D}_{0^+}^{r_{2k};\varphi}\big(x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma;\varphi}g_k(\mathrm{t},\mathrm{x}(\mathrm{t}))\big)\Big] = h_k\big(\mathrm{t},\mathrm{x}(\mathrm{t})\big), \quad k = \overline{1,3}, \\ \psi_p\Big[\mathcal{D}_{0^+}^{r_{2k};\varphi}\big(x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma;\varphi}g_k\big(\mathrm{t},\mathrm{x}(\mathrm{t})\big)\big)\Big]\Big|_{\mathrm{t}=0} = 0, \end{cases}$$

for $\mathrm{t} \in L$, where $\mathrm{x}(\mathrm{t}) = (x_1(\mathrm{t}), x_2(\mathrm{t}), x_3(\mathrm{t}))$, with $x_k(0) = 0$,

$$x_k(1) = \sum_{i=1}^{3} \lambda_i x_i(\zeta_{ik}), \qquad \zeta_{ik} \in L,$$

$\varphi(1) - \varphi(0) = K > 0$ where $\mathcal{D}_{0^+}^{r_{ik};\varphi}$, $i,k = \overline{1,3}$ as the $\varphi-$Caputo derivatives of fractional orders r_{ik}, $0 \le r_{1k} < 1 < r_{2k} < 2$, $\mathcal{I}_{0^+}^{\sigma;\varphi}$, the integral of fractional order $0 < \sigma$, $\lambda_i \in \mathbb{R}_+^*$, and $\varphi : L \to \mathbb{R}$ is an increasing function such that $\varphi'(\mathrm{t}) \neq 0$, and $\psi_{p_m}(s) = |s|^{p-2}\, s$ denotes the $p-$Laplacian operator and for $\mathrm{t} \in L$, $g_k, h_k : L \times \mathbb{R}^3 \to \mathbb{R}$ is a given functions [8].

In the research work, first, we recall basic notions and lemmas in Section 2. Then, in Section 3, we analyze the existence tripled system of hybrid FDEs with ψ_{P_k} Laplacian operator for the following problem: for $\mathrm{t} \in L$, and $k = \overline{1,\ 3}$,

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha_k;\varphi}\psi_{p_k}\left[\mathcal{D}_{0^+}^{\beta_k;\varphi}\left(\frac{x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma_k;\varphi}f_k(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_k(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)\right] = h_k\big(\mathrm{t},\mathrm{x}(\mathrm{t})\big), \\ \psi_{p_k}\left[\mathcal{D}_{0^+}^{\beta_k;\varphi}\left(\frac{x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma_k;\varphi}f_k(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_k(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)\right]\Bigg|_{\mathrm{t}=0} = 0, \\ \frac{x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma_k;\varphi}f_k(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_k(\mathrm{t},\mathrm{x}(\mathrm{t}))}\Bigg|_{\mathrm{t}=b} = \sum_{i=1}^{3}\lambda_{ik}x_i\left(\zeta_{ik}\right), \quad \zeta_{ik} \in L, \end{cases} \tag{1}$$

here, $\mathrm{x}(\mathrm{t}) = (x_1(\mathrm{t}), x_2(\mathrm{t}), x_3(\mathrm{t}))$ and we take $\mathcal{D}_{0^+}^{\alpha_k;\varphi}, \mathcal{D}_{0^+}^{\beta_k;\varphi}, k = \overline{1,3}$ as the $\varphi-$Caputo derivatives of fractional orders α_k, β_k, $0 \le \alpha_k < 1 < \beta_k < 2$, $\lambda_{ik} \in \mathbb{R}_+^*$, $\mathcal{I}_{0^+}^{\sigma_k;\varphi}$, $0 < \sigma_k$, the fractional integral of order σ_k,, an increasing function $\varphi : L \to \mathbb{R}$ with $\varphi'(\mathrm{t}) \neq 0$,

$\hat{\varphi}_0(b) := \varphi(b) - \varphi(0) > 0$, and $\psi_{p_k}(s) = s\,|s|^{p_k-2}$ denotes the p_k-Laplacian operators and satisfies

$$p_k + q_k = p_k \cdot q_k, \quad (\psi_{p_k})^{-1} = \psi_{q_k}\,(q_k \geq 2),$$

and $f_k, h_k : L \times \mathbb{R}^3 \to \mathbb{R}$, $g_k : L \times \mathbb{R}^3 \to \mathbb{R}^*$ is a given functions. Next, in Section 4, practical examples are shown. Finally, we present the conclusions of the obtained results for use in future research in in Section 5.

2. Basic notions and preliminaries

Consider an increasing function $\varphi : L \to \mathbb{R}$ with $\varphi'(\mathfrak{t}) \neq 0$, for $\mathfrak{t} \in L$ and the Gamma function $\Gamma(\cdot)$. We pose

$$\mathfrak{\varphi}_r(\mathfrak{t}, \varrho) = \frac{\varphi'(\varrho)(\hat{\varphi}_\varrho(\mathfrak{t}))^{r-1}}{\Gamma(r)}, \qquad (\mathfrak{t} > \varrho), \, r > 0, \, \mathfrak{t} \in L,$$

where $\hat{\varphi}_\varrho(\mathfrak{t}) := \varphi(\mathfrak{t}) - \varphi(\varrho)$. The left-sided $\varphi-$RL fractional integral of order α for an integrable function $x : L \to \mathbb{R}$ with respect to function φ, is defined as follows

$$\mathcal{I}_{a^+}^{r;\varphi} x(\mathfrak{t}) = \int_a^{\mathfrak{t}} \mathfrak{\varphi}_r(\mathfrak{t}, \varrho)\, x(\varrho)\, \mathrm{d}\varrho. \tag{2}$$

Note that Eq. (2) is reduced to the RL and Hadamard fractional integrals when $\varphi(\mathfrak{t}) = \mathfrak{t}$ and $\varphi(\mathfrak{t}) = \ln \mathfrak{t}$, respectively. Now, assume that φ, $x \in C^n(J)$ too. The left-sided $\varphi-$RL and left-sided $\varphi-$Caputo derivative of a function x of fractional order r are defined by

$$\begin{aligned}\mathcal{D}_{a^+}^{r;\varphi} x(\mathfrak{t}) &= \left(\tfrac{1}{\varphi'(\mathfrak{t})} \tfrac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}\right)^n \mathcal{I}_{a^+}^{n-r;\varphi} x(\mathfrak{t}) \\ &= \left(\tfrac{1}{\varphi'(\mathfrak{t})} \tfrac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}\right)^n \int_a^{\mathfrak{t}} \mathfrak{\varphi}_{n-r}(\mathfrak{t}, \mathfrak{s})\, x(\mathfrak{s})\, \mathrm{d}\mathfrak{s}, \qquad n = [r] + 1,\end{aligned}$$

and ${}^c\mathcal{D}_{a^+}^{r;\varphi} x(\mathfrak{t}) = \mathcal{I}_{a^+}^{n-r;\varphi} x_\varphi^{[n]}(\mathfrak{t})$, where $n = [r] + 1$ for $r \notin \mathbb{N}, n = r$ for $r \in \mathbb{N}$, respectively, and

$$x_\varphi^{[n]} x(\mathfrak{t}) = \left(\tfrac{1}{\varphi'(\mathfrak{t})} \tfrac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}\right)^n x(\mathfrak{t}).$$

Thus,

$${}^c\mathcal{D}_{a^+}^{r;\varphi} u(\mathfrak{t}) = \begin{cases} \displaystyle\int_a^{\mathfrak{t}} \mathfrak{\varphi}_{n-r}(\mathfrak{t}, \mathfrak{s}) x_\varphi^{[n]}(\mathfrak{s})\, \mathrm{d}\mathfrak{s}, & r \notin \mathbb{N}, \\ x_\varphi^{[n]}(\mathfrak{t}), & r \in \mathbb{N}. \end{cases} \tag{3}$$

Lemma 2.1. *Let $r > 0$. The following holds:* (i) *If $x \in C(L)$, then ${}^c\mathcal{D}_{a^+}^{r;\varphi} \mathcal{I}_{a^+}^{r;\varphi} x(\varrho) = x(\varrho)$, for $\varrho \in L$;* (ii) *If $x \in C^n(L)$, $n - 1 < r < n$, then*

$$\mathcal{I}_{a^+}^{\alpha;\varphi} \, {}^c\mathcal{D}_{a^+}^{\alpha;\varphi} x(\varrho) = x(\varrho) - \sum_{k=0}^{n-1} \frac{x_\varphi^{[k]}(a)}{k!} \left[\hat{\varphi}_a(\varrho)\right]^k.$$

In special case, we have $\mathcal{I}_{a^+}^{\alpha;\varphi} \, {}^c\mathcal{D}_{a^+}^{\alpha;\varphi} x(\varrho) = x(\varrho) - x(a)$, when $0 < r < 1$.

3. Existence results

We consider Banach spaces $C(L)$ and $\mathcal{E} = [C(L)]^3$ endowed with the norms $\|x\|_{C(J)} = \sup_{\mathrm{t}\in J} |x(\mathrm{t})|$ and

$$\|\mathrm{x}\|_{\mathcal{E}} = \|(x_1, x_2, x_3)\|_{\mathcal{E}} = \sum_{i=1}^{3} \|x_i\|_{C(J)},$$

respectively.

Lemma 3.1. *For a given $\widetilde{f}_k, \widetilde{h}_k, \widetilde{g}_k \in L^1(J, \mathbb{R}^3)$ $(k = \overline{1,\ 3})$, the solution of hybrid fractional problem*

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha_k;\varphi} \psi_{p_k} \left[\mathcal{D}_{0^+}^{\beta_k;\varphi} \left(\frac{x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma_k;\varphi} \widetilde{f}_k(\mathrm{t})}{\widetilde{g}_k(\mathrm{t})} \right) \right] = \widetilde{h}_k(\mathrm{t}), \quad \mathrm{t} \in L, \\ \psi_{p_k} \left[\mathcal{D}_{0^+}^{\beta_k;\varphi} \left(\frac{x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma_k;\varphi} \widetilde{f}_k(t)}{\widetilde{g}_k(\mathrm{t})} \right) \right] \Big|_{\mathrm{t}=0} = 0, \\ \frac{x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma_k;\varphi} \widetilde{f}_k(\mathrm{t})}{\widetilde{g}_k(\mathrm{t})} \Big|_{\mathrm{t}=b} = \sum_{i=1}^{3} \lambda_{ik} x_i(\zeta_{ik}), \qquad \zeta_{ik} \in L, \end{cases}$$

is given by

$$\begin{aligned} x_k(\mathrm{t}) = {} & \mathcal{I}_{0^+}^{\sigma_k;\varphi} \widetilde{f}_k(\mathrm{t}) + \widetilde{g}_k(\mathrm{t}) \mathcal{I}_{0^+}^{\beta_k;\varphi} \left(\psi_{q_k} \mathcal{I}_{0^+}^{\alpha_k;\varphi} \widetilde{h}_k(\mathrm{t}) \right) \\ & + \frac{\hat{\varphi}_0(\mathrm{t}) \widetilde{g}_k(\mathrm{t})}{\hat{\varphi}_0(b)} \left[\mathcal{I}_{0^+}^{\beta_k;\varphi} \left(\psi_{q_k} \mathcal{I}_{0^+}^{\alpha_k;\varphi} \widetilde{h}_k(b) \right) - \sum_{i=1}^{3} \lambda_{ik} x_i(\zeta_{ik}) \right]. \end{aligned}$$

Proof. For $0 \le \alpha_k < 1 < \beta_k < 2$, Lemma 2.1 yields,

$$\psi_{p_k} \mathcal{D}_{0^+}^{\beta_k;\varphi} \left(\frac{x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma_k;\varphi} \widetilde{f}_k(\mathrm{t})}{\widetilde{g}_k(\mathrm{t})} \right) = \mathcal{I}_{0^+}^{\alpha_k;\varphi} \widetilde{h}_k(\mathrm{t}) + d_{1k},$$

by conditions

$$\psi_{p_k} \left[\mathcal{D}_{0^+}^{\beta_k;\varphi} \left(\frac{x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma_k;\varphi} \widetilde{f}_k(\mathrm{t})}{\widetilde{g}_k(\mathrm{t})} \right) \right] \Big|_{\mathrm{t}=0} = 0,$$

we get $d_{1k} = 0$, and

$$\frac{x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma_k;\varphi} \widetilde{f}_k(\mathrm{t})}{\widetilde{g}_k(\mathrm{t})} = \mathcal{I}_{0^+}^{\beta_k;\varphi} \left(\psi_{q_k} \mathcal{I}_{0^+}^{\alpha_k;\varphi} \widetilde{h}_k(\mathrm{t}) \right) + d_{2k} \hat{\varphi}_0(\mathrm{t}),$$

and so by conditions

$$\frac{x_k(\mathrm{t}) - \mathcal{I}_{0^+}^{\sigma_k;\varphi} \widetilde{f}_k(\mathrm{t})}{\widetilde{g}_k(\mathrm{t})} \Big|_{\mathrm{t}=b} = \sum_{i=1}^{3} \lambda_{ik} x_i(\zeta_{ik}),$$

then

$$d_{2k} = \frac{1}{\hat{\varphi}_0(b)} \left(\mathcal{I}_{0^+}^{\beta_k;\varphi} \left(\psi_{q_k} \mathcal{I}_{0^+}^{\alpha_k;\varphi} \widetilde{h}_k(b) \right) - \sum_{i=1}^{3} \lambda_{ik} x_i(\zeta_{ik}) \right).$$

Thus,

$$x_k(\mathfrak{t}) = \mathcal{I}_{0^+}^{\sigma_k;\varphi}\widetilde{f}_k(\mathfrak{t}) + \widetilde{g}_k(\mathfrak{t})\mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k}\mathcal{I}_{0^+}^{\alpha_k;\varphi}\widetilde{h}_k(\mathfrak{t})\right) + \frac{\hat{\varphi}_0(\mathfrak{t})\widetilde{g}_k(\mathfrak{t})}{\hat{\varphi}_0(b)}\left\{\mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k}\mathcal{I}_{0^+}^{\alpha_k;\varphi}\widetilde{h}_k(b)\right) - \sum_{i=1}^{3}\lambda_{ik}x_i\left(\zeta_{ik}\right)\right\}.$$

This completes the proof. □

The following hypotheses are required for further discussion. We need the following to prove our main theorem.

Required hypotheses:

$\mathbb{H}_1$) The functions f_k, g_k and h_k are continuous;

$\mathbb{H}_2$) There exist three positive functions $\mathcal{T}_{fk}, \mathcal{T}_{gk}, \mathcal{T}_{hk}$ with bounds $\|\mathcal{T}_{fk}\|, \|\mathcal{T}_{gk}\|$, $\|\mathcal{T}_{hk}\|$, respectively, with

$$|f_k(\varrho, \mathrm{x}) - f_k(\varrho, \mathrm{y})| \leq \mathcal{T}_{fk}(\varrho)\,\Sigma,$$
$$|g_k(\varrho, \mathrm{x}) - g_k(\varrho, \mathrm{y})| \leq \mathcal{T}_{gk}(\varrho)\,\Sigma,$$
$$|h_k(\varrho, \mathrm{x}) - h_k(\varrho, \mathrm{y})| \leq \mathcal{T}_{hk}(\varrho)\,\Sigma,$$

where $\Sigma := \sum_{i=1}^{3}|x_i - y_i|$, for all $\mathfrak{t} \in L$ and $\mathrm{x} = (x_1, x_2, x_3)$, $\mathrm{y} = (y_1, y_2, y_3)$, $x_i, y_i \in \mathbb{R}, i, k = \overline{1,\ 3}$;

$\mathbb{H}_3$) There exist three functions $\Lambda_k \in L^\infty(L, \mathbb{R}_+)$ and a nondecreasing function $\vartheta_k \in C([0,\infty), \mathbb{R}^{>0})$ with

$$|h_k(\mathfrak{t}, \mathrm{x})| \leq \Lambda_k(\mathfrak{t})\,\vartheta_k\left(\|\mathrm{x}\|_{\mathcal{E}}\right), \quad \forall\,\mathfrak{t} \in L,\ x_i \in \mathbb{R},\ i, k = \overline{1,\ 3}. \tag{4}$$

$\mathbb{H}_4$) There exists $r > 0$ such that

$$\frac{3(\mathcal{K}_1\mathcal{K}_2 + \mathcal{K}_3)}{1 - \mathcal{K}_1\sum\limits_{k=1}^{3}\|\mathcal{T}_{gk}\| - \sum_{k=1}^{3}\frac{(\hat{\varphi}_0(b))^{\sigma_k}\left\|\mathcal{T}_{fk}\right\|}{\Gamma(1+\sigma_k)}} \leq r, \tag{5}$$

and

$$\max\left\{\frac{(\hat{\varphi}_0(b))^{\alpha_k}\|\psi_k\|\vartheta_k(r)}{\Gamma(1+\alpha_k)}\ :\ k = \overline{1,3}\right\} < \Delta, \tag{6}$$

where

$$\mathcal{K}_1 = \sum_{k=1}^{3}\left[\frac{2(\hat{\varphi}_0(b))^{\beta_k}\Delta^{q_k-1}}{\Gamma(1+\beta_k)} + r\sum_{i=1}^{3}\lambda_{ik}\right],$$
$$\mathcal{K}_2 = \sum_{k=1}^{3}\sup_{\mathfrak{t}\in L}|g_k(\mathfrak{t}, 0, 0, 0)|,$$
$$\mathcal{K}_3 = \sum_{k=1}^{3}\frac{(\hat{\varphi}_0(b))^{\alpha_k}}{\Gamma(1+\alpha_k)}\sup_{\mathfrak{t}\in L}|f_k(\mathfrak{t}, 0, 0, 0)|. \tag{7}$$

Theorem 3.2. *Assuming hypotheses* ($\mathbb{H}_1$)–($\mathbb{H}_4$) *are met. Then, the hybrid fractional problem* (1) *has at least one solution defined on L, if*

$$\mathcal{K}_1\sum_{k=1}^{3}\|\mathcal{T}_{gk}\| + \sum_{k=1}^{3}\frac{(\hat{\varphi}_0(b))^{\sigma_k}\|\mathcal{T}_{fk}\|}{\Gamma(1+\sigma_k)} < 1. \tag{8}$$

In order to prove the theorem, we adopt the following Lemmas.

Lemma 3.3 ([33]). *Given a function $x \in C^n(L)$ and $0 < r < 1$, we have*

$$\left|\mathcal{I}_{a^+}^{r;\varphi}x(\acute{\mathfrak{t}}) - \mathcal{I}_{a^+}^{r;\varphi}x(\mathfrak{t})\right| \leq 2\,\|x\|_{C(L)}\,\frac{\left(\hat{\varphi}_{\mathfrak{t}}(\acute{\mathfrak{t}})\right)^r}{\Gamma(r+1)}, \quad \mathfrak{t} < \acute{\mathfrak{t}}. \tag{9}$$

Lemma 3.4 ([24]). *For the $p-$Laplacian operator ψ_p, the following conditions hold true:*
(i) *If $|\acute{\delta}|, \delta \geq \rho > 0$, with $\delta\,\acute{\delta} > 0$, $1 < p \leq 2$, then*

$$\left|\psi_p(\delta) - \psi_p(\acute{\delta})\right| \leq \tfrac{p-1}{\rho^{2-p}}|\delta - \acute{\delta}|; \tag{10}$$

(ii) *If $p > 2$, $|\delta|, |\acute{\delta}| \leq \rho_* > 0$, then*

$$\left|\psi_p(\delta) - \psi_p(\acute{\delta})\right| \leq \tfrac{p-1}{\rho_*^{2-p}}|\delta - \acute{\delta}|\,. \tag{11}$$

Lemma 3.5 ([14]). *Assume $\emptyset \neq \mathcal{X} \subseteq \mathcal{E}$ be a closed convex bounded, and consider three operators $\mathcal{A}, \mathcal{C} : \mathcal{E} \to \mathcal{E}$ and $\mathcal{B} : \mathcal{X} \to \mathcal{E}$ with:*

(l1) *$\mathcal{A}$, $\mathcal{C}$ are Lipschitzian with a Lipschitz constants τ and σ, respectively;*
(l2) *$\mathcal{C}$ is compact and continuous;*
(l3) *$x = \mathcal{A}x\,\mathcal{B}y + \mathcal{C}x \implies x \in \mathcal{X}$ for each $y \in \mathcal{X}$;*

Then the equation $\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x$ has a solution in $\mathcal{X}$, if $\tau K + \sigma < 1$, with $K = \|\mathcal{C}(\mathcal{X})\|_{\mathcal{E}}$.

Proof of Theorem 3.2: We define the closed convex bounded $\mathcal{X} \subseteq \mathcal{E}$ by

$$\mathcal{X} = \left\{\mathrm{x} \in \mathcal{E} \;:\; \|\mathrm{x}\|_{\mathcal{E}} \leq r\right\}.$$

Thanks to Lemma 3.5, the fractional hybrid problem (1) is equivalent to the equation

$$\begin{aligned} x_k(\mathfrak{t}) = \mathcal{I}_{0^+}^{\sigma_k;\varphi} f_k(\mathfrak{t},\mathrm{x}) + g_k(\mathfrak{t},\mathrm{x})\Big\{&\mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k}\mathcal{I}_{0^+}^{\alpha_k;\varphi}h_k(\mathfrak{t},\mathrm{x})\right) \\ &+ \tfrac{\hat{\varphi}_0(\mathfrak{t})}{\hat{\varphi}_0(b)}\left[\mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k}\mathcal{I}_{0^+}^{\alpha_k;\varphi}h_k(b,\mathrm{x})\right) - \sum_{3}^{i=1}\lambda_{ik}x_i\left(\zeta_{ik}\right)\right]\Big\}. \end{aligned} \tag{12}$$

Taking into account Lemma 3.1, we define three operators $\mathcal{A}, \mathcal{C} : \mathcal{E} \to \mathcal{E}$ and $\mathcal{B} : \mathcal{X} \to \mathcal{E}$

$$\begin{aligned} \mathcal{A}\left(\mathrm{x}\right)\left(\mathfrak{t}\right) &= \left(\mathcal{A}_1(\mathrm{x})(\mathfrak{t}), \mathcal{A}_2(\mathrm{x})(\mathfrak{t}), \mathcal{A}_3(\mathrm{x})(\mathfrak{t})\right), \\ \mathcal{B}\left(\mathrm{x}\right)\left(\mathfrak{t}\right) &= \left(\mathcal{B}_1(\mathrm{x})(\mathfrak{t}), \mathcal{B}_2(\mathrm{x})(\mathfrak{t}), \mathcal{B}_3(\mathrm{x})(\mathfrak{t})\right), \\ \mathcal{C}\left(\mathrm{x}\right)\left(\mathfrak{t}\right) &= \left(\mathcal{C}_1(\mathrm{x})(\mathfrak{t}), \mathcal{C}_2(\mathrm{x})(\mathfrak{t}), \mathcal{C}_3(\mathrm{x})(\mathfrak{t})\right), \end{aligned} \tag{13}$$

where

$$\begin{aligned} \mathcal{A}_k\left(\mathrm{x}\right)\left(\mathfrak{t}\right) &= g_k(\mathfrak{t},\mathrm{x}), \\ \mathcal{B}_k\left(\mathrm{x}\right)\left(\mathfrak{t}\right) &= \mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k}\mathcal{I}_{0^+}^{\alpha_k;\varphi}h_k(\mathfrak{t},\mathrm{x})\right) \\ &\quad + \tfrac{\hat{\varphi}_0(\mathfrak{t})}{B_\varphi}\left[\mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k}\mathcal{I}_{0^+}^{\alpha_k;\varphi}h_k(b,\mathrm{x})\right) - \sum_{i=1}^{3}\lambda_{ik}x_i\left(\zeta_{ik}\right)\right], \\ \mathcal{C}_k\left(\mathrm{x}\right)\left(\mathfrak{t}\right) &= \mathcal{I}_{0^+}^{\sigma_k;\varphi} f_k(\mathfrak{t},\mathrm{x}), \end{aligned}$$

and

$$\mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k}\mathcal{I}_{0^+}^{\alpha_k;\varphi}h_k(\mathfrak{t},\mathrm{x})\right) = \int_0^{\mathfrak{t}} \varphi_{\beta_k}\left(\mathfrak{t},\mathfrak{s}\right)\psi_{q_k}\left[\int_0^{\mathfrak{s}}\varphi_{\alpha_k}\left(\mathfrak{s},z\right)h_k(z,\mathrm{x})\,\mathrm{d}z\right]\mathrm{d}\mathfrak{s}.$$

In this case, Eq. (12) can be written as

$$x_k(\mathrm{t}) = \mathcal{A}_k\,(\mathrm{x})\,(\mathrm{t})\,\mathcal{B}_k\,(\mathrm{x})\,(\mathrm{t}) + \mathcal{C}_k\,(\mathrm{x})\,(\mathrm{t}), \quad \mathrm{t} \in L.$$

In the following series of claims, we show the operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$ satisfy all the conditions of Lemma 3.5.

Claim (a). Let $\mathrm{x} = (x_1, x_2, x_3)$, $\mathrm{y} = (y_1, y_2, y_3) \in \mathcal{E}$. Then by $(\mathbb{H}_2)$, for $\mathrm{t} \in L$, we have

$$\left|\mathcal{A}_k(\mathrm{x})(\mathrm{t}) - \mathcal{A}_k(\mathrm{y})(\mathrm{t})\right| \le \Sigma\, \mathcal{T}_{gk}(\mathrm{t}), \qquad \mathrm{t} \in L,$$

and so

$$\left\|\mathcal{A}_k(\mathrm{x}) - \mathcal{A}_k(\mathrm{y})\right\|_{C(L)} \le \|\mathcal{T}_{gk}\| \sum_{i=1}^{3} \|x_i - y_i\|_{C(L)}.$$

Thus,

$$\|\mathcal{A}(\mathrm{x}) - \mathcal{A}(\mathrm{y})\|_{\mathcal{E}} \le \left(\sum_{k=1}^{3} \|\mathcal{T}_{gk}\|\right) \left[\sum_{i=1}^{3} \|x_i - y_i\|_{C(L)}\right].$$

Therefore $\mathcal{A}$ is Lipschitzian on $\mathcal{E}$ with constant $\sum_{k=1}^{3} \|\mathcal{T}_{gk}\|$, for $\mathrm{x}, \mathrm{y} \in \mathcal{E}$. Also

$$\|\mathcal{C}(\mathrm{x}) - \mathcal{C}(\mathrm{y})\|_{\mathcal{E}} \le \left(\sum_{k=1}^{3} \tfrac{(\hat{\varphi}_0(b))^{\sigma_k} \|\mathcal{T}_{fk}\|}{\Gamma(1+\sigma_k)}\right) \left[\sum_{i=1}^{3} \|x_i - y_i\|_{C(L)}\right].$$

Hence $\mathcal{C}$ is Lipschitzian on $\mathcal{E}$ with constant

$$\sum_{k=1}^{3} \tfrac{(\hat{\varphi}_0(b))^{\sigma_k} \|\mathcal{T}_{fk}\|}{\Gamma(1+\sigma_k)}, \qquad \forall\, \mathrm{x}, \mathrm{y} \in \mathcal{E}.$$

Claim (b). We show that $\mathcal{B}$ is is a completely continuous operator from $\mathcal{X}$ into $\mathcal{E}$. First, by $(\mathbb{H}_1)$ we have $\mathcal{B}$ is a continuous operator $\mathcal{X}$ for all $\mathrm{t} \in [0, b]$. Next, we will prove that the set $\mathcal{B}(\mathcal{X})$ is a uniformly bounded in $\mathcal{X}$. For any $\mathrm{x} \in \mathcal{X}$, and by

$$\left|\tfrac{\hat{\varphi}_0(\mathrm{t})}{\hat{\varphi}_0(b)}\right| \le 1, \qquad \psi_{q_k}(s) = s\,|s|^{q_k - 2}\ (q_k \ge 2),$$

we have

$$\begin{aligned}
|\mathcal{B}_k\,(\mathrm{x})\,(\mathrm{t})| &\le \left|\mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k} \mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(\mathrm{t}, \mathrm{x})\right)\right| \\
&\quad + \left|\mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k} \mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(b, \mathrm{x})\right)\right| + \sum_{i=1}^{3} \lambda_{ik}\,|x_i\,(\zeta_{ik})| \\
&\le 2\left|\mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k} \mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(b, \mathrm{x})\right)\right| + \sum_{i=1}^{3} \lambda_{ik}\,|x_i\,(\zeta_{ik})| \\
&\le \tfrac{2(\hat{\varphi}_0(b))^{\beta_k}}{\Gamma(1+\beta_k)} \left|\mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(b, \mathrm{x})\right|^{q_k - 1} + r \sum_{i=1}^{3} \lambda_{ik} \\
&\le \tfrac{2(\hat{\varphi}_0(b))^{\beta_k} \Delta^{q_k - 1}}{\Gamma(1+\beta_k)} + r \sum_{i=1}^{3} \lambda_{ik},
\end{aligned}$$

and so

$$|\mathcal{B}\,(\mathrm{x})\,(\mathrm{t})| \le \sum_{k=1}^{3} \left[\tfrac{2(\hat{\varphi}_0(b))^{\beta_k} \Delta^{q_k - 1}}{\Gamma(1+\beta_k)} + r \sum_{i=1}^{3} \lambda_{ik}\right] \le \mathcal{K}_1. \tag{14}$$

Thus $\|\mathcal{B}(\mathrm{x})\|_{\mathcal{E}} \leq \mathcal{K}_1$ for all $\mathrm{x} \in \mathcal{X}$ with $\mathcal{K}_1$ is given by (7). Indeed, $\mathcal{B}$ is uniformly bounded on $\mathcal{X}$. Now, we will show that $\mathcal{B}(\mathcal{X})$ is an equicontinuous set in $\mathcal{E}$. Let $\mathrm{t}_1, \mathrm{t}_2 \in L$. Then for any $\mathrm{x} \in \mathcal{X}$, by (6), (4) and Lemma 3.3, we get

$$\begin{aligned}
\left|\mathcal{B}_k(\mathrm{x})(\mathrm{t}_2) - \mathcal{B}_k(\mathrm{x})(\mathrm{t}_1)\right| \leq & \Big| \mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k}\mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(t_2,\mathrm{x})\right) \\
& - \mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k}\mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(t_1,\mathrm{x})\right)\Big| + \frac{\hat{\varphi}_{\mathrm{t}_1}(\mathrm{t}_2)}{\hat{\varphi}_0(b)} \\
& \times \left[\left|\mathcal{I}_{0^+}^{\beta_k;\varphi}\left(\psi_{q_k}\mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(b,\mathrm{x})\right)\right| + \sum_{i=1}^{3}\lambda_{ik}\left|x_i(\zeta_{ik})\right|\right] \\
\leq & \frac{2(\hat{\varphi}_0(b))^{\beta_k}}{\Gamma(1+\beta_k)}\left|\psi_{q_k}\left(\mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(t_2,\mathrm{x})\right) - \psi_{q_k}\left(\mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(t_1,\mathrm{x})\right)\right| \\
& + \frac{\hat{\varphi}_{\mathrm{t}_1}(\mathrm{t}_2)}{\hat{\varphi}_0(b)}\left[\frac{(\hat{\varphi}_0(b))^{\beta_k}\Delta^{q_k-1}}{\Gamma(1+\beta_k)} + r\sum_{i=1}^{3}\lambda_{ik}\right],
\end{aligned}$$

and by (6), Lemmas 3.3 and 3.4, we get

$$\begin{aligned}
&\left|\mathcal{B}_k(\mathrm{x})(\mathrm{t}_2) - \mathcal{B}_k(\mathrm{x})(\mathrm{t}_1)\right| \\
&\quad \leq \frac{2(\hat{\varphi}_0(b))^{\beta_k}(q_k-1)\Delta^{q_k-2}}{\Gamma(1+\beta_k)}\Big|\mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(\mathrm{t}_2,\mathrm{x}) \\
&\quad - \mathcal{I}_{0^+}^{\alpha_k;\varphi} h_k(\mathrm{t}_1,\mathrm{x})\Big| + \frac{\hat{\varphi}_{\mathrm{t}_1}(\mathrm{t}_2)}{\hat{\varphi}_0(b)} \\
&\quad \times \left[\frac{(\hat{\varphi}_0(b))^{\beta_k+(q_k-1)\alpha_k}\|\psi_k\|^{q_k-1}(\vartheta_k(r))^{q_k-1}}{\Gamma(1+\beta_k)(\Gamma(1+\alpha_k))^{q_k-1}} + r\sum_{i=1}^{3}\lambda_{ik}\right] \\
&\quad \leq \frac{4\|\mathcal{T}_{hk}\|(\hat{\varphi}_0(b))^{\alpha_k+\beta_k}(q_k-1)\Delta^{q_k-2}}{\Gamma(1+\alpha_k)\Gamma(1+\beta_k)}\left(\hat{\varphi}_{\mathrm{t}_1}(\mathrm{t}_2)\right)^{\alpha_k} \\
&\quad + \left[\frac{(\hat{\varphi}_0(b))^{\beta_k}\Delta^{q_k-1}}{\Gamma(1+\beta_k)} + r\sum_{i=1}^{3}\lambda_{ik}\right]\frac{\hat{\varphi}_{\mathrm{t}_1}(\mathrm{t}_2)}{\hat{\varphi}_0(b)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\mathcal{B}(\mathrm{x})(\mathrm{t}_2) - \mathcal{B}(\mathrm{x})(\mathrm{t}_1)\right| \leq & \sum_{k=1}^{3}\frac{4\|\mathcal{T}_{hk}\|(\hat{\varphi}_0(b))^{\alpha_k+\beta_k}(q_k-1)\Delta^{q_k-2}}{\Gamma(1+\alpha_k)\Gamma(1+\beta_k)}\left(\hat{\varphi}_{\mathrm{t}_1}(\mathrm{t}_2)\right)^{\alpha_k} \\
& + \sum_{k=1}^{3}\left[\frac{(\hat{\varphi}_0(b))^{\beta_k}\Delta^{q_k-1}}{\Gamma(1+\beta_k)} + r\sum_{i=1}^{3}\lambda_{ik}\right]\frac{\hat{\varphi}_{\mathrm{t}_1}(\mathrm{t}_2)}{\hat{\varphi}_0(b)}. \qquad (15)
\end{aligned}$$

The right-hand side of (15) tends to zero freely of $\mathrm{x} \in \mathcal{X}$ as $\mathrm{t}_2 \to \mathrm{t}_1$. So, the Ascoli–Arzelá theorem implies that $\mathcal{B}$ is a completely continuous operator on $\mathcal{X}$.
Claim (c). Hypothesis (13) of Lemma 3.5 is satisfied. For any $\mathrm{x} \in \mathcal{X}$, $\mathrm{t} \in L$, using the condition (7), we have

$$\begin{aligned}
\left|\mathcal{C}_k(\mathrm{x})(\mathrm{t})\right| &= \left|\mathcal{I}_{0^+}^{\sigma_k;\varphi} f_k(\mathrm{t},\mathrm{x})\right| \\
&\leq \mathcal{I}_{0^+}^{\sigma_k;\varphi}\left|(f_k(\mathrm{t},\mathrm{x}) - f_k(\mathrm{t},0,0,0))\right| + \mathcal{I}_{0^+}^{\sigma_k;\varphi}\left|f_k(\mathrm{t},0,0,0)\right| \\
&\leq \frac{(\hat{\varphi}_0(b))^{\alpha_k}\|\mathcal{T}_{fk}\|}{\Gamma(1+\alpha_k)}\sum_{i=1}^{3}|x_i| + \frac{(\hat{\varphi}_0(b))^{\alpha_k}}{\Gamma(1+\alpha_k)}\sup_{\mathrm{t}\in L}\left|f_k(\mathrm{t},0,0,0)\right|,
\end{aligned}$$

$$\begin{aligned}|\mathcal{A}_k(\mathrm{x})(\mathfrak{t})| &= |g_k(\mathfrak{t},\mathrm{x})|\\ &\le |g_k(\mathfrak{t},\mathrm{x}) - g_k(\mathfrak{t},0,0,0)| + |g_k(\mathfrak{t},0,0,0)|\\ &\le \|\mathcal{T}_{gk}\|\left(\sum_{i=1}^{3}|x_i|\right) + \sup_{\mathfrak{t}\in L}|g_k(\mathfrak{t},0,0,0)|,\end{aligned}$$

and so

$$\begin{aligned}|\mathcal{C}(\mathrm{x})(\mathfrak{t})| &\le \sum_{k=1}^{3}\tfrac{(\hat{\varphi}_0(b))^{\alpha_k}\|\mathcal{T}_{fk}\|}{\Gamma(1+\alpha_k)}\|\mathrm{x}\|_{\mathcal{E}} + \sum_{k=1}^{3}\tfrac{(\hat{\varphi}_0(b))^{\alpha_k}}{\Gamma(1+\alpha_k)}\sup_{\mathfrak{t}\in L}|f_k(\mathfrak{t},0,0,0)|\\ &= \|\mathrm{x}\|_{\mathcal{E}}\sum_{k=1}^{3}\tfrac{(\hat{\varphi}_0(b))^{\alpha_k}\|\mathcal{T}_{fk}\|}{\Gamma(1+\alpha_k)} + \mathcal{K}_3,\\ |\mathcal{A}(\mathrm{x})(\mathfrak{t})| &\le \|\mathrm{x}\|_{\mathcal{E}}\sum_{k=1}^{3}\|\mathcal{T}_{gk}\| + \sum_{k=1}^{3}\sup_{\mathfrak{t}\in L}|g_k(\mathfrak{t},0,0,0)|\\ &= \|\mathrm{x}\|_{\mathcal{E}}\sum_{k=1}^{3}\|\mathcal{T}_{gk}\| + \mathcal{K}_2.\end{aligned}$$

Consider two elements $\mathrm{x}\in\mathcal{E}$ and $\mathrm{y}\in\mathcal{X}$ with $\mathrm{x} = \mathcal{A}(\mathrm{x})\mathcal{B}(\mathrm{y}) + \mathcal{C}(\mathrm{x})$. Then by (14) we have

$$|x_k(\mathfrak{t})| \le |\mathcal{A}_k(\mathrm{x})(\mathfrak{t})|\,|\mathcal{B}_k(\mathrm{x})(\mathfrak{t})| + |\mathcal{C}_k(\mathrm{x})(\mathfrak{t})|,$$

and so

$$\begin{aligned}\sup_{\mathfrak{t}\in L}|\mathrm{x}(\mathfrak{t})| &\le \sum_{k=1}^{3}\left(\sup_{\mathfrak{t}\in L}|x_k(\mathfrak{t})|\right) \le \|\mathrm{x}\|_{\mathcal{E}}\left(\mathcal{K}_1\sum_{k=1}^{3}\|\mathcal{T}_{gk}\|\right.\\ &\quad\left.+\sum_{k=1}^{3}\tfrac{(\hat{\varphi}_0(b))^{\sigma_k}\|\mathcal{T}_{fk}\|}{\Gamma(1+\sigma_k)}\right) + 3(\mathcal{K}_1\mathcal{K}_2 + \mathcal{K}_3),\\ &\Rightarrow \|\mathrm{x}\|_{\mathcal{E}} \le \frac{3(\mathcal{K}_1\mathcal{K}_2+\mathcal{K}_3)}{1-\mathcal{K}_1\sum_{k=1}^{3}\|\mathcal{T}_{gk}\|-\sum_{k=1}^{3}\frac{(\hat{\varphi}_0(b))^{\sigma_k}\left\|\mathcal{T}_{fk}\right\|}{\Gamma(1+\sigma_k)}} \le r.\end{aligned}$$

Claim (d). Finally, we show that $\tau K + \sigma < 1$, that is, (d) of Lemma 3.5 holds. Since

$$K = \|\mathcal{B}(\mathcal{X})\| = \sup_{\mathrm{x}\in\mathcal{X}}\left\{\sup_{\mathfrak{t}\in L}|\mathcal{B}(\mathrm{x})(\mathfrak{t})|\right\} \le \mathcal{K}_1,$$

we have

$$K\sum_{k=1}^{3}\|\mathcal{T}_{gk}\| + \sum_{k=1}^{3}\tfrac{(\hat{\varphi}_0(b))^{\sigma_k}\|\mathcal{T}_{fk}\|}{\Gamma(1+\sigma_k)} \le \mathcal{K}_1\sum_{k=1}^{3}\|\mathcal{T}_{gk}\| + \sum_{k=1}^{3}\tfrac{(\hat{\varphi}_0(b))^{\sigma_k}\|\mathcal{T}_{fk}\|}{\Gamma(1+\sigma_k)} < 1,$$

with

$$\tau = \sum_{k=1}^{3}\|\mathcal{T}_{gk}\|, \qquad \sigma = \sum_{k=1}^{3}\tfrac{1}{\Gamma(1+\sigma_k)}(\hat{\varphi}_0(b))^{\sigma_k}\|\mathcal{T}_{fk}\|.$$

Thus, all the conditions of Lemma 3.5 are satisfied, so the operator equation $\mathrm{x} = \mathcal{A}(\mathrm{x})\mathcal{B}(\mathrm{x}) + \mathcal{C}(\mathrm{x})$, has a solution in $\mathcal{X}$. Indeed, problem (1) has a solution on L.

4. Illustrative examples

We present, a few numerical examples that show the accuracy of the results well.

Example 4.1. Base on the system (1), consider the following tripled system of hybrid FDEs with ψ_{P_k} Laplacian operator of nonlinear equation for all $\mathrm{t} \in L = [0, \ \ln 3]$,

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha_1;e^{\mathrm{t}}} \psi_{8/5}\left[\mathcal{D}_{0^+}^{16/13;e^{\mathrm{t}}}\left(\frac{x_1(\mathrm{t})-\mathcal{I}_{0^+}^{5/4;e^{\mathrm{t}}} f_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)\right] \\ \quad = \frac{\left(2+e^{2\mathrm{t}}\right)\sin(x_1(\mathrm{t}))}{1100} + \frac{e^{\mathrm{t}}\sin^{-1}(x_2(\mathrm{t}))}{275} + \frac{(1+|\mathrm{t}|)x_3(\mathrm{t})}{550}, \\ \mathcal{D}_{0^+}^{\alpha_2;e^{\mathrm{t}}} \psi_{8/5}\left[\mathcal{D}_{0^+}^{16/13;e^{\mathrm{t}}}\left(\frac{x_2(\mathrm{t})-\mathcal{I}_{0^+}^{5/4;e^{\mathrm{t}}} f_2(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_2(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)\right] \\ \quad = \frac{\left(1+e^{\mathrm{t}}\right)x_1(\mathrm{t})}{50} + \frac{e^{\mathrm{t}}\tan(x_2(\mathrm{t}))}{300} + \frac{e^{\mathrm{t}}\sin^{-1}(x_3(\mathrm{t}))}{100}, \\ \mathcal{D}_{0^+}^{\alpha_3;e^{\mathrm{t}}} \psi_{8/5}\left[\mathcal{D}_{0^+}^{16/13;e^{\mathrm{t}}}\left(\frac{x_3(\mathrm{t})-\mathcal{I}_{0^+}^{5/4;e^{\mathrm{t}}} f_3(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_3(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)\right] \\ \quad = \frac{\tan(x_1(\mathrm{t}))}{25(1+e^{2\mathrm{t}})} + \frac{\sin(x_2(\mathrm{t}))}{2+e^{3\mathrm{t}}} + \frac{x_3(\mathrm{t})}{50(1+e^{\mathrm{t}})}, \end{cases} \tag{16}$$

with three different values $\alpha_1 = \alpha_2 = \alpha_3 = \frac{3}{11}, \ \frac{1}{2}, \ \frac{8}{9}$, under conditions

$$\psi_{8/5}\mathcal{D}_{0^+}^{16/13;e^{\mathrm{t}}}\left(\frac{x_1(\mathrm{t})-\mathcal{I}_{0^+}^{5/4;e^{\mathrm{t}}} f_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)\Bigg|_{\mathrm{t}=0} = 0,$$
$$\frac{x_1(\mathrm{t})-\mathcal{I}_{0^+}^{5/4;e^{\mathrm{t}}} f_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}\Bigg|_{\mathrm{t}=1} = \sum_{i=1}^{3}\frac{i}{100}x_i\left(\zeta_{i1}\right),$$

$$\psi_{8/5}\mathcal{D}_{0^+}^{16/13;e^{\mathrm{t}}}\left(\frac{x_2(\mathrm{t})-1\mathcal{I}_{0^+}^{3/2;e^{\mathrm{t}}} f_2(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_2(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)\Bigg|_{\mathrm{t}=0} = 0,$$
$$\frac{x_2(\mathrm{t})-\mathcal{I}_{0^+}^{5/4;e^{\mathrm{t}}} f_2(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_2(\mathrm{t},\mathrm{x}(\mathrm{t}))}\Bigg|_{\mathrm{t}=1} = \sum_{i=1}^{3}\frac{i}{200}x_i\left(\zeta_{i2}\right),$$

and

$$\psi_{8/5}\mathcal{D}_{0^+}^{16/13;e^{\mathrm{t}}}\left(\frac{x_3(\mathrm{t})-\mathcal{I}_{0^+}^{5/4;e^{\mathrm{t}}} f_3(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_3(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)\Bigg|_{\mathrm{t}=0} = 0,$$
$$\frac{x_3(\mathrm{t})-\mathcal{I}_{0^+}^{5/4;e^{\mathrm{t}}} f_3(\mathrm{t},\mathrm{x}(\mathrm{t}))}{g_3(\mathrm{t},\mathrm{x}(\mathrm{t}))}\Bigg|_{\mathrm{t}=1} = \sum_{i=1}^{3}\frac{i}{300}x_i\left(\zeta_{i3}\right),$$

with

$$\begin{aligned} f_1(\mathrm{t},\mathrm{x}) &= \tfrac{9}{100(8+e^{2\mathrm{t}})}\left(x_1(\mathrm{t}) + x_2(\mathrm{t}) + x_3(\mathrm{t}) + \tfrac{8}{19}\right), \\ f_2(\mathrm{t},\mathrm{x}) &= \tfrac{e^{2\mathrm{t}}}{900}\left(x_1(\mathrm{t}) - x_2(\mathrm{t}) + x_3(\mathrm{t}) + \tfrac{5}{19}\right), \\ f_3(\mathrm{t},\mathrm{x}) &= \tfrac{e^{\mathrm{t}}}{100(3+\mathrm{t}^2)}\left(x_1(\mathrm{t}) + x_2(\mathrm{t}) - x_3(\mathrm{t}) + \tfrac{6}{19}\right), \end{aligned} \tag{17}$$

and

$$\begin{aligned} g_1(\mathrm{t},\mathrm{x}) &= \tfrac{7-e^{\mathrm{t}}}{6000}\left(x_1(\mathrm{t})+x_2(\mathrm{t})+x_3(\mathrm{t})+\tfrac{5}{12}\right),\\ g_2(\mathrm{t},\mathrm{x}) &= \tfrac{e^{2\mathrm{t}}}{9000}\left(x_1(\mathrm{t})-x_2(\mathrm{t})+x_3(\mathrm{t})+\tfrac{7}{12}\right),\\ g_3(\mathrm{t},\mathrm{x}) &= \tfrac{3+e^{3\mathrm{t}}}{30000}\left(x_1(\mathrm{t})-x_2(\mathrm{t})-x_3(\mathrm{t})+\tfrac{9}{16}\right). \end{aligned} \tag{18}$$

Set

$$\begin{aligned} h_1(\mathrm{t},\mathrm{x}) &= \frac{\left(2+e^{2\mathrm{t}}\right)\sin(x_1(\mathrm{t}))}{1100}+\frac{e^{\mathrm{t}}\sin^{-1}(x_2(\mathrm{t}))}{275}+\frac{(1+|\mathrm{t}|)x_3(\mathrm{t})}{550},\\ h_2(\mathrm{t},\mathrm{x}) &= \frac{\left(1+e^{\mathrm{t}}\right)x_1(\mathrm{t})}{50}+\frac{e^{\mathrm{t}}\tan(x_2(\mathrm{t}))}{300}+\frac{e^{\mathrm{t}}\sin^{-1}(x_3(\mathrm{t}))}{100},\\ h_3(\mathrm{t},\mathrm{x}) &= \frac{\tan(x_1(\mathrm{t}))}{25(1+e^{2\mathrm{t}})}+\frac{\sin(x_2(\mathrm{t}))}{2+e^{3\mathrm{t}}}+\frac{x_3(\mathrm{t})}{50(1+e^{\mathrm{t}})}. \end{aligned} \tag{19}$$

Clearly, f_k, g_k, h_k, $k=1,2,3$, are continuous. The condition $(\mathbb{H}_2)$ is satisfied, if we consider $\|\mathcal{T}_{fk}\| = 10\,\|\mathcal{T}_{gk}\| = \frac{1}{100}$, $k=1,2,3,$, $\|\mathcal{T}_{h1}\| = \frac{3}{275}$, $\|\mathcal{T}_{h2}\| = \frac{3}{50}$, $\|\mathcal{T}_{h3}\| = \frac{1}{29}$ as follows:

$$\begin{aligned} \left|f_1\left(\mathrm{t},\mathrm{x}\right)-f_1\left(\mathrm{t},\mathrm{y}\right)\right| &= \Bigg|\tfrac{9}{100(8+e^{2\mathrm{t}})}\left(x_1(\mathrm{t})+x_2(\mathrm{t})+x_3(\mathrm{t})+\tfrac{8}{19}\right)\\ &\qquad -\left(\tfrac{9}{100(8+e^{2\mathrm{t}})}\left(y_1(\mathrm{t})+y_2(\mathrm{t})+y_3(\mathrm{t})+\tfrac{8}{19}\right)\right)\Bigg|\\ &\le \left|\tfrac{9}{100(8+e^{2\mathrm{t}})}\right|\sum_{i=1}^{3}|x_i-y_i|\\ &\Rightarrow \quad \mathcal{T}_{f1}(\mathrm{t}) = \tfrac{9}{100(8+e^{2\mathrm{t}})} \Rightarrow \|\mathcal{T}_{f1}(\mathrm{t})\| = \tfrac{1}{100},\\ &\Rightarrow \quad \mathcal{T}_{f2}(\mathrm{t}) = \tfrac{e^{2\mathrm{t}}}{900} \Rightarrow \|\mathcal{T}_{f2}(\mathrm{t})\| = \tfrac{1}{100},\\ &\Rightarrow \quad \mathcal{T}_{f3}(\mathrm{t}) = \tfrac{e^{\mathrm{t}}}{100(3+\mathrm{t}^2)} \Rightarrow \|\mathcal{T}_{f3}(\mathrm{t})\| = \tfrac{1}{100}, \end{aligned}$$

$$\begin{aligned} \left|g_1\left(\mathrm{t},\mathrm{x}\right)-g_1\left(\mathrm{t},\mathrm{y}\right)\right| &= \Bigg|\tfrac{7-e^{\mathrm{t}}}{6000}\left(x_1(\mathrm{t})+x_2(\mathrm{t})+x_3(\mathrm{t})+\tfrac{5}{12}\right)\\ &\qquad -\tfrac{7-e^{\mathrm{t}}}{6000}\left(x_1(\mathrm{t})+x_2(\mathrm{t})+x_3(\mathrm{t})+\tfrac{5}{12}\right)\Bigg|\\ &\le \left|\tfrac{7-e^{\mathrm{t}}}{6000}\right|\sum_{i=1}^{3}|x_i-y_i|,\\ &\Rightarrow \quad \mathcal{T}_{g1}(\mathrm{t}) = \tfrac{7-e^{\mathrm{t}}}{6000} \Rightarrow \|\mathcal{T}_{g1}(\mathrm{t})\| = \tfrac{1}{1000},\\ &\Rightarrow \quad \mathcal{T}_{g2}(\mathrm{t}) = \tfrac{e^{2\mathrm{t}}}{9000} \Rightarrow \|\mathcal{T}_{g2}(\mathrm{t})\| = \tfrac{1}{1000},\\ &\Rightarrow \quad \mathcal{T}_{g3}(\mathrm{t}) = \tfrac{3+e^{3\mathrm{t}}}{30000} \Rightarrow \|\mathcal{T}_{g3}(\mathrm{t})\| = \tfrac{1}{1000}, \end{aligned}$$

and

$$\begin{aligned} \left|h_1(\mathrm{t},\mathrm{x})-h_1(\mathrm{t},\mathrm{y})\right| &= \Bigg|\frac{\left(2+e^{2\mathrm{t}}\right)\sin(x_1(\mathrm{t}))}{1100}+\frac{e^{\mathrm{t}}\sin^{-1}(x_2(\mathrm{t}))}{275}+\frac{(1+|\mathrm{t}|)x_3(\mathrm{t})}{550}\\ &\qquad -\left(\frac{\left(2+e^{2\mathrm{t}}\right)\sin(y_1(\mathrm{t}))}{1100}+\frac{e^{\mathrm{t}}\sin^{-1}(y_2(\mathrm{t}))}{275}+\frac{(1+|\mathrm{t}|)y_3(\mathrm{t})}{550}\right)\Bigg| \end{aligned}$$

$$\leq \left|\frac{2+e^{2t}}{1100}\right| \left|\sin\left(x_1(t)\right)-\sin\left(y_1(t)\right)\right| + \left|\frac{e^{t}}{275}\right| \left|\sin^{-1}\left(x_2(t)\right)-\sin^{-1}\left(y_2(t)\right)\right|$$

$$+\left|\frac{1+|t|}{550}\right| \left|y_3(t)-x_3(t)\right| \leq \left|\frac{e^{t}}{275}\right| \sum_{i=1}^{3}|x_i-y_i|$$

$$\Rightarrow \quad \mathcal{T}_{h1}(t)=\tfrac{e^{t}}{275} \Rightarrow \|\mathcal{T}_{h1}(t)\|=\tfrac{3}{275},$$

$$\Rightarrow \quad \mathcal{T}_{h2}(t)=\tfrac{1+e^{t}}{50} \Rightarrow \|\mathcal{T}_{h2}(t)\|=\tfrac{3}{50},$$

$$\Rightarrow \quad \mathcal{T}_{h3}(t)=\tfrac{1}{2+e^{3t}} \Rightarrow \|\mathcal{T}_{h3}(t)\|=\tfrac{1}{29}.$$

Furthermore,

$$|h_1(t,x)|=\left|\frac{\left(2+e^{2t}\right)\sin(x_1(t))}{1100}+\frac{e^{t}\sin^{-1}(x_2(t))}{275}+\frac{(1+|t|)x_3(t)}{550}\right|$$

$$\leq \left|\frac{4+e^{2t}+4e^{t}+2|t|}{1100}\right| \sum_{i=1}^{3}\|x_i\| = \psi_1(t)\vartheta_1\left(\|x\|\right),$$

$$|h_2(t,x)|=\left|\frac{\left(1+e^{t}\right)x_1(t)}{50}+\frac{e^{t}\tan(x_2(t))}{300}+\frac{e^{t}\sin^{-1}(x_3(t))}{100}\right|$$

$$\leq \left|\frac{6+10e^{t}}{300}\right| \sum_{i=1}^{3}\|x_i\| = \psi_2(t)\vartheta_2\left(\|x\|\right),$$

$$|h_3(t,x)|=\left|\frac{\tan(x_1(t))}{25(1+e^{2t})}+\frac{\sin(x_2(t))}{2+e^{3t}}+\frac{x_3(t)}{50(1+e^{t})}\right|$$

$$\leq \left|\frac{1}{50(1+e^{t})}\right| \sum_{i=1}^{3}\|x_i\| = \psi_3(t)\vartheta_3\left(\|(x)\|\right).$$

Assume that $r=2.1$ is given. Now, by employing Eq. (6), we have

$$\begin{aligned}
\Delta_i &> \max\left\{\tfrac{(\hat{\varphi}_0(b))^{\alpha_k}\|\psi_k\|\vartheta_k(r)}{\Gamma(1+\alpha_k)} \,:\, k=\overline{1,3}\right\}\\
&= \tfrac{(\varphi(\ln 3)-\varphi(0))^{3/11}}{\Gamma\left(1+\frac{3}{11}\right)} \max\left\{\left\|\tfrac{4+e^{2t}+4e^{t}+2|t|}{1100}\right\|\vartheta_1(r),\right.\\
&\quad \left.\left\|\tfrac{6+10e^{t}}{300}\right\|\vartheta_2(r), \left\|\tfrac{1}{50(1+e^{t})}\right\|\vartheta_3(r)\right\}\\
&= \tfrac{2^{3/11}r}{\Gamma\left(\frac{14}{11}\right)} \max\left\{\tfrac{21+\ln 9}{1100}, \tfrac{12}{100}, \tfrac{1}{100}\right\}\\
&= \tfrac{0.12\, r\, 2^{3/11}}{\Gamma\left(\frac{14}{11}\right)} \simeq \begin{cases} 2.013, & \alpha_1=\alpha_2=\alpha_3={}^{3}/_{11},\\ 2.206, & \alpha_1=\alpha_2=\alpha_3={}^{1}/_{2},\\ 2.461, & \alpha_1=\alpha_2=\alpha_3={}^{8}/_{9},\end{cases}
\end{aligned}$$

and by using Eq. (7), we obtain

$$\begin{aligned}
\mathcal{K}_1 &= \sum_{k=1}^{3}\left(\tfrac{2(\hat{\varphi}_0(b))^{\beta_k}\Delta^{q_k-1}}{\Gamma(1+\beta_k)}+r\sum_{i=1}^{3}\lambda_{ik}\right)\\
&= \tfrac{6(\varphi(\ln 3)-\varphi(0))^{3/2}\Delta^{5/3}}{\Gamma\left(\frac{5}{2}\right)}+\tfrac{6r}{100}+\tfrac{6r}{200}+\tfrac{6r}{300}\\
&= \tfrac{3(2)^{5/2}\Delta^{5/3}}{\Gamma\left(\frac{5}{2}\right)}+\tfrac{11r}{100} \simeq \begin{cases} 81.353, & \alpha_1=\alpha_2=\alpha_3={}^{3}/_{11},\\ 103.900, & \alpha_1=\alpha_2=\alpha_3={}^{1}/_{2},\\ 138.988, & \alpha_1=\alpha_2=\alpha_3={}^{8}/_{9},\end{cases}
\end{aligned}$$

TABLE 1. Numerical results of Δ, $\mathcal{K}_1$ and Eqs. (20), (21) in Example 4.1 for three cases of $\alpha_1 = \alpha_2 = \alpha_3 = \frac{3}{11}, \frac{1}{2}, \frac{9}{8}$.

t	Δ	$\mathcal{K}_1$	w_1	w_2
	$\boldsymbol{\alpha_1 = \alpha_2 = \alpha_3 = \frac{3}{11}}$			
0.000	1.5000	0.2310	0.0007	0.4034
0.110	1.5320	1.4108	0.0060	0.4111
0.220	1.5716	3.4080	0.0148	0.4243
0.330	1.6485	6.6104	0.0280	0.4455
0.439	1.7127	11.0462	0.0457	0.4755
0.549	1.7698	16.9442	0.0688	0.5170
0.659	1.8225	24.5973	0.0981	0.5736
0.769	1.8723	34.3687	0.1349	0.6509
0.879	1.9202	46.7039	0.1807	0.7579
0.989	1.9668	62.1455	0.2374	0.9091
1.099	2.0125	81.3533	0.3070	1.1304
	$\boldsymbol{\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}}$			
0.000	1.5000	0.2310	0.0007	0.4034
0.110	1.5200	1.3863	0.0060	0.4109
0.220	1.5290	3.1838	0.0141	0.4229
0.330	1.5366	5.5201	0.0247	0.4387
0.439	1.6337	9.7668	0.0419	0.4674
0.549	1.7299	15.9579	0.0658	0.5104
0.659	1.8241	24.6540	0.0982	0.5740
0.769	1.9178	36.6295	0.1417	0.6684
0.879	2.0123	52.8921	0.1993	0.8117
0.989	2.1083	74.7498	0.2752	1.0381
1.099	2.2064	103.8997	0.3747	1.4217
	$\boldsymbol{\alpha_1 = \alpha_2 = \alpha_3 = \frac{8}{9}}$			
0.000	1.5000	0.2310	0.0007	0.4034
0.110	1.5080	1.3622	0.0059	0.4108
0.220	1.5156	3.1149	0.0139	0.4225
0.330	1.5235	5.4006	0.0244	0.4380
0.439	1.5320	8.2641	0.0374	0.4579
0.549	1.5981	12.9617	0.0568	0.4906
0.659	1.7421	21.8356	0.0898	0.5541
0.769	1.8988	35.6746	0.1388	0.6610
0.879	2.0698	57.0008	0.2116	0.8488
0.989	2.2567	89.5738	0.3197	1.2080
1.099	2.4613	138.9877	0.4799	2.0257

$$\begin{aligned}
\mathcal{K}_2 &= \sum_{k=1}^{3} \sup_{\mathfrak{t}\in L} |g_k(\mathfrak{t},0,0,0)| = \sup_{\mathfrak{t}\in L}\left|\frac{5(7-e^{\mathfrak{t}})}{12\times 6000}\right| + \sup_{\mathfrak{t}\in L}\left|\frac{7e^{2\mathfrak{t}}}{12\times 9000}\right| + \sup_{\mathfrak{t}\in L}\left|\frac{9(3-e^{3\mathfrak{t}})}{16\times 30000}\right| \\
&= \frac{5}{12000} + \frac{7}{12000} + \frac{9}{16000} = \frac{1}{640}, \\
\mathcal{K}_3 &= \sum_{k=1}^{3} \frac{(\hat{\varphi}_0(b))^{\alpha_k}}{\Gamma(1+\alpha_k)} \sup_{\mathfrak{t}\in L} |f_k(\mathfrak{t},0,0,0)| \\
&= \frac{(\varphi(\ln 3)-\varphi(0))^{3/11}}{\Gamma\left(\frac{14}{11}\right)} \left[\sup_{\mathfrak{t}\in L}\left|\frac{72}{1900(8+e^{2\mathfrak{t}})}\right| + \sup_{\mathfrak{t}\in L}\left|\frac{5e^{2\mathfrak{t}}}{9\times 1900}\right| + \sup_{\mathfrak{t}\in L}\left|\frac{6e^{\mathfrak{t}}}{1900(3+\mathfrak{t}^2)}\right|\right] \\
&= \frac{2^{3/11}}{\Gamma\left(\frac{14}{11}\right)}\left[\frac{8}{1900} + \frac{5}{1900} + \frac{6}{1900}\right] = 0.134.
\end{aligned}$$

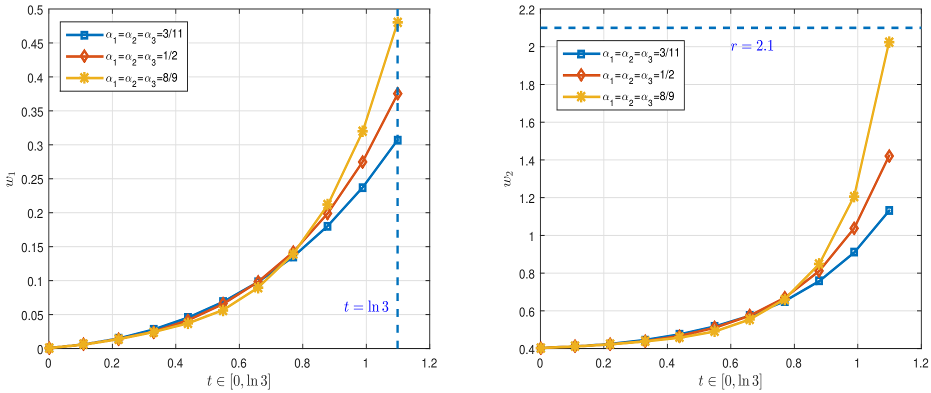
These results are shown in Table 1 for three cases of α. Inequality (8) implies that

$$\begin{aligned} w_1 &:= \mathcal{K}_1 \sum_{k=1}^{3} \|\mathcal{T}_{gk}\| + \sum_{k=1}^{3} \frac{(\hat{\varphi}_0(b))^{\sigma_k} \|\mathcal{T}_{fk}\|}{\Gamma(1+\sigma_k)} = \frac{3\mathcal{K}_1}{1000} + \frac{3(\varphi(\ln 3)-\varphi(0))^{3/2}}{100\Gamma(\frac{5}{2})} \\ &= \frac{3\mathcal{K}_1}{1000} + \frac{3(2)^{3/2}}{100\Gamma(\frac{5}{2})} \simeq \left\{ \begin{array}{ll} 0.307, & \alpha_1 = \alpha_2 = \alpha_3 = 3/11, \\ 0.375, & \alpha_1 = \alpha_2 = \alpha_3 = 1/2, \\ 0.480, & \alpha_1 = \alpha_2 = \alpha_3 = 8/9, \end{array} \right\} < 1. \end{aligned} \tag{20}$$

Finally, inequality (5) implies that

$$\begin{aligned} w_2 &:= 3\left(\mathcal{K}_1\mathcal{K}_2 + \mathcal{K}_3\right) \left[1 - \mathcal{K}_1 \sum_{k=1}^{3} \|\mathcal{T}_{gk}\| - \sum_{k=1}^{3} \frac{(\hat{\varphi}_0(b))^{\sigma_k} \|\mathcal{T}_{fk}\|}{\Gamma(1+\sigma_k)}\right]^{-1} \\ &= 3\left(\mathcal{K}_1\mathcal{K}_2 + \mathcal{K}_3\right) \left[1 - \mathcal{K}_1 \frac{3}{1000} - \frac{3(2)^{3/2}}{\gamma(\frac{5}{2})}\right]^{-1} \\ &\simeq \left\{ \begin{array}{ll} 1.130, & \alpha_1 = \alpha_2 = \alpha_3 = 3/11, \\ 1.422, & \alpha_1 = \alpha_2 = \alpha_3 = 1/2, \\ 2.026, & \alpha_1 = \alpha_2 = \alpha_3 = 8/9, \end{array} \right\} \leq 2.1 = r. \end{aligned} \tag{21}$$

These results are shown in Table 1 and in Figures 1a and 1b, we have plotted w_1 and w_2 for three cases of $\alpha_1 = \alpha_2 = \alpha_3 = \frac{3}{11}, \frac{1}{2}, \frac{8}{9}$. Since, $(\mathbb{H}_1) - (\mathbb{H}_3)$ are satisfied,



(A) w_1, Eq. (20)

(B) w_2, Eq. (21)

FIGURE 1. Representation of w_1 and w_2 for three cases of α in Example 4.1.

Theorem 3.2 implies that (16) has at least one solution on L.

Example 4.2. In this example, we consider the system (16) for all $\mathfrak{t} \in L = [0,\ \ln 3]$, with $\alpha_1 = \frac{1}{7}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{7}{8}$ and for four cases $\varphi_i(\mathfrak{t})$,

$$\varphi_1(\mathfrak{t}) = e^{\mathfrak{t}}, \quad \varphi_2(\mathfrak{t}) = \mathfrak{t}, \quad \varphi_3(\mathfrak{t}) = \ln(\mathfrak{t} + 0.01), \quad \varphi_4(\mathfrak{t}) = \sqrt{\mathfrak{t}}, \tag{22}$$

under the same conditions and functions $f_k(\mathfrak{t},\mathrm{x})$, $g_k(\mathfrak{t},\mathrm{x})$, $h_k(\mathfrak{t},\mathrm{x})$ which are defined by (17), (18) and (19). We saw that, f_k, g_k, h_k, $k=1,2,3$, were continuous and condition $(\mathbb{H}_2)$ satisfied, if we considered $\|\mathcal{T}_{fk}\| = 10\,\|\mathcal{T}_{gk}\| = \frac{1}{100}$, $k=1,2,3,$, $\|\mathcal{T}_{h1}\| = \frac{3}{275}$, $\|\mathcal{T}_{h2}\| = \frac{3}{50}$, $\|\mathcal{T}_{h3}\| = \frac{1}{29}$. Assum that $r = 3.05$ is given. We have tried to consider a specific number r for all states of the function φ. Now, by employing Eq. (6), we have

$$\begin{aligned}
\Delta_i &> \max\left\{\tfrac{(\hat{\varphi}_0(b))^{\alpha_k}\|\psi_k\|\vartheta_k(r)}{\Gamma(1+\alpha_k)} \;:\; k=\overline{1,3}\right\}\\
&= \tfrac{(\varphi_i(\ln 3)-\varphi_i(0))^{\frac{3}{11}}}{\Gamma\left(1+\frac{3}{11}\right)}\max\left\{\left\|\tfrac{4+e^{2\mathfrak{t}}+4e^{\mathfrak{t}}+2|\mathfrak{t}|}{1100}\right\|\vartheta_1(r)\,,\left\|\tfrac{6+10e^{\mathfrak{t}}}{300}\right\|\vartheta_2(r)\,,\left\|\tfrac{1}{50(1+e^{\mathfrak{t}})}\right\|\vartheta_3(r)\right\}\\
&\simeq \begin{cases} 2.4705, & \varphi_1(\mathfrak{t}) = e^{\mathfrak{t}},\\ 2.2986, & \varphi_2(\mathfrak{t}) = \mathfrak{t},\\ 1.8783, & \varphi_3(\mathfrak{t}) = \ln(\mathfrak{t}+1),\\ 2.1950, & \varphi_4(\mathfrak{t}) = \sqrt{\mathfrak{t}}. \end{cases}
\end{aligned}$$

Table 2 shows the results for four cases of φ. One can see the curve of Δ for the cases of φ_i in Figure 2. In the sequel, by using Eq. (7), we obtain

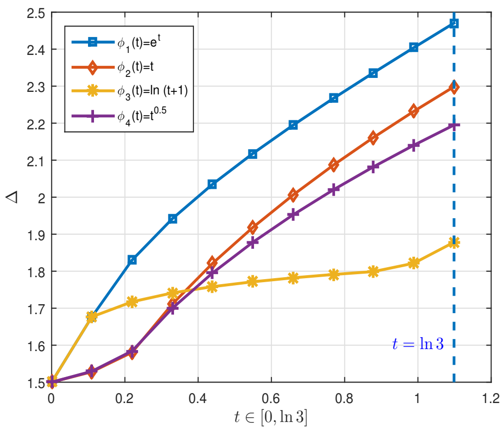


FIGURE 2. Representation of Δ for four cases of φ in Example 4.2.

$$\begin{aligned}
\mathcal{K}_1 &= \sum_{k=1}^{3}\left(\tfrac{2(\hat{\varphi}_0(b))^{\beta_k}\Delta^{q_k-1}}{\Gamma(1+\beta_k)} + r\sum_{i=1}^{3}\lambda_{ik}\right) = \tfrac{6(\varphi_i(\ln 3)-\varphi_i(0))^{3/2}\Delta^{5/3}}{\Gamma\left(\frac{5}{2}\right)} + \tfrac{6r}{100} + \tfrac{6r}{200} + \tfrac{6r}{300}\\
&\simeq \begin{cases} 140.4957, & \varphi_1(\mathfrak{t}) = e^{\mathfrak{t}},\\ 55.6496, & \varphi_2(\mathfrak{t}) = \mathfrak{t},\\ 204.8407, & \varphi_3(\mathfrak{t}) = \ln(\mathfrak{t}+1),\\ 46.5002, & \varphi_4(\mathfrak{t}) = \sqrt{\mathfrak{t}}, \end{cases}
\end{aligned}$$

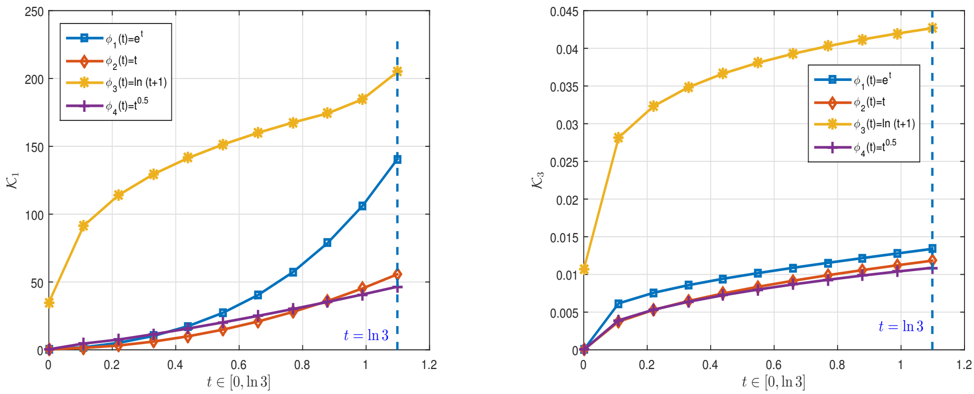
$$\begin{aligned}\mathcal{K}_2 &= \sum_{k=1}^{3} \sup_{t\in L} |g_k(t,0,0,0)| \\ &= \sup_{t\in L}\left|\frac{5(7-e^{t})}{12\times 6000}\right| + \sup_{t\in L}\left|\frac{7e^{2t}}{12\times 9000}\right| + \sup_{t\in L}\left|\frac{9(3-e^{3t})}{16\times 30000}\right| \\ &= \frac{5}{12000} + \frac{7}{12000} + \frac{9}{16000} = \frac{1}{640},\end{aligned}$$

TABLE 2. Numerical results of Δ, $\mathcal{K}_1$, $\mathcal{K}_3$ and Eqs. (23), (24) in Example 4.2 for four cases of φ.

t	Δ	$\mathcal{K}_1$	$\mathcal{K}_2$	Δ	$\mathcal{K}_1$	$\mathcal{K}_2$
	$\varphi_1(t)=e^t$			$\varphi_2(t)=t$		
0.0000	1.5000	0.2310	0.3355	1.5000	0.3355	0.0000
0.1099	1.6767	1.4108	1.8363	1.5282	1.4302	0.0037
0.2197	1.8301	3.4080	5.1044	1.5808	3.1472	0.0053
0.3296	1.9419	6.6104	10.2088	1.7113	6.0578	0.0065
0.4394	2.0351	11.0462	17.4666	1.8213	9.9630	0.0075
0.5493	2.1180	16.9442	27.3182	1.9183	14.8850	0.0084
0.6592	2.1945	24.5973	40.3248	2.0059	20.8491	0.0092
0.7690	2.2669	34.3687	57.1845	2.0865	27.8816	0.0099
0.8789	2.3364	46.7039	78.7571	2.1615	36.0087	0.0106
0.9888	2.4041	62.1455	106.0974	2.2320	45.2562	0.0112
1.0986	2.4706	81.3533	140.4958	2.2986	55.6496	0.0118
	$\varphi_3(t)=\ln(t+0.01)$			$\varphi_4(t)=\sqrt{t}$		
0.0000	1.5000	34.8034	0.0107	1.5000	0.3355	0.0000
0.1099	1.6767	91.4099	0.0282	1.5295	4.6066	0.0039
0.2197	1.7173	113.9983	0.0323	1.5844	7.5242	0.0053
0.3296	1.7412	129.4109	0.0348	1.6998	11.4641	0.0064
0.4394	1.7582	141.4112	0.0367	1.7953	15.7028	0.0072
0.5493	1.7714	151.3644	0.0381	1.8782	20.2204	0.0080
0.6592	1.7821	159.9336	0.0393	1.9523	25.0003	0.0087
0.7690	1.7911	167.4955	0.0403	2.0199	30.0288	0.0093
0.8789	1.7990	174.2864	0.0412	2.0822	35.2945	0.0099
0.9888	1.8215	184.6495	0.0420	2.1404	40.7878	0.0104
1.0986	1.8783	204.8408	0.0427	2.1951	46.5003	0.0109

$$\begin{aligned}\mathcal{K}_3 &= \sum_{k=1}^{3} \frac{(\hat{\varphi}_0(b))^{\alpha_k}}{\Gamma(1+\alpha_k)} \sup_{t\in L} |f_k(t,0,0,0)| \\ &= \frac{(\varphi_i(\ln 3)-\varphi_i(0))^{3/11}}{\Gamma\left(\frac{14}{11}\right)} \sup_{t\in L}\left|\frac{72}{1900(8+e^{2t})}\right| + \frac{(\varphi_i(\ln 3)-\varphi_i(0))^{3/11}}{\Gamma\left(\frac{14}{11}\right)} \sup_{t\in L}\left|\frac{5e^{2t}}{9\times 1900}\right| \\ &\quad + \frac{(\varphi_i(\ln 3)-\varphi_i(0))^{3/11}}{\Gamma\left(\frac{14}{11}\right)} \sup_{t\in L}\left|\frac{6e^{t}}{1900(3+t^2)}\right| \\ &= \frac{(\varphi_i(\ln 3)-\varphi_i(0))^{3/11}}{\Gamma\left(\frac{14}{11}\right)} \left[\frac{8}{1900} + \frac{5}{1900} + \frac{6}{1900}\right] \simeq \begin{cases} 0.0133, & \varphi_1(t)=e^{t}, \\ 0.0118, & \varphi_2(t)=t, \\ 0.0426, & \varphi_3(t)=\ln(t+1), \\ 0.0108, & \varphi_4(t)=\sqrt{t}. \end{cases}\end{aligned}$$

These results are shown in Table 2 for four cases of φ and also, Figures 3a and 3b show the curves of $\mathcal{K}_1$ and $\mathcal{K}_3$. Inequality (8) implies that



(A) $\mathcal{K}_1$ (B) $\mathcal{K}_3$

FIGURE 3. Representation of $\mathcal{K}_1$ and $\mathcal{K}_3$ for four cases of φ in Example 4.2.

$$w_1 := \mathcal{K}_1 \sum_{k=1}^{3} \|\mathcal{T}_{gk}\| + \sum_{k=1}^{3} \frac{(\hat{\varphi}_0(b))^{\sigma_k} \|\mathcal{T}_{fk}\|}{\Gamma(1+\sigma_k)} = \frac{3\mathcal{K}_1}{1000} + \frac{3(\varphi_i(\ln 3)-\varphi_i(0))^{3/2}}{100\Gamma\left(\frac{5}{2}\right)}$$
$$\simeq \left\{ \begin{array}{ll} 0.4844, & \varphi_1(\mathfrak{t}) = e^{\mathfrak{t}}, \\ 0.1967, & \varphi_2(\mathfrak{t}) = \mathfrak{t}, \\ 0.8087, & \varphi_3(\mathfrak{t}) = \ln(\mathfrak{t}+1), \\ 0.1675, & \varphi_4(\mathfrak{t}) = \sqrt{\mathfrak{t}}. \end{array} \right\} < 1. \tag{23}$$

Finally, inequality (5) implies that

$$w_2 := 3\left(\mathcal{K}_1\mathcal{K}_2 + \mathcal{K}_3\right) \left[1 - \mathcal{K}_1 \sum_{k=1}^{3} \|\mathcal{T}_{gk}\| - \sum_{k=1}^{3} \frac{(\hat{\varphi}_0(b))^{\sigma_k} \|\mathcal{T}_{fk}\|}{\Gamma(1+\sigma_k)}\right]^{-1}$$
$$\simeq \left\{ \begin{array}{ll} 1.3553, & \varphi_1(\mathfrak{t}) = e^{\mathfrak{t}}, \\ 0.3689, & \varphi_2(\mathfrak{t}) = \mathfrak{t}, \\ 2.8531, & \varphi_3(\mathfrak{t}) = \ln(\mathfrak{t}+1),\ \mathfrak{t} \in \left[0, \frac{\ln 3}{7}\right], \\ 0.3010, & \varphi_4(\mathfrak{t}) = \sqrt{\mathfrak{t}}. \end{array} \right\} \leq 3.05 = r. \tag{24}$$

These results are shown in Table 3 and in Figures 3a and 3b, we have plotted the results for the tripled system of hybrid FDEs (16) with new conditions in this example and four cases of φ_i are defined by (22).

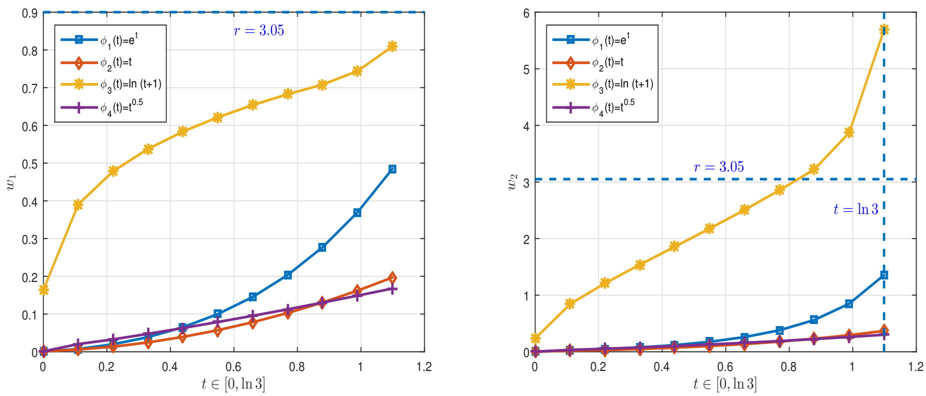
Thus, the assumptions $(\mathbb{H}_1) - (\mathbb{H}_3)$ are satisfied and Theorem 3.2 implies that (16) with four cases of φ_i (22) has at least one solution on $[0,\ \ln 3]$.

5. Conclusion and future results

In this paper, we have defined a new triple system of nonlinear hybrid FDEs under with nonlocal integro multi point boundary conditions involving the $p-$Laplacian operator and the $\varphi-$Caputo derivative. Base on the hybrid Dhage fixed point theorem

TABLE 3. Numerical results of w_1 and w_2, Eqs. (23), (24) in Example 4.2 for four cases of φ.

t	w_1	w_2	w_1	w_2
	$\varphi_1(t) = e^t$		$\varphi_2(t) = t$	
0.0000	0.0010	0.0016	0.0010	0.0016
0.1099	0.0073	0.0273	0.0060	0.0180
0.2197	0.0199	0.0476	0.0134	0.0310
0.3296	0.0388	0.0766	0.0248	0.0490
0.4394	0.0650	0.1178	0.0394	0.0720
0.5493	0.0999	0.1762	0.0572	0.1006
0.6592	0.1453	0.2593	0.0783	0.1358
0.7690	0.2033	0.3799	0.1027	0.1787
0.8789	0.2769	0.5610	0.1306	0.2306
0.9888	0.3692	0.8493	0.1619	0.2933
1.0986	0.4845	1.3554	0.1967	0.3689
	$\varphi_1(t) = e^t$		$\varphi_2(t) = t$	
0.0000	0.1640	0.2334	0.0010	0.0016
0.1099	0.3905	0.8416	0.0205	0.0340
0.2197	0.4784	1.2103	0.0328	0.0530
0.3296	0.5377	1.5381	0.0476	0.0765
0.4394	0.5835	1.8557	0.0629	0.1017
0.5493	0.6214	2.1759	0.0789	0.1289
0.6592	0.6539	2.5064	0.0954	0.1583
0.7690	0.6825	2.8532	0.1126	0.1900
0.8789	0.7081	3.2215	0.1303	0.2242
0.9888	0.7438	3.8705	0.1487	0.2612
1.0986	0.8087	5.6887	0.1676	0.3011



(A) w_1, Eq. (20) (B) w_2, Eq. (21)

FIGURE 4. Representation of w_1 and w_2 for four cases of φ in Example 4.2.

for a sum of three operators, the results are obtained. Two examples are presented at the end to show the applicability of the obtained results which are generalization for different states with derivative definition.

References

[1] M. I. Abbas, M. A. Ragusa, On the Hybrid Fractional Differential Equations with Fractional Proportional Derivatives of a Function with Respect to a Certain Function, *Symmetry* **13** (2021), no. 2, 264.

[2] A. Aghajani, E. Pourhadi, J.J. Trujillo, Application of measure of noncompactness to a Cauchy problem for fractional differential equations in Banach spaces, *Fractional Calculus and Applied Analysis* **16** (2013), 962-977.

[3] O.P. Agrawal, Some generalized fractional calculus operators and their applications in integral equations, *Fractional Calculus and Applied Analysis* **15** (2012), 700-711.

[4] R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Communications in Nonlinear Science and Numerical Simulation* **44** (2017), 460-481.

[5] A. Amara, S. Etemad, S. Rezapour, Approximate solutions for a fractional hybrid initial value problem via the Caputo conformable derivative, *Advances in Difference Equations* **2020** (2020), 608.

[6] P. Amiri, M.E. Samei, Existence of Urysohn and Atangana-Baleanu fractional integral inclusion systems solutions via common fixed point of multi-valued operators, *Chaos, Solitons & Fractals* **165** (2022), no. 2, 112822.

[7] H. Beddani, M. Beddani, Solvability for a differential systems via $\Phi-$Caputo approach, *Journal of Science and Arts* **3** (2021), no. 3, 749-762.

[8] H. Beddani, M. Beddani, Z. Dahmani, An existence study for a tripled system with $p-$Laplacian involving $\varphi-$Caputo derivatives, *Miskolc Mathematical Notes* **56** (2021), no. 2, 1-17.

[9] H. Beddani, Z. Dahmani, Solvability for nonlinear differential problem of Langevin type via $\phi-$Caputo approch, *European Journal of Mathematics and Applications* **1** (2021), 11.

[10] B. Belhadji, J. Alzabut, M.E. Samei, N. Fatima, On the global behaviour of solutions for a delayed viscoelastic type Petrovesky wave equation with p-Laplace and logarithmic source, *Mathematics* **10** (2022), 4194.

[11] F. Chabanea, M. Benbachir, M. Hachama, M.E. Samei, Existence of positive solutions for p-Laplacian boundary value problems of fractional differential equations, *Boundary Value Problems* **2022** (2022), 65.

[12] J. Dehong, G. Weigao, A nonlocal boundary value problems for hybrid $\varphi-$Caputo fractional integro-differential equations, *AIMS Mathematics* **5** (2020), no. 6, 7175-7190.

[13] A. Devi, A. Kumar, D. Baleanu, A. Khan, On stability analysis and existence of positive solutions for a general non-linear fractional differential equations, *Advances in Difference Equations* **2020** (2020), 300.

[14] B.C. Dhage, A fixed point theorem in Banach algebras with applications to functional integral equations, *Kyungpook Mathematical Journal* **44** (2004), 145-155.

[15] B.C. Dhage, On a Fixed Point Theorem in Banach Algebras with Applications, *Applied Mathematics Letters* **18** (2005), no. 3, 273-280.

[16] B.C. Dhage, Quadratic Perturbations Of Periodic Boundary Value Problems Of Second Order Ordinary Differential Equations, *Differential Equations and Applications* **2** (2010), no. 4, 465-486.

[17] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-oriented Exposition Using Differential Operators of Caputo Type*, Springer-Verlag, Berlin, 2010.

[18] S. Etemad, S. Rezapour, M.E. Samei, On fractional hybrid and non-hybrid multi-term integro-differential inclusions with three-point integral hybrid boundary conditions, *Advances in Difference Equations* **2020** (2020), 161.

[19] S. Ferraoun, Z. Dahmani, Existence and stability of solutions of a class of hybrid fractional differential equations involving RL-operator, *Journal of Interdisciplinary Mathematics* **23** (2020), no. 4, 885-903.

[20] H.A. Hammad, R.A. Rashwan, A. Nafea, M.E. Samei, M. De la Sen, Stability and existence of solutions for a tripled problem of fractional hybrid delay differential equations, *Symmetry* **14**, no. 12, (2022), 2579.
[21] R. Herrmann, *Fractional Calculus for Physicist*, World Scientific, Berlin, 2014.
[22] M.D. Kassim, N.E. Tatar, Stability of logarithmic type for a Hadamard fractional differential problem, *Journal of Pseudo-Differential Operators and Applications* **11** (2020), 447-466.
[23] H. Khan, T. Abdeljawad, M. Aslam, R.A. Khan, A. Khan, Existence of positive solution and Hyers–Ulam stability for a nonlinear singular-delay-fractional differential equation, *Advances in Difference Equations* **2019** (2019), 104.
[24] H. Khan, W. Chen, H. Sun, Analysis of positive solution and Hyers-Ulam stability for a class of singular fractional differential equations with $p-$Laplacian in Banach space, *Mathematical Methods in the Applied Sciences* **41** (2018), no. 9, 3430-3440.
[25] A. Khan, M.I. Syam, A. Zada, H. Khan, Stability analysis of nonlinear fractional differential equations with Caputo and Riemann–Liouville derivatives, *The European Physical Journal Plus* **133** (2018), 264.
[26] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science, North-Holland Mathematics Studies, Amsterdam, 2006.
[27] Y. Li, Existence of positive solutions for fractional differential equation involving integral boundary conditions with $p-$Laplacian operator, *Advances in Difference Equations* **2017** (2017), no. 2, 135.
[28] T.J. Osler, Fractional derivatives of a composite function, *SIAM Journal on Mathematical Analysis* **1** (1970), no. 2, 288-293.
[29] A. Parvate, A.D. Gangal, Fractal differential equations and fractal-time dynamical systems, *Pramana* **64** (2005), 389-409.
[30] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1993.
[31] H. Qin, X. Zuo, J. Liu, Existence and controllability results for fractional impulsive integro-differential systems in Banach spaces, *Abstract and Applied Analysis* **2013** (2013), no. 295837, 12 pages.
[32] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, Switzerland; Philadelphia, USA, 1993.
[33] A. Seemab, M.U. Rehman, J. Alzabut, M. Rehman, Y. Adjabi, M.S. Abdo, Langevin equation with nonlocal boundary conditions involving a $\psi-$Caputo fractional operator, *AIMS Mathematics* **6** (2021), no. 7, 6749-6780.
[34] Y. Wang, Existence and nonexistence of positive solutions for mixed fractional boundary value problem with parameter and $p-$Laplacian operator, *Journal of Function Spaces* **62** (2018), no. 1462825, 6 pages.
[35] Y. Zhao, S. Sun, Z. Han, Q. Li, Theory of fractional hybrid differential equations, *Computers & Mathematics with Applications* **62** (2011), no. 3, 1312-1324.
[36] A. Boutiara, M.M. Matar, J. Alzabut, M.E. Samei, H. Khan, Investigation of $\mathcal{ABC}$ coupled Langevin fractional differential equations constrained by Perov's fixed point in generalized Banach spaces, *AIMS Mathematics* **8** (2023), no. 5, 12109-12132.

(Hamid Beddani) Laboratory of Complex Systems of the Higher School of Electrical and Energy Engineering of Oran, 31000, Algeria
E-mail address: `beddanihamid@gmail.com`

(Moustafa Beddani) E. N. S of Mostaganem, Department of Mathematics, 27000 Mostaganem, Algeria
E-mail address: `beddani2004@yahoo.fr`

(Mohammad Esmael Samei) Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Hamedan, Iran
E-mail address: `mesamei@basu.ac.ir, mesamei@gmail.com`