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Asymptotic Modeling of the Effect of a Thin Slab in the Framework of Linear Elasticity with Voids

Athmane Abdallaoui, Abdelkarim Kelleche, and Amirouche Berkani

Abstract. The aim of this paper is to model the effect of a planar thin layer in the framework of linear elasticity with voids by using the notion of impedance boundary condition. We start from a transmission model problem which models the wave propagation between an elastic body with small distributed voids Ω_- and a thin coating slab Ω_+^δ (δ is supposed to be small enough). We show how to model the effect of the thin coating by an impedance boundary condition on the junction of the elastic two bodies. To this end, we use the technique of asymptotic expansion with scaling. We also prove an error estimate.

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1. Introduction

1.1. Physical and numerical motivations. The theory of linear elasticity with voids or modified elasticity can be viewed as a generalization of the theory of linear classical elasticity. It is adequate to describe the behavior of solids with small distributed voids or pores such as granular and manufactured porous bodies where the theory of classical elasticity is inadequate. The three-dimensional model is characterized by four independent variables: The components of the displacement vector u_i $(i = 1, 2, 3)$ and the change in volume fraction ω. The linear theory has been developed by Nunziato and Cowin [10] as a specialization of the non-linear theory [20].

This paper deals with the study of a transmission model problem in the context of linear elasticity with voids set in a fixed domain (i.e. not depending on δ) bonded with a planar thin layer of thickness δ. From a numerical point of view, the resolution of this problem can not be computed accurately since the small thickness δ of the thin layer creates instabilities related to the parameter δ. To avoid these numerical instabilities, we will use the concept of impedance condition which allows us to replace the initial transmission problem by an equivalent one which doesn't take into account any more the thin layer. This impedance condition is defined on the junction between the fixed domain and the thin layer and given through an operator called the impedance operator which is better known in English literature as the Dirichlet-to-Neumann operator.

In addition to the numerical motivation, the importance of this paper also comes from the fact that the results of this paper can be served as a guide in the case of a

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non planar thin layer (curved thin layer), where the formulas of the impedance will contain the curvature [1] , so this study will be useful when one considers the general case of a non planar thin layer.

The concept of impedance boundary condition [27] is largely used in numerous studies, mainly in electro-magnetics and mechanics, see for instance [9, 12, 25] for the Helmholtz equation in acoustics, [8, 14, 15] for Maxwell equations, and [18, 21] in structure mechanics, see also [17, 22].

This paper falls within the framework of applications of the technique of asymptotic expansion with scaling for modeling the effect of a planar thin layer in linear elasticity with voids. The asymptotic technique is used in vast literature for studying the asymptotic behavior in thin layer, see for instance [26, 24, 19, 13]. This paper is a continuation of [2, 3, 4], where the authors have derived first order approximations of the impedance in asymmetric elasticity. To begin with, we consider a three-dimensional model, of linear elasticity with voids in a domain $\Omega^\delta = \mathbb{R}^2 \times]-1,\delta[$ consisting of two bonded porous elastic bodies, $\Omega_- = \mathbb{R}^2 \times]-1,0[$ and a slab $\Omega_+^\delta = \mathbb{R}^2 \times]0,\delta[$, we also set $\Gamma_- = \mathbb{R}^2 \times \{-1\}$, $\Sigma = \mathbb{R}^2 \times \{0\}$ and $\Gamma_+^\delta = \mathbb{R}^2 \times \{\delta\}$ (see Fig. 1). We assume that Ω_- and Ω_+^δ are homogeneous and isotropic. We restrict our consideration to the case of elastostatics and a planar geometry, and we denote by the index $+$ (resp. $-$) to the restriction on Ω_+^δ (resp. on Ω_-). The transmission problem (P^δ) given by the

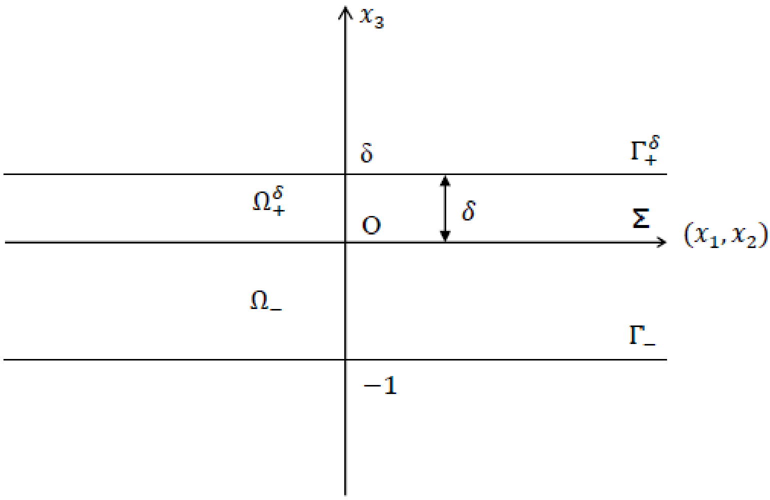

FIGURE 1. Domain of the transmission problem.

model for the displacement $u_\pm^\delta$ and the change in volume fraction $\omega_\pm^\delta$ read as follows (see [10, 11]):

(1) Equilibrium equations in Ω_-

$$\begin{cases} \sum_{j=1}^{3} D_j \sigma_{-ij}\left(u_-^\delta, \omega_-^\delta\right) = -p_{-i}, \ i = 1,2,3, \\ \sum_{j=1}^{3} D_j h_{-j}\left(\omega_-^\delta\right) - g_-\left(u_-^\delta, \omega_-^\delta\right) = -q_-. \end{cases} \tag{1}$$

(2) Equilibrium equations in Ω_+^δ

$$\begin{cases} \sum_{j=1}^3 D_j \sigma_{+ij}\left(u_+^\delta, \omega_+^\delta\right) = 0, \ i = 1,2,3, \\ \sum_{j=1}^3 D_j h_{+j}\left(\omega_+^\delta\right) - g_+\left(u_+^\delta, \omega_+^\delta\right) = 0. \end{cases} \tag{2}$$

(3) Dirichlet boundary conditions on Γ_-

$$\begin{cases} u_{-i}^\delta = 0, \ i = 1,2,3, \\ \omega_-^\delta = 0. \end{cases}$$

(4) Neumann boundary conditions on Γ_+^δ

$$\begin{cases} \sum_{j=1}^3 \sigma_{+ij}\left(u_+^\delta, \omega_+^\delta\right)\nu_j = 0, \ i = 1,2,3, \\ D_3\omega_+^\delta = 0. \end{cases} \tag{3a}$$

(5) Transmission conditions at the interface Σ

$$\begin{cases} u_{-i}^\delta = u_{+i}^\delta, , \ i = 1,2,3, \\ \omega_-^\delta = \omega_+^\delta, \\ \sum_{j=1}^3 \sigma_{-ij}\left(u_-^\delta, \omega_-^\delta\right)\nu_j = \sum_{j=1}^3 \sigma_{+ij}\left(u_+^\delta, \omega_+^\delta\right)\nu_j, \ i = 1,2,3, \\ \alpha_- D_3\omega_-^\delta = \alpha_+ D_3\omega_+^\delta. \end{cases} \tag{4}$$

where $\nu = (\nu_1, \nu_2, \nu_3) = (0,0,1)$ is the unit normal vector to Σ, $\sigma_{\pm ij}$ is the stress tensor, p_- is the body force vector, $g_\pm$ is the intrinsic equilibrated body force, $h_\pm$ is the equilibrated stress vector, q_- is the extrinsic equilibrated body force and $D_j = \frac{\partial}{\partial x_j}$. For the sake of simplicity in the next sections, we adopt the following writings:

$$\left(u_\pm^\delta, \omega_\pm^\delta\right) = \left(u_{\pm 1}^\delta, u_{\pm 2}^\delta, u_{\pm 3}^\delta, \omega_\pm^\delta\right),$$

$$\sigma_\pm\left(u_\pm^\delta, \omega_\pm^\delta\right)\nu = \left(\sigma_{\pm 13}\left(u_\pm^\delta, \omega_\pm^\delta\right), \sigma_{\pm 23}\left(u_\pm^\delta, \omega_\pm^\delta\right), \sigma_{\pm 33}\left(u_\pm^\delta, \omega_\pm^\delta\right)\right),$$

$$\left(\sigma_\pm\left(u_\pm^\delta, \omega_\pm^\delta\right)\nu, \alpha_\pm D_3\omega_\pm^\delta\right) = \left(\sigma_{\pm 13}\left(u_\pm^\delta, \omega_\pm^\delta\right), \sigma_{\pm 23}\left(u_\pm^\delta, \omega_\pm^\delta\right), \sigma_{\pm 33}\left(u_\pm^\delta, \omega_\pm^\delta\right), \alpha_\pm D_3\omega_\pm^\delta\right).$$

The constitutive equations for the linear isotropic elasticity with voids are defined by:

$$\sigma_{\pm ij}\left(u_\pm^\delta, \omega_\pm^\delta\right) = 2\mu_\pm e_{\pm ij}\left(u_\pm^\delta\right) + \lambda_\pm e_{\pm pp}\left(u_\pm^\delta\right)\delta_{ij} + \beta_\pm \omega_\pm^\delta \delta_{ij}, \ i,j,p = 1,2,3,$$

$$h_{\pm j}\left(\omega_\pm^\delta\right) = \alpha_\pm D_j \omega_\pm^\delta, \ j = 1,2,3,$$

$$g_\pm\left(u_\pm^\delta, \omega_\pm^\delta\right) = \beta_\pm e_{\pm pp}\left(u_\pm^\delta\right) + \zeta_\pm \omega_\pm^\delta, p = 1,2,3,$$

where δ_{ij} is the Kronecker delta, $e_{\pm ij}$ is the strain tensor defined by:

$$e_{\pm ij}\left(u_\pm^\delta\right) = \frac{1}{2}\left(D_i u_{\pm j}^\delta + D_j u_{\pm i}^\delta\right), \ i,j = 1,2,3,$$

and $\mu_\pm$, $\alpha_\pm$, $\zeta_\pm$, $\lambda_\pm$ and $\beta_\pm$ are material constants satisfying the inequalities:

$$\mu_\pm > 0, \ \alpha_\pm > 0, \ \zeta_\pm > 0, \ \lambda_\pm > 0, \ \beta_\pm > 0,$$

$$\mu_\pm + 3\lambda_\pm > 0, \ (\mu_\pm + 3\lambda_\pm)\zeta_\pm > 3\beta_\pm^2.$$

As was already pointed out, our aim in this paper is to derive an approximate impedance boundary condition on the interface Σ that incorporates in an approximate way the effect of the thin slab Ω_+^δ on Ω_- to reduce the transmission problem (P^δ) to a boundary value problem set in the fixed domain Ω_-, i.e. the equilibrium equations in Ω_+^δ, the transmission conditions on Σ and the Neumann boundary conditions on Γ_+^δ are embodied in the form of an impedance boundary condition on Σ and depending on δ.

1.2. Concept of impedance in modeling. Our goal is to reduce the transmission problem set in $\Omega^\delta = \Omega_- \cup \Sigma \cup \Omega_+^\delta$ to a boundary value problem set only on the fixed domain Ω_-. The exact effect of the thin slab Ω_+^δ on the domain Ω_- is given by the impedance operator T_δ defined by:

$$T_\delta \left(v^\delta, \psi^\delta\right) := \left(\sigma_+ \left(u_+^\delta, \omega_+^\delta\right) \nu_{|\Sigma},\ \alpha_+ D_3 \omega_{+|\Sigma}^\delta\right),$$

where $\left(u_+^\delta, \omega_+^\delta\right)$ is the solution of the following boundary value problem:

$$(P_+^\delta) : \begin{cases} \text{Equations } (2) \ \text{ in } \Omega_+^\delta, \\ \text{Boundary conditions on } \ \Gamma_+^\delta, \\ u_+^\delta = v^\delta \ \text{ on } \Sigma,\ \omega_+^\delta = \psi^\delta \ \text{ on } \Sigma, \end{cases}$$

from the transmission conditions (4), it follows that:

$$\left(\sigma_- \left(u_-^\delta, \omega_-^\delta\right) \nu_{|\Sigma},\ \alpha_- D_3 \omega_{-|\Sigma}^\delta\right) = T_\delta \left(u_{-|\Sigma}^\delta, \omega_{-|\Sigma}^\delta\right),$$

and the transmission problem (P^δ) is then equivalent to the following impedance problem set in Ω_-:

$$(P_-^\delta) : \begin{cases} \text{Equations } (1) \ \text{ in } \Omega_-, \\ u_-^\delta = 0_{1\times 3} \ \text{ on } \Gamma_-, \\ \omega_-^\delta = 0 \ \text{ on } \Gamma_-, \\ \left(\sigma_- \left(u_-^\delta, \omega_-^\delta\right) \nu_{|\Sigma},\ \alpha_- D_3 \omega_{-|\Sigma}^\delta\right) = T_\delta \left(u_{-|\Sigma}^\delta, \omega_{-|\Sigma}^\delta\right) \ \text{ on } \Sigma. \end{cases}$$

Since an explicit expression of the exact impedance operator T_δ is not reachable for the general case, we will just derive an effective approximation $T_{*\delta}$ of T_δ with:

$$T_{*\delta} = \delta T_* \ \text{ and } \ T_* \left(v^\delta, \psi^\delta\right) = \left(C_1, C_2, C_3, C_4\right) \left(v^\delta, \psi^\delta\right),$$

where

$$\begin{aligned} C_1 \left(v^\delta, \psi^\delta\right) &= \frac{4\mu_+ \left(\mu_+ + \lambda_+\right)}{\left(\lambda_+ + 2\mu_+\right)} D_1^2 v_1^\delta + \mu_+ D_2^2 v_1^\delta + \frac{2\mu_+ \lambda_+}{2\mu_+ + \lambda_+} D_1 D_2 v_2^\delta + \mu_+ D_1 D_2 v_2^\delta \\ &\quad + \frac{2\mu_+ \beta_+}{2\mu_+ + \lambda_+} D_1 \psi^\delta, \\ C_2 \left(v^\delta, \psi^\delta\right) &= \frac{4\mu_+ \left(\mu_+ + \lambda_+\right)}{\left(\lambda_+ + 2\mu_+\right)} D_2^2 v_2^\delta + \mu_+ D_1^2 v_2^\delta + \frac{2\mu_+ \lambda_+}{2\mu_+ + \lambda_+} D_1 D_2 v_1^\delta + \mu_+ D_1 D_2 v_1^\delta \\ &\quad + \frac{2\mu_+ \beta_+}{2\mu_+ + \lambda_+} D_2 \psi^\delta, \\ C_3 \left(v^\delta, \psi^\delta\right) &= 0, \end{aligned}$$

and

$$C_4 \left(v^\delta, \psi^\delta\right) = \alpha_+ \left(D_1^2 \psi^\delta + D_2^2 \psi^\delta\right) - \frac{2\mu_+ \beta_+}{2\mu_+ + \lambda_+} \left(D_1 v_1^\delta + D_2 v_2^\delta\right) - \frac{\zeta_+ \left(2\mu_+ + \lambda_+\right) - \beta_+^2}{2\mu_+ + \lambda_+} \psi^\delta.$$

The solution $(u_-^\delta, \omega_-^\delta)$ of the transmission problem (P^δ) in Ω_- is then approximated by the solution $(u_{-*}^\delta, \omega_{-*}^\delta)$ of the following approximate impedance problem:

$$\left(P_{-*}^\delta\right) : \begin{cases} \sum_{j=1}^3 D_j \sigma_{-ij}\left(u_{-*}^\delta, \omega_{-*}^\delta\right) = -p_{-i},\ i = 1,2,3 \text{ in } \Omega_-, \\ \sum_{j=1}^3 D_j h_{-j}\left(\omega_{-*}^\delta\right) - g_-\left(u_{-*}^\delta, \omega_{-*}^\delta\right) = -q_- \text{ in } \Omega_-, \\ u_{-*}^\delta = 0_{1\times 3} \text{ on } \Gamma_-,\ \omega_{-*}^\delta = 0 \text{ on } \Gamma_-, \\ \left(\sigma_-\left(u_{-*}^\delta, \omega_{-*}^\delta\right)\nu_{|\Sigma},\ \alpha_- D_3 \omega_{-*|\Sigma}^\delta\right) = T_{*\delta}\left(u_{-*|\Sigma}^\delta, \omega_{-*|\Sigma}^\delta\right) \text{ on } \Sigma, \end{cases}$$

and we prove the following main result of the paper:

Theorem 1.1. For given (p_-, q_-) in $\left[L^2(\Omega_-)\right]^4$, the boundary value problem $\left(P_{-*}^\delta\right)$ has a unique solution in the space

$$W_*(\Omega_-) = \left\{ \begin{array}{c} (v_-, \varphi_-) \in \left[H^1(\Omega_-)\right]^4 : \\ (D_1 v_{-1}, D_1 v_{-2}, D_1 \varphi_-) \in \left[L^2(\Sigma)\right]^3, \\ (D_2 v_{-1}, D_2 v_{-2}, D_2 \varphi_-) \in \left[L^2(\Sigma)\right]^3, \\ v_- = 0_{1\times 3} \text{ on } \Gamma_-,\ \varphi_- = 0 \text{ on } \Gamma_-, \end{array} \right\}$$

and the following error estimate holds

$$\left\|u_-^\delta - u_{-*}^\delta\right\|_{[H^1(\Omega_-)]^3} + \left\|\omega_-^\delta - \omega_{-*}^\delta\right\|_{H^1(\Omega_-)} \le C\delta^2,$$

where the constant C depends only on p_-, q_- and the elasticity coefficients.

We will prove Theorem 1.1 according to the following scheme: In Section 2, we prove the well-posedness of problem (P^δ). In section 3, we derive an approximate impedance boundary condition for the thin slab Ω_+^δ by using a formal Taylor expansion. In section 4, by using the techniques of asymptotic expansion with scaling we construct and recuperate the same approximate impedance boundary condition derived by the technique of Taylor expansion in section 3. In section 5, we state and prove a stability result for the scaled transmission problem. In section 6, we prove the well-posedness of the approximate impedance problem $\left(P_{-*}^\delta\right)$. Finally, in section 7, we prove error estimates in an appropriate space.

2. Well-posedness of the transmission problem

In this section, we will prove the existence and uniqueness of the solution to the transmission problem $\left(P^\delta\right)$. We consider the space $W\left(\Omega^\delta\right)$ defined by:

$$W\left(\Omega^\delta\right) = \left\{ \begin{array}{c} (v, \varphi) \in \left[L^2\left(\Omega^\delta\right)\right]^4 : \\ (v, \varphi)_{|\Omega_-} = (v_-, \varphi_-) \in \left[H^1(\Omega_-)\right]^4, \\ (v, \varphi)_{|\Omega_+^\delta} = (v_+, \varphi_+) \in \left[H^1\left(\Omega_+^\delta\right)\right]^4, \\ v_- = v_+ \text{ on } \Sigma,\ \varphi_- = \varphi_+ \text{ on } \Sigma, \\ v_- = 0_{1\times 3} \text{ on } \Gamma_-,\ \varphi_- = 0 \text{ on } \Gamma_-. \end{array} \right\},$$

endowed with the norm

$$\|(v, \varphi)\|_{W(\Omega^\delta)} = \left[\|(v_-, \varphi_-)\|_{[H^1(\Omega_-)]^4}^2 + \|(v_+, \varphi_+)\|_{\left[H^1\left(\Omega_+^\delta\right)\right]^4}^2\right]^{1/2},$$

which is equivalent to the norm

$$|(v, \varphi)|_{W(\Omega^\delta)} = \left[|(v_-, \varphi_-)|_{[H_0^1(\Omega_-)]^4}^2 + |(v_+, \varphi_+)|_{[H_0^1(\Omega_+^\delta)]^4}^2\right]^{1/2},$$

with

$$\left\|\left(v_{\pm},\varphi_{\pm}\right)\right\|^2_{\left[H^1\left(\Omega^\delta_{\pm}\right)\right]^4}=\left\|\varphi_{\pm}\right\|^2_{H^1\left(\Omega^\delta_{\pm}\right)}+\sum_{i=1}^{3}\left\|v_{\pm i}\right\|^2_{H^1\left(\Omega^\delta_{\pm}\right)},$$

and

$$\left|\left(v_{\pm},\varphi_{\pm}\right)\right|^2_{\left[H^1_0\left(\Omega^\delta_{\pm}\right)\right]^4}=\left|\varphi_{\pm}\right|^2_{H^1_0\left(\Omega^\delta_{\pm}\right)}+\sum_{i=1}^{3}\left|v_{\pm i}\right|^2_{H^1_0\left(\Omega^\delta_{\pm}\right)}.$$

For all $\left[\left(u^\delta,\omega^\delta\right),(v,\varphi)\right]\in\left[W\left(\Omega^\delta\right)\right]^2$, we set

$$a^-\left[\left(u^\delta_-,\omega^\delta_-\right),\left(v_-,\varphi_-\right)\right]=\int_{\Omega_-}\left[\begin{array}{c}\mathrm{tr}_3\left(\sigma_-\left(u^\delta_-,\omega^\delta_-\right)e_-\left(v_-\right)\right)+\alpha_-\nabla\omega^\delta_-.\nabla\varphi_-\\+\zeta_-\omega^\delta_-\varphi_-+\beta_-\left(D_1u^\delta_{-1}+D_2u^\delta_{-2}+D_3u^\delta_{-3}\right)\varphi_-\end{array}\right]d\Omega_-,$$

$$a^+\left[\left(u^\delta_+,\omega^\delta_+\right),\left(v_+,\varphi_+\right)\right]=\int_{\Omega^\delta_+}\left[\begin{array}{c}\mathrm{tr}_3\left(\sigma_+\left(u^\delta_+,\omega^\delta_+\right)e_+\left(v_+\right)\right)+\alpha_+\nabla\omega^\delta_+.\nabla\varphi_+\\+\zeta_+\omega^\delta_+\varphi_++\beta_+\left(D_1u^\delta_{+1}+D_2u^\delta_{+2}+D_3u^\delta_{+3}\right)\varphi_+\end{array}\right]d\Omega^\delta_+,$$

where tr_3 is the trace of a matrix of size 3.

Theorem 2.1. For given (p_-,q_-) in $\left[L^2\left(\Omega_-\right)\right]^4$, there exists a unique solution $\left(u^\delta,\omega^\delta\right)$ in $W\left(\Omega^\delta\right)$ to the transmission problem (P^δ). Its weak formulation is given by

$$a\left[\left(u^\delta,\omega^\delta\right),(v,\varphi)\right]=L\left(v,\varphi\right),\ \forall\left(v,\varphi\right)\in W\left(\Omega^\delta\right),$$

with

$$a\left[\left(u^\delta,\omega^\delta\right),(v,\varphi)\right]=a^-\left[\left(u^\delta_-,\omega^\delta_-\right),\left(v_-,\varphi_-\right)\right]+a^+\left[\left(u^\delta_+,\omega^\delta_+\right),\left(v_+,\varphi_+\right)\right],$$

and

$$L\left(v,\varphi\right)=\int_{\Omega_-}\left(p_-.v_-+q_-\varphi_-\right)d\Omega_-.$$

Proof. To prove the existence and uniqueness of the solution to the transmission problem $\left(P^\delta\right)$ in $W\left(\Omega^\delta\right)$, it suffices to check that the linear form L and the bilinear form a satisfy the hypothesis of Lax-Milgram theorem. It is clear that L is continuous on $W\left(\Omega^\delta\right)$. For the continuity of a, we have

$$\begin{aligned}&\left|a\left[\left(u^\delta,\omega^\delta\right),(v,\varphi)\right]\right|\le\left|a^-\left[\left(u^\delta_-,\omega^\delta_-\right),\left(v_-,\varphi_-\right)\right]\right|+\left|a^+\left[\left(u^\delta_+,\omega^\delta_+\right),\left(v_+,\varphi_+\right)\right]\right|\\&\le\int_{\Omega_-}\left[\begin{array}{c}\left|\lambda_-\right|\left(\sum_{k=1}^3\left|D_ku^\delta_{-k}\right|\right)\left(\sum_{k=1}^3\left|D_kv_{-k}\right|\right)+2\left|\mu_-\right|\sum_{i,j=1}^3\left|e_{ij}\left(u^\delta_-\right)\right|\left|e_{ij}\left(v_-\right)\right|\\+\left|\beta_-\right|\left|\left(D_1u^\delta_{-1}+D_2u^\delta_{-2}+D_3u^\delta_{-3}\right)\right|\left|\varphi_-\right|+\left|\zeta_-\right|\left|\omega^\delta_-\right|\left|\varphi_-\right|+\left|\alpha_-\right|\left|\nabla\omega^\delta_-\right|\left|\nabla\varphi_-\right|\end{array}\right]d\Omega_-\\&+\int_{\Omega^\delta_+}\left[\begin{array}{c}\left|\lambda_+\right|\left(\sum_{k=1}^3\left|D_ku^\delta_{+k}\right|\right)\left(\sum_{k=1}^3\left|D_kv_{+k}\right|\right)+2\left|\mu_+\right|\sum_{i,j=1}^3\left|e_{ij}\left(u^\delta_+\right)\right|\left|e_{ij}\left(v_+\right)\right|\\+\left|\beta_+\right|\left|\left(D_1u^\delta_{+1}+D_2u^\delta_{+2}+D_3u^\delta_{+3}\right)\right|\left|\varphi_+\right|+\left|\zeta_+\right|\left|\omega^\delta_+\right|\left|\varphi_+\right|+\left|\alpha_+\right|\left|\nabla\omega^\delta_+\right|\left|\nabla\varphi_+\right|\end{array}\right]d\Omega^\delta_+.\end{aligned}$$

Using the Cauchy-Schwarz inequality and from the definition of the space $W\left(\Omega^\delta\right)$, we get

$$\begin{aligned}\sum_{i,j=1}^{3}\left(\int_{\Omega_-}e_{ij}\left(u^\delta_-\right)e_{ij}\left(v_-\right)d\Omega_-\right)&\le\left(\sum_{i,j=1}^{3}\left\|e_{ij}\left(u^\delta_-\right)\right\|_{L^2(\Omega_-)}\right)\left(\sum_{i,j=1}^{3}\left\|e_{ij}\left(v_-\right)\right\|_{L^2(\Omega_-)}\right)\\&\le c\left\|u^\delta_-\right\|_{[H^1(\Omega_-)]^3}\left\|v_-\right\|_{[H^1(\Omega_-)]^3}\\&\le c\left\|\left(u^\delta,\omega^\delta\right)\right\|_{W(\Omega^\delta)}\left\|(v,\varphi)\right\|_{W(\Omega^\delta)},\end{aligned}$$

where c is a positive constant independent of δ, the remaining terms of a^{-} can also be majorated by $\left\|\left(u^{\delta},\omega^{\delta}\right)\right\|_{W(\Omega^{\delta})}\|(v,\varphi)\|_{W(\Omega^{\delta})}$, we get

$$\left|a^{-}\left[\left(u_{-}^{\delta},\omega_{-}^{\delta}\right),\left(v_{-},\varphi_{-}\right)\right]\right|\leq C_{1}\left\|\left(u^{\delta},\omega^{\delta}\right)\right\|_{W(\Omega^{\delta})}\|(v,\varphi)\|_{W(\Omega^{\delta})},$$

where C_1 is a positive constant independent of δ. In the same manner, we prove that

$$\left|a_{+}\left[\left(u_{+}^{\delta},\omega_{+}^{\delta}\right),\left(v_{+},\varphi_{+}\right)\right]\right|\leq C_{2}\left\|\left(u^{\delta},\omega^{\delta}\right)\right\|_{W(\Omega^{\delta})}\|(v,\varphi)\|_{W(\Omega^{\delta})},$$

where C_2 is a positive constant independent of δ. Therefore for all $\left(u^{\delta},\omega^{\delta}\right)$, (v,φ) in $W\left(\Omega^{\delta}\right)$, there exists a positive constant C independent of δ such that

$$\left|a\left[\left(u^{\delta},\omega^{\delta}\right),(v,\varphi)\right]\right|\leq C\left\|\left(u^{\delta},\omega^{\delta}\right)\right\|_{W(\Omega^{\delta})}\|(v,\varphi)\|_{W(\Omega^{\delta})},$$

which prove the continuity of a on $\left[W\left(\Omega^{\delta}\right)\right]^{2}$. Let us show that a is coercive, we have

$$\begin{aligned} a\left[\left(u^{\delta},\omega^{\delta}\right),\left(u^{\delta},\omega^{\delta}\right)\right] &= a^{-}\left[\left(u_{-}^{\delta},\omega_{-}^{\delta}\right),\left(v_{-}^{\delta},\varphi_{-}^{\delta}\right)\right]+a^{+}\left[\left(u_{+}^{\delta},\omega_{+}^{\delta}\right),\left(v_{+}^{\delta},\varphi_{+}^{\delta}\right)\right] \\ =\int_{\Omega_{-}} & \left[\begin{array}{c} 2\mu_{-}\left[e_{-11}\left(u_{-}^{\delta}\right)\right]^{2}+2\mu_{-}\left[e_{-22}\left(u_{-}^{\delta}\right)\right]^{2}+2\mu_{-}\left[e_{-33}\left(u_{-}^{\delta}\right)\right]^{2} \\ +4\mu_{-}\left[e_{-21}\left(u_{-}^{\delta}\right)\right]^{2}+4\mu_{-}\left[e_{-13}\left(u_{-}^{\delta}\right)\right]^{2}+4\mu_{-}\left[e_{-23}\left(u_{-}^{\delta}\right)\right]^{2} \\ +\lambda_{-}\left[e_{-22}\left(u_{-}^{\delta}\right)+e_{-11}\left(u_{-}^{\delta}\right)+e_{-33}\left(u_{-}^{\delta}\right)\right]^{2} \\ +2\beta_{-}\omega_{-}^{\delta}\left[e_{-11}\left(u_{-}^{\delta}\right)+e_{-22}\left(u_{-}^{\delta}\right)+e_{-33}\left(u_{-}^{\delta}\right)\right]+\zeta_{-}\left(\omega_{-}^{\delta}\right)^{2}+\alpha_{-}\left|\nabla\omega_{-}^{\delta}\right|^{2} \end{array}\right]d\Omega_{-} \\ +\int_{\Omega_{+}^{\delta}} & \left[\begin{array}{c} 2\mu_{+}\left[e_{+11}\left(u_{+}^{\delta}\right)\right]^{2}+2\mu_{+}\left[e_{+22}\left(u_{-}^{\delta}\right)\right]^{2}+2\mu_{+}\left[e_{+33}\left(u_{+}^{\delta}\right)\right]^{2} \\ +4\mu_{+}\left[e_{+21}\left(u_{+}^{\delta}\right)\right]^{2}+4\mu_{+}\left[e_{+13}\left(u_{+}^{\delta}\right)\right]^{2}+4\mu_{+}\left[e_{+23}\left(u_{+}^{\delta}\right)\right]^{2} \\ +\lambda_{+}\left[e_{+22}\left(u_{+}^{\delta}\right)+e_{+11}\left(u_{+}^{\delta}\right)+e_{+33}\left(u_{+}^{\delta}\right)\right]^{2} \\ +2\beta_{+}\omega_{+}^{\delta}\left[e_{+11}\left(u_{+}^{\delta}\right)+e_{+22}\left(u_{+}^{\delta}\right)+e_{+33}\left(u_{+}^{\delta}\right)\right]+\zeta_{+}\left(\omega_{+}^{\delta}\right)^{2}+\alpha_{+}\left|\nabla\omega_{+}^{\delta}\right|^{2} \end{array}\right]d\Omega_{+}^{\delta}, \end{aligned}$$

for all $\left(u^{\delta},\omega^{\delta}\right)$ in $W\left(\Omega^{\delta}\right)$, and let's try to minorate $a\left[\left(u^{\delta},\omega^{\delta}\right),\left(u^{\delta},\omega^{\delta}\right)\right]$ by $\left\|\left(u^{\delta},\omega^{\delta}\right)\right\|_{W(\Omega^{\delta})}^{2}$. We start with $a^{-}\left[\left(u_{-}^{\delta},\omega_{-}^{\delta}\right),\left(u_{-}^{\delta},\omega_{-}^{\delta}\right)\right]$. Using the inequality

$$\left(3a^{2}+3b^{2}+3c^{2}\geq(a+b+c)^{2},\ \forall a,b,c\in\mathbb{R}\right),$$

we get

$$\begin{aligned} & 2\mu_{-}\left[e_{-11}\left(u_{-}^{\delta}\right)\right]^{2}+2\mu_{-}\left[e_{-22}\left(u_{-}^{\delta}\right)\right]^{2}+2\mu_{-}\left[e_{-33}\left(u_{-}^{\delta}\right)\right]^{2} \\ & \geq\mu_{-}\left[e_{-11}\left(u_{-}^{\delta}\right)\right]^{2}+\mu_{-}\left[e_{-22}\left(u_{-}^{\delta}\right)\right]^{2}+\mu_{-}\left[e_{-33}\left(u_{-}^{\delta}\right)\right]^{2} \\ & \quad+\frac{\mu_{-}}{3}\left[e_{-11}\left(u_{-}^{\delta}\right)+e_{-22}\left(u_{-}^{\delta}\right)+e_{-33}\left(u_{-}^{\delta}\right)\right]^{2}, \end{aligned}$$

which implies that

$$\begin{aligned} & a^{-}\left[\left(u_{-}^{\delta},\omega_{-}^{\delta}\right),\left(u_{-}^{\delta},\omega_{-}^{\delta}\right)\right] \\ & \geq\int_{\Omega_{-}}\left[\begin{array}{c} \mu_{-}\left[e_{-11}\left(u_{-}^{\delta}\right)\right]^{2}+\mu_{-}\left[e_{-22}\left(u_{-}^{\delta}\right)\right]^{2}+\mu_{-}\left[e_{-33}\left(u_{-}^{\delta}\right)\right]^{2} \\ +4\mu_{-}\left[e_{-21}\left(u_{-}^{\delta}\right)\right]^{2}+4\mu_{-}\left[e_{-13}\left(u_{-}^{\delta}\right)\right]^{2}+4\mu_{-}\left[e_{-23}\left(u_{-}^{\delta}\right)\right]^{2} \\ +\left(\lambda_{+}+\frac{\mu_{-}}{3}\right)\left[e_{-11}\left(u_{-}^{\delta}\right)+e_{-22}\left(u_{-}^{\delta}\right)+e_{-33}\left(u_{-}^{\delta}\right)\right]^{2}+2\beta_{-}\widetilde{\omega}_{-}^{\delta} \\ \times\left[e_{-11}\left(u_{-}^{\delta}\right)+e_{-22}\left(u_{-}^{\delta}\right)+e_{-33}\left(u_{-}^{\delta}\right)\right]+\zeta_{-}\left(\omega_{-}^{\delta}\right)^{2}+\alpha_{-}\left|\nabla\omega_{-}^{\delta}\right|^{2} \end{array}\right]d\Omega_{-}, \end{aligned}$$

for all $\left(u_{-}^{\delta},\omega_{-}^{\delta}\right)$ in $\left[H^{1}\left(\Omega_{-}\right)\right]^{4}$. Using the inequality $\left(a^{2}+2ab\geq-b^{2},\ \forall(a,b)\in\mathbb{R}^{2}\right)$ with

$$a=\sqrt{\left(\lambda_{+}+\frac{\mu_{-}}{3}\right)}\left[e_{-11}\left(u_{-}^{\delta}\right)+e_{-22}\left(u_{-}^{\delta}\right)+e_{-33}\left(u_{-}^{\delta}\right)\right]\text{ and }b=\frac{\beta_{-}\omega_{-}^{\delta}}{\sqrt{\left(\lambda_{+}+\frac{\mu_{-}}{3}\right)}},$$

we get

$$\begin{aligned}&\left(\lambda_+ + \frac{\mu_-}{3}\right)\left[e_{-11}\left(u_-^\delta\right)+e_{-22}\left(u_-^\delta\right)+e_{-33}\left(u_-^\delta\right)\right]^2\\ &\quad +2\beta_-\omega_-^\delta\left[e_{-11}\left(u_-^\delta\right)+e_{-22}\left(u_-^\delta\right)+e_{-33}\left(u_-^\delta\right)\right]^2+\zeta_-\left(\omega_-^\delta\right)^2\\ &\geq -\frac{\beta_-^2\left(\omega_-^\delta\right)^2}{\left(\lambda_+ + \frac{\mu_-}{3}\right)}+\zeta_-\left(\omega_-^\delta\right)^2=\frac{\zeta_-\left(\mu_-+3\lambda_+\right)-3\beta_-^2}{\left(\mu_-+3\lambda_+\right)}\left(\omega_-^\delta\right)^2,\end{aligned}$$

which implies that

$$\begin{aligned}&a^-\left[\left(u_-^\delta,\omega_-^\delta\right),\left(u_-^\delta,\omega_-^\delta\right)\right]\\ &\geq\int_{\Omega_-}\left[\begin{array}{c}\mu_-\left[e_{-11}\left(u_-^\delta\right)\right]^2+\mu_-\left[e_{-22}\left(u_-^\delta\right)\right]^2+\mu_-\left[e_{-33}\left(u_-^\delta\right)\right]^2\\ +4\mu_-\left[e_{-21}\left(u_-^\delta\right)\right]^2+4\mu_-\left[e_{-13}\left(u_-^\delta\right)\right]^2\\ +4\mu_-\left[e_{-23}\left(u_-^\delta\right)\right]^2+\frac{\zeta_-(\mu_-+3\lambda_+)-3\beta_-^2}{(\mu_-+3\lambda_+)}\left(\omega_-^\delta\right)^2+\alpha_-\left|\nabla\omega_-^\delta\right|^2\end{array}\right]d\Omega_-,\end{aligned}$$

for all $\left(u_-^\delta,\omega_-^\delta\right)$ in $\left[H^1\left(\Omega_-\right)\right]^4$. Since we have

$$\mu_->0 \text{ and } \frac{\zeta_-\left(\mu_-+3\lambda_+\right)-3\beta_-^2}{\left(\mu_-+3\lambda_+\right)}>0$$

because $\zeta_-\left(\mu_-+3\lambda_+\right)\geq 3\beta_-^2$ and $\left(\mu_-+3\lambda_+\right)>0$, then there exists a positive constant $C_1=\min\left(\mu_-,\alpha_-,\frac{\zeta_-(\mu_-+3\lambda_-)-3\beta_-^2}{(\mu_-+3\lambda_-)}\right)$ independent of δ such that

$$a^-\left[\left(u_-^\delta,\omega_-^\delta\right),\left(u_-^\delta,\omega_-^\delta\right)\right]\geq C_1\int_{\Omega_-}\left[\begin{array}{c}\left[e_{-11}\left(u_-^\delta\right)\right]^2+\left[e_{-22}\left(u_-^\delta\right)\right]^2+\left[e_{-33}\left(u_-^\delta\right)\right]^2\\ +\left[e_{-21}\left(u_-^\delta\right)\right]^2+\left[e_{-13}\left(u_-^\delta\right)\right]^2\\ +\left[e_{-23}\left(u_-^\delta\right)\right]^2+\left(\omega_-^\delta\right)^2+\left|\nabla\omega_-^\delta\right|^2\end{array}\right]d\Omega_-.$$

In the same manner, we prove that for all $\left(u_+^\delta,\omega_+^\delta\right)$ in $\left[H^1\left(\Omega_+^\delta\right)\right]^4$, we have

$$a^+\left[\left(u_+^\delta,\omega_+^\delta\right),\left(u_+^\delta,\omega_+^\delta\right)\right]\geq C_2\int_{\Omega_+^\delta}\left[\begin{array}{c}\left[e_{+11}\left(u_+^\delta\right)\right]^2+\left[e_{+22}\left(u_+^\delta\right)\right]^2+\left[e_{+33}\left(u_+^\delta\right)\right]^2\\ +\left[e_{+21}\left(u_+^\delta\right)\right]^2+\left[e_{+13}\left(u_+^\delta\right)\right]^2\\ +\left[e_{+23}\left(u_+^\delta\right)\right]^2+\left(\omega_+^\delta\right)^2+\left|\nabla\omega_+^\delta\right|^2\end{array}\right]d\Omega_+^\delta,$$

where $C_2=\min\left(\mu_+,\alpha_+,\frac{\zeta_+(\mu_++3\lambda_+)-3\beta_+^2}{(\mu_++3\lambda_+)}\right)$ is a positive constant independent of δ. Thus

$$\begin{aligned}a\left[\left(u^\delta,\omega^\delta\right),\left(u^\delta,\omega^\delta\right)\right]\geq{}& C_1\int_{\Omega_-}\left[\begin{array}{c}\left[e_{-11}\left(u_-^\delta\right)\right]^2+\left[e_{-22}\left(u_-^\delta\right)\right]^2+\left[e_{-33}\left(u_-^\delta\right)\right]^2\\ +\left[e_{-21}\left(u_-^\delta\right)\right]^2+\left[e_{-13}\left(u_-^\delta\right)\right]^2\\ +\left[e_{-23}\left(u_-^\delta\right)\right]^2+\left(\omega_-^\delta\right)^2+\alpha_-\left|\nabla\omega_-^\delta\right|^2\end{array}\right]d\Omega_-\\ &+C_2\int_{\Omega_+^\delta}\left[\begin{array}{c}\left[e_{+11}\left(u_+^\delta\right)\right]^2+\left[e_{+22}\left(u_+^\delta\right)\right]^2+\left[e_{+33}\left(u_+^\delta\right)\right]^2\\ +\left[e_{+21}\left(u_+^\delta\right)\right]^2+\left[e_{+13}\left(u_+^\delta\right)\right]^2\\ +\left[e_{+23}\left(u_+^\delta\right)\right]^2+\left(\omega_+^\delta\right)^2+\alpha_+\left|\nabla\omega_+^\delta\right|^2\end{array}\right]d\Omega_+^\delta.\end{aligned}$$

Thanks to the Korn's inequality (see [23], page 51), we obtain

$$a\left[\left(u^\delta,\omega^\delta\right),\left(u^\delta,\omega^\delta\right)\right]\geq C\left\|\left(u^\delta,\omega^\delta\right)\right\|_{W(\Omega^\delta)}^2,\quad\text{for all }\left(u^\delta,\omega^\delta\right)\in W\left(\Omega^\delta\right),$$

where C is a positive constant independent of δ, which proves the coercivity of a. Therefore by Lax-Milgram theorem, the transmission problem (P^δ) has a unique solution in $W\left(\Omega^\delta\right)$. □

3. Derivation of the approximate impedance operator by Taylor expansion

As in [5, 6], we construct a first-order approximation of the impedance by using Taylor expansion. The first step to obtaining an approximate impedance condition is to express explicitly the normal derivatives of u_+^δ, ω_+^δ, σ_{+13}, σ_{+23}, σ_{+33} and $D_3\omega_+^\delta$ on Σ in terms of traces on Σ of σ_{+13}, σ_{+23}, σ_{+33}, u_+^δ and ω_+^δ. Since we have

$$\sigma_{+13} = \mu_+ \left(D_3 u_{+1}^\delta + D_1 u_{+3}^\delta\right), \quad \sigma_{+23} = \mu_+ \left(D_3 u_{+2}^\delta + D_2 u_{+3}^\delta\right),$$

and

$$\sigma_{+33} = \left(2\mu_+ + \lambda_+\right) D_3 u_{+3}^\delta + \lambda_+ \left(D_1 u_{+1}^\delta + D_2 u_{+2}^\delta\right) + \beta_+ \omega_+^\delta,$$

it follows that

$$D_3 u_{+1}^\delta = \frac{1}{\mu_+}\left[\sigma_{+13} - \mu_+ D_1 u_{+3}^\delta\right], \quad D_3 u_{+2}^\delta = \frac{1}{\mu_+}\left[\sigma_{+23} - \mu_+ D_2 u_{+3}^\delta\right], \tag{5}$$

and

$$D_3 u_{+3}^\delta = \frac{1}{2\mu_+ + \lambda_+}\left[\sigma_{+33} - \lambda_+ \left(D_1 u_{+1}^\delta + D_2 u_{+2}^\delta\right) - \beta_+ \omega_+^\delta\right], \tag{6}$$

and then by using (5) and (6), we get

$$\begin{aligned} \sigma_{+11} &= \left(2\mu_+ + \lambda_+\right) D_1 u_{+1}^\delta + \lambda_+ D_2 u_{+2}^\delta + \lambda_+ D_3 u_{+3}^\delta + \beta_+ \omega_+^\delta \\ &= \tfrac{4\mu_+(\mu_+ + \lambda_+)}{(\lambda_+ + 2\mu_+)} D_1 u_{+1}^\delta + \tfrac{2\mu_+\lambda_+}{2\mu_+ + \lambda_+} D_2 u_{+2}^\delta + \tfrac{2\mu_+\beta_+}{2\mu_+ + \lambda_+} \omega_+^\delta + \tfrac{\lambda_+}{2\mu_+ + \lambda_+} \sigma_{+33}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} \sigma_{+22} &= \left(2\mu_+ + \lambda_+\right) D_2 u_{+2}^\delta + \lambda_+ D_1 u_{+1}^\delta + \lambda_+ D_3 u_{+3}^\delta + \beta_+ \omega_+^\delta \\ &= \tfrac{4\mu_+(\mu_+ + \lambda_+)}{(\lambda_+ + 2\mu_+)} D_2 u_{+2}^\delta + \tfrac{2\mu_+\lambda_+}{2\mu_+ + \lambda_+} D_1 u_{+1}^\delta + \tfrac{2\mu_+\beta_+}{2\mu_+ + \lambda_+} \omega_+^\delta + \tfrac{\lambda_+}{2\mu_+ + \lambda_+} \sigma_{+33}. \end{aligned} \tag{8}$$

Using the equations (2) and the formulas (7) and (8) of σ_{+11} and σ_{+22} and the fact that $\sigma_{+13} = \sigma_{+31}$, $\sigma_{+23} = \sigma_{+32}$, we get

$$\left\{ \begin{aligned} D_3\sigma_{+13} = &-\tfrac{4\mu_+(\mu_+ + \lambda_+)}{(\lambda_+ + 2\mu_+)} D_1^2 u_{+1}^\delta - \mu_+ D_2^2 u_{+1}^\delta - \tfrac{2\mu_+\lambda_+}{2\mu_+ + \lambda_+} D_1 D_2 u_{+2}^\delta - \mu_+ D_1 D_2 u_{+2}^\delta \\ &- \tfrac{2\mu_+\beta_+}{2\mu_+ + \lambda_+} D_1 \omega_+^\delta - \tfrac{\lambda_+}{2\mu_+ + \lambda_+} D_1 \sigma_{+33}, \\ D_3\sigma_{+23} = &-\tfrac{4\mu_+(\mu_+ + \lambda_+)}{(\lambda_+ + 2\mu_+)} D_2^2 u_{+2}^\delta - \mu_+ D_1^2 u_{+2}^\delta - \tfrac{2\mu_+\lambda_+}{2\mu_+ + \lambda_+} D_1 D_2 u_{+1}^\delta - \mu_+ D_1 D_2 u_{+1}^\delta \\ &- \tfrac{2\mu_+\beta_+}{2\mu_+ + \lambda_+} D_2 \omega_+^\delta - \tfrac{\lambda_+}{2\mu_+ + \lambda_+} D_2 \sigma_{+33}, \\ D_3\sigma_{+33} = &-D_1\sigma_{+13} - D_2\sigma_{+23}, \end{aligned} \right. \tag{9}$$

and

$$\begin{aligned} \alpha_+ D_3^2 \omega_+^\delta = &-\alpha_+ \left(D_1^2 \omega_+^\delta + D_2^2 \omega_+^\delta\right) + \frac{2\mu_+\beta_+}{2\mu_+ + \lambda_+}\left(D_1 u_{+1}^\delta + D_2 u_{+2}^\delta\right) \\ &+ \frac{\zeta_+ \left(2\mu_+ + \lambda_+\right) - \beta_+^2}{2\mu_+ + \lambda_+} \omega_+^\delta + \frac{\beta_+}{2\mu_+ + \lambda_+} \sigma_{+33}. \end{aligned} \tag{10}$$

The second step for obtaining an approximate impedance condition is to use Taylor expansion of σ_{+13}, σ_{+23}, σ_{+33} and $D_3\omega_+^\delta$ in Ω_+^δ with respect to δ, we write then the boundary conditions on Γ_+^δ as follows:

$$\left\{ \begin{aligned} \sigma_{+13}(\delta) &= \sigma_{+13}(0) + \delta D_3 \sigma_{+13}(0) + ..., \\ \sigma_{+23}(\delta) &= \sigma_{+23}(0) + \delta D_3 \sigma_{+23}(0) + ..., \\ \sigma_{+33}(\delta) &= \sigma_{+33}(0) + \delta D_3 \sigma_{+33}(0) + ..., \end{aligned} \right. \tag{11}$$

and

$$D_3\omega_+^\delta(\delta) = D_3\omega_+^\delta(0) + \delta D_3^2\omega_+^\delta(0) + \dots . \tag{12}$$

Using (11), (12), (9), (10) and the boundary conditions (3a) on Γ_+^δ, we obtain at order one the following approximate impedance $T_{*\delta}$ defined by its expression:

$$T_{*\delta}\left(v^\delta,\psi^\delta\right) = \delta\left(C_1\left(v^\delta,\psi^\delta\right), C_2\left(v^\delta,\psi^\delta\right), C_3\left(v^\delta,\psi^\delta\right), C_4\left(v^\delta,\psi^\delta\right)\right),$$

and thus we get problem $\left(P_{-*}^\delta\right)$ in Ω_-.

Remark 3.1. In the case of the two-dimensional model of the linear elasticity with voids $\left(v^\delta,\psi^\delta\right) = \left(v_1^\delta, v_2^\delta,\psi^\delta\right)$, $\nu = (0,1)$ and in the same way, we can show that the expression of the impedance will be of the form:

$$\begin{aligned} T_{*\delta}\left(v^\delta,\psi^\delta\right) &= \left(\sigma_-\left(v^\delta,\psi^\delta\right)\nu,\ \alpha_- D_2\psi^\delta\right) \\ &= \left(\sigma_{-12}\left(v^\delta,\psi^\delta\right), \sigma_{-22}\left(v^\delta,\psi^\delta\right),\ \alpha_- D_2\psi^\delta\right) \\ &= \delta\left(C_1\left(v^\delta,\psi^\delta\right), C_2\left(v^\delta,\psi^\delta\right), C_3\left(v^\delta,\psi^\delta\right)\right), \end{aligned}$$

where

$$\begin{aligned} C_1\left(v^\delta,\psi^\delta\right) &= \tfrac{4\mu_+(\mu_+ + \lambda_+)}{(\lambda_+ + 2\mu_+)} D_1^2 v_1^\delta + \tfrac{2\mu_+\beta_+}{2\mu_+ + \lambda_+} D_1\psi^\delta, \\ C_2\left(v^\delta,\psi^\delta\right) &= 0, \end{aligned}$$

and

$$C_3\left(v^\delta,\psi^\delta\right) = \alpha_+ D_1^2\psi^\delta - \tfrac{\zeta_+(2\mu_+ + \lambda_+) - \beta_+^2}{2\mu_+ + \lambda_+}\psi^\delta - \tfrac{2\beta_+\mu_+}{2\mu_+ + \lambda_+} D_1 v_1^\delta.$$

4. Construction of the asymptotic expansion and the approximate impedance operator

Here, we construct a first approximation of the impedance by using the techniques of asymptotic expansion with scaling as follows:

4.1. Scaling and the scaled transmission problem. When the parameter δ varies, the domain Ω_+^δ also varies. The solution $\left(u_\pm^\delta,\omega_\pm^\delta\right)$ of problem $\left(P^\delta\right)$ depends on δ and we cannot compare the solutions corresponding to different values of the parameter δ. We therefore make a change of scale to bring back the transmission problem set in Ω^δ to a transmission problem set on a fixed domain, let us symbolize it with Ω. So we perform a dilatation in the normal direction of Ω_+^δ of ratio δ^{-1} to get a fixed geometry. Accordingly, we set $\Omega_+ = \mathbb{R}^2\times]0,1[$, $\Gamma_+ = \mathbb{R}^2\times\{1\}$, $\Omega = \mathbb{R}^2\times]-1,1[$ and for each point $(x_1,x_2,x_3)\in\Omega$, we associate the point $\chi^\delta(x_1,x_2,x_3)\in\Omega^\delta$ as the following:

$$\begin{aligned} \chi^\delta :\ \Omega &\to \Omega^\delta \\ (x_1,x_2,x_3) &\mapsto \chi^\delta(x_1,x_2,x_3) = \begin{cases} (x_1,x_2,x_3) & \text{if } x_3 \le 0 \\ (x_1,x_2,\delta x_3) & \text{if } x_3 > 0 \end{cases} . \end{aligned}$$

and we define the function $\left(\widetilde{u}_\pm^\delta,\widetilde{\omega}_\pm^\delta\right)$ by:

$$\begin{aligned} \widetilde{u}_-^\delta(x_1,x_2,x_3) &= u_-^\delta(x_1,x_2,x_3), \text{ for all } (x_1,x_2,x_3)\in\Omega_-, \\ \widetilde{\omega}_-^\delta(x_1,x_2,x_3) &= \omega_-^\delta(x_1,x_2,x_3), \text{ for all } (x_1,x_2,x_3)\in\Omega_-, \end{aligned}$$

and

$$\begin{array}{llll} \widetilde{u}_{+}^{\delta}: & \Omega_{+}=\ \mathbb{R}^{2}\times\,]0,1[& \rightarrow & \mathbb{R}^{3} \\ & (x_{1},x_{2},x_{3}) & \mapsto & \left(u_{+1}^{\delta}\left(x_{1},x_{2},\delta x_{3}\right),u_{+2}^{\delta}\left(x_{1},x_{2},\delta x_{3}\right),\delta u_{+3}^{\delta}\left(x_{1},x_{2},\delta x_{3}\right)\right), \\ \widetilde{\omega}_{+}^{\delta}: & \Omega_{+}=\ \mathbb{R}^{2}\times\,]0,1[& \rightarrow & \mathbb{R} \\ & (x_{1},x_{2},x_{3}) & \mapsto & \omega_{+}^{\delta}\left(x_{1},x_{2},\delta x_{3}\right), \end{array}$$

Then we obtain the following scaled problem:
(1) The equations in Ω_{+}^{δ} are rewritten in Ω_{+} as:

$$\begin{aligned} (E_{+1}):0=&\ D_{1}\left[\left(2\mu_{+}+\lambda_{+}\right)D_{1}\widetilde{u}_{+1}^{\delta}+\lambda_{+}\left(D_{2}\widetilde{u}_{+2}^{\delta}+\frac{1}{\delta^{2}}D_{3}\widetilde{u}_{+3}^{\delta}\right)+\beta_{+}\widetilde{\omega}_{+}^{\delta}\right] \\ &+\mu_{+}D_{2}\left[D_{1}\widetilde{u}_{+2}^{\delta}+D_{2}\widetilde{u}_{+1}^{\delta}\right]+\frac{\mu_{+}}{\delta^{2}}D_{3}\left[D_{1}\widetilde{u}_{+3}^{\delta}+D_{3}\widetilde{u}_{+1}^{\delta}\right], \\ (E_{+2}):0=&\ D_{2}\left[\left(2\mu_{+}+\lambda_{+}\right)D_{2}\widetilde{u}_{+2}^{\delta}+\lambda_{+}\left(D_{1}\widetilde{u}_{+1}^{\delta}+\frac{1}{\delta^{2}}D_{3}\widetilde{u}_{+3}^{\delta}\right)+\beta_{+}\widetilde{\omega}_{+}^{\delta}\right] \\ &+\frac{\mu_{+}}{\delta^{2}}D_{3}\left[D_{2}\widetilde{u}_{+3}^{\delta}+D_{3}\widetilde{u}_{+2}^{\delta}\right]+\mu_{+}D_{1}\left[D_{1}\widetilde{u}_{+2}^{\delta}+D_{2}\widetilde{u}_{+1}^{\delta}\right], \\ (E_{+3}):0=&\ \frac{1}{\delta}D_{1}\left[\mu_{+}\left(D_{1}\widetilde{u}_{+3}^{\delta}+D_{3}\widetilde{u}_{+1}^{\delta}\right)\right]+\frac{1}{\delta}D_{2}\left[\mu_{+}\left(D_{2}\widetilde{u}_{+3}^{\delta}+D_{3}\widetilde{u}_{+2}^{\delta}\right)\right] \\ &+\frac{1}{\delta}D_{3}\left[\frac{\left(2\mu_{+}+\lambda_{+}\right)}{\delta^{2}}D_{3}\widetilde{u}_{+3}^{\delta}+\lambda_{+}\left(D_{1}\widetilde{u}_{+1}^{\delta}+D_{2}\widetilde{u}_{+2}^{\delta}\right)+\beta_{+}\widetilde{\omega}_{+}^{\delta}\right], \\ (E_{+4}):0=&\ \alpha_{+}\left(D_{1}^{2}\widetilde{\omega}_{+}^{\delta}+D_{2}^{2}\widetilde{\omega}_{+}^{\delta}+\frac{1}{\delta^{2}}D_{3}^{2}\widetilde{\omega}_{+}^{\delta}\right)-\beta_{+}\left(D_{1}\widetilde{u}_{+1}^{\delta}+D_{2}\widetilde{u}_{+2}^{\delta}+\frac{1}{\delta^{2}}D_{3}\widetilde{u}_{+3}^{\delta}\right) \\ &-\zeta_{+}\widetilde{\omega}_{+}^{\delta}. \end{aligned}$$

(2) The boundary conditions on Γ_{+}^{δ} are rewritten in Γ_{+} as:

$$\begin{array}{lll} (BC\Gamma_{+1}) & : & 0=\mu_{+}\frac{1}{\delta}D_{3}\widetilde{u}_{+1}^{\delta}+\mu_{+}\frac{1}{\delta}D_{1}\widetilde{u}_{+3}^{\delta}, \\ (BC\Gamma_{+2}) & : & 0=\mu_{+}\frac{1}{\delta}D_{3}\widetilde{u}_{+2}^{\delta}+\mu_{+}\frac{1}{\delta}D_{2}\widetilde{u}_{+3}^{\delta}, \\ (BC\Gamma_{+3}) & : & 0=\frac{\left(2\mu_{+}+\lambda_{+}\right)}{\delta^{2}}D_{3}\widetilde{u}_{+3}^{\delta}+\lambda_{+}\left(D_{1}\widetilde{u}_{+1}^{\delta}+D_{2}\widetilde{u}_{+2}^{\delta}\right)+\beta_{+}\widetilde{\omega}_{+}^{\delta}, \\ (BC\Gamma_{+4}) & : & 0=\frac{1}{\delta}D_{3}\widetilde{\omega}_{+}^{\delta}. \end{array}$$

(3) The transmission conditions on Σ are rewritten as:

$$\begin{array}{lll} (CT\Sigma_{1}) & : & \widetilde{u}_{+1}^{\delta}=\widetilde{u}_{-1}^{\delta}, \\ (CT\Sigma_{2}) & : & \widetilde{u}_{+2}^{\delta}=\widetilde{u}_{-2}^{\delta}, \\ (CT\Sigma_{3}) & : & \frac{1}{\delta}\widetilde{u}_{+3}^{\delta}=\widetilde{u}_{-3}^{\delta}, \\ (CT\Sigma_{4}) & : & \widetilde{\omega}_{+}^{\delta}=\widetilde{\omega}_{-}^{\delta}, \\ (CT\Sigma_{5}) & : & \sigma_{-13}\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right)=\frac{\mu_{+}}{\delta}\left(D_{3}\widetilde{u}_{+1}^{\delta}+D_{1}\widetilde{u}_{+3}^{\delta}\right), \\ (CT\Sigma_{6}) & : & \sigma_{-23}\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right)=\frac{\mu_{+}}{\delta}\left(D_{3}\widetilde{u}_{+2}^{\delta}+D_{2}\widetilde{u}_{+3}^{\delta}\right), \end{array}$$

$$(CT\Sigma_7) \quad : \quad \sigma_{-33}\left(\widetilde{u}_-^\delta, \widetilde{\omega}_-^\delta\right) = \frac{(2\mu_+ + \lambda_+)}{\delta^2} D_3 \widetilde{u}_{+3}^\delta + \lambda_+ \left(D_1 \widetilde{u}_{+1}^\delta + D_2 \widetilde{u}_{+2}^\delta\right) + \beta_+ \widetilde{\omega}_+^\delta,$$

$$(CT\Sigma_8) \quad : \quad \alpha_- D_3 \widetilde{\omega}_-^\delta = \frac{\alpha_+}{\delta} D_3 \widetilde{\omega}_+^\delta.$$

4.2. Asymptotic expansion of the scaled transmission problem. We expand the solution $\left(\widetilde{u}_\pm^\delta, \widetilde{\omega}_\pm^\delta\right)$ of the transmission problem after scaling in the form:

$$\left(\widetilde{u}_\pm^\delta, \widetilde{\omega}_\pm^\delta\right) = \left(\widetilde{u}_\pm^0, \widetilde{\omega}_\pm^0\right) + \delta \left(\widetilde{u}_\pm^1, \widetilde{\omega}_\pm^1\right) + \delta^2 \left(\widetilde{u}_\pm^2, \widetilde{\omega}_\pm^2\right) + ...,$$

with $\left(\widetilde{u}_\pm^k, \widetilde{\omega}_\pm^k\right)$ for all $k \in \mathbb{N}$ are independent of δ. Inserting these asymptotic expansions in the transmission problem after scaling and identifying the terms with the same power of δ, we obtain the following hierarchy of boundary value problems:

$$\left\{\begin{array}{l}
(EE_{+1})_{k-2} : \ 0 = D_1 \left[\begin{array}{c} (2\mu_+ + \lambda_+) D_1 \widetilde{u}_{+1}^{k-2} + \lambda_+ \left(D_2 \widetilde{u}_{+2}^{k-2} + D_3 \widetilde{u}_{+3}^k\right) \\ + \beta_+ \widetilde{\omega}_+^{k-2} \end{array}\right] \\
\qquad + \mu_+ D_2 \left(D_1 \widetilde{u}_{+2}^{k-2} + D_2 \widetilde{u}_{+1}^{k-2}\right) + \mu_+ D_3 \left(D_1 \widetilde{u}_{+3}^k + D_3 \widetilde{u}_{+1}^k\right) \ \text{ in } \Omega_+, \\
(EBC\Gamma_{+1})_{k-1} : \ \mu_+ D_3 \widetilde{u}_{+1}^k + \mu_+ D_1 \widetilde{u}_{+3}^k = 0 \ \text{ on } \Gamma_+, \\
(ECT\Sigma_5)_{k-1} : \ \sigma_{-13}\left(\widetilde{u}_-^{k-1}, \widetilde{\omega}_-^{k-1}\right) = \mu_+ \left(D_3 \widetilde{u}_{+1}^k + D_1 \widetilde{u}_{+3}^k\right) \ \text{ on } \Sigma, \\
(ECT\Sigma_1)_k : \ \widetilde{u}_{+1}^k = \widetilde{u}_{-1}^k \ \text{ on } \Sigma,
\end{array}\right. \tag{13}$$

$$\left\{\begin{array}{l}
(EE_{+2})_{k-2} : \ 0 = D_2 \left[\begin{array}{c} (2\mu_+ + \lambda_+) D_2 \widetilde{u}_{+2}^{k-2} + \lambda_+ \left(D_1 \widetilde{u}_{+1}^{k-2} + D_3 \widetilde{u}_{+3}^k\right) \\ + \beta_+ \widetilde{\omega}_+^{k-2} \end{array}\right] \\
\qquad + \mu_+ D_1 \left[\left(D_1 \widetilde{u}_{+2}^{k-2} + D_2 \widetilde{u}_{+1}^{k-2}\right)\right] + \mu_+ D_3 \left[\left(D_2 \widetilde{u}_{+3}^k + D_3 \widetilde{u}_{+2}^k\right)\right] \ \text{ in } \Omega_+, \\
(EBC\Gamma_{+2})_{k-1} : \ \mu_+ D_3 \widetilde{u}_{+2}^k + \mu_+ D_2 \widetilde{u}_{+3}^k = 0 \ \text{ on } \Gamma_+, \\
(ECT\Sigma_6)_{k-1} : \ \sigma_{-23}\left(\widetilde{u}_-^{k-1}, \widetilde{\omega}_-^{k-1}\right) = \mu_+ \left(D_3 \widetilde{u}_{+2}^k + D_2 \widetilde{u}_{+3}^k\right) \ \text{ on } \Sigma, \\
(ECT\Sigma_2)_{k-1} : \widetilde{u}_{+2}^k = \widetilde{u}_{-2}^k \ \text{ on } \Sigma,
\end{array}\right. \tag{14}$$

$$\left\{\begin{array}{l}
(EE_{+3})_{k-3} : \ 0 = D_3 \left[\begin{array}{c} (2\mu_+ + \lambda_+) D_3 \widetilde{u}_{+3}^k + \lambda_+ \left(D_1 \widetilde{u}_{+1}^{k-2} + D_2 \widetilde{u}_{+2}^{k-2}\right) \\ + \beta_+ \widetilde{\omega}_+^{k-2} \end{array}\right] \\
\qquad + \mu_+ D_1 \left[\left(D_1 \widetilde{u}_{+3}^{k-2} + D_3 \widetilde{u}_{+1}^{k-2}\right)\right] + \mu_+ D_2 \left[\left(D_2 \widetilde{u}_{+3}^{k-2} + D_3 \widetilde{u}_{+2}^{k-2}\right)\right] \ \text{ in } \Omega_+, \\
(EBC\Gamma_{+3})_{k-2} : \left[\begin{array}{c} (2\mu_+ + \lambda_+) D_3 \widetilde{u}_{+3}^k \\ + \lambda_+ \left(D_1 \widetilde{u}_{+1}^{k-2} + D_2 \widetilde{u}_{+2}^{k-2}\right) + \beta_+ \widetilde{\omega}_+^{k-2} \end{array}\right] = 0 \ \text{ on } \Gamma_+, \\
(ECT\Sigma_7)_{k-2} : \ \sigma_{-33}\left(\widetilde{u}_-^{k-2}, \widetilde{\omega}_-^{k-2}\right) \\
\quad = \left[(2\mu_+ + \lambda_+) D_3 \widetilde{u}_{+3}^k + \lambda_+ \left(D_1 \widetilde{u}_{+1}^{k-2} + D_2 \widetilde{u}_{+2}^{k-2}\right) + \beta_+ \widetilde{\omega}_+^{k-2}\right] \ \text{ on } \Sigma, \\
(ECT\Sigma_3)_{k-1} : \ \widetilde{u}_{+3}^k = \widetilde{u}_{-3}^{k-1} \ \text{ on } \Sigma,
\end{array}\right. \tag{15}$$

and

$$\begin{cases} (EE_{+4})_{k-2}: \ 0 = \left[\begin{array}{c} \alpha_+ \left(D_1^2 \widetilde{\omega}_+^{k-2} + D_2^2 \widetilde{\omega}_+^{k-2} + D_3^2 \widetilde{\omega}_+^{k} \right) \\ -\beta_+ \left(D_1 \widetilde{u}_{+1}^{k-2} + D_2 \widetilde{u}_{+2}^{k-2} + D_3 \widetilde{u}_{+3}^{k} \right) - \zeta_+ \widetilde{\omega}_+^{k-2} \end{array} \right] \text{ in } \Omega_+, \\ (EBC\Gamma_{+4})_{k-1}: \ D_3 \widetilde{\omega}_+^k = 0 \ \text{ on } \Gamma_+, \\ (ECT\Sigma_8)_{k-1}: \ \alpha_- D_3 \widetilde{\omega}_-^{k-1} = \alpha_+ D_3 \widetilde{\omega}_+^k \ \text{ on } \Sigma, \\ (ECT\Sigma_4)_k: \ \widetilde{\omega}_+^k = \widetilde{\omega}_-^k \ \text{ on } \Sigma, \end{cases} \tag{16}$$

where we set

$$\begin{aligned} \widetilde{u}_+^{-2} &= \widetilde{u}_+^{-1} = 0_{1\times 2} \text{ in } \Omega_+, \ \widetilde{\omega}_+^{-2} = \widetilde{\omega}_+^{-1} = 0 \ \text{ in } \Omega_+, \\ \widetilde{u}_{+1}^{-2} &= \widetilde{u}_{+1}^{-1} = \widetilde{u}_{+2}^{-2} = \widetilde{u}_{+2}^{-1} = \widetilde{\omega}_+^{-2} = \widetilde{\omega}_+^{-1} = 0 \ \text{ on } \Gamma_+, \\ \widetilde{u}_{+1}^{-2} &= \widetilde{u}_{+1}^{-1} = \widetilde{u}_{-2}^{-1} = \widetilde{\omega}_+^{-2} = \widetilde{\omega}_+^{-1} = \widetilde{u}_{-3}^{-1} = 0 \ \text{ on } \Sigma, \\ \sigma_{-13}\left(\widetilde{u}_-^{-1}, \widetilde{\omega}_-^{-1}\right) &= \sigma_{-23}\left(\widetilde{u}_-^{-1}, \widetilde{\omega}_-^{-1}\right) = \sigma_{-33}\left(\widetilde{u}_-^{-2}, \widetilde{\omega}_-^{-2}\right) = D_3 \widetilde{\omega}_-^{-1} = 0 \ \text{ on } \Sigma. \end{aligned}$$

Remark 4.1.

(1) Thanks to the technique of scaling in the thin slab Ω_+^δ, the terms of the asymptotic expansion of $\widetilde{u}_+^\delta$ and $\widetilde{\omega}_+^\delta$ can be calculated explicitly by recurrence in function of terms of the asymptotic expansion of $\widetilde{u}_-^\delta$ and $\widetilde{\omega}_-^\delta$ on Σ.

(2) Equations in problems (13)−(16) are second order linear differential equations with respect to the variable x_3.

(3) By integrating equations in problems (13)−(16) and using the transmission conditions on Σ and the boundary conditions on Γ_+, we can calculate $\widetilde{u}_+^\delta$ and $\widetilde{\omega}_+^\delta$ in terms of u_-^k, and ω_-^k on Σ.

Calculation of terms of order 0 in Ω_+. For $k = 0$, an integration by part in x_3 in problems (13)−(16) gives the following results:

$$\begin{aligned} \widetilde{u}_{+3}^0 &= 0 \ \text{ in } \Omega_+, \\ \widetilde{u}_{+1}^0 &= u_{-1|\Sigma}^0 \ \text{ in } \ \Omega_+, \\ \widetilde{u}_{+2}^0 &= u_{-2|\Sigma}^0 \ \text{ in } \ \Omega_+, \\ \widetilde{\omega}_+^0 &= \omega_{-|\Sigma}^0 \ \text{ in } \Omega_+. \end{aligned}$$

Calculation of terms of order 1 in Ω_+. For $k = 1$, also an integration by part in x_3 in problems (13)−(16) gives the following results:

$$\begin{aligned} \widetilde{u}_{+3}^1 &= u_{-3|\Sigma}^0 \ \text{ in } \Omega_+, \\ \widetilde{u}_{+1}^1 &= u_{-1|\Sigma}^1 - x_3 D_1 u_{-3|\Sigma}^0 \ \text{ in } \Omega_+, \\ \widetilde{u}_{+2}^1 &= u_{-2|\Sigma}^1 - x_3 D_2 u_{-3|\Sigma}^0 \ \text{ in } \Omega_+, \\ \widetilde{\omega}_+^1 &= \omega_{-|\Sigma}^1 \ \text{ in } \Omega_+. \end{aligned}$$

Calculation of terms of order 2 in Ω_+. For $k = 2$, in the same way, we get

$$\widetilde{u}_{+3}^2 = u_{-3|\Sigma}^1 - \frac{x_3}{(2\mu_+ + \lambda_+)} \left[\lambda_+ \left(D_1 u_{-1|\Sigma}^0 + D_2 u_{-2|\Sigma}^0 \right) + \beta_+ \omega_{-|\Sigma}^0 \right] \ \text{ in } \Omega_+,$$

$$\widetilde{u}_{+1}^2 = u_{-1|\Sigma}^2 + x_3 \left[\begin{array}{c} \frac{4(\mu_+ + \lambda_+)}{(2\mu_+ + \lambda_+)} D_1^2 u_{-1|\Sigma}^0 + \frac{2\mu_+ + 3\lambda_+}{(2\mu_+ + \lambda_+)} D_1 D_2 u_{-2|\Sigma}^0 \\ + \frac{2\beta_+}{(2\mu_+ + \lambda_+)} D_1 \omega_{-|\Sigma}^0 + D_2^2 u_{-1|\Sigma}^0 - D_1 u_{-3|\Sigma}^1 \end{array} \right] - \frac{x_3^2}{2} \left[\frac{(4\mu_+ + 2\lambda_+)}{(2\mu_+ + \lambda_+)} D_1^2 u_{-1|\Sigma}^0 + D_1 D_2 u_{-2|\Sigma}^0 + D_2^2 u_{-1|\Sigma}^0 \right] \text{ in } \Omega_+,$$

$$\widetilde{u}_{+2}^2 = u_{-2|\Sigma}^2 + x_3 \left[\begin{array}{c} \frac{4(\mu_+ + \lambda_+)}{(2\mu_+ + \lambda_+)} D_2^2 u_{-2|\Sigma}^0 + \frac{2\mu_+ + 3\lambda_+}{(2\mu_+ + \lambda_+)} D_1 D_2 u_{-1|\Sigma}^0 \\ + \frac{2\beta_+}{(2\mu_+ + \lambda_+)} D_2 \omega_{-|\Sigma}^0 + D_1^2 u_{-2|\Sigma}^0 - D_2 u_{-3|\Sigma}^1 \end{array} \right] - \frac{x_3^2}{2} \left[\begin{array}{c} \frac{(4\mu_+ + 3\lambda_+)}{(2\mu_+ + \lambda_+)} D_2^2 u_{-2|\Sigma}^0 + \frac{2\mu_+ + 2\lambda_+}{(2\mu_+ + \lambda_+)} D_1 D_2 u_{-1|\Sigma}^0 \\ + \frac{\beta_+}{(2\mu_+ + \lambda_+)} D_2 \omega_{-|\Sigma}^0 + D_1^2 u_{-2|\Sigma}^0 \end{array} \right] \text{ in } \Omega_+,$$

$$\widetilde{\omega}_+^2 = \omega_{-|\Sigma}^2 + x_3 \left[\begin{array}{c} D_1^2 \omega_{-|\Sigma}^0 + D_2^2 \omega_{-|\Sigma}^0 - \frac{2\mu_+ \beta_+}{\alpha_+ (2\mu_+ + \lambda_+)} \left(D_1 \widetilde{u}_{-1|\Sigma}^0 + D_2 u_{-2|\Sigma}^0 \right) \\ - \frac{\zeta_+ (2\mu_+ + \lambda_+) - \beta_+^3}{\alpha_+ (2\mu_+ + \lambda_+)} \omega_{-|\Sigma}^0 \end{array} \right] + \frac{x_3^2}{2} \left[\begin{array}{c} \frac{2\mu_+ \beta_+}{\alpha_+ (2\mu_+ + \lambda_+)} \left(D_1 \widetilde{u}_{-1|\Sigma}^0 + D_2 u_{-2|\Sigma}^0 \right) + \frac{\zeta_+ (2\mu_+ + \lambda_+) - \beta_+^3}{\alpha_+ (2\mu_+ + \lambda_+)} \omega_{-|\Sigma}^0 \\ - D_1^2 \omega_{-|\Sigma}^0 - D_2^2 \omega_{-|\Sigma}^0 \end{array} \right] \text{ in } \Omega_+,$$

Calculation of terms of order 3 in Ω_+. For $k = 3$, problem (15) gives the following relation:

$$\sigma_{-33} \left(u_{-|\Sigma}^1, \omega_{-|\Sigma}^1 \right) = (2\mu_+ + \lambda_+) D_3 \widetilde{u}_{+3}^3 + \lambda_+ \left(D_1 \widetilde{u}_{+1}^1 + D_2 \widetilde{u}_{+2}^1 \right) + \beta_+ \widetilde{\omega}_+^1 = 0 \text{ on } \Sigma.$$

Then, the asymptotic expansion of the equations in Ω_-, the boundary conditions on Γ_- and the transmission conditions $(CT\Sigma_5) - (CT\Sigma_8)$ on Σ, allow us to obtain the following results:

Terms of order 0 in Ω_-: At order 0, the terms $\left(u_-^0, \omega_-^0 \right)$ satisfy the following boundary value problem in Ω_-**:**

$$(P_-)_0 : \left\{ \begin{array}{l} \sum_{j=1}^3 D_j \sigma_{-ij} \left(u_-^0, \omega_-^0 \right) = -p_{-i}, \ i = 1,2,3 \text{ in } \Omega_-, \\ \sum_{j=1}^3 D_j h_{-j} \left(\omega_-^0 \right) - g_- \left(u_-^0, \omega_-^0 \right) = -q_- \text{ in } \Omega_-, \\ u_-^0 = \omega_-^0 = 0 \text{ on } \Gamma_-, \\ \left(\sigma_- \left(u_-^0, \omega_-^0 \right)_{|\Sigma} \nu, \ \alpha_- D_3 \omega_{-|\Sigma}^0 \right) = 0 \text{ on } \Sigma. \end{array} \right.$$

which means that at order 0, the thin slab Ω_+^δ has no effect on Ω_-

Terms of order 1 in Ω_-: At order 1, the terms $\left(u_-^1, \omega_-^1 \right)$ satisfy the following boundary value problem in Ω_-**:**

$$(P_-)_1 : \left\{ \begin{array}{l} \sum_{j=1}^3 D_j \sigma_{-ij} \left(u_-^1, \omega_-^1 \right) = 0, \ i = 1,2,3 \text{ in } \Omega_-, \\ \sum_{j=1}^3 D_j h_{-j} \left(\omega_-^1 \right) - g_- \left(u_-^1, \omega_-^1 \right) = 0 \text{ in } \Omega_-, \\ u_-^1 = \omega_-^1 = 0 \text{ on } \Gamma_-, \\ \left(\sigma_- \left(u_-^1, \omega_-^1 \right)_{|\Sigma} \nu, \ \alpha_- D_3 \omega_{-|\Sigma}^1 \right) = T_* \left(u_{-|\Sigma}^0, \omega_{-|\Sigma}^0 \right) \text{ on } \Sigma. \end{array} \right.$$

which means that at order 1, the effect of the thin slab Ω_+^δ on Ω_- is represented by forces and equilibrated forces exerted on Σ.

First order approximation of the impedance. By linearity of the constitutive equations, we find that the function $\left(u_{-*}^\delta, \omega_{-*}^\delta \right)$ defined by: $\left(u_{-*}^\delta, \omega_{-*}^\delta \right) =$

$\left(u_-^0 + \delta u_-^1, \omega_-^0 + \delta\omega_-^1\right)$ satisfies:

$$\begin{aligned}\left(\sigma_-\left(u_{-*}^\delta, \omega_{-*}^\delta\right)\nu_{|\Sigma},\ \alpha_- D_3\omega_{-*|\Sigma}^\delta\right) &= \left(\sigma_-\left(u_-^0, \omega_-^0\right)\nu_{|\Sigma},\ \alpha_- D_3\omega_{-|\Sigma}^0\right) \\ &\quad + \delta\left(\sigma_-\left(u_-^1, \omega_-^1\right)\nu_{|\Sigma},\ \alpha_- D_3\omega_{-|\Sigma}^1\right). \quad (17)\end{aligned}$$

Using the boundary conditions on Σ of problems $(P_-)_0$ and $(P_-)_1$, we get from (17) the following relation:

$$\begin{aligned}\left(\sigma_-\left(u_{-*}^\delta, \omega_{-*}^\delta\right)\nu_{|\Sigma},\ \alpha_- D_3\omega_{-*|\Sigma}^\delta\right) &= \delta\left(\sigma_-\left(u_-^1, \omega_-^1\right)\nu_{|\Sigma},\ \alpha_- D_3\omega_{-|\Sigma}^1\right) \\ &= \delta T_*\left(u_{-|\Sigma}^0, \omega_{-|\Sigma}^0\right), \quad (18)\end{aligned}$$

and as we have $\left(u_{-|\Sigma}^0, \omega_{-|\Sigma}^0\right) = \left(u_{-*|\Sigma}^\delta - \delta u_{-|\Sigma}^1, \omega_{-*|\Sigma}^\delta - \delta\omega_{-|\Sigma}^1\right)$, then the relation (18) can be written as follows:

$$\left(\sigma_-\left(u_{-*}^\delta, \omega_{-*}^\delta\right)\nu_{|\Sigma},\ \alpha_- D_3\omega_{-*|\Sigma}^\delta\right) = \delta T_*\left(u_{-*|\Sigma}^\delta, \omega_{-*|\Sigma}^\delta\right) - \delta^2 T_*\left(u_{-|\Sigma}^1, \omega_{-|\Sigma}^1\right),$$

thus, if we neglect the terms of order 2 with respect to δ, we recuperate the same approximate impedance that we have obtained by Taylor formula in section 3.

5. Stability for the scaled transmission problem

After scaling, the space for studying the transmission problem $\left(P^\delta\right)$ in the fixed domain $\Omega_- \cup \Sigma \cup \Omega_+$ becomes:

$$W_\delta(\Omega) = \left\{\begin{array}{c}(v,\varphi) \in \left[L^2(\Omega)\right]^4 : \\ (v,\varphi)_{|\Omega_-} = (v_-, \varphi_-) \in \left[H^1(\Omega_-)\right]^4, \\ (v,\varphi)_{|\Omega_+} = (v_+, \varphi_+) \in \left[H^1(\Omega_+)\right]^4, \\ v_- = 0_{1\times 3} \text{ on } \Gamma_-,\ \varphi_- = 0 \text{ on } \Gamma_-, \\ (v_{-1}, v_{-2}, \delta v_{-3}) = (v_{+1}, v_{+2}, v_{+3}) \text{ on } \Sigma, \text{ and } \varphi_- = \varphi_+ \text{ on } \Sigma,\end{array}\right\}$$

and the variational formulation of the scaled transmission problem is written:

$$\begin{aligned}\text{Find } \left(\widetilde{u}^\delta, \widetilde{\omega}^\delta\right) &\in W_\delta(\Omega), \text{ such that } \forall (v,\varphi) \in W_\delta(\Omega): \\ L_\delta(v,\varphi) &= a^-\left[\left(\widetilde{u}_-^\delta, \widetilde{\omega}_-^\delta\right), (v_-, \varphi_-)\right] + a^+\left[\left(\widetilde{u}_+^\delta, \widetilde{\omega}_+^\delta\right), (v_+, \varphi_+)\right],\end{aligned}$$

with

$$a^-\left[\left(\widetilde{u}_-^\delta, \widetilde{\omega}_-^\delta\right), (v_-, \varphi_-)\right] = \int_{\Omega_-}\left[\begin{array}{c}\mathrm{tr}_3\left(\sigma_-\left(\widetilde{u}_-^\delta, \widetilde{\omega}_-^\delta\right)e_-(v_-)\right) + \alpha_-\nabla\widetilde{\omega}_-^\delta.\nabla\varphi_- \\ +\zeta_-\widetilde{\omega}_-^\delta\varphi_- + \beta_-\left(D_1\widetilde{u}_{-1}^\delta + D_2\widetilde{u}_{-2}^\delta + D_3\widetilde{u}_{-3}^\delta\right)\varphi_-\end{array}\right]d\Omega_-,$$

and

$$\begin{aligned}a^+\left[\left(\widetilde{u}_+^\delta, \widetilde{\omega}_+^\delta\right), (v_+, \varphi_+)\right] &= \delta a_{+1}^+\left[\left(\widetilde{u}_+^\delta, \widetilde{\omega}_+^\delta\right), (v_+, \varphi_+)\right] + \frac{1}{\delta}a_{-1}^+\left[\left(\widetilde{u}_+^\delta, \widetilde{\omega}_+^\delta\right), (v_+, \varphi_+)\right] \\ &\quad + \frac{1}{\delta^3}a_{-3}^+\left[\left(\widetilde{u}_+^\delta, \widetilde{\omega}_+^\delta\right), (v_+, \varphi_+)\right],\end{aligned}$$

where

$$a_{+1}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),(v_{+},\varphi_{+})\right]=\int_{\Omega_{+}}\left[\begin{array}{c}(2\mu_{+}+\lambda_{+})\,D_{1}\widetilde{u}_{+1}^{\delta}D_{1}v_{+1}\\+\lambda_{+}\left(D_{2}\widetilde{u}_{+2}^{\delta}D_{1}v_{+1}+D_{1}\widetilde{u}_{+1}^{\delta}D_{2}v_{+2}\right)\\+\beta_{+}\left(\widetilde{\omega}_{+}^{\delta}D_{1}v_{+1}+\widetilde{\omega}_{+}^{\delta}D_{2}v_{+2}\right)\\+\mu_{+}\left[\left(D_{1}\widetilde{u}_{+2}^{\delta}+D_{2}\widetilde{u}_{+1}^{\delta}\right)\right]\left[D_{1}v_{+2}+D_{2}v_{+1}\right]+\\+\left(2\mu_{+}+\lambda_{+}\right)D_{2}\widetilde{u}_{+2}^{\delta}D_{2}v_{+2}\\+\alpha_{+}D_{1}\widetilde{\omega}_{+}^{\delta}D_{1}\varphi_{+}+\alpha_{+}D_{2}\widetilde{\omega}_{+}^{\delta}D_{2}\varphi_{+}+\zeta_{+}\widetilde{\omega}_{+}^{\delta}\varphi_{+}\\+\beta_{+}\left(D_{1}\widetilde{u}_{+1}^{\delta}+D_{2}\widetilde{u}_{+2}^{\delta}\right)\varphi_{+}\end{array}\right]d\Omega,$$

$$a_{-1}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),(v_{+},\varphi_{+})\right]=\int_{\Omega_{+}}\left[\begin{array}{c}\lambda_{+}\left[D_{3}\widetilde{u}_{+3}^{\delta}\left(D_{1}v_{+1}+D_{2}v_{+2}\right)\right.\\\left.+\left(D_{1}\widetilde{u}_{+1}^{\delta}+D_{2}\widetilde{u}_{+2}^{\delta}\right)D_{3}v_{+3}\right]\\+\beta_{+}\left[\widetilde{\omega}_{+}^{\delta}D_{3}v_{+3}+D_{3}\widetilde{u}_{+3}^{\delta}\varphi_{+}\right]\\+\mu_{+}\left(D_{1}\widetilde{u}_{+3}^{\delta}+D_{3}\widetilde{u}_{+1}^{\delta}\right)\left(D_{1}v_{+3}+D_{3}v_{+1}\right)\\+\mu_{+}\left(D_{2}\widetilde{u}_{+3}^{\delta}+D_{3}\widetilde{u}_{+2}^{\delta}\right)\left(D_{2}v_{+3}+D_{3}v_{+2}\right)\\+\alpha_{+}D_{3}\widetilde{\omega}_{+}^{\delta}D_{3}\varphi_{+}\end{array}\right]d\Omega_{+},$$

$$a_{-3}^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),(v_{+},\varphi_{+})\right]=\int_{\Omega_{+}}(2\mu_{+}+\lambda_{+})\,D_{3}\widetilde{u}_{+3}^{\delta}D_{2}v_{+3}d\Omega_{+},$$

and

$$L_{\delta}(v,\varphi)=\int_{\Omega_{-}}(p_{-}.v_{-}d\Omega_{-}+q_{-}\varphi_{-})\,d\Omega_{-}.$$

Now, in order to estimate the error between the solution of the transmission problem in Ω_{-} and its approximation of order 1, which will be in section 7, we first set

$$\begin{aligned}\mathbb{A}(v_{-},\varphi_{-})&=\|v_{-}\|_{[H^{1}(\Omega_{-})]^{3}}+\|\varphi_{-}\|_{H^{1}(\Omega_{-})},\\\mathbb{B}(v_{+},\varphi_{+})&=\left[\begin{array}{c}\|D_{1}v_{+1}\|_{L^{2}(\Omega_{+})}+\|D_{2}v_{+2}\|_{L^{2}(\Omega_{+})}+\|D_{1}v_{+2}+D_{2}v_{+1}\|_{L^{2}(\Omega_{+})}\\+\|D_{1}\varphi_{+}\|_{L^{2}(\Omega_{+})}+\|D_{2}\varphi_{+}\|_{L^{2}(\Omega_{+})}+\|\varphi_{+}\|_{L^{2}(\Omega_{+})}\end{array}\right],\\\mathbb{C}(v_{+},\varphi_{+})&=\|D_{1}v_{+3}+D_{3}v_{+1}\|_{L^{2}(\Omega_{+})}+\|D_{2}v_{+3}+D_{3}v_{+2}\|_{L^{2}(\Omega_{+})}+\|D_{3}\varphi_{+}\|_{L^{2}(\Omega_{+})},\\\mathbb{D}(v_{+},\varphi_{+})&=\|D_{3}v_{+3}\|_{L^{2}(\Omega_{+})},\end{aligned}$$

for all $(v,\varphi)\in W_{\delta}(\Omega)$. After that, we state and prove the following stability result:
Theorem 5.1. Let L_{δ} be a continuous linear form on $W_{\delta}(\Omega)$ such that

$$|L_{\delta}(v,\varphi)|\leq l_{\delta}\left[\mathbb{A}(v_{-},\varphi_{-})+\sqrt{\delta}\,\mathbb{B}(v_{+},\varphi_{+})+\frac{1}{\sqrt{\delta}}\,\mathbb{C}(v_{+},\varphi_{+})+\frac{1}{\delta\sqrt{\delta}}\,\mathbb{D}(v_{+},\varphi_{+})\right],$$

where l_{δ} is any function of $\delta>0$. Then there exists a constant $C>0$ (not depending on δ) such that the solution $\left(\widetilde{u}_{\pm}^{\delta},\widetilde{\omega}_{\pm}^{\delta}\right)$ of the problem

$$\begin{aligned}\text{Find }\left(\widetilde{u}^{\delta},\widetilde{\omega}^{\delta}\right)&\in W_{\delta}(\Omega),\text{ such that }\forall(v,\varphi)\in W_{\delta}(\Omega):\\L_{\delta}(v,\varphi)&=a^{-}\left[\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right),(v_{-},\varphi_{-})\right]+a^{+}\left[\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right),(v_{+},\varphi_{+})\right],\end{aligned}\tag{19}$$

satisfies the estimates

$$\mathbb{A}\left(\widetilde{u}_{-}^{\delta},\widetilde{\omega}_{-}^{\delta}\right)\leq Cl_{\delta},\tag{20}$$

$$\mathbb{B}\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)\leq C\delta^{-\frac{1}{2}}l_{\delta},\tag{21}$$

$$\mathbb{C}\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)\leq C\delta^{\frac{1}{2}}l_{\delta},\tag{22}$$

$$\mathbb{D}\left(\widetilde{u}_{+}^{\delta},\widetilde{\omega}_{+}^{\delta}\right)\leq C\delta^{\frac{3}{2}}l_{\delta}.\tag{23}$$

Proof. Thanks to the Korn's inequality, the expression $\mathbb{A}^2\left(\widetilde{u}_-^\delta,\widetilde{\omega}_-^\delta\right)$ is equivalent to $a^-\left[\left(\widetilde{u}_-^\delta,\widetilde{\omega}_-^\delta\right),\left(\widetilde{u}_-^\delta,\widetilde{\omega}_-^\delta\right)\right]$, and we have

$$a_{-1}^{+}\left[\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right),\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\right]=\int_{\Omega_+}\left[\begin{array}{c}2\lambda_+D_3\widetilde{u}_{+3}^\delta\left(D_1\widetilde{u}_{+1}^\delta+D_2\widetilde{u}_{+2}^\delta\right)+2\beta_+\widetilde{\omega}_+^\delta D_3\widetilde{u}_{+3}^\delta\\+\mu_+\left[D_1\widetilde{u}_{+3}^\delta+D_3\widetilde{u}_{+1}^\delta\right]^2+\mu_+\left[D_2\widetilde{u}_{+3}^\delta+D_3\widetilde{u}_{+2}^\delta\right]^2\\+\alpha_+\left(D_3\widetilde{\omega}_+^\delta\right)^2\end{array}\right]d\Omega_+,$$

and from the expressions of $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$, we get

$$a_{-1}^{+}\left[\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right),\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\right]\leq(2\lambda_++2\beta_+)\,\mathbb{B}\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\mathbb{D}\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)+(2\mu_++\alpha_+)\,\mathbb{C}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right).$$

As we have

$$(2\lambda_++2\beta_+)\frac{1}{\delta}\left(\mathbb{B}.\mathbb{D}\right)\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\leq(\lambda_++\beta_+)\left(\delta\mathbb{B}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)+\frac{1}{\delta^3}\mathbb{D}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\right),$$

then by taking $\left(\widetilde{v}^\delta,\widetilde{\varphi}^\delta\right)=\left(\widetilde{u}^\delta,\widetilde{\omega}^\delta\right)$ in the variational formulation (19), we get the following estimate:

$$\left(\begin{array}{c}\mathbb{A}^2\left(\widetilde{u}_-^\delta,\widetilde{\omega}_-^\delta\right)+\delta\,\mathbb{B}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\\+\frac{1}{\delta}\mathbb{C}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)+\frac{1}{\delta^3}\mathbb{D}^2\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\end{array}\right)\leq Cl_\delta\left(\begin{array}{c}\mathbb{A}\left(\widetilde{u}_-^\delta,\widetilde{\omega}_-^\delta\right)+\sqrt{\delta}\,\mathbb{B}\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\\+\frac{1}{\sqrt{\delta}}\,\mathbb{C}\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)+\frac{1}{\delta\sqrt{\delta}}\,\mathbb{D}\left(\widetilde{u}_+^\delta,\widetilde{\omega}_+^\delta\right)\end{array}\right),$$

where $C>0$ is a positive constant independent of δ. This leads to the estimates (20)-(23). ☐

6. Well-posedness of the approximate impedance problem

6.1. Existence and uniqueness of the solution to the impedance problem.

We consider the space:

$$W_*\left(\Omega_-\right)=\left\{\begin{array}{c}(v_-,\varphi_-)\in\left[H^1\left(\Omega_-\right)\right]^4:\\(D_1v_{-1},D_1v_{-2},D_1\varphi_-)\in\left[L^2\left(\Sigma\right)\right]^3,\\(D_2v_{-1},D_2v_{-2},D_2\varphi_-)\in\left[L^2\left(\Sigma\right)\right]^3,\\v_-=0_{1\times3}\text{ on }\Gamma_-,\ \varphi_-=0\text{ on }\Gamma_-.\end{array}\right\},$$

endowed with the norm

$$\|(v_-,\varphi_-)\|_{W_*(\Omega_-)}=\left[\|(v_-,\varphi_-)\|^2_{[H^1(\Omega_-)]^4}+\left\|\left(v_{-1|\Sigma},v_{-2|\Sigma},\varphi_{-|\Sigma}\right)\right\|^2_{[H^1(\Sigma)]^3}\right]^{1/2},$$

with

$$\|(v_-,\varphi_-)\|^2_{[H^1(\Omega_-)]^4}=\|\varphi_-\|^2_{H^1(\Omega_-)}+\sum_{i=1}^{3}\|v_{-i}\|^2_{H^1(\Omega_-)},$$

and

$$\left\|\left(v_{-1|\Sigma},v_{-2|\Sigma},\varphi_{-|\Sigma}\right)\right\|^2_{[H^1(\Sigma)]^3}=\left\|\varphi_{-|\Sigma}\right\|^2_{H^1(\Sigma)}+\sum_{i=1}^{2}\left\|v_{-i|\Sigma}\right\|^2_{H^1(\Sigma)}.$$

For all $\left(u_{-*}^\delta,\omega_{-*}^\delta\right)$ and (v_-,φ_-) in $W_*\left(\Omega_-\right)$, we set

$$a^-\left[\left(u_{-*}^\delta,\omega_{-*}^\delta\right),(v_-,\varphi_-)\right]=\int_{\Omega_-}\left[\begin{array}{c}\mathrm{tr}_3\left(\sigma_-\left(u_{-*}^\delta,\omega_{-*}^\delta\right)e_-\left(v_-\right)\right)+\alpha_-\nabla\omega_{-*}^\delta.\nabla\varphi_-\\+\zeta_-\omega_{-*}^\delta\varphi_-+\beta_-\left(D_1u_{-*1}^\delta+D_2u_{-*2}^\delta+D_3u_{-*3}^\delta\right)\varphi_-\end{array}\right]d\Omega_-,$$

$$a_\Sigma\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right]$$

$$=\int_\Sigma\left[\begin{array}{c}\frac{4\mu_+(\mu_++\lambda_+)}{(\lambda_++2\mu_+)}D_1u^\delta_{-*1}D_1v_{-1}+\mu_+D_2u^\delta_{-*1}D_2v_{-1}+\frac{2\mu_+\lambda_+}{2\mu_++\lambda_+}D_2u^\delta_{-*2}D_1v_{-1}\\ +\mu_+D_1u^\delta_{-*1}D_2v_{-2}+\frac{2\mu_+\beta_+}{2\mu_++\lambda_+}\omega^\delta_{-*}D_1v_{-1}+\frac{2\mu_+\beta_+}{2\mu_++\lambda_+}\omega^\delta_{-*}D_2v_{-2}\\ +\frac{4\mu_+(\mu_++\lambda_+)}{(\lambda_++2\mu_+)}D_2u^\delta_{-*2}D_2v_{-2}+\mu_+D_1u^\delta_{-*2}D_1v_{-2}+\frac{2\mu_+\lambda_+}{2\mu_++\lambda_+}D_1u^\delta_{-*1}D_2v_{-2}\\ +\mu_+D_1u^\delta_{-*1}D_2v_{-2}+\alpha_+D_1\omega^\delta_{-*}D_1\varphi_-+\alpha_+D_2\omega^\delta_{-*}D_2\varphi_-\\ -\frac{2\mu_+\beta_+}{2\mu_++\lambda_+}\left(D_1u^\delta_{-*1}+D_2u^\delta_{-*2}\right)\varphi_-+\frac{\zeta_+(2\mu_++\lambda_+)-\beta_+^2}{2\mu_++\lambda_+}\omega^\delta_{-*}\varphi_-\end{array}\right]d\Sigma,$$

and we state and prove the following theorem:

Theorem 6.1. For given (p_-,q_-) in $\left[L^2(\Omega_-)\right]^4$, there exists a unique solution $\left(u^\delta_{-*},\omega^\delta_{-*}\right)$ in $W_*(\Omega_-)$ to the impedance problem $\left(P^\delta_{-*}\right)$. Its weak formulation is given by

$$a^-_\delta\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right]=L(v_-,\varphi_-),\ \forall(v_-,\varphi_-)\in W_*(\Omega_-)$$

with

$$a^-_\delta\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right]=a^-\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right]+\delta a_\Sigma\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right],$$

and

$$L(v_-,\varphi_-)=\int_{\Omega_-}(p_-.v_-+q_-\varphi_-)\,d\Omega_-.$$

Proof. It is clear that L is continuous on $W_*(\Omega_-)$. For the continuity of a^-_δ, as in the proof of Theorem 2.1, we have

$$\left|a^-\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right]\right|\le C_1\left\|\left(u^\delta_{-*},\omega^\delta_{-*}\right)\right\|_{W_*(\Omega_-)}\|(v_-,\varphi_-)\|_{W_*(\Omega_-)},$$

where C_1 is a positive constant independent of δ, and by using the cauchy-Schawarz inequality and the definition of the space $W_*(\Omega_-)$, we get

$$\left|a_\Sigma\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right]\right|\le C_2\left\|\left(u^\delta_{-*},\omega^\delta_{-*}\right)\right\|_{W_*(\Omega_-)}\|(v_-,\varphi_-)\|_{W_*(\Omega_-)},$$

where C_2 is a positive constant independent of δ, and then

$$\begin{aligned}\left|a^-_\delta\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right]\right|&\le\left|a^-\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right]\right|+\delta\left|a_\Sigma\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right]\right|\\&\le C_1\left\|\left(u^\delta_{-*},\omega^\delta_{-*}\right)\right\|_{W_*(\Omega_-)}\|(v_-,\varphi_-)\|_{W_*(\Omega_-)}\\&\quad+C_2\delta\left\|\left(u^\delta_{-*},\omega^\delta_{-*}\right)\right\|_{W_*(\Omega_-)}\|(v_-,\varphi_-)\|_{W_*(\Omega_-)},\end{aligned}$$

as $\delta<<1$, then for all $\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)$ in $W_*(\Omega_-)$, we get

$$\left|a^-_\delta\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),(v_-,\varphi_-)\right]\right|\le C\left\|\left(u^\delta_{-*},\omega^\delta_{-*}\right)\right\|_{W_*(\Omega_-)}\|(v_-,\varphi_-)\|_{W_*(\Omega_-)}.$$

with $C=C_1+C_2$, which prove the continuity of a^-_δ on $[W_*(\Omega_-)]^2$. For the coercivity of a^-_δ, as in the proof of Theorem 2.1, for all $\left(u^\delta_{-*},\omega^\delta_{-*}\right)$ in $W_*(\Omega_-)$, we have

$$a^-\left[\left(u^\delta_{-*},\omega^\delta_{-*}\right),\left(u^\delta_{-*},\omega^\delta_{-*}\right)\right]\ge C_1\left\|\left(u^\delta_{-*},\omega^\delta_{-*}\right)\right\|^2_{[H^1(\Omega_-)]^4},$$

with C_1 is a positive constant independent of δ, and since we have

$$2D_2u^\delta_{-*2}D_1u^\delta_{-*1}\ge-\left(D_2u^\delta_{-*2}\right)^2-\left(D_1u^\delta_{-*1}\right)^2,$$

then for all $\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)$ in $W_*\left(\Omega_-\right)$, we get

$$\begin{aligned}
&a_{\Sigma}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right]\\
&=\int_{\Sigma}\left[\begin{array}{c}\frac{4\mu_-(\mu_-+\lambda_-)}{(\lambda_-+2\mu_-)}\left(D_1u_{-*1}^{\delta}\right)^2+\frac{4\mu_-(\mu_-+\lambda_-)}{(\lambda_-+2\mu_-)}\left(D_2u_{-*2}^{\delta}\right)^2\\ +\frac{2\mu_-(2\mu_-+3\lambda_-)}{2\mu_-+\lambda_-}D_2u_{-*2}^{\delta}D_1u_{-*1}^{\delta}+\mu_-\left(D_1u_{-*2}^{\delta}\right)^2+\mu_-\left(D_2u_{-*1}^{\delta}\right)^2\\ +\alpha_-\left(D_1\omega_{-*}^{\delta}\right)^2+\alpha_-\left(D_2\omega_{-*}^{\delta}\right)^2+\frac{\zeta_-(2\mu_-+\lambda_-)-\beta_-^2}{2\mu_-+\lambda_-}\left(\omega_{-*}^{\delta}\right)^2\end{array}\right]d\Sigma\\
&\geq\int_{\Sigma}\left[\begin{array}{c}\mu_-\left(D_1u_{-*1}^{\delta}\right)^2+\mu_-\left(D_2u_{-*2}^{\delta}\right)^2+\mu_-\left(D_1u_{-*2}^{\delta}\right)^2+\mu_-\left(D_2u_{-*1}^{\delta}\right)^2\\ +\alpha_-\left(D_1\omega_{-*}^{\delta}\right)^2+\alpha_-\left(D_2\omega_{-*}^{\delta}\right)^2+\frac{\zeta_-(2\mu_-+\lambda_-)-\beta_-^2}{2\mu_-+\lambda_-}\left(\omega_{-*}^{\delta}\right)^2\end{array}\right]d\Sigma\\
&\geq C_2\left\|\left(u_{-*1|\Sigma}^{\delta},u_{-*2|\Sigma}^{\delta},\omega_{-*|\Sigma}^{\delta}\right)\right\|_{[H^1(\Sigma)]^3}^2,
\end{aligned}$$

with $C_2=\min\left(\mu_-,\alpha_-,\frac{\zeta_-(2\mu_-+\lambda_-)-\beta_-^2}{2\mu_-+\lambda_-}\right)$ is a positive constant independent of δ, and then for all $\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)$ in $W_*\left(\Omega_-\right)$, we get

$$\begin{aligned}
&a_{\delta}^{-}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right]\\
&=a^{-}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right]+\delta a_{\Sigma}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right]\\
&\geq C_1\left\|\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right\|_{[H^1(\Omega_-)]^4}^2+\delta C_2\left\|\left(u_{-*1|\Sigma}^{\delta},u_{-*2|\Sigma}^{\delta},\omega_{-*|\Sigma}^{\delta}\right)\right\|_{[H^1(\Sigma)]^3}^2\\
&\geq C\delta\left\|\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right\|_{W_*(\Omega_-)}^2,
\end{aligned}$$

where $C=\min\left(C_1,C_2\delta\right)$ is a positive constant not depending on δ, which prove the coercivity of a_{δ}^{-}. Therefore by Lax-Milgram theorem, the impedance problem (P_{-*}^{δ}) has a unique solution in $W_*\left(\Omega_-\right)$. □

6.2. Stability for the impedance problem. In order to estimate the error between the solution of the approximate impedance and its approximation of order 1, which will be in section 7, we first set

$$\begin{aligned}
A\left(v_-,\varphi_-\right)&=\left\|\left(v_-,\varphi_-\right)\right\|_{[H^1(\Omega_-)]^4},\\
B\left(v_-,\varphi_-\right)&=\left[\begin{array}{c}\left\|D_1v_{-1}\right\|_{L^2(\Sigma)}+\left\|D_2v_{-1}\right\|_{L^2(\Sigma)}+\left\|D_1v_{-2}\right\|_{L^2(\Sigma)}+\\ \left\|D_2v_{-2}\right\|_{L^2(\Sigma)}+\left\|D_1\varphi_-\right\|_{L^2(\Sigma)}+\left\|D_2\varphi_-\right\|_{L^2(\Sigma)}+\left\|\varphi_-\right\|_{L^2(\Sigma)}\end{array}\right].
\end{aligned}$$

for all $\left(v_-,\varphi_-\right)\in W_*\left(\Omega_-\right)$. After that, we state and prove the following stability result.

Theorem 6.2. Let L_δ be a given linear form on $W_*\left(\Omega_-\right)$ satisfying the following bound in δ:

$$\left|L_\delta\left(v_-,\varphi_-\right)\right|\leq m_\delta\left(A\left(v_-,\varphi_-\right)+\sqrt{\delta}\,B\left(v_-,\varphi_-\right)\right),\quad\text{for all }\left(v_-,\varphi_-\right)\in W_*\left(\Omega_-\right),$$

where m_δ is any function of $\delta>0$. Then there exists $C>0$ (not depending on δ) such that the solution $\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)$ of the problem

$$\left\{\begin{array}{c}\text{Find }\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\in W_*\left(\Omega_-\right),\text{ such that for all }\left(v_-,\varphi_-\right)\in W_*\left(\Omega_-\right):\\ a^{-}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(v_-,\varphi_-\right)\right]+\delta a_{\Sigma}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(v_-,\varphi_-\right)\right]=L_\delta\left(v_-,\varphi_-\right),\end{array}\right.\tag{24}$$

satisfies the following estimates

$$A\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right) \leq Cm_{\delta}, \tag{25}$$

$$B\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right) \leq C\,\delta^{-1/2}m_{\delta}. \tag{26}$$

Proof. Thanks to Korn's inequality (see [7], [23]), the expression $A^2\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)$ is equivalent to $a^{-}\left[\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right]$, and we have

$$a_{\Sigma}\left(\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right) \geq C\;B^2\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right),$$

where $C>0$ is a positive constant not depending on δ. Then by taking $(v_{-},\varphi_{-})=\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)$ in the variational formulation (24), we get the estimate:

$$A^2\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)+\delta\;B^2\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right) \leq Km_{\delta}\left(A\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)+\sqrt{\delta}\;B\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)\right),$$

where $K>0$ is a positive constant not depending on δ. This gives directly the estimates (25) and (26).

6.3. Asymptotic expansion for the approximate impedance problem. By setting

$$\begin{aligned} u_{-*}^{\delta} &= u_{-*}^{0}+\delta u_{-*}^{1}+\delta^2 u_{-*}^{2}+\ldots \\ \omega_{-*}^{\delta} &= \omega_{-*}^{0}+\delta \omega_{-*}^{1}+\delta^2 \omega_{-*}^{2}+\ldots \end{aligned}$$

and inserting these expansions in the approximate impedance problem $\left(P_{-*}^{\delta}\right)$, we get a hierarchy of equations and boundary conditions. At order 0, we get

$$(P_{-*})_0 : \left\{ \begin{array}{l} \sum_{j=1}^{3} D_j\sigma_{-ij}\left(u_{-*}^{0},\omega_{-*}^{0}\right)=-p_{-i},\;\; i=1,2,3\;\text{ in }\Omega_{-}, \\ \sum_{j=1}^{3} D_j h_{-j}\left(\omega_{-*}^{0}\right)-g_{-}\left(u_{-*}^{0},\omega_{-*}^{0}\right)=-q_{-}\;\text{in }\Omega_{-}, \\ u_{-*}^{0}=0_{1\times 3}\text{ on }\Gamma_{-},\;\;\omega_{-*}^{0}=0\text{ on }\Gamma_{-}, \\ \left(\sigma_{-}\left(u_{-*}^{0},\omega_{-*}^{0}\right)_{|\Sigma}\nu,\;\alpha_{-}D_3\omega_{-*|\Sigma}^{0}\right)=0_{1\times 4}\;\text{ on }\Sigma. \end{array}\right.$$

At order 1, we get

$$(P_{-*})_1 : \left\{ \begin{array}{l} \sum_{j=1}^{3} D_j\sigma_{-ij}\left(u_{-*}^{1},\omega_{-*}^{1}\right)=0,\;\; i=1,2,3\;\text{ in }\Omega_{-}, \\ \sum_{j=1}^{3} D_j h_{-j}\left(\omega_{-*}^{1}\right)-g_{-}\left(u_{-*}^{1},\omega_{-*}^{1}\right)=0\text{ in }\Omega_{-}, \\ u_{-*}^{1}=0_{1\times 3}\text{ on }\Gamma_{-},\;\;\omega_{-*}^{1}=0\text{ on }\Gamma_{-}, \\ \left(\sigma_{-}\left(u_{-*}^{1},\omega_{-*}^{1}\right)_{|\Sigma}\nu,\;\alpha_{-}D_3\omega_{-*|\Sigma}^{1}\right)=T_{*}\left(u_{-*|\Sigma}^{0},\omega_{-*|\Sigma}^{0}\right)\;\text{ on }\Sigma, \end{array}\right.$$

Remark 6.1. The terms $\left(u_{-*}^{0},\omega_{-*}^{0}\right)$ and $\left(u_{-}^{0},\omega_{-}^{0}\right)$ (respectively $\left(u_{-*}^{1},\omega_{-*}^{1}\right)$ and $\left(u_{-}^{1},\omega_{-}^{1}\right)$) of the expansion of $\left(u_{-*}^{\delta},\omega_{-*}^{\delta}\right)$ and $\left(u_{-}^{\delta},\omega_{-}^{\delta}\right)$ are solutions of the same boundary value problem at order 0 (respectively at order 1). Then by uniqueness, we have

$$\left(u_{-*}^{0},\omega_{-*}^{0}\right)=\left(u_{-}^{0},\omega_{-}^{0}\right)\quad\text{and}\quad\left(u_{-*}^{1},\omega_{-*}^{1}\right)=\left(u_{-}^{1},\omega_{-}^{1}\right).$$

7. Error estimates and optimality

7.1. Error estimate for the transmission problem. By setting

$$u_-^{(\delta,1)} = u_-^0 + \delta u_-^1, \quad u_+^{(\delta,1)} = \widetilde{u}_+^0 + \delta \widetilde{u}_+^1 + \delta^2 \left(0, 0, \widetilde{u}_{+3}^2\right),$$
$$\omega_-^{(\delta,1)} = \omega_-^0 + \delta \omega_-^1, \quad \omega_+^{(\delta,1)} = \widetilde{\omega}_+^0 + \delta \widetilde{\omega}_+^1,$$

and making use of the problems $(P_-)_0$ and $(P_-)_1$ satisfied by $\left(u_-^0, \omega_-^0\right)$ and $\left(u_-^1, \omega_-^1\right)$ and also the problems (13)−(15) for $k \in \{0, 1, 2, 3\}$, we get

$$\begin{aligned} L_\delta(v,\varphi) &= a_{-3}^+ \left[\left(\widetilde{u}_+^3, \widetilde{\omega}_+^3\right), (v_+, \varphi_+)\right] + \delta \int_{\Omega_+} \mu_+ D_3 \widetilde{u}_{+1}^2 D_3 v_{+1} d\Omega_+ \\ &+ \delta \int_{\Omega_+} \mu_+ D_3 \widetilde{u}_{+2}^2 D_3 v_{+2} d\Omega_+ + \delta \int_{\Omega_+} \alpha_+ D_3 \widetilde{\omega}_+^2 D_3 \varphi_+ d\Omega_+ \\ &- \delta \int_{\Omega_+} \left[\mu_+ D_1 \widetilde{u}_{+3}^2 D_1 v_{+3} + \mu_+ D_2 \widetilde{u}_{+3}^2 D_2 v_{+3}\right] d\Omega_+ - \delta^2 a_{+1}^+ \left[\left(\widetilde{u}_+^1, \widetilde{\omega}_+^1\right), (v_+, \varphi_+)\right] \end{aligned}$$

where

$$\begin{aligned} L_\delta(v,\varphi) \quad = \quad & a^- \left[\left(\widetilde{u}_-^\delta - u_-^{(\delta,1)}, \widetilde{\omega}_-^\delta - \omega_-^{(\delta,1)}\right), (v_-, \varphi_-)\right] \\ & + a^+ \left[\left(\widetilde{u}_+^\delta - u_+^{(\delta,1)}, \widetilde{\omega}_+^\delta - \omega_+^{(\delta,1)}\right), (v_+, \varphi_+)\right], \end{aligned}$$

and thus

$$|L_\delta(v,\varphi)| \le C\delta^{3/2} \left[\mathbb{A}(v_-, \varphi_-) + \sqrt{\delta}\, \mathbb{B}(v_+, \varphi_+) + \tfrac{1}{\sqrt{\delta}}\, \mathbb{C}(v_+, \varphi_+) + \tfrac{1}{\delta\sqrt{\delta}}\, \mathbb{D}(v_+, \varphi_+)\right],$$

which implies, by the stability result (see Theorem 5.1), the following error estimate:

$$\left\| u_-^\delta - u_-^{(\delta,1)} \right\|_{[H^1(\Omega_-)]^3} + \left\| \omega_-^\delta - \omega_-^{(\delta,1)} \right\|_{H^1(\Omega_-)} \le C\delta^{3/2}, \tag{27}$$

where C is a positive constant independent of δ.

Remark 7.1. We have taken

$$u_{+3}^{(\delta,1)} = \widetilde{u}_{+3}^0 + \delta \widetilde{u}_{+3}^1 + \delta^2 \widetilde{u}_{+3}^2,$$

in order to satisfy the transmission condition

$$u_{+3}^{(\delta,1)} = \delta u_{-3}^{(\delta,1)} \quad \text{on } \Sigma,$$

and, consequently, we have $\left(u_\pm^{(\delta,1)}, \omega_\pm^{(\delta,1)}\right)$ in the space $W_\delta(\Omega)$.

Remark 7.2. If p_- and q_- are smooth we can obtain all the terms of the asymptotic expansion $\left(u_-^k, \omega_-^k\right)$ for $k \ge 2$, and prove an analogous error estimate. For instance we can prove

$$\left\| u_-^\delta - u_-^{(\delta,2)} \right\|_{[H^1(\Omega_-)]^3} + \left\| \omega_-^\delta - \omega_-^{(\delta,2)} \right\|_{H^1(\Omega_-)} \le C\delta^{5/2}, \tag{28}$$

where

$$u_-^{(\delta,2)} = u_-^0 + \delta u_-^1 + \delta^2 u_-^2 \quad \text{and} \quad \omega_-^{(\delta,2)} = \omega_-^0 + \delta \omega_-^1 + \delta^2 \omega_-^2.$$

7.2. Error estimate for the impedance problem. By setting

$$\begin{aligned} u_{-*}^{(\delta,1)} &= u_{-*}^0 + \delta u_{-*}^1, \\ \omega_{-*}^{(\delta,1)} &= \omega_{-*}^0 + \delta \omega_{-*}^1, \end{aligned}$$

and

$$\begin{aligned} &a^- \left[\left(u_{-*}^\delta - u_{-*}^{(\delta,1)}, \omega_{-*} - \omega_{-*}^{(\delta,1)} \right), (v_-, \varphi_-) \right] \\ &+ \delta a_\Sigma \left[\left(u_{-*}^\delta - u_{-*}^{(\delta,1)}, \omega_{-*} - \omega_{-*}^{(\delta,1)} \right), (v_-, \varphi_-) \right] = L_\delta (v_-, \varphi_-), \end{aligned}$$

and making use of the problems $(P_{-*})_0$ and $(P_{-*})_1$ satisfied by $(u_{-*}^0, \omega_{-*}^0)$ and $(u_{-*}^1, \omega_{-*}^1)$, we get

$$L_\delta (v_-, \varphi_-) = -\delta^2 a_\Sigma \left[\left(u_{-*}^1, \omega_{-*}^1 \right), (v_-, \varphi_-) \right]$$

and thus

$$|L_\delta (v_-, \varphi_-)| \leq C \delta^{3/2} \left[A (v_-, \varphi_-) + \sqrt{\delta}\, B (v_-, \varphi_-) \right],$$

which implies, by the stability result (see Theorem 6.2), the following error estimate:

$$\left\| u_{-*}^\delta - u_{-*}^{(\delta,1)} \right\|_{[H^1(\Omega_-)]^3} + \left\| \omega_{-*}^\delta - \omega_{-*}^{(\delta,1)} \right\|_{H^1(\Omega_-)} \leq C \delta^{3/2}, \tag{29}$$

where C is a positive constant independent of δ.

Remark 7.3. If p_- and q_- are smooth we can obtain all the terms of the asymptotic expansion $\left(u_{-*}^k, \omega_{-*}^k\right)$ for $k \geq 2$, and prove an analogous error estimate. For instance we can prove

$$\left\| u_{-*}^\delta - u_{-*}^{(\delta,2)} \right\|_{[H^1(\Omega_-)]^3} + \left\| \omega_{-*}^\delta - \omega_{-*}^{(\delta,2)} \right\|_{H^1(\Omega_-)} \leq C \delta^{5/2}, \tag{30}$$

where

$$u_{-*}^{(\delta,2)} = u_{-*}^0 + \delta u_{-*}^1 + \delta^2 u_{-*}^2 \text{ and } \omega_{-*}^{(\delta,2)} = \omega_{-*}^0 + \delta \omega_{-*}^1 + \delta^2 \omega_{-*}^2.$$

7.3. Final error estimate and optimality. This subsection is devoted to the error estimate between the solution of the transmission problem in Ω_-, and the solution of the approximate impedance problem. As we have $(\left(u_{-*}^0, \omega_{-*}^0\right) = \left(u_-^0, \omega_-^0\right)$ and $\left(u_{-*}^1, \omega_{-*}^1\right) = \left(u_-^1, \omega_-^1\right)$ (see Remark 6.1) then by triangular inequality, we can write

$$\begin{aligned} &\left\| \widetilde{u}_-^\delta - u_{-*}^\delta \right\|_{[H^1(\Omega_-)]^3} + \left\| \widetilde{\omega}_-^\delta - \omega_{-*}^\delta \right\|_{H^1(\Omega_-)} \\ &\leq \left\| \widetilde{u}_-^\delta - u_-^0 - \delta u_-^1 \right\|_{[H^1(\Omega_-)]^3} + \left\| u_{-*}^\delta - u_{-*}^0 - \delta u_{-*}^1 \right\|_{[H^1(\Omega_-)]^3} \\ &\quad + \left\| \widetilde{\omega}_-^\delta - \omega_-^0 - \delta \omega_-^1 \right\|_{H^1(\Omega_-)} + \left\| \omega_{-*}^\delta - \omega_{-*}^0 - \delta \omega_{-*}^1 \right\|_{H^1(\Omega_-)}, \end{aligned}$$

and in virtue of (29) and (27), we find

$$\left\| \widetilde{u}_-^\delta - u_{-*}^\delta \right\|_{[H^1(\Omega_-)]^3} + \left\| \widetilde{\omega}_-^\delta - \omega_{-*}^\delta \right\|_{H^1(\Omega_-)} \leq C \delta^{3/2},$$

where the constant C depends only on p_-, q_- and the elasticity coefficients. Indeed, if the data p_- and q_- are smooth enough such that we can determinate $\left(u_-^2, \omega_-^2\right)$ and $\left(u_{-*}^2, \omega_{-*}^2\right)$, then by (28) the last error estimate which is not optimal may be ameliorated as follows:

$$\begin{aligned} \left\| \widetilde{u}_-^\delta - u_-^0 - \delta u_-^1 \right\|_{[H^1(\Omega_-)]^3} &\leq \left\| \widetilde{u}_-^\delta - u_-^0 - \delta u_-^1 - \delta^2 u_-^2 \right\|_{[H^1(\Omega_-)]^3} + \delta^2 \left\| u_-^2 \right\|_{[H^1(\Omega_-)]^3} \\ &\leq C_1 \delta^{5/2} + C_2 \delta^2 \leq C \delta^2, \end{aligned}$$

and

$$\begin{aligned}\left\|\widetilde{\omega}_{-}^{\delta}-\omega_{-}^{0}-\delta\omega_{-}^{1}\right\|_{H^{1}(\Omega_{-})} &\leq \left\|\widetilde{\omega}_{-}^{\delta}-\omega_{-}^{0}-\delta\omega_{-}^{1}-\delta^{2}\omega_{-}^{2}\right\|_{H^{1}(\Omega_{-})}+\delta^{2}\left\|\omega_{-}^{2}\right\|_{H^{1}(\Omega_{-})} \\ &\leq C_{1}\delta^{5/2}+C_{2}\delta^{2}\leq C\delta^{2}.\end{aligned}$$

Using (30), similar estimates for $\left(u_{-*}^{\delta}-u_{-*}^{0}-\delta u_{-*}^{1},\omega_{-*}^{\delta}-\omega_{-*}^{0}-\delta\omega_{-*}^{1}\right)$ can be proved in the same manner as outlined above.

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(Athmane Abdallaoui) Laboratoire de Mathématiques et Physique Appliquées, École Normale Supérieure de Bou Saâda, Bou Saâda, Algeria
E-mail address: abdallaoui.athmane@ens-bousaada.dz, a.abdallaoui18@gmail.com

(Abdelkarim Kelleche) Department of Mathematics, Faculty of Sciences of Matter and Computer Sciences, University Djilali Bounaama of Khemis Miliana, Algeria
E-mail address: a.kelleche@univ-dbkm.dz

(Amirouche Berkani) Faculty of Exact Sciences, University Akli Mohand Oulhadj of Bouira, Bouira, Algeria
E-mail address: aberkanid@gmail.com