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On Integer Sequences Associated with Admissible Prime k-tuples

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Abstract. It is conjectured that every admissible k-tuple matches infinitely many positions in the sequence of prime numbers. The investigations of this hypothesis have led to many important results; however, the problem remains unsolved. In this work, the problem is studied from the experimental mathematics side. Integer sequences associated with the distribution of admissible prime k-tuples in the intervals $(s^n; s^{n+1}]$ are studied experimentally, using the probabilistic Miller-Rabin primality test and parallel computing technologies. The obtained result gives us one more argument in favor of the infinitude of admissible prime k-tuples conjecture.

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Key words and phrases. prime k-tuples, integer sequences, Hardy-Littlewood conjecture.

1. Introduction

One of the most famous unsolved tasks in mathematics (after the Riemann hypothesis and already proved Fermat theorem) is the problem of the twin primes (pairs of prime numbers with a difference of two). The conjecture states that there are infinitely many twin primes. The paradoxical contrast between the elementary formulation and the deep underlying content, resisting the efforts of mathematicians for decades, is striking. This old problem remains relevant nowadays. Its (or its generalizations) analytical (see [1, 2, 3]) or numerical (cf. [4, 5]) studies appear annually.

The Polignac conjecture (stating that there are infinitely many primes with difference n) and the Hardy-Littlewood conjecture on admissible prime k-tuples generalize the twin primes conjecture. If $p+h_j \in \mathbb{P}$, $0 \leqslant j \leqslant k$, then the sequence $H = \{p+h_j\}$ is a prime k-tuple and $\#H = k+1$. Next, if the sequence H does not form a complete residue class with respect to any prime, i.e., $\#\{H \bmod p\} < p$, for all $p \in \mathbb{P}$, then the prime k-tuple is called admissible. Note that $\#\{H \bmod p\}$ equals the number of distinct residues of $0, h_1, ..., h_k \pmod p$.

Let $\mathrm{Li}(m,x)$ denote m-th offset logarithmic integral, $m > 0$,

$$\begin{aligned}\mathrm{Li}(m,x) &= \int_2^x \frac{dt}{(\log t)^m} = \frac{x}{(\log x)^m} - \frac{2}{(\log 2)^m} + m\mathrm{Li}(m+1,x) \\ &= \sum_{j=0}^{r} \frac{m^{(j)}x}{(\log x)^{m+j}} + O\left(\frac{x}{(\log x)^{m+r+1}}\right), \qquad r \in \mathbb{N}_0. \qquad (1)\end{aligned}$$

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Here $m^{(j)}$ stands for the rising factorial, $m^{(j)} = \Gamma(m+j)/\Gamma(m)$. Hardy and Littlewood [6] considered the asymptotic density of prime constellations. In its most optimistic form of the error term, the conjecture is formulated as follows (cf. [2, 5]).

Proposition 1.1. *(Hardy and Littlewood).* Let $\pi_H(x)$ be the admissible prime k-tuple counting function, then

$$\pi_H(x) = \#\{p \leqslant x \mid p + h_j \in \mathbb{P};\ j \in [1,k]\} = C_H \mathrm{Li}(k+1, x) + O(x^{1/2+\varepsilon}), \qquad (2)$$

where ε is a small positive quantity and C_H is the prime k-tuple constant,

$$C_H = 2^k \prod_{p \geqslant 3} \left(1 - \frac{\#\{H \bmod p\}}{p}\right) \left(1 - \frac{1}{p}\right)^{-(k+1)}. \qquad (3)$$

Remark 1.1. Calculating the constant C_H by the formula (3), we get, e.g.,

$$C_{(0,2,6)} = C_{(0,4,6)} = \frac{9}{2} \prod_{p \geqslant 5} \frac{p^2(p-3)}{(p-1)^3} \approx 2.858248596,$$

$$C_{(0,4,6,10)} = 2C_{(0,2,6,8)} = 27 \prod_{p \geqslant 5} \frac{p^3(p-4)}{(p-1)^4} \approx 8.302361726,$$

$$C_{(0,4,6,10,12)} = C_{(0,2,6,8,12)} = \frac{50625}{2048} \prod_{p \geqslant 7} \frac{p^4(p-5)}{(p-1)^5} \approx 10.131794949,$$

$$C_{(0,4,6,10,12,16)} = \frac{759375}{8192} \prod_{p \geqslant 7} \frac{p^5(p-6)}{(p-1)^6} \approx 17.298612309.$$

All limits in the paper, unless specified, are taken as $x \to \infty$ or $n \to \infty$.

2. Sequences, associated with admissible prime k-tuples

Let us consider integer sequences $\{a_{n,H}\}$, associated with admissible prime k-tuples,

$$a_{n,H} = \#\{s^n < p \leqslant s^{n+1} \mid p + h_j \in \mathbb{P};\ j \in [1,k]\}, \qquad n \in \mathbb{N}, \quad s \geqslant 2. \qquad (4)$$

It is possible to formulate a result about the ratio of the adjacent elements of the sequence $\{a_{n,H}\}$.

Proposition 2.1. Let $\theta_s = 1/\log s$. Then under Proposition 1.1 we have

$$\rho_n := \frac{a_{n+1,H}}{a_{n,H}} = \underbrace{s - s(k+1) \sum_{j=0}^{3} \frac{\omega_j}{n^{j+1}}}_{:=\hat{\rho}_n} + O\left(\frac{1}{n^5}\right). \qquad (5)$$

Here

$$\begin{aligned}
\omega_0 =&1, \qquad \omega_1 = \theta_s - 2 - \frac{k}{2} - \frac{1}{s-1},\\
\omega_2 =&(\theta_s^2 - \theta_s)k + (3\theta_s^2 - 4\theta_s) + \frac{(k+8)(k+3)}{6} - \frac{k+1}{(s-1)^2} + \frac{2-2\theta_s}{s-1},\\
\omega_3 =& \left(\theta_s^3 - \theta_s^2 + \frac{1}{2}\theta_s\right)k^2 + \left(8\theta_s^3 - 9\theta_s^2 + \frac{11}{2}\theta_s\right)k\\
&+ \left(13\theta_s^3 - 17\theta_s^2 + 12\theta_s\right) - \frac{(k+6)(k^2+15k+32)}{24} - \frac{(k+1)^2}{(s-1)^3}\\
&- \frac{k^2/2 + (3\theta_s - 3/2)k - 6\theta_s + 7}{(s-1)^2} - \frac{(3\theta_s^2 + 1/2)k + 9\theta_s^2 - 6\theta_s + 4}{s-1}.
\end{aligned} \tag{6}$$

Proof. Using formulas (2) and (1) we obtain

$$\pi_H(x) = C_H x \left(\sum_{j=0}^{r} (k+1)^{(j)} (\log x)^{-k-j-1} + O\left((\log x)^{-k-r-2}\right)\right).$$

Let us consider the ratio

$$\begin{aligned}
r_n = \frac{\pi_H(s^{n+1})}{\pi_H(s^n)} &= s\frac{\sum_{j=0}^{r}(k+1)^{(j)}(n+1)^{-k-j-1}\theta_s^{k+j+1} + O(n^{-k-r-2})}{\sum_{j=0}^{r}(k+1)^{(j)}n^{-k-j-1}\theta_s^{k+j+1} + O(n^{-k-r-2})}\\
&= s\frac{\sum_{j=0}^{r}(k+1)^{(j)}\theta_s^j n^{-j}(1+n^{-1})^{-k-j-1} + O(n^{-r-1})}{\sum_{j=0}^{r}(k+1)^{(j)}\theta_s^j n^{-j} + O(n^{-r-1})}\\
&= s\left(\sum_{j=0}^{r}(k+1)^{(j)}\theta_s^j \sum_{q=0}^{r-j}\binom{-k-j-1}{q}\frac{1}{n^{q+j}} + O\left(\frac{1}{n^{r+1}}\right)\right)\\
&\quad\times\left(1 + \sum_{q=1}^{r}(-1)^q\left(\sum_{j=1}^{r}\frac{(k+1)^{(j)}\theta_s^j}{n^j}\right)^q + O\left(\frac{1}{n^{r+1}}\right)\right)\\
&= s\left(\sum_{i=0}^{r}\frac{\alpha_i}{n^i} + O\left(\frac{1}{n^{r+1}}\right)\right)\left(\sum_{i=0}^{r}\frac{\beta_i}{n^i} + O\left(\frac{1}{n^{r+1}}\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_i &= (k+1)^{(i)}\theta_s^i \underbrace{\sum_{j=0}^{i}\frac{(-1)^j}{j!}\theta_s^{-j}}_{:=\exp_i(-1/\theta_s)},\\
\beta_i &= \theta_s^i\sum_{q=0}^{r}(-1)^q \sum_{\substack{k_1+\ldots+k_r=q\\ k_1+\ldots+rk_r=i}}\binom{i}{k_1,\ldots,k_r}\prod_{j=1}^{r}\left((k+1)^{(j)}\right)^{k_j}.
\end{aligned} \tag{7}$$

Thus

$$
\begin{aligned}
r_n &= s\sum_{j=0}^{r}\frac{\gamma_j}{n^j} + O\left(\frac{1}{n^{r+1}}\right),\\
\gamma_j &= \sum_{q=0}^{j}\alpha_q\beta_{j-q} = \theta_s^j\sum_{q=0}^{j}(k+1)^{(q)}\exp_q\left(-\frac{1}{\theta_s}\right)\\
&\quad\times\sum_{i=0}^{r}(-1)^i\sum_{\substack{k_1+\ldots+k_r=i\\ k_1+\ldots+rk_r=j-q}}\binom{i}{k_1,\ldots,k_r}\prod_{t=1}^{r}\left((k+1)^{(t)}\right)^{k_t}.
\end{aligned}\tag{8}
$$

We have

$$
\begin{aligned}
r_n &= s - s\frac{k+1}{n} + s\frac{k+1}{n^2}\left(1-\theta_s+\frac{k}{2}\right)\\
&\quad + s\frac{k+1}{n^3}\left((\theta_s-\theta_s^2)k + (2\theta_s-3\theta_s^2) - \frac{(k+2)(k+3)}{6}\right)\\
&\quad + s\frac{k+1}{n^4}\left(\left(\theta_s^2-\theta_s^3-\frac{1}{2}\theta_s\right)k^2 + \left(6\theta_s^2-8\theta_s^3-\frac{5}{2}\theta_s\right)k\right.\\
&\quad \left. + \left(8\theta_s^2-13\theta_s^3-3\theta_s\right) + \frac{(k+2)(k+3)(k+4)}{24}\right) + O\left(\frac{1}{n^5}\right)\\
&= s - \frac{s(k+1)}{n}\left(\eta_0 + \frac{\eta_1}{n} + \frac{\eta_2}{n^2} + \frac{\eta_3}{n^3} + O\left(\frac{1}{n^4}\right)\right).
\end{aligned}
$$

Here

$$
\begin{aligned}
\eta_0 &= 1, \qquad \eta_1 = \theta_s - 1 - \frac{k}{2},\\
\eta_2 &= (\theta_s^2-\theta_s)k + (3\theta_s^2-2\theta_s) + \frac{(k+2)(k+3)}{6},\\
\eta_3 &= \left(\theta_s^3-\theta_s^2+\frac{1}{2}\theta_s\right)k^2 + \left(8\theta_s^3-6\theta_s^2+\frac{5}{2}\theta_s\right)k\\
&\quad + \left(13\theta_s^3-8\theta_s^2+3\theta_s\right) - \frac{(k+2)(k+3)(k+4)}{24}.
\end{aligned}
$$

Returning to the ratio of adjacent elements of the sequence $\{a_{n,H}\}$, we receive that

$$
\begin{aligned}
\rho_n &= \frac{a_{n+1,H}}{a_{n,H}} = \frac{\pi_H(s^{n+2})-\pi_H(s^{n+1})}{\pi_H(s^{n+1})-\pi_H(s^n)} = \frac{r_{n+1}-1}{1-r_n^{-1}}\\
&= \left((s^2-s) - \sum_{j=0}^{3}\frac{s^2(k+1)}{(n+1)^{j+1}}\eta_j + O\left(\frac{1}{n^5}\right)\right)\\
&\quad\times\left(s-\left(1-\sum_{j=0}^{3}\frac{k+1}{n^{j+1}}\eta_j + O\left(\frac{1}{n^5}\right)\right)^{-1}\right)^{-1}.
\end{aligned}
$$

Thus,

$$\rho_n = s\left(1 - \frac{s(k+1)}{s-1}\sum_{j=0}^{3}\frac{\eta_j}{n^{j+1}}\left(1+\frac{1}{n}\right)^{-j-1} + O\left(\frac{1}{n^5}\right)\right)$$
$$\times\left(1 - \frac{k+1}{n(s-1)}\sum_{j=0}^{3}\frac{\eta_j}{n^j} - \frac{(k+1)^2}{n^2(s-1)}\left(1+\frac{2\eta_1}{n}+\frac{\eta_1^2+2\eta_2}{n^2}\right)\right.$$
$$\left. - \frac{(k+1)^3}{n^3(s-1)}\left(1+\frac{3\eta_1}{n}\right) - \frac{(k+1)^4}{n^4(s-1)} + O\left(\frac{1}{n^5}\right)\right)^{-1}$$
$$= s\left(1 - \frac{s(k+1)}{s-1}\left(\frac{1}{n}+\frac{\eta_1-1}{n^2}+\frac{\eta_2-2\eta_1+1}{n^3}\right.\right.$$
$$\left.\left. + \frac{\eta_3-3\eta_2+3\eta_1-1}{n^4} + O\left(\frac{1}{n^5}\right)\right)\right)$$
$$\times\left(1 - \frac{k+1}{s-1}\left(\frac{1}{n}+\frac{\eta_1+k+1}{n^2}+\frac{\eta_2+2(k+1)\eta_1+(k+1)^2}{n^3}\right.\right.$$
$$\left.\left. + \frac{\eta_3+(k+1)(\eta_1^2+2\eta_2)+3(k+1)^2\eta_1+(k+1)^3}{n^4} + O\left(\frac{1}{n^5}\right)\right)\right)^{-1},$$

yielding the statement of the proposition. □

Figure 1 illustrates the approximation (5) with the ratios of the adjacent elements of the sequences $\{a_{n,H}\}$ taken in the intervals $(3^n, 3^{n+1}]$.

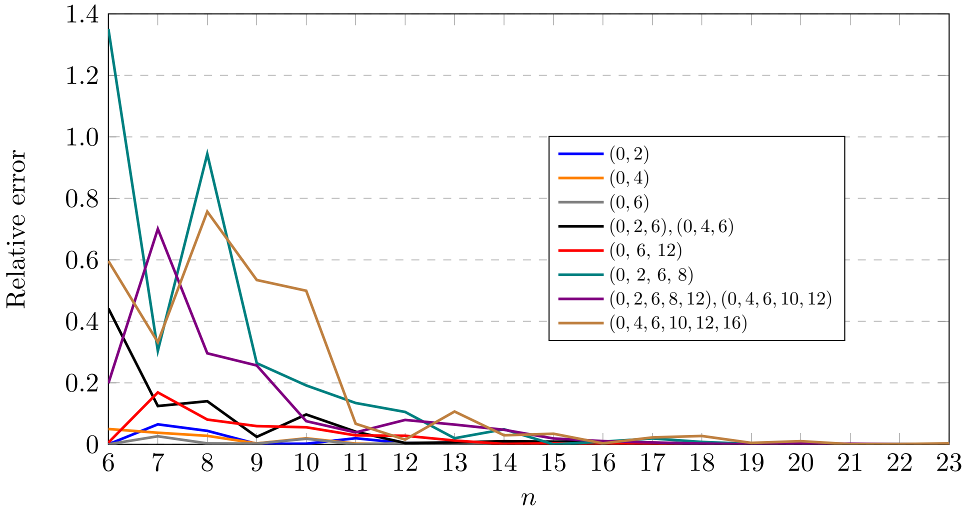

FIGURE 1. Relative error of the approximation, $|1 - \hat{\rho}_n/\rho_n|$, for $s = 3$.

The coefficients $(\omega_1, \omega_2, \omega_3)^T$ of the expansion (5), values of the ratios of the adjacent elements ρ_n, their estimates $\hat{\rho}_n$, and the relative error of the approximation $|1 - \hat{\rho}_n/\rho_n|$ are presented in Section 5 (see Tables 2-6).

Corollary 2.1. Let ρ_1 and ρ_2 be the ratios of the adjacent elements of integer sequences $\{a_{n,H_1}\}$ and $\{a_{n,H_2}\}$, associated with admissible prime k-tuples (cf. (4)). Then, under conditions of Proposition 2.1, we have

$$\frac{\rho_n(s_1)}{\rho_n(s_2)} = \begin{cases} \frac{s_1}{s_2}\left(1 + \frac{k+1}{n^2}\left(\frac{\log(s_1/s_2)}{\log s_1 \log s_2} + \frac{s_2-s_1}{(s_2-1)(s_1-1)}\right) + O\left(\frac{1}{n^3}\right)\right), & k = k_1 = k_2, \\ \frac{s_1}{s_2}\left(1 + \frac{k_2-k_1}{n} + \frac{B(s_1,s_2,k_1,k_2)}{n^2} + O\left(\frac{1}{n^3}\right)\right), & k_1 \neq k_2, s_1 \neq s_2. \end{cases}$$

Here

$$B(s_1,s_2,k_1,k_2) = (k_2+1)\left(\frac{1}{\log s_2} - \frac{1}{s_2-1}\right) - (k_1+1)\left(\frac{1}{\log s_1} - \frac{1}{s_1-1}\right) + \frac{3}{2}(k_1-k_2) + \frac{1}{2}(k_1-k_2)^2.$$

3. Algorithms and software

We have chosen probabilistic primality tests instead of classical sieve methods for calculating integer sequences associated with admissible prime k-tuples to process more significant amounts of data in a given time. The most efficient probabilistic primality test is Miller-Rabin's with $O(wq^2 \log q \log\log q)$ complexity. It uses the fast Fourier transform method of q-digit integer multiplication (w is the number of iterations). In our numerical experiments, we have chosen the value $w = 9$, since it has been shown (cf. [7]) that the selection of the value $w = 5$ gives 2 misidentifications of $\sim 1.5 \times 10^{12}$ potential twin prime pairs.

[7] showed that *C/C++ OpenMP* technology performs the Miller-Rabin test about 9 times faster than *Python* multiprocessing. Hence we use MinGW-w64 4.3.5 *C/C++* compilator, 32 GB DDR4 RAM 4266 MHz, AMD Ryzen 9 5950x 16C/16T 4.9 GHz CPU assigning 30 threads for calculations. Figure 2 demonstrates that the running time of k-tuples is exponential.

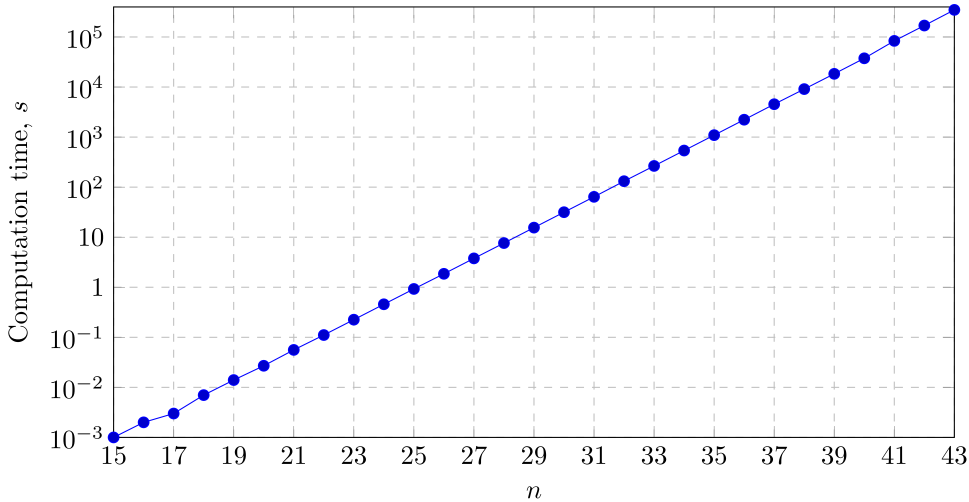

FIGURE 2. Computation time of k-tuples (shown on a logarithmic scale).

4. New integer sequences, associated with admissible prime k-tuples

Investigating the distribution of admissible prime k-tuples in exponentially growing intervals $(2^n; 2^{n+1}]$, we have obtained the following sequences $\{a_{n,\hat{H}}\}$, defined by (4) (see Table 1).

TABLE 1. First terms of sequences, associated with the distribution of admissible prime k-tuples in $(2^n; 2^{n+1}]$ intervals.

$\hat{H}$	$\{a_{n,\hat{H}}\}$
$(0, 2)$	1, 1, 1, 2, 2, 3, 7, 7, 12, 26, 45, 70, 113, 215, 355, 666, ...
$(0, 4)$	1, 1, 1, 1, 2, 6, 4, 11, 15, 23, 44, 64, 131, 197, 359, 658, ...
$(0, 6)$	0, 2, 2, 3, 5, 7, 11, 17, 28, 55, 79, 142, 241, 434, 719, ...
$(0, 2, 6)$ $(0, 4, 6)$	0, 2, 2, 1, 2, 5, 4, 6, 8, 20, 22, 28, 49, 93, 126, 248, 427, ...
$(0, 6, 12)$	0, 2, 1, 2, 3, 3, 4, 4, 9, 22, 22, 38, 56, 93, 134, 255, 405, ...
$(0, 2, 6, 8)$	0, 1, 1, 0, 0, 1, 1, 0, 1, 2, 3, 1, 5, 7, 5, 15, 19, 44, 71, ...
$(0, 6, 12, 18)$	0, 1, 1, 0, 2, 0, 1, 0, 2, 5, 5, 6, 9, 13, 10, 37, 37, 84, 127, ...
$(0, 2, 6, 8, 12)$ $(0, 4, 6, 10, 12)$	0, 2, 1, 0, 0, 2, 0, 0, 0, 2, 1, 1, 3, 4, 3, 3, 8, 18, 21, 46, 67, ...
$(0, 4, 6, 10, 12, 16)$	0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 3, 4, 5, ...

The first sequence of the table is known as the number of lesser twin primes in range $(2^n, 2^{n+1}]$ and is included in the OEIS database as A095017 entry (see [8]). The rest of the sequences in Table 1 are new to the authors' knowledge. Table 3 in the next section provides a more comprehensive composition of these sequences. Information on their properties is also presented in Tables 4 and 5 of Section 5.

5. Tables of coefficients and ratios

This section illustrates the numerical results of the study. First, the coefficients of the principal of Proposition 1.1 (see expansion (5)) are given in Table 2.

Sequences associated with admissible prime k-tuples. Table 3 shows the sequences $\{a_{n,H}\}$, which are associated with the distribution of admissible prime k-tuples in the intervals $(2^n; 2^{n+1}]$. These sequences provide a numerical insight into the distribution of prime k-tuples as n increases.

The computation of each subsequent term took, on average, twice as long as the previous one. Furthermore, our computed data can be used for further theoretical research and hypothesis testing related to the distribution of primes and their k-tuples.

TABLE 2. Coefficients of $\rho_n(s,k)$ expansion, $\Omega = (\omega_1, \omega_2, \omega_3)^T$.

s \ k	Ω	1	2	3	4	5
2	ω_1	−2.057305	−2.557305	−3.057305	−3.557305	−4.057305
	ω_2	+4.226611	+6.198618	+8.503959	+11.142632	+14.114640
	ω_3	−4.659449	−10.578615	−18.462264	−28.560393	−41.123005
3	ω_1	−2.089761	−2.589761	−3.089761	−3.589761	−4.089761
	ω_2	+4.352706	+6.354336	+8.689299	+11.357595	+14.359225
	ω_3	−9.202326	−15.277740	−23.341654	−33.644068	−46.434984
4	ω_1	−2.111986	−2.611986	−3.111986	−3.611986	−4.111986
	ω_2	+4.438177	+6.459394	+8.813945	+11.501828	+14.523045
	ω_3	−9.520378	−15.698872	−23.881193	−34.317341	−47.257316
5	ω_1	−2.128665	−2.628665	−3.128665	−3.628665	−4.128665
	ω_2	+4.501886	+6.537442	+8.906331	+11.608553	+14.644108
	ω_3	−9.634525	−15.888455	−24.157173	−34.690679	−47.738973
6	ω_1	−2.141889	−2.641889	−3.141889	−3.641889	−4.141889
	ω_2	+4.552153	+6.598863	+8.978906	+11.692283	+14.738993
	ω_3	−9.722456	−16.035108	−24.370935	−34.979938	−48.112116

Ratios and estimates of adjacent elements. Table 4 presents the values of ratios of the adjacent elements of the sequences $\{a_{n,H}\}$, for $s = 2$. These ratios provide insight into the growth rate of the sequences as n increases.

Estimates and relative errors. Table 5 provides the estimates of the ratios of the adjacent elements $\hat{\rho}_n$, for $s = 2$. The relative error of these approximations, $|1-\hat{\rho}_n/\rho_n|$, is presented in Table 6. These tables help understand the approximations' accuracy and the sequences' behavior.

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TABLE 3. Sequences $\{a_{n,H}\}$, associated with the distribution of admissible prime k-tuples in $(2^n; 2^{n+1}]$ intervals.

n	$(0,2)$	$(0,4)$	$(0,6)$	$(0,2,6)$, $(0,4,6)$	$(0,6,12)$	$(0,2,6,8)$	$(0,6,12,18)$	$(0,2,6,8,12)$, $(0,4,6,10,12)$	$(0,4,6,10,12,16)$
1	1	1	0	0	0	0	0	0	0
2	1	1	2	2	2	1	1	2	1
3	1	1	2	2	1	1	1	1	0
4	2	1	3	1	2	0	0	0	0
5	2	2	5	2	3	0	2	0	0
6	3	6	7	5	3	1	0	2	1
7	7	4	11	4	4	1	1	0	0
8	7	11	17	6	4	0	0	0	0
9	12	15	28	8	9	1	2	0	0
10	26	23	55	20	22	2	5	2	0
11	45	44	79	22	22	3	5	1	0
12	70	64	142	28	38	1	6	1	0
13	113	131	241	49	56	5	9	3	1
14	215	197	434	93	93	7	13	4	1
15	355	359	719	126	134	5	10	3	1
16	666	658	1291	248	255	15	37	3	0
17	1153	1160	2319	427	405	19	37	8	0
18	2071	2071	4171	727	738	44	84	18	0
19	3785	3751	7538	1188	1218	71	127	21	0
20	6965	6820	13569	2121	2072	132	222	46	3
21	12495	12445	24824	3602	3550	174	330	67	4
22	22643	22561	45263	6265	6288	312	589	104	5
23	41608	41670	82579	11028	10866	510	973	148	10
24	76371	76289	152839	19391	19338	800	1603	238	17
25	140944	141009	282165	34418	34280	1420	2826	402	27
26	261752	262183	523710	61909	61711	2398	4817	638	24
27	484968	485670	970747	109704	109838	4173	8398	1071	54
28	904799	904901	1808934	197133	197631	7163	14535	1746	62
29	1689477	1688468	3378523	357691	356480	12745	25318	3076	128
30	3160113	3159596	6318344	644999	645909	22028	44343	5075	209
31	5928904	5930172	11857968	1173390	1175337	39006	78147	8615	331
32	11139071	11135222	22269858	2134316	2134721	68563	137689	14716	564
33	20970782	20966894	41934802	3903603	3904019	121966	243218	25596	943
34	39535081	39542450	79077648	7150609	7148315	216777	433223	43999	1538
35	74697745	74704771	149384651	13123864	13121139	387063	773260	76772	2607
36	141342490	141326344	282669548	24153689	24148320	692082	1383986	133163	4430
37	267812262	267809109	535618852	44554336	44544051	1244485	2485964	233878	7848
38	508194094	508203585	1016436932	82351638	82350169	2238795	4477993	407964	13035
39	965623233	965648995	1931330869	152516401	152508600	4041399	8080760	718976	22370
40	1837147717	1837225037	3674340893	283007306	282970529	7309042	14618433	1269874	38331
41	3499726481	3499687076	6999275265	526130626	526071622	13262623	26534704	2249196	66314
42	6674251373	6674155761	13348507170	979745175	979782916	24120000	48249844	3995303	115580
43	12742529417	12742483159	25484958671	1827631459	1827601133	43958007	87921567	7104747	201046

[8] OEIS Foundation Inc., Entry A095017 in The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A095017, [on-line; accessed 2023-11-23] (2023).

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TABLE 4. Values of ρ_n, the ratio of the adjacent elements of the sequences $\{a_{n,H}\}$, $s = 2$.

n	$(0,2)$	$(0,4)$	$(0,6)$	$(0,2,6)$, $(0,4,6)$	$(0,6,12)$	$(0,2,6,8)$	$(0,6,12,18)$	$(0,2,6,8,12)$, $(0,4,6,10,12)$	$(0,4,6,10,12,16)$
1	1	1	undefined	undefined	undefined	undefined	undefined	undefined	undefined
2	1	1	1	1	0.5	1	1	0.5	0
3	2	1	1.5	0.5	2	0	0	0	undefined
4	1	2	1.66666667	2	1.5	undefined	undefined	undefined	undefined
5	1.5	3	1.4	2.5	1	undefined	0	undefined	undefined
6	2.33333333	0.66666667	1.57142857	0.8	1.33333333	1	undefined	0	0
7	1	2.75	1.54545455	1.5	1	0	0	undefined	undefined
8	1.71428571	1.36363636	1.64705882	1.33333333	2.25	undefined	undefined	undefined	undefined
9	2.16666667	1.53333333	1.96428571	2.5	2.44444444	2	2.5	undefined	undefined
10	1.73076923	1.91304348	1.43636364	1.1	1	1.5	1	0.5	undefined
11	1.55555556	1.45454545	1.79746835	1.27272727	1.72727273	0.33333333	1.2	1	undefined
12	1.61428571	2.04687500	1.69718310	1.75	1.47368421	5	1.5	3	undefined
13	1.90265487	1.50381679	1.80082988	1.89795918	1.66071429	1.4	1.44444444	1.33333333	1
14	1.65116279	1.82233503	1.65668203	1.35483871	1.44086022	0.71428571	0.76923077	0.75	1
15	1.87605634	1.83286908	1.79554937	1.96825397	1.90298507	3	3.7	1	0
16	1.73123123	1.76291793	1.79628195	1.72177419	1.58823529	1.26666667	1	2.66666667	undefined
17	1.79618387	1.78534483	1.79862009	1.70257611	1.82222222	2.31578947	2.27027027	2.25	undefined
18	1.82761951	1.81120232	1.80724047	1.63411279	1.65040650	1.61363636	1.51190476	1.16666667	undefined
19	1.84015852	1.81818182	1.80007960	1.78535354	1.70114943	1.85915493	1.74803150	2.19047619	undefined
20	1.79396985	1.82478006	1.82946422	1.69825554	1.71332046	1.31818182	1.48648649	1.45652174	1.33333333
21	1.81216487	1.81285657	1.82335643	1.73931149	1.77126761	1.79310345	1.78484848	1.55223881	1.25
22	1.83756569	1.84699260	1.82442613	1.76025539	1.72805344	1.63461538	1.65195246	1.42307692	2
23	1.83548837	1.83078954	1.85082164	1.75834240	1.77967973	1.56862745	1.64748201	1.60810811	1.7
24	1.84551728	1.84835297	1.84615838	1.77494714	1.77267556	1.775	1.76294448	1.68907563	1.58823529
25	1.85713475	1.85933522	1.85604168	1.79873903	1.80020420	1.68873239	1.70452937	1.58706468	0.88888889
26	1.85277667	1.85240843	1.85359646	1.77202022	1.77987717	1.74020017	1.74340876	1.67868339	2.25
27	1.86568805	1.86320135	1.86344537	1.79695362	1.79929533	1.71651090	1.73076923	1.63025210	1.14814815
28	1.86724013	1.86591461	1.86768727	1.81446536	1.80376560	1.77928242	1.74186447	1.76174112	2.06451613
29	1.87046820	1.87127976	1.87014977	1.80322960	1.81190810	1.72836406	1.75144166	1.64986996	1.63281250
30	1.87616835	1.87687666	1.87675252	1.81921212	1.81966345	1.77074632	1.76233002	1.69753695	1.58373206
31	1.87877405	1.87772328	1.87805010	1.81893147	1.81626291	1.75775522	1.76192304	1.70818340	1.70392749
32	1.88263294	1.88293453	1.88302961	1.82897144	1.82881932	1.77888949	1.76643014	1.73933134	1.67198582
33	1.88524591	1.88594696	1.88572842	1.83179719	1.83101440	1.77735598	1.78121274	1.71897953	1.63096501
34	1.88940412	1.88922970	1.88908819	1.83534913	1.83555691	1.78553537	1.78490062	1.74485784	1.69505852
35	1.89219219	1.89179810	1.89222618	1.84044036	1.84041340	1.78803451	1.78980679	1.73452561	1.69927119
36	1.89477532	1.89496948	1.89485870	1.84461827	1.84460248	1.79817565	1.79623493	1.75632871	1.77155756
37	1.89757590	1.89763368	1.89768700	1.84834172	1.84873551	1.79897307	1.80131048	1.74434534	1.66093272
38	1.90010715	1.90012236	1.90009907	1.85201418	1.85195248	1.80516707	1.80454949	1.76235158	1.71614883
39	1.90255128	1.90258059	1.90249167	1.85558605	1.85543982	1.80854254	1.80904185	1.76622585	1.71350022
40	1.90497827	1.90487665	1.90490634	1.85907083	1.85910393	1.81455011	1.81515379	1.77119620	1.73003574
41	1.90707800	1.90707215	1.90712705	1.86217096	1.86245157	1.81864477	1.81836752	1.77632496	1.74292005
42	1.90920730	1.90922772	1.90919916	1.86541512	1.86531231	1.82247127	1.82221453	1.77827489	1.73945319

TABLE 5. Estimates $\hat{\rho}_n$, for $s = 2$.

n	$(0,2)$	$(0,4)$	$(0,6)$	$(0,2,6)$, $(0,4,6)$	$(0,6,12)$	$(0,2,6,8)$	$(0,6,12,18)$	$(0,2,6,8,12)$, $(0,4,6,10,12)$	$(0,4,6,10,12,16)$
1	7.96057168	7.96057168	7.96057168	37.6238145	37.6238145	98.1248807	98.1248807	201.750659	362.788039
2	1.10886179	1.10886179	1.10886179	2.15397477	2.15397477	4.84178323	4.84178323	9.81521772	17.8422088
3	1.18495611	1.18495611	1.18495611	1.11100046	1.11100046	1.35467978	1.35467978	2.01830132	3.22886373
4	1.32296696	1.32296696	1.32296696	1.12580523	1.12580523	1.04260340	1.04260340	1.09791965	1.32412462
5	1.42373772	1.42373772	1.42373772	1.21777424	1.21777424	1.07040122	1.07040122	0.98847768	0.98206266
6	1.49803322	1.49803322	1.49803322	1.30300873	1.30300873	1.14507093	1.14507093	1.02598439	0.94905690
7	1.55498728	1.55498728	1.55498728	1.37400142	1.37400142	1.21946688	1.21946688	1.09150316	0.99106274
8	1.60011141	1.60011141	1.60011141	1.43260330	1.43260330	1.28534788	1.28534788	1.15792688	1.05041036
9	1.63680028	1.63680028	1.63680028	1.48141999	1.48141999	1.34225678	1.34225678	1.21874471	1.11062268
10	1.66724954	1.66724954	1.66724954	1.52259376	1.52259376	1.39132254	1.39132254	1.27286456	1.16684852
11	1.69294464	1.69294464	1.69294464	1.55774649	1.55774649	1.43383792	1.43383792	1.32069212	1.21791889
12	1.71492902	1.71492902	1.71492902	1.58809234	1.58809234	1.47093622	1.47093622	1.36299224	1.26388845
13	1.73395862	1.73395862	1.73395862	1.61454735	1.61454735	1.50354549	1.50354549	1.40054295	1.30519963
14	1.75059545	1.75059545	1.75059545	1.63781181	1.63781181	1.53241121	1.53241121	1.43403668	1.34238333
15	1.76526648	1.76526648	1.76526648	1.65842880	1.65842880	1.55813081	1.55813081	1.46406213	1.37595189
16	1.77830223	1.77830223	1.77830223	1.67682533	1.67682533	1.58118518	1.58118518	1.49111150	1.40636453
17	1.78996268	1.78996268	1.78996268	1.69334155	1.69334155	1.60196416	1.60196416	1.51559449	1.43402046
18	1.80045524	1.80045524	1.80045524	1.70825162	1.70825162	1.62078630	1.62078630	1.53785244	1.45926227
19	1.80994747	1.80994747	1.80994747	1.72177893	1.72177893	1.63791402	1.63791402	1.55817078	1.48238260
20	1.81857623	1.81857623	1.81857623	1.73410731	1.73410731	1.65356525	1.65356525	1.57678936	1.50363141
21	1.82645445	1.82645445	1.82645445	1.74538939	1.74538939	1.66792234	1.66792234	1.59391080	1.52322255
22	1.83367614	1.83367614	1.83367614	1.75575297	1.75575297	1.68113895	1.68113895	1.60970725	1.54133955
23	1.84032021	1.84032021	1.84032021	1.76530583	1.76530583	1.69334545	1.69334545	1.62432575	1.55814054
24	1.84645337	1.84645337	1.84645337	1.77413953	1.77413953	1.70465314	1.70465314	1.63789258	1.57376227
25	1.85213245	1.85213245	1.85213245	1.78233235	1.78233235	1.71515758	1.71515758	1.65051674	1.58832351
26	1.85740613	1.85740613	1.85740613	1.78995159	1.78995159	1.72494131	1.72494131	1.66229277	1.60192782
27	1.86231635	1.86231635	1.86231635	1.79705545	1.79705545	1.73407593	1.73407593	1.67330305	1.61466585
28	1.86689948	1.86689948	1.86689948	1.80369453	1.80369453	1.74262389	1.74262389	1.68361968	1.62661730
29	1.87118716	1.87118716	1.87118716	1.80991299	1.80991299	1.75063988	1.75063988	1.69330602	1.63785243
30	1.87520709	1.87520709	1.87520709	1.81574959	1.81574959	1.75817203	1.75817203	1.70241797	1.64843346
31	1.87898361	1.87898361	1.87898361	1.82123844	1.82123844	1.76526281	1.76526281	1.71100504	1.65841564
32	1.88253818	1.88253818	1.88253818	1.82640974	1.82640974	1.77194989	1.77194989	1.71911122	1.66784822
33	1.88588982	1.88588982	1.88588982	1.83129026	1.83129026	1.77826676	1.77826676	1.72677572	1.67677522
34	1.88905544	1.88905544	1.88905544	1.83590386	1.83590386	1.78424332	1.78424332	1.73403362	1.68523608
35	1.89205012	1.89205012	1.89205012	1.84027186	1.84027186	1.78990632	1.78990632	1.74091640	1.69326628
36	1.89488733	1.89488733	1.89488733	1.84441335	1.84441335	1.79527981	1.79527981	1.74745235	1.70089780
37	1.89757918	1.89757918	1.89757918	1.84834552	1.84834552	1.80038542	1.80038542	1.75366702	1.70815953
38	1.90013658	1.90013658	1.90013658	1.85208383	1.85208383	1.80524267	1.80524267	1.75958349	1.71507766
39	1.90256935	1.90256935	1.90256935	1.85564229	1.85564229	1.80986926	1.80986926	1.76522271	1.72167596
40	1.90488638	1.90488638	1.90488638	1.85903357	1.85903357	1.81428122	1.81428122	1.77060368	1.72797606
41	1.90709575	1.90709575	1.90709575	1.86226917	1.86226917	1.81849316	1.81849316	1.77574375	1.73399768
42	1.90920479	1.90920479	1.90920479	1.86535958	1.86535958	1.82251835	1.82251835	1.78065870	1.73975887

TABLE 6. Relative error of the approximation, $|1 - \hat{\rho}_n/\rho_n|$, for $s = 2$.

n	$(0,2)$	$(0,4)$	$(0,6)$	$(0,2,6)$, $(0,4,6)$	$(0,6,12)$	$(0,2,6,8)$	$(0,6,12,18)$	$(0,2,6,8,12)$, $(0,4,6,10,12)$	$(0,4,6,10,12,16)$
1	6.96057168	6.96057168	undefined	undefined	undefined	undefined	undefined	undefined	undefined
2	0.10886179	0.10886179	0.10886179	1.15397477	3.30794954	3.84178323	3.84178323	18.6304354	undefined
3	0.40752194	0.18495611	0.21002926	1.22200093	0.44449977	undefined	undefined	undefined	undefined
4	0.32296696	0.33851652	0.20621982	0.43709739	0.24946318	undefined	undefined	undefined	undefined
5	0.05084152	0.52542076	0.01695552	0.51289030	0.21777424	undefined	undefined	undefined	undefined
6	0.35798576	1.24704983	0.04670613	0.62876092	0.02274345	0.14507093	undefined	undefined	undefined
7	0.55498728	0.43455008	0.00616824	0.08399906	0.37400142	undefined	undefined	undefined	undefined
8	0.06660168	0.17341503	0.02850379	0.07445248	0.36328742	undefined	undefined	undefined	undefined
9	0.24455372	0.06747844	0.16671986	0.40743201	0.39396455	0.32887161	0.46309729	undefined	undefined
10	0.03670027	0.12848320	0.16074335	0.38417614	0.52259376	0.07245164	0.39132254	1.54572913	undefined
11	0.08832156	0.16389944	0.05815052	0.22394367	0.09814677	3.30151375	0.19486493	0.32069212	undefined
12	0.06234541	0.16217208	0.01045610	0.09251866	0.07763409	0.70581276	0.01937586	0.54566925	undefined
13	0.08866361	0.15303847	0.03713358	0.14932452	0.02779945	0.07396106	0.04091611	0.05040721	0.30519963
14	0.06021978	0.03936684	0.05668765	0.20886110	0.13669028	1.14537569	0.99213457	0.91204891	0.34238333
15	0.05905465	0.03688349	0.01686553	0.15741118	0.12851193	0.48062306	0.57888357	0.46406213	undefined
16	0.02718932	0.00872661	0.01000941	0.02610613	0.05577891	0.24830409	0.58118518	0.44083319	undefined
17	0.00346356	0.00258653	0.00481337	0.00542388	0.07072720	0.30824275	0.29437293	0.32640245	undefined
18	0.01486320	0.00593367	0.00375447	0.04536947	0.03504901	0.00443095	0.07201613	0.31815924	undefined
19	0.01641764	0.00452889	0.00548191	0.03560897	0.01212680	0.11900079	0.06299513	0.28866116	undefined
20	0.01371616	0.00339977	0.00595146	0.02111094	0.01213249	0.25442881	0.11239844	0.08257180	0.12772356
21	0.00788537	0.00750081	0.00169908	0.00349443	0.01461000	0.06981254	0.06551040	0.02684638	0.21857804
22	0.00211669	0.00720980	0.00507009	0.00255782	0.01602933	0.02846148	0.01766788	0.13114563	0.22933022
23	0.00263245	0.00520577	0.00567393	0.00396022	0.00807668	0.07950772	0.02783851	0.01008492	0.08344674
24	0.00050723	0.00102773	0.00015979	0.00045501	0.00082585	0.03963204	0.03306476	0.03030240	0.00911264
25	0.00269356	0.00387384	0.00210622	0.00912122	0.00992768	0.01564794	0.00623528	0.03998077	0.78686395
26	0.00249866	0.00269794	0.00205529	0.01011916	0.00566018	0.00876845	0.01059273	0.00976397	0.28803208
27	0.00180721	0.00047499	0.00060587	0.00005667	0.00124486	0.01023298	0.00191054	0.02640754	0.40632187
28	0.00018243	0.00052782	0.00042180	0.00593609	0.00003940	0.02060299	0.00043598	0.04434332	0.21210725
29	0.00038437	0.00004949	0.00055471	0.00370635	0.00110111	0.01288838	0.00045778	0.02632696	0.00308665
30	0.00051235	0.00088955	0.00082346	0.00190331	0.00215087	0.00710113	0.00235937	0.00287536	0.04085375
31	0.00011154	0.00067120	0.00049706	0.00126831	0.00273943	0.00427113	0.00189552	0.00165184	0.02670997
32	0.00005034	0.00021050	0.00026098	0.00140062	0.00131756	0.00390109	0.00312480	0.01162523	0.00247466
33	0.00034155	0.00003030	0.00008559	0.00027674	0.00015066	0.00051243	0.00165392	0.00453536	0.02808779
34	0.00018454	0.00009224	0.00001733	0.00030225	0.00018902	0.00072362	0.00036826	0.00620349	0.00579475
35	0.00007509	0.00013321	0.00009305	0.00009156	0.00007691	0.00104686	0.00005561	0.00368446	0.00353382
36	0.00005911	0.00004335	0.00001511	0.00011109	0.00010253	0.00161043	0.00053174	0.00505393	0.03988567
37	0.00000173	0.00002872	0.00005681	0.00000205	0.00021095	0.00078508	0.00051355	0.00534394	0.02843391
38	0.00001549	0.00000748	0.00001974	0.00003761	0.00007093	0.00004188	0.00038413	0.00157068	0.00062417
39	0.00000950	0.00000591	0.00004083	0.00003031	0.00010912	0.00073359	0.00045738	0.00056796	0.00477137
40	0.00004824	0.00000511	0.00001048	0.00002004	0.00003785	0.00014819	0.00048071	0.00033453	0.00119054
41	0.00000931	0.00001237	0.00001641	0.00005274	0.00009793	0.00008337	0.00006909	0.00032720	0.00511920
42	0.00000131	0.00001201	0.00000295	0.00002977	0.00002534	0.00002583	0.00016673	0.00134052	0.00017573