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Set-valued integration in seminorm. I

Anca Croitoru and Christiane Godet-Thobie

Abstract. The purpose of this paper is to define an integral for multifunctions with respect to an additive multimeasure. If $\mathcal{P}_k(X)$ is the family of nonempty compact subsets of a locally convex algebra X, then both the multifunction and the multimeasure take values in a subset $\widetilde{X}$ of $\mathcal{P}_k(X)$ which satisfies certain conditions. This integral is weaker than that introduced in Croitoru [6] and contains, as particular cases, the integrals defined in Sambucini [11] and Croitoru [5]. The method used for integration is an extension of that introduced by Blondia [1].

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1. Preliminaries

Let S be a nonempty set, $\mathcal{A}$ an algebra of subsets of S. Let X be a Hausdorff locally convex vector space and let Q be a filtering family of seminorms which defines the topology of X. We consider $(x,y) \mapsto xy$ having the following properties for every $x,y,z \in X$, $\alpha,\beta \in \mathbb{R}$, $p \in Q$:

(i) $x(yz) = (xy)z$,
(ii) $xy = yx$,
(iii) $x(y+z) = xy + xz$,
(iv) $(\alpha x)(\beta y) = (\alpha\beta)(xy)$,
(v) $p(xy) \leq p(x)p(y)$.

Example 1.1. We can give some examples:

(a) $X = \{f \mid f : T \to \mathbb{R}$ *is bounded*$\}$ *where* T *is a topological space.*
Let $\mathcal{K} = \{K \subset T | K$ *is compact*$\}$ *and* $Q = \{p_K | K \in \mathcal{K}\}$ *where, for every* $f \in X$, $p_K(f) = \sup_{t\in K} |f(t)|$.
(b) $X = \{f \mid f : T \to \mathbb{R}\}$ *where* T *is a nonempty set.*
Let $Q = \{p_t | t \in T\}$ *where* $p_t(f) = |f(t)|$, *for every* $f \in X$.

We denote by $\mathcal{P}_k(X)$ or $\mathcal{P}_k$, if there is no ambiguity, the family of all nonempty compact subsets of X. If $A, B \in \mathcal{P}_k$ and $\alpha \in \mathbb{R}$, then:

$$A + B = \{x + y | x \in A, y \in B\},$$
$$\alpha A = \{\alpha x | x \in A\},$$
$$A \cdot B = \{xy | x \in A, y \in B\}.$$

For every $p \in Q$ and every $A, B \in \mathcal{P}_k$, let $e_p(A,B) = \sup_{x\in A} \inf_{y\in B} p(x-y)$ be the p-excess of A over B and let $h_p(A,B) = \max\{e_p(A,B), e_p(B,A)\}$ be the Hausdorff - Pompeiu semimetric defined by p on $\mathcal{P}_k$. If we note $O = \{0\}$, we define, for every $A \in \mathcal{P}_k$,

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$\|A\|_p = h_p(A, O) = \sup_{x\in A} p(x)$. Then $\{h_p\}_{p\in Q}$ is a filtering family of semimetrics on $\mathcal{P}_k$ which defines a Hausdorff topology on $\mathcal{P}_k$.

Let $\widetilde{X} \subset \mathcal{P}_k$ satisfy the following conditions:

(x_1) $\widetilde{X}$ is complete with respect to $\{h_p\}_{p\in Q}$,

(x_2) $O \in \widetilde{X}$,

(x_3) $A + B, A \cdot B \in \widetilde{X}$ for every $A, B \in \widetilde{X}$,

(x_4) $A \cdot (B + C) = A \cdot B + A \cdot C$ for every $A, B, C \in \widetilde{X}$.

Example 1.2. *We can give some examples:*

(a) $\widetilde{X} = \{\{f\} | f \in X\}$ *for* X *like in examples (a) and (b) of 1.1.*

(b) $\widetilde{X} = \{A \mid A \subset [0, +\infty), A$ *is nonempty compact convex*$\}$ *for* $X = \mathbb{R}$.

(c) $\widetilde{X} = \{[f, g] \mid f, g \in X, 0 \leq f \leq g\}$ *for* X *like in example 1.1-(b), where* $[f, g] = \{u \in X \mid f \leq u \leq g\}$ *and* $[f, f] = \{f\}$.

Definition 1.1. $\varphi : \mathcal{A} \to \mathcal{P}_k$ *is said to be an* **additive multimeasure** *if:*

(i) $$\varphi(\emptyset) = 0,$$

(ii) $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ *for all* A *and* $B \in \mathcal{A}$, *such that* $A \cap B = \emptyset$.

Definition 1.2. *Let* $\varphi : \mathcal{A} \to \mathcal{P}_k$. *For every* $p \in Q$, *the* p**-variation** *of* φ *is the non-negative (possibly infinite) set function* $v_p(\varphi, \cdot)$ *defined on* $\mathcal{A}$ *as follows:*

$$v_p(\varphi, A) = \sup\left\{\sum_{i=1}^{n} \|\varphi(E_i)\|_p \;\middle|\; (E_i)_{i=1}^{n} \subset \mathcal{A}, E_i \cap E_j = \emptyset \text{ for } i \neq j, \bigcup_{i=1}^{n} E_i = A, n \in \mathbb{N}^*\right\}, \text{for every } A \in \mathcal{A}.$$

Such a family $(E_i)_{i=1}^{n}$ is called an $\mathcal{A}-$partition of E.

We denote $v_p(\varphi, \cdot)$ by ν_p if there is no ambiguity.

We say that φ is *with bounded* $p-$*variation* or $p-$*variation-bounded* iff ν_p is bounded for every $p \in Q$.

If φ is an additive multimeasure, then ν_p is finitely additive for every $p \in Q$.

In the sequel, $\varphi : \mathcal{A} \to \widetilde{X}$ is an additive and $p-$variation-bounded multimeasure.

2. Integration in seminorm

In this part, we define the notions of measurability and integrability which are weaker than those introduced in [6].

Definition 2.1. *A multifunction* $F : S \to \widetilde{X}$ *is said to be a* **simple multifunction** *if* $F = \sum_{i=1}^{n} B_i \cdot \chi_{A_i}$, *where* $B_i \in \widetilde{X}$, $A_i \in \mathcal{A}, i \in \{1, 2, ..., n\}$, $A_i \cap A_j = \emptyset$ *for* $i \neq j$, $\bigcup_{i=1}^{n} A_i = S$ *and* χ_{A_i} *is the characteristic function of* A_i.

The integral of F *over* $E \in \mathcal{A}$ *with respect to* φ *is:*

$$\int_E F d\varphi = \sum_{i=1}^{n} B_i \cdot \varphi(A_i \cap E) \in \widetilde{X}.$$

It is easy to see that the integral is correctly defined, thanks to conditions $(x_3), (x_4)$ and to obtain the following theorem.

Theorem 2.1. *Let F and $G : S \to \widetilde{X}$ be simple multifunctions, $\alpha \in \mathbb{R}$ and, for every $E \in \mathcal{A}$, $\Gamma(E) = \int_E F d\varphi$. Then we have:*
(a) Γ is a multimeasure .
(b) $h_p\left(\int_E F d\varphi, \int_E G d\varphi\right) \leq \int_E h_p(F,G) d\nu_p$, for every $E \in \mathcal{A}$ and every $p \in Q$.
(c) $\left\| \int_E F d\varphi \right\|_p \leq \int_E \|F\|_p d\nu_p$, for every $E \in \mathcal{A}$ and every $p \in Q$.
(d) $v_p(\Gamma, E) = \int_E \|F\|_p d\nu_p$, for every $E \in \mathcal{A}$ and every $p \in Q$.
(e) αF is simple and $\int_E (\alpha F) d\varphi = \alpha \int_E F d\varphi$, for every $E \in \mathcal{A}$.
(f) $F + G$ is simple and $\int_E (F+G) d\varphi = \int_E F d\varphi + \int_E G d\varphi$, for every $E \in \mathcal{A}$.

Before defining the notion of $\varphi-$integrability in seminorm, we precise the φ-totally measurability in seminorm.

Definition 2.2. *A multifunction $F : S \to \widetilde{X}$ is said to be* φ**-totally measurable in seminorm** *if, for every $p \in Q$, there is a sequence $(F_n^p)_n$ of simple multifunctions $F_n^p : S \to \widetilde{X}$ such that $h_p(F_n, F)$ converges to 0 in ν_p-measure (that is denoted by $h_p(F_n, F) \xrightarrow{\nu_p} 0$).*

Remark 2.1. .
(a) It is obvious that every simple multifunction, every φ-totally measurable multifunction are φ-totally measurable in seminorm.
(b) Let $F : S \to \widetilde{X}$ be a multifunction, $p \in Q$ and a sequence $(F_n^p)_n$ of simple multifunctions $F_n^p : S \to \widetilde{X}$ such that $h_p(F_n^p, F) \xrightarrow{\nu_p} 0$. Then, for every $n \in \mathbb{N}$, $h_p(F_n^p, F)$ and $\|F\|_p$ are ν_p-measurable.

Particularly, if F is $\varphi-$totally measurable in seminorm, then, for every $n \in \mathbb{N}$, $h_p(F_n^p, F)$ and $\|F\|_p$ are ν_p-measurable for every $p \in Q$.

Proof. Let us fix $m \in \mathbb{N}$. From the inequality:

$$|h_p(F_m^p, F_n^p) - h_p(F_m^p, F)| \leq h_p(F_n^p, F)$$

and (i), it follows that $h_p(F_m^p, F_n^p) \xrightarrow[n\to\infty]{\nu_p} h_p(F_m^p, F)$. Since the functions $h_p(F_m^p, F_n^p)$ are ν_p-simple, according to Definition III.2.10 of [8], $h_p(F_m^p, F)$ is ν_p-measurable.

Since $|\|F_n^p\|_p - \|F\|_p| \leq h_p(F_n^p, F)$ and (i), it results: $\|F_n^p\|_p \xrightarrow{\nu_p} \|F\|_p$. But the functions $\|F_n^p\|_p$ are ν_p-simple, so by Definition III.2.10 of [8], $\|F\|_p$ is ν_p-measurable. □

Theorem 2.2. *If F and $G : S \to \widetilde{X}$ are $\varphi-$totally measurable in seminorm, then, for every $p \in Q$, $h_p(F,G)$ is ν_p-measurable.*

Proof. Since F and G are $\varphi-$totally measurable in seminorm, for every $p \in Q$, there exist $(F_n^p)_n, (G_n^p)_n$ sequences of simple multifunctions such that:

(i) $$h_p(F_n^p, F) \xrightarrow{\nu_p} 0, \quad h_p(G_n^p, G) \xrightarrow{\nu_p} 0.$$

Since
$$h_p(F_n^p, G_n^p) \le h_p(F_n^p, F) + h_p(F, G) + h_p(G, G_n^p) \text{ and}$$
$$h_p(F, G) \le h_p(F, F_n^p) + h_p(F_n^p, G_n^p) + h_p(G_n^p, G),$$
we have the inequality:
$$|h_p(F_n^p, G_n^p) - h_p(F, G)| \le h_p(F_n^p, F) + h_p(G_n^p, G), \quad \forall n \in \mathbb{N}.$$
Now, from (i) and the last inequality, it follows:
$$h_p(F_n^p, G_n^p) \xrightarrow{\nu_p} h_p(F, G).$$
But $h_p(F_n^p, G_n^p)$ are simple functions, so $h_p(F, G)$ is ν_p-measurable. □

Theorem 2.3. *Let $F, G : S \to \widetilde{X}$ be φ-totally measurable in seminorm multifunctions and let $\alpha \in \mathbb{R}$. Then $F + G$ and αF are φ-totally measurable in seminorm.*

Proof. For every $p \in Q$, let $(F_n^p)_n$ and $(G_n^p)_n$ be sequences of simple multifunctions satisfying:

(i) $$h_p(F_n^p, F) \xrightarrow{\nu_p} 0, \quad h_p(G_n^p, G) \xrightarrow{\nu_p} 0.$$

From the relations:
$$h_p(F_n^p + G_n^p, F + G) \le h_p(F_n^p, F) + h_p(G_n^p, G),$$
$$h_p(\alpha F_n^p, \alpha F) = |\alpha| h_p(F_n^p, F)$$
and from (i), it results that the sequences $(F_n^p + G_n^p)_n$ and $(\alpha F_n^p)_n$ of simple multifunctions satisfy the following condition for every $p \in Q$:
$$h_p(F_n^p + G_n^p, F + G) \xrightarrow{\nu_p} 0, \quad h_p(\alpha F_n^p, \alpha F) \xrightarrow{\nu_p} 0,$$
which shows that $F + G$ and αF are φ-totally measurable in seminorm. □

Definition 2.3. *A multifunction $F : S \to \widetilde{X}$ is said to be $\varphi-$***integrable in seminorm** *if for every $p \in Q$, there exists a sequence $(F_n^p)_n$ of simple multifunctions, $F_n^p : S \to \widetilde{X}$, satisfying the following conditions:*

(i) $h_p(F_n^p, F) \xrightarrow{\nu_p} 0$ *(that is: F is φ-totally measurable in seminorm),*

(ii) $h_p(F_n^p, F)$ *is ν_p-integrable, for every $n \in \mathbb{N}$,*

(iii) $\lim\limits_{n\to\infty} \int_E h_p(F_n^p, F)\, d\nu_p = 0$, *for every $E \in \mathcal{A}$,*

(iv) *For every $E \in \mathcal{A}$, there exists $I_E \in \widetilde{X}$ such that, for every $p \in Q$,*
$$\lim_{n\to\infty} h_p\left(\int_E F_n^p\, d\varphi, I_E\right) = 0.$$

*The sequence $(F_n^p)_n$ is said to be a $p-$***defining sequence** *for F.*

We denote $I_E = \int_E F\, d\varphi$ and we call it the integral of F over E with respect to φ.

This definition will be justified by next Theorem 2.5.

It is clear that (iii) is equivalent to $\lim\limits_{n\to\infty} \int_S h_p(F_n^p, F)\, d\nu_p = 0$.

Now, we show that we can replace (ii) and (iii) by the following condition (*).

Theorem 2.4. *A multifunction $F : S \to \widetilde{X}$ is $\varphi-$integrable in seminorm if and only if for every $p \in Q$, there exists a sequence $(F_n^p)_n$ of simple multifunctions $F_n^p : S \to \widetilde{X}$ which satisfy the three following conditions:*

(i) $h_p(F_n^p, F) \xrightarrow{\nu_p} 0$;

(*) $\lim\limits_{n,m\to\infty} \int_S h_p(F_n^p, F_m^p) d\nu_p = 0$;

(iv) *For every* $E \in \mathcal{A}$, *there exists* $I_E \in \widetilde{X}$ *such that, for every* $p \in Q$,
$\lim\limits_{n\to\infty} h_p\left(\int_E F_n^p d\varphi, I_E\right) = 0.$

Proof. Let us suppose F is $\varphi-$integrable in seminorm. Then, for every $p \in Q$, there exists a sequence $(F_n^p)_n$ of simple multifunctions $F_n^p : S \to \widetilde{X}$, satisfying the conditions (i)-(iv) of Definition 2.3. We have to prove only the relation (*).

We have for every $n, m \in \mathbb{N}$:

$$h_p(F_n^p, F_m^p) \le h_p(F_n^p, F) + h_p(F, F_m^p) \Rightarrow \int_S h_p(F_n^p, F_m^p) d\nu_p \le$$

$$\le \int_S h_p(F_n^p, F) d\nu_p + \int_S h_p(F, F_m^p) d\nu_P$$

and from (iii) it follows (*).

Conversely, suppose for every $p \in Q$, there exists a sequence $(F_n^p)_n$ of simple multifunctions $F_n^p : S \to \widetilde{X}$, satisfying the conditions (i), (*) and (iv). We have to prove (ii) and (iii) of Definition 2.3. Let's fix $m \in \mathbb{N}$.

Since $|h_p(F_m^p, F_n^p) - h_p(F_m^p, F)| \le h_p(F_n^p, F)$ and (i),

$$h_p(F_m^p, F_n^p) \xrightarrow[n\to\infty]{\nu_p} h_p(F_m^p, F).$$

Also, from the inequality $\int_S |h_p(F_m^p, F_n) - h_p(F_m^p, F_l^p)| d\nu_p \le \int_S h_p(F_n^p, F_l^p) d\nu_p$ and (*), it results $\lim\limits_{n,l\to\infty} \int_S |h_p(F_m^p, F_n^p) - h_p(F_m^p, F_l^p)| d\nu_p = 0$. Since Definition III.2.17 of [8], $h_p(F_m^p, F)$ is ν_p-integrable and

(1) $\lim\limits_{n\to\infty} \int_S h_p(F_m^p, F_n^p) d\nu_p = \int_S h_p(F_m^p, F) d\nu_p$.

Let $\varepsilon > 0$. By (*), there is $n_0(\varepsilon) = n_0 \in \mathbb{N}$ such that:

(2) $\int_S h_p(F_m^p, F_n^p) d\nu_p < \varepsilon$, $\forall m, n \ge n_0$.

Let $n \ge n_0$. Since (1), $\lim\limits_{k\to\infty} \int_S h_p(F_n^p, F_k^p) d\nu_p = \int_S h_p(F_n^p, F) d\nu_p$.

Then, there exists $n_1(n) \in \mathbb{N}$ such that:

(3) $\left|\int_S h_p(F_n^p, F_k^p) d\nu_p - \int_S h_p(F_n^p, F) d\nu_p\right| < \varepsilon$, $\forall k \ge n_1(n)$.

Choosing $k \ge \max\{n_0, n_1(n)\}$, from (2) and (3) we have:

$$\int_S h_p(F_n^p, F) d\nu_p \le \left|\int_S h_p(F_n^p, F) d\nu_p - \int_S h_p(F_n^p, F_k^p) d\nu_p\right| + \int_S h_p(F_n^p, F_k^p) d\nu_p < 2\varepsilon$$

that is $\lim\limits_{n\to\infty} \int_S h_p(F_n^p, F) d\nu_p = 0$. □

Remark 2.2. *(a) Every simple multifunction is φ-integrable in seminorm. Every φ-integrable multifunction* (*see* [6]) *is φ-integrable in seminorm.*

(b) If X is a real Banach algebra, then we obtain the integral defined in [5].
(c) If $X = \mathbb{R}, \widetilde{X} = \{A \subset [0,+\infty) \mid A$ is a non-empty compact convex set$\}$ and $\varphi = \{\mu\}$ (where μ is finitely additive), then we get the integral, defined in [11], of the multifunction F with respect to μ.

The next result shows that the integral $I_E = \int_E F d\varphi$, for every $E \in \mathcal{A}$, given in Definition 2.3 is correctly defined.

Theorem 2.5. *Let $F : S \to \widetilde{X}$ be a $\varphi-$integrable in seminorm multifunction with two $p-$defining sequences $(F_n^p)_n$, $(G_n^p)_n$ and for every $E \in \mathcal{A}$, let $I'_E, I''_E \in \widetilde{X}$ such that $\lim_{n\to\infty} h_p\left(\int_E F_n^p d\varphi, I'_E\right) = 0 = \lim_{n\to\infty} h_p\left(\int_E G_n^p d\varphi, I''_E\right)$ for every $p \in Q$. Then $I'_E = I''_E$.*

Proof. From Theorem 2.1-(b) we have:

$$h_p(I'_E, I''_E) \le h_p\left(I'_E, \int_E F_n^p d\varphi\right) +$$
$$+ \int_E h_p(F_n^p, F) d\nu_p + \int_E h_p(F, G_n^p) d\nu_p + h_p\left(\int_E G_n^p d\varphi, I''_E\right)$$

for every $n \in \mathbb{N}, p \in Q$. Since (iii) and (iv) of Definition 2.3, $I'_E = I''_E$. □

In the following theorems, we show that integrals of $\varphi-$integrable in seminorm multifunctions have also properties (a)-(f) of Theorem 2.1 already given for simple multifunctions.

Theorem 2.6. *If $F : S \to \widetilde{X}$ is $\varphi-$integrable in seminorm and $(F_n^p)_n$ is a $p-$defining sequence for F, then, for every $p \in Q$, the scalar function $\|F\|_p$ is ν_p-integrable and $(\|F_n^p\|_p)_n$ is a defining sequence for $\|F\|_p$ and, for every $E \in \mathcal{A}$,*

$$\int_E \|F\|_p d\nu_p = \lim_{n\to\infty} \int_E \|F_n^p\|_p d\nu_p.$$

Proof. If $(F_n^p)_n$ is a $p-$defining sequence for F, then the sequence $(\|F_n^p\|_p)_n$, of ν_p-integrable simple functions, satisfies the conditions

$$\|F_n^p\|_p \xrightarrow{\nu_p} \|F\|_p$$

and

$$\lim_{n,m\to\infty} \int_S \left| \|F_n^p\|_p - \|F_m^p\|_p \right| d\nu_p = 0.$$

It follows that $\|F\|_p$ is ν_p-integrable and moreover,

$$\int_E \|F\|_p d\nu_p = \lim_{n\to\infty} \int_E \|F_n^p\|_p d\nu_p, \forall E \in \mathcal{A}.$$

□

Theorem 2.7. *Let $F, G : S \to \widetilde{X}$ be $\varphi-$integrable in seminorm multifunctions. Then, for every $p \in Q$ and every $E \in \mathcal{A}$, we have:*

(a) $h_p\left(\int_E F d\varphi, \int_E G d\varphi\right) \le \int_E h_p(F, G) d\nu_p,$

(b) $\| \int_E F d\varphi \|_p \le \int_E \|F\|_p d\nu_p.$

Proof. (a) Let $(F_n^p)_n, (G_n^p)_n$ be $p-$defining sequences for F, G respectively. Then, for every $n \in \mathbb{N}$ we have:

(4) $$h_p\left(\int_E Fd\varphi, \int_E Gd\varphi\right) \leq h_p\left(\int_E Fd\varphi, \int_E F_n^p d\varphi\right) + h_p\left(\int_E F_n^p d\varphi, \int_E G_n^p d\varphi\right) + h_p\left(\int_E G_n^p d\varphi, \int_E Gd\varphi\right) \leq h_p\left(\int_E Fd\varphi, \int_E F_n^p d\varphi\right) + \int_E h_p(F_n^p, G_n^p)d\nu_p + h_p\left(\int_E G_n^p d\varphi, \int_E Gd\varphi\right).$$

Since

(5) $|h_p(F_n^p, G_n^p) - h_p(F, G)| \leq h_p(F_n^p, F) + h_p(G_n^p, G)$, for each $n \in \mathbb{N}$

and (i) of Definition 2.3, $h_p(F_n^p, G_n^p) \xrightarrow{\nu_p} h_p(F, G)$, so the function $h_p(F, G)$ is ν_p-measurable. From the inequality

$$h_p(F, G) \leq h_p(F, F_n^p) + h_p(F_n^p, G_n^p) + h_p(G_n^p, G)$$

and (ii) of Definition 2.3, it follows that $h_p(F, G)$ is ν_p-integrable.

From (5) we have:

(6) $$\lim_{n\to\infty} \int_E h_p(F_n^p, G_n^p)d\nu_p = \int_E h_p(F, G)d\nu_p.$$

Using (iv) of Definition 2.3 and (6), we make $n \to \infty$ in (4) and obtain:

$$h_p\left(\int_E Fd\varphi, \int_E Gd\varphi\right) \leq \int_E h_p(F, G)d\nu_p.$$

(b) It follows from (a) with $G(s) = O$, for every $s \in S$. □

Theorem 2.8. *Let $F : S \to \widetilde{X}$ be a $\varphi-$integrable in seminorm multifunction and let $\Gamma(E) = \int_E Fd\varphi$, for every $E \in \mathcal{A}$. Then Γ has the following properties:*

(a) Γ is a multimeasure.

(b) $v_p(\Gamma, E) = \int_E \|F\|_p d\nu_p$, for every $E \in \mathcal{A}$.

(c) $\Gamma \ll \varphi$ (i.e. for every $p \in Q$ and $\varepsilon > 0$, there exists $\delta(p, \varepsilon) = \delta > 0$ such that $v_p(\Gamma, E) < \varepsilon$ for all $E \in \mathcal{A}$ satisfying $\nu_p(E) < \delta$).

Proof. (a) Let $p \in Q$ and $E_1, E_2 \in \mathcal{A}$ such that $E_1 \cap E_2 = \emptyset$. Let $(F_n^p)_n$ be a $p-$defining sequence for F. It follows from Theorem 2.1-(a) that

$$h_p\left(\int_{E_1\cup E_2} F_n^p d\varphi, \int_{E_1} Fd\varphi + \int_{E_2} Fd\varphi\right) \leq h_p\left(\int_{E_1} F_n^p d\varphi, \int_{E_1} Fd\varphi\right) + h_p\left(\int_{E_2} F_n^p d\varphi, \int_{E_2} Fd\varphi\right)$$

and the statement is proved.

(b) Let $(F_n^p)_n$ be a $p-$defining sequence for F and $\Gamma_n^p(E) = \int_E F_n^p d\varphi$, $E \in \mathcal{A}$, $n \in \mathbb{N}$.

According to Theorem 2.6 and Theorem 2.1-(d),

(7) $$\int_E \|F\|_p d\nu_p = \lim_{n\to\infty} \int_E \|F_n^p\|_p d\nu_p = \lim_{n\to\infty} v_p(\Gamma_n^p, E).$$

Let $\varepsilon > 0$. Thanks to property (iii) of Definition 2.3, there is $n_0 \in \mathbb{N}$ such that $\int_E h_p(F_n^p, F)d\nu_p < \varepsilon$, for every $n \geq n_0$. Now, we can choose $(A_i)_{i=1}^n$ and $(B_j)_{j=1}^q$ two

$\mathcal{A}-$partitions of E such that:

$$v_p(\Gamma, E) - \sum_{i=1}^{m} \|\Gamma(A_i)\|_p < \varepsilon$$

and

$$v_p(\Gamma_n^p, E) - \sum_{j=1}^{q} \|\Gamma_n^p(B_j)\|_p < \varepsilon.$$

Let $(E_k)_{k=1}^{l}$ be the $\mathcal{A}-$partition of E consisting of $(A_i \cap B_j)_{\substack{i=\overline{1,m}\\ j=\overline{1,q}}}$. Then we have:

$$\left| v_p(\Gamma_n^p, E) - v_p(\Gamma, E) \right| \leq \left| v_p(\Gamma_n^p, E) - \sum_{k=1}^{l} \|\Gamma_n^p(E_k)\|_p \right| +$$

$$+ \left| \sum_{k=1}^{l} \|\Gamma_n^p(E_k)\|_p - \sum_{k=1}^{l} \|\Gamma(E_k)\|_p \right| +$$

$$+ \left| \sum_{k=1}^{l} \|\Gamma(E_k)\|_p - v_p(\Gamma, E) \right| < \varepsilon + \sum_{k=1}^{l} \int_{E_k} h_p(F_n^p, F) d\nu_p + \varepsilon < 3\varepsilon.$$

So, $v_p(\Gamma, E) = \lim_{n\to\infty} v_p(\Gamma_n^p, E)$ and since (7), $v_p(\Gamma, E) = \displaystyle\int_E \|F\|_p d\nu_p$.

(c) It results straightforward from Theorem 2.6 and (b). □

Theorem 2.9. *Let $F, G : S \to \widetilde{X}$ be φ-integrable in seminorm multifunctions and $\alpha \in \mathbb{R}$. Then $F + G, \alpha F$ are φ-integrable in seminorm and for all $E \in \mathcal{A}$:*

$$\int_E (F+G) d\varphi = \int_E F f d\varphi + \int_E G d\varphi,$$

$$\int_E (\alpha F) d\varphi = \alpha \int_E F d\varphi.$$

Proof. Since the multifunctions F and G are φ-integrable in seminorm and from Theorem 2.4, for every $p \in Q$, there exist $(F_n^p)_n$ and $(G_n^p)_n$ two sequences of simple multifunctions $F_n^p, G_n^p : S \to \widetilde{X}$, satisfying the conditions:

$$h_p(F_n^p, F) \xrightarrow{\nu_p} 0, \ \ h_p(G_n^p, G) \xrightarrow{\nu_p} 0, \tag{8}$$

$$\lim_{n,m\to\infty} \int_S h_p(F_n^p, F_m^p) d\nu_p = 0 = \lim_{n,m\to\infty} \int_S h_p(G_n^p, G_m^p) d\nu_p, \tag{9}$$

(10)

$$\lim_{n\to\infty} h_p\left(\int_E F_n^p d\varphi, \int_E F d\varphi\right) = 0 = \lim_{n\to\infty} h_p\left(\int_E G_n^p d\varphi, \int_E G d\varphi\right), \ \ \forall p \in Q, E \in \mathcal{A}.$$

Since $h_p(\alpha F_n^p, \alpha F) = |\alpha| h_p(F_n^p, F)$ and from (8) it results:

$$h_p(\alpha F_n^p, \alpha F) \xrightarrow{\nu_p} 0. \tag{11}$$

Since $\int_S h_p(\alpha F_n^p, \alpha F_m^p) d\nu_p = |\alpha| \int_S h_p(F_n^p, F_m^p) d\nu_p$ and from (9) it results:

$$\lim_{n,m\to\infty} \int_S h_p(\alpha F_n^p, \alpha F_m^p) d\nu_p = 0. \tag{12}$$

Since $h_p(\int_E(\alpha F_n^p)d\varphi, \alpha\int_E Fd\varphi) = |\alpha|h_p(\int_E F_n^p d\varphi, \int_E Fd\varphi)$, $\forall p \in Q, \forall E \in \mathcal{A}$ and from (10), it results

$$\lim_{n\to\infty} h_p\left(\int_E(\alpha F_n^p)d\varphi, \alpha\int_E Fd\varphi\right) = 0, \quad \forall p \in Q, \forall E \in \mathcal{A}. \tag{13}$$

So, from (11), (12) and (13), according to Theorem 2.4, it follows that αF is φ-integrable in seminorm and $\int_E(\alpha F)d\varphi = \alpha\int_E Fd\varphi$, $\forall E \in \mathcal{A}$.

Since $h_p(F_n^p + G_n^p, F+G) \leq h_p(F_n^p, F) + h_p(G_n^p, G)$ and from (8) it results:

$$h_p(F_n^p + G_n^p, F+G) \xrightarrow{\nu_p} 0. \tag{14}$$

Since $\int_S h_p(F_n^p+G_n^p, F_m^p+G_m^p)d\nu_p \leq \int_S h_p(F_n^p, F_m^p)d\nu_p + \int_S h_p(G_n^p, G_m^p)d\nu_p$ and from (9) it results

$$\lim_{n,m\to\infty}\int_S h_p(F_n^p + G_n^p, F_m^p + G_m^p)d\nu_p = 0. \tag{15}$$

By Theorem 2.1-(f), we have:

$$\begin{aligned} &h_p\left(\int_E(F_n^p + G_n^p)d\varphi, \int_E Fd\varphi + \int_E Gd\varphi\right) = \\ &= h_p\left(\int_E F_n^p d\varphi + \int_E G_n^p d\varphi, \int_E Fd\varphi + \int_E Gd\varphi\right) \leq \\ &\leq h_p\left(\int_E F_n^p d\varphi, \int_E Fd\varphi\right) + h_p\left(\int_E G_n^p d\varphi, \int_E Gd\varphi\right). \end{aligned} \tag{16}$$

From (16) and (10) it results:

$$\lim_{n\to\infty} h_p\left(\int_E(F_n^p + G_n^p)d\varphi, \int_E Fd\varphi + \int_E Gd\varphi\right) = 0, \quad \forall p \in Q, E \in \mathcal{A}. \tag{17}$$

Thus, since (14), (15) and (17), according to Theorem 2.4, it follows that $F+G$ is φ-integrable in seminorm and $\int_E(F+G)d\varphi = \int_E Fd\varphi + \int_E Gd\varphi$, $\forall E \in \mathcal{A}$. □

Remark 2.3. *The study of the properties of this integral is continued in an article entitled "Set-valued integration in seminorm II". In particular, theorems of Vitali and Lebesgue type are given.*

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(Anca Croitoru) Department of Mathematics, "Al.I. Cuza" University,
Carol I Street, No. 11, Iaşi 700506, Romania
E-mail address: croitoru@uaic.ro

(Christiane Godet-Thobie) Université de Bretagne Occidentale,
Laboratoire de Mathématiques unité CNRS UMR 6205,
6 Av. Victor Le Gorgeu - CS 93837,
29238 Brest cedex3, France
E-mail address: christiane.godet-thobie@univ-brest.fr