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# On Edge Metric Dimension of Extended Kayak Paddle Graph

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Abstract. Graph with no edge intersection other than endpoints and that can be embedded on a plane is termed as a planar graph. A subset of vertices of a graph $G$ is said to be an edge resolving set, if each edge in the graph is uniquely identified by its distances to the vertices in the edge resolving set. The smallest cardinality of an edge resolving set for $G$ is called the edge metric dimension and is denoted by $edim(G)$. In this article, we determine the edge metric dimension and independent edge metric dimensions for 3-cycle and 4-cycle extended kayak paddle graphs.

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## 1. Introduction

A graph consists of vertices connected by edges each of which is connected to a pair of unique vertices. Also, graph theory, a significant branch of discrete mathematics that studies graphs, has a wide array of applications across various fields. That includes internet search: search engines use graphs to understand the relationships between web pages, which can determine the importance and relevance of a webpage for a particular search query. Navigation Apps: navigation apps uses graph theory to model road networks. Also, graph algorithms helps in finding the shortest or fastest route between two points, considering factors like traffic conditions and road closures. Medical Imaging: graph theory also helps in medical imaging to analyze complex structures and improves tasks like tumor detection, and organ segmentation which helps in treatment [1]. There are many types of graphs one such type of graph is a planar graph, which has a unique set of properties since it can be drawn on a flat surface without any edges intersecting there are many applications to it as well [2]. A planar graph is a graph that can be drawn on a plane surface, such as a piece of paper, without edges crossing, and edge intersection happens only at their vertices. Also, planar graphs are simpler compared to non-planar graphs, and they have some prominent applications, such as Image processing and computer vision: planar graph helps build relationships between pixels in an image, which helps in image segmentation and shape matching [2]. Fingerprint classification: planar graphs help in fingerprint recognition using algorithms, which are useful in optimizing and finding

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structural similarities that enhance the matching process for fingerprints from database [3]. Road networks: the construction of bridges and underpasses will be a major factor in traffic management. Planar graphs help in the design of road networks that solve issues like the crowdedness of the streets associated with urban communities [4].

Slater first gave the concept of metric dimension [5] in 1975, using the term locating set for certain location problems in graphs. Harary and Melter [6] used the term resolving set when they studied the same concept. After these two introductory papers, more work on this concept has been presented by several authors regarding applications as well as theoretical properties. The families of planar graphs, convex polytopes, molecular graphs, etc. with constant metric dimension have been explored by many researchers. There are many variants of the metric dimension, such as edge-metric dimension, local metric dimension, fault-tolerant metric dimension, strong metric dimension, and many more that have been introduced and studied from time to time [7, 8, 9, 10].

Further, the concept of edge metric dimension (EMD) was introduced by Kelenc et al. [11]. They have demonstrated several characteristics of this graph parameter and computed it for several families of the planar and non-planar graphs. Wei et al. studied this notion for certain classes of complex planar graphs [12]. They have also computed edge metric dimension for connected bipartite graphs in [13]. Peterin et al. determined the edge metric dimension of the join, lexicographic, and corona product of graphs [14]. Nasir et al. computed it for prism graph and $n$-sunlet graph [15]. Zhang and Gao computed edge metric dimension for antiprism graph, web graph, and certain convex polytope graphs [16]. Ahsan et al. determined these notions for flower graph, and its related graphs [17]. Singh et al. studied edge metric dimension for Dutch windmill graph and French windmill graph of two classes of windmill graphs [18]. Sharma et al. determined the edge metric dimension for a complex molecular graph [19]. Masmali et al. calculated the edge metric dimension for some antiviral drug structures [20]. Farooq et al. computed the edge metric dimension for certain chemical networks [21].

In this paper, we have determined the edge metric dimension and independent edge metric dimension for 3 & 4-cycle extended kayak paddle graphs. To attain this, we present the structure of the manuscript as follows: the introduction of planar graphs, metric dimensions, and edge metric dimensions, has been presented in Section 1; Section 2 is dedicated to basic definitions and results for edge metric dimensions in planar graphs; the edge metric dimension and independent edge metric dimension of 3-cycle and 4-cycle extended kayak paddle graph has been calculated in Section 3 and Section 4 respectively; lastly the conclusion and the future scope of study has been presented.

## 2. Preliminaries

A graph $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, \ldots, v_n\}$ called vertices and $E \subseteq \{\{v_i, v_j\} \,|v_i, v_j \in V \text{ and } i \neq j\}$, whose elements are called edges. Let $a, b \in V(G)$ such that the distance $d(a, b)$ between two vertices is the shortest path connecting $a$ and $b$. In a path graph $P_n$ on $n$ vertices and $n-1$ edges, each vertex is connected to exactly two others, except two endpoints where they are connected to only one vertex.

The distance between an edge $e$ and vertex $v$, defined as $d(v, e) = min\{d(v, a), d(v, b)\}$, where $e = ab$. A vertex $v$ distinguish two edges $e_1$ and $e_2$, if $d(e_1, v) \neq d(e_2, v)$.
A graph $G$ is said to be a complete graph $K_n$, if every vertex in $G$ is adjacent to every other vertex present in it. If a path in which the initial and terminal vertex are the same, that path is known as the closed path, which is also called as a cycle. It is denoted by $C_n$. In a graph $G$, the vertex set can be partitioned into two subsets $U$ and $W$, called partite sets, such that every edge in $G$ connects a vertex of $U$ and a vertex of $W$. If every vertex of $U$ is adjacent to every vertex of $W$, then $G$ is called a complete bipartite graph. A subset in a vertex set of a graph $G$ is said to be an independent set in $G$, if no two vertices in the subset are adjacent to each other in $G$.

**Definition 2.1.** [5] Let $v$ be a vertex in $G$ and let $S = \{s_1, s_2, \ldots, s_k\}$ be an ordered set of vertices of $G$. Then, the representation of a vertex $v$ with respect to the set $S$, denoted by $r(v|S)$, is a $k-$tuple $(d(v, s_1), d(v, s_2), d(v, s_3) \ldots, d(v, s_k))$. If distinct vertices of $G$ have distinct representations with respect to $S$, then $S$ is called a resolving set in $G$. The minimum cardinality resolving set in $G$ is called the metric basis and the cardinality of the metric basis is called the metric dimension of $G$, denoted by $dim(G)$.

**Definition 2.2.** [11] Let $e \in E(G)$ and let $S_E = \{x_1, x_2, \ldots, x_k\}$ be an ordered set of vertices in $G$. Then, the representation of an edge $e$ with respect to the set $S_e$, denoted by $r_e(e|S_E)$, is the $k-$tuple $(d(e, x_1), d(e, x_2), d(e, x_3), \ldots, d(e, x_k))$. If distinct edges of $G$ have distinct edge representations with respect to $S_E$, then $S_E$ is called an edge resolving set in $G$. An edge resolving set with minimum cardinality is termed as an edge metric basis for $G$, and its cardinality is called the edge metric dimension of $G$, denoted by $edim(G)$.

**Definition 2.3.** [22] A set $S_E$ of vertices in a connected graph $G$ is said to be an independent edge resolving set if $S_E$ is both an edge resolving and satisfies the independent set property. The minimum cardinality of an independent edge resolving set is called the independent edge metric dimension, denoted by $iedim(G)$.

For our convenience, we can write "*distance from*" as "*DF.*" Next, regarding metric dimension and an edge metric dimension of certain basic graph families, we have the following result:

**Proposition 2.1.** [11] *For $K_n$, $P_n$, $C_n$, and $K_{m,n}$ we have*
(1) $edim(P_n) = 1 = dim(P_n)$; $\forall\ n \geq 1$.
(2) $edim(G) = 1$ *if and only if* $G$ *is a path* $P_n$; $\forall\ n \geq 1$.
(3) $edim(K_n) = n - 1 = dim(K_n)$; $\forall\ n \geq 2$.
(4) $edim(C_n) = 2 = dim(C_n)$; $\forall\ n \geq 3$.
(5) $edim(K_{m,n}) = m + n - 2 = dim(K_{m,n})$; $\forall\ m \geq 1$ *and* $n \geq 1$.

**$n$-Cycle extended kayak paddle graph** [23]**:** The $n-$cycle extended kayak paddle graph $KP(C^n, P^{n-1})$ is a graph with $n-$cycles $(C_1, C_2, \ldots, C_n)$ and $(n-1)$-paths $(P_1, P_2, \ldots, P_{n-1})$ as shown in Fig. 1. Graph $KP(C^n, P^{n-1})$ has vertices of degree 2 and 3. The cycles $C_i$, where $1 \leq i \leq n$, consists of $m_j$ vertices, $m_j \in N$ and $m_j \geq 6$ for each $i$ and note that $a^j_{n+1} = a^j_1$ where $1 \leq j \leq n$. Also the paths $P_j$, where $1 \leq j \leq n-1$, consists of $l_j$ vertices; $l_j \in N$ and $l_j \geq 1$ for each $j$. Next, the vertex set and edge set of $KP(C^n, P^{n-1})$ are denoted by $V(KP(C^n, P^{n-1}))$ and

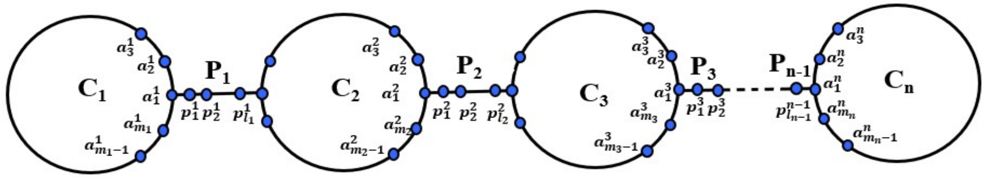


FIGURE 1. $KP(C^n, P^{n-1})$.

$E(KP(C^n, P^{n-1}))$, respectively. These sets are presented as follows:

$$V(KP(C^n, P^{n-1})) = \left\{a_i^j \ : \ 1 \leq j \leq n, \ 1 \leq i \leq m_j; \ m_j \geq \ 6\right\} \cup$$
$$\left\{p_i^j : \ 1 \leq j \leq n-1, \quad 1 \leq i \leq l_j; \ l_j \geq 1\right\}$$

and

$$\begin{aligned} E(KP(C^n, P^{n-1})) = & \left\{a_i^j a_{i+1}^j : 1 \leq j \leq n, 1 \leq i \leq m_j; \text{ where } m_j \in \{6, 7, \ldots, n\}\right\} \cup \\ & \left\{p_i^j p_{i+1}^j : 1 \leq j \leq n-1, 1 \leq i \leq l_j; \text{ where } \ l_j \geq 1 \ \right\} \cup \\ & \left\{a_1^j p_1^j; \text{where } 1 \leq j \leq n-1\right\} \cup \left\{p_{l_{n-1}}^{n-1} a_1^n\right\} \\ & \cup \left\{p_{l_j}^j a_f^{j+1}; \text{where } 1 \leq j \leq n-2\right\}, \\ & \text{where } f = \begin{cases} (\frac{m_j}{2}) + 1 & \text{when } m_j \text{ is even,} \\ \frac{(m_j-1)}{2} + 1 & \text{when } m_j \text{ is odd.} \end{cases} \end{aligned}$$

## 3. Edge Metric Dimension of 3-Cycle Extended Kayak Paddle Graph

A 3-cycle extended kayak paddle is a graph made up of three cycles $(C_1, C_2, \ and \ C_3)$ and two paths $(P_1 \ and \ P_2)$ as shown in Fig. 2. We denote it by $KP(C^3, P^2)$. It has vertices of degree 2 and 3. The cycles $C_1$, $C_2$, and $C_3$ consist of $l, m,$ and $n$ vertices, where $l, m, \ and \ n$ are greater than or equal to 6. Also $P_1 \ and \ P_2$ consists of $o$ and $p$ number of vertices, where $o$ and $p$ both are greater than or equal to 2. The vertex set and edge set of $KP(C^3, P^2)$ are denoted by $V(KP(C^3, P^2))$ and $E(KP(C^3, P^2))$, respectively. These sets are given as follows

$$V(KP(C^3, P^2)) = \{a_1, a_2, \ldots, a_l, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_n, w_1, w_2, \ldots, w_o, x_1, x_2, \ldots, x_p\}$$

and

$$\begin{aligned} E(KP(C^3, P^2)) = & \{a_i a_{i+1} : 1 \leq i \leq l\} \cup \{b_j b_{j+1} : 1 \leq j \leq m\} \cup \{c_k c_{k+1} : 1 \leq k \leq n\} \\ & \cup \{w_f w_{f+1} : 1 \leq f \leq o-1\} \cup \{x_g x_{g+1} : 1 \leq g \leq p-1\} \cup \\ & \{a_1 w_1, b_1 x_1\} \cup \{w_o b_{e+1}, x_p c_1\}, \end{aligned}$$

where $A = \{a_i a_{i+1} : 1 \leq i \leq l\}$, $B = \{b_j b_{j+1} : 1 \leq j \leq m\}$, $C = \{c_k c_{k+1} : 1 \leq k \leq n\}$,
$D = \{w_f w_{f+1} : 1 \leq f \leq o-1\}$, $F = \{x_g x_{g+1} : 1 \leq g \leq p-1\}$,
$G = \{a_1 w_1, b_1 x_1\}$, and $H = \{w_o b_{e+1}, x_p c_1\}$.

Note that, $a_{l+1} = a_1, \quad b_{m+1} = b_1, \quad$ and $\quad c_{n+1} = c_1$. As a convention, we will label the vertices of $C_1$ and $C_2$ in the counterclockwise direction and $C_3$ in a clockwise direction. Now, we determine its minimum edge resolving set and the respective edge metric dimension. Therefore, we have the following result:

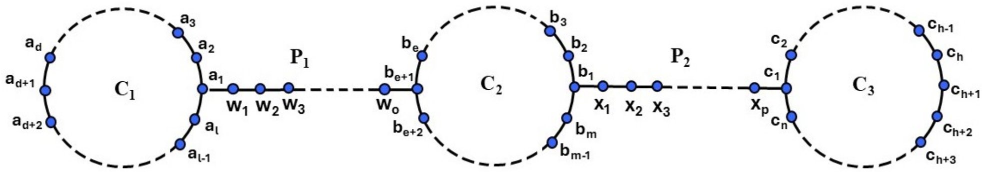


FIGURE 2. $KP(C^3, P^2)$.

**Theorem 3.1.** *Let $KP(C^3, P^2)$ be a 3-cycle extended kayak paddle graph. Then,*

$$edim(KP(C^3, P^2)) = \begin{cases} 2; & if \;\; m \;\; is \;\; odd, \\ 3; & if \;\; m \;\; is \;\; even. \end{cases}$$

*Proof.* In $KP(C^3, P^2)$, we consider the even-odd criteria based on the vertices present in the cycles of $KP(C^3, P^2)$. Thus, we have the following cases:

**Case 1: When $m$ is odd.**
Then, we have the following subcases:

**Subcase 1: When $l$ and $n$ are odd.**
Suppose $l = 2d + 1$, $m = 2e + 1$, and $n = 2h + 1$, where $l, m, n \geq 6$; $o, p \geq 2$; and $d, e, h \geq 3$. Consider the set $S_E = \{a_3, c_3\}$. Then, the representations of each edge of $KP(C^3, P^2)$ with respect to $S_E$ are shown below from Table no. 1 to 7:

TABLE 1. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $i = 1$ | $1$ | $i + o + e + p + 3$ |
| $2 \leq i \leq 3$ | $0$ | $i + o + e + p + 3$ |
| $4 \leq i \leq d + 1$ | $i - 3$ | $i + o + e + p + 3$ |
| $d + 2 \leq i \leq d + 3$ | $i - 3$ | $l - i + o + e + p + 4$ |
| $d + 4 \leq i \leq l$ | $l - i + 2$ | $l - i + o + e + p + 4$ |

TABLE 2. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \leq j \leq e$ | $o + e + 3 - j$ | $2 + p + j$ |
| $e + 1 \leq j \leq m$ | $o + j - e + 2$ | $3 + p + m - j$ |

From the above shown edge metric codes with respect to edge resolving set $S_E$, we found that $r_e(a|S_E) \neq r_e(b|S_E)$; $\forall\ a, b\ (a \neq b) \in E(KP(C^3, P^2))$. Thus, $S_E$ is a edge resolving set for $KP(C^3, P^2)$. Hence $edim(KP(C^3, P^2)) \leq 2$. By using Proposition

Table 3. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $k=1$ | $e+k+o+p+3$ | $1$ |
| $2\leq k\leq 3$ | $e+k+o+p+3$ | $0$ |
| $4\leq k\leq h+1$ | $e+k+o+p+3$ | $k-3$ |
| $h+2\leq k\leq h+3$ | $n-k+o+e+p+4$ | $k-3$ |
| $h+4\leq k\leq n$ | $n-k+o+e+p+4$ | $n-k+2$ |

Table 4. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1\leq f\leq o-1$ | $2+f$ | $p+e+o-f+3$ |

Table 5. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1\leq g\leq p-1$ | $o+e+3+g$ | $p-g+2$ |

Table 6. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $E=a_1w_1$ | $2$ | $3+p+e+o$ |
| $E=b_1x_1$ | $o+e+3$ | $2+p$ |

Table 7. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $3+p+e$ |
| $E=x_pc_1$ | $o+e+3+p$ | $2$ |

1, we conclude that $edim(KP(C^3,P^2))=2$.

**Subcase 2: When $l$ is odd and $n$ is even.**
Suppose $l=2d+1$, $m=2e+1$, and $n=2h$, where $l,m,n\geq 6$; $o,p\geq 2$; and $d,e,h\geq 3$. Consider the set $S_E=\{a_3,c_3\}$. Then, the representations of each edge of $KP(C^3,P^2)$ with respect to $S_E$ are shown below from Table no. 8 to 14:

Table 8. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $i=1$ | $1$ | $i+o+e+p+3$ |
| $2\leq i\leq 3$ | $0$ | $i+o+e+p+3$ |
| $4\leq i\leq d+1$ | $i-3$ | $i+o+e+p+3$ |
| $d+2\leq i\leq d+3$ | $i-3$ | $l-i+o+e+p+4$ |
| $d+4\leq i\leq l$ | $l-i+2$ | $l-i+o+e+p+4$ |

From the above shown edge metric codes with respect to edge resolving set $S_E$, we found that $r_e(a|S_E)\neq r_e(b|S_E)$; $\forall\ a,b\ (a\neq b)\in E(KP(C^3,P^2))$. Thus, $S_E$ is a edge resolving set for $KP(C^3,P^2)$. Hence $edim(KP(C^3,P^2))\leq 2$. By using Proposition 1, we conclude that $edim(KP(C^3,P^2))=2$.

TABLE 9. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \leq j \leq e$ | $o+e+3-j$ | $2+p+j$ |
| $e+1 \leq j \leq m$ | $o+j-e+2$ | $3+p+m-j$ |

TABLE 10. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $k=1$ | $e+k+o+p+3$ | $1$ |
| $2 \leq k \leq 3$ | $e+k+o+p+3$ | $0$ |
| $4 \leq k \leq h$ | $e+k+o+p+3$ | $k-3$ |
| $h+1 \leq k \leq h+2$ | $n-k+o+e+p+4$ | $k-3$ |
| $h+3 \leq k \leq n$ | $n-k+o+e+p+4$ | $n-k+2$ |

TABLE 11. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \leq f \leq o-1$ | $2+f$ | $p+e+o-f+3$ |

TABLE 12. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \leq g \leq p-1$ | $o+e+3+g$ | $p-g+2$ |

TABLE 13. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $E=a_1w_1$ | $2$ | $3+p+e+o$ |
| $E=b_1x_1$ | $o+e+3$ | $2+p$ |

TABLE 14. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $3+p+e$ |
| $E=x_pc_1$ | $o+e+3+p$ | $2$ |

**Subcase 3: When $l$ is even and $n$ is odd.**
Suppose $l=2d$, $m=2e+1$, and $n=2h+1$, where $l,m,n \geq 6$; $o,p \geq 2$; and $d,e,h \geq 3$. Consider the set $S_E=\{a_3,c_3\}$. Then, the representations of each edge of $KP(C^3,P^2)$ with respect to $S_E$ are shown below from Table no. 15 to 21:

TABLE 15. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $i=1$ | $1$ | $i+o+e+p+3$ |
| $2 \leq i \leq 3$ | $0$ | $i+o+e+p+3$ |
| $4 \leq i \leq d$ | $i-3$ | $i+o+e+p+3$ |
| $d+1 \leq i \leq d+2$ | $i-3$ | $l-i+o+e+p+4$ |
| $d+3 \leq i \leq l$ | $l-i+2$ | $l-i+o+e+p+4$ |

From the above shown edge metric codes with respect to edge resolving set $S_E$, we found that $r_e(a|S_E) \neq r_e(b|S_E)$; $\forall\ a,b\ (a \neq b) \in E(KP(C^3,P^2))$. Thus, $S_E$ is a edge

TABLE 16. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \leq j \leq e$ | $o+e+3-j$ | $2+p+j$ |
| $e+1 \leq j \leq m$ | $o+j-e+2$ | $3+p+m-j$ |

TABLE 17. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $k=1$ | $e+k+o+p+3$ | $1$ |
| $2 \leq k \leq 3$ | $e+k+o+p+3$ | $0$ |
| $4 \leq k \leq h+1$ | $e+k+o+p+3$ | $k-3$ |
| $h+2 \leq k \leq h+3$ | $n-k+o+e+p+4$ | $k-3$ |
| $h+4 \leq k \leq n$ | $n-k+o+e+p+4$ | $n-k+2$ |

TABLE 18. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \leq f \leq o-1$ | $2+f$ | $p+e+o-f+3$ |

TABLE 19. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \leq g \leq p-1$ | $o+e+3+g$ | $p-g+2$ |

TABLE 20. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $E=a_1w_1$ | $2$ | $3+p+e+o$ |
| $E=b_1x_1$ | $o+e+3$ | $2+p$ |

TABLE 21. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $3+p+e$ |
| $E=x_pc_1$ | $o+e+3+p$ | $2$ |

resolving set for $KP(C^3,P^2)$. Hence $edim(KP(C^3,P^2)) \leq 2$. By using Proposition 1, we conclude that $edim(KP(C^3,P^2)) = 2$.

**Subcase 4: When $l$ and $n$ are even.**
Suppose $l=2d$, $m=2e+1$ and $n=2h$, where $l,m,n \geq 6$; $o,p \geq 2$; and $d,e,h \geq 3$. Consider the set $S_E=\{a_3,c_3\}$. Then the representations of each edge of $KP(C^3,P^2)$ with respect to $S_E$ are shown below from Table no. 22 to 28:
From the above shown edge metric codes with respect to edge resolving set $S_E$, we found that $r_e(a|S_E) \neq r_e(b|S_E)$; $\forall\ a,b\ (a \neq b) \in E(KP(C^3,P^2))$. Thus, $S_E$ is a edge resolving set for $KP(C^3,P^2)$. Hence, $edim(KP(C^3,P^2)) \leq 2$. By using Proposition 1, we conclude that $edim(KP(C^3,P^2)) = 2$.

**Case 2: When $m$ is even.**

TABLE 22. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $i=1$ | $1$ | $i+o+e+p+3$ |
| $2\le i\le 3$ | $0$ | $i+o+e+p+3$ |
| $4\le i\le d$ | $i-3$ | $i+o+e+p+3$ |
| $d+1\le i\le d+2$ | $i-3$ | $l-i+o+e+p+4$ |
| $d+3\le i\le l$ | $l-i+2$ | $l-i+o+e+p+4$ |

TABLE 23. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1\le j\le e$ | $o+e+3-j$ | $2+p+j$ |
| $e+1\le j\le m$ | $o+j-e+2$ | $3+p+m-j$ |

TABLE 24. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $k=1$ | $e+k+o+p+3$ | $1$ |
| $2\le k\le 3$ | $e+k+o+p+3$ | $0$ |
| $4\le k\le h$ | $e+k+o+p+3$ | $k-3$ |
| $h+1\le k\le h+2$ | $n-k+o+e+p+4$ | $k-3$ |
| $h+3\le k\le n$ | $n-k+o+e+p+4$ | $n-k+2$ |

TABLE 25. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1\le f\le o-1$ | $2+f$ | $p+e+o-f+3$ |

TABLE 26. Edge Metric Codes for $F$

| $r_e(x_gx_{g+1}\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1\le g\le p-1$ | $o+e+3+g$ | $p-g+2$ |

TABLE 27. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $E=a_1w_1$ | $2$ | $3+p+e+o$ |
| $E=b_1x_1$ | $o+e+3$ | $2+p$ |

TABLE 28. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $3+p+e$ |
| $E=x_pc_1$ | $o+e+3+p$ | $2$ |

Then, we have the following subcases:

**Subcase 1: When $l$ and $n$ are even.**
Suppose $l=2d$, $m=2e$ and $n=2h$, where $l,m,n\ge 6$; $o,p\ge 2$; and $d,e,h\ge 3$. Consider the set $S_E=\{a_3,b_3,c_3\}$. Then the representations of each edge of $KP(C^3,P^2)$ with respect to $S_E$ are shown below from Table no. 29 to 35:

TABLE 29. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF$ $a_3$ | $DF$ $b_3$ | $DF$ $c_3$ |
|---|---|---|---|
| $i=1$ | $1$ | $e+o+i-2$ | $i+o+e+p+3$ |
| $2\le i\le 3$ | $0$ | $e+o+i-2$ | $i+o+e+p+3$ |
| $4\le i\le d$ | $i-3$ | $e+o+i-2$ | $i+o+e+p+3$ |
| $d+1\le i\le d+2$ | $i-3$ | $e+o+l-i-1$ | $l-i+o+e+p+4$ |
| $d+3\le i\le l$ | $l-i+2$ | $e+o+l-i-1$ | $l-i+o+e+p+4$ |

TABLE 30. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF$ $a_3$ | $DF$ $b_3$ | $DF$ $c_3$ |
|---|---|---|---|
| $j=1$ | $o+e+3-j$ | $1$ | $2+p+j$ |
| $2\le j\le 3$ | $o+e+3-j$ | $0$ | $2+p+j$ |
| $4\le j\le e$ | $o+e+3-j$ | $j-3$ | $2+p+j$ |
| $e+1\le j\le e+2$ | $o+j-e+2$ | $j-3$ | $3+p+m-j$ |
| $e+3\le j\le m$ | $o+j-e+2$ | $m-j+2$ | $3+p+m-j$ |

TABLE 31. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF$ $a_3$ | $DF$ $b_3$ | $DF$ $c_3$ |
|---|---|---|---|
| $k=1$ | $e+k+o+p+3$ | $p+k+2$ | $1$ |
| $2\le k\le 3$ | $e+k+o+p+3$ | $p+k+2$ | $0$ |
| $4\le k\le h$ | $e+k+o+p+3$ | $p+k+2$ | $k-3$ |
| $h+1\le k\le h+2$ | $n-k+o+e+p+4$ | $3+p+n-k$ | $k-3$ |
| $h+3\le k\le n$ | $n-k+o+e+p+4$ | $3+p+n-k$ | $n-k+2$ |

TABLE 32. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF$ $a_3$ | $DF$ $b_3$ | $DF$ $c_3$ |
|---|---|---|---|
| $1\le f\le o-1$ | $2+f$ | $e+o-2-f$ | $p+e+o-f+3$ |

TABLE 33. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF$ $a_3$ | $DF$ $b_3$ | $DF$ $c_3$ |
|---|---|---|---|
| $1\le g\le p-1$ | $o+e+3+g$ | $2+g$ | $p-g+2$ |

TABLE 34. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF$ $a_3$ | $DF$ $b_3$ | $DF$ $c_3$ |
|---|---|---|---|
| $E=a_1w_1$ | $2$ | $e+o-2$ | $3+p+e+o$ |
| $E=b_1x_1$ | $o+e+3$ | $2$ | $2+p$ |

TABLE 35. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF$ $a_3$ | $DF$ $b_3$ | $DF$ $c_3$ |
|---|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $e-3$ | $3+p+e$ |
| $E=x_pc_1$ | $o+e+3+p$ | $2+p$ | $2$ |

From the above shown edge metric codes, we have determined that each edge of $KP(C^3,P^2)$ has a unique metric. Thus, we conclude that $edim(KP(C^3,P^2))\le 3$. Further, we consider the condition $edim(KP(C^3,P^2))\ge 3$.

Suppose on the converse, $edim(KP(C^3, P^2)) \leq 2$. Then by Proposition 1, we know that $edim(KP(C^3, P^2)) \neq 1$, as it is not a path graph. Next, we are left with $edim(KP(C^3, P^2)) = 2$. So, we have the following observations:

| S.no. | **Edge resolving set** | **Contradiction** |
|---|---|---|
| 1 | $S_E = \{a_i, a_j\};\ 1 \leq i \leq l;\ 1 \leq j \leq l$ and $i \neq j$ | $r_e(c_n c_1\|S_E) = r_e(c_1 c_2\|S_E)$ |
| 2 | $S_E = \{w_i, w_j\};\ 1 \leq i \leq o;\ 1 \leq j \leq o$ and $i \neq j$ | $r_e(c_n c_1\|S_E) = r_e(c_1 c_2\|S_E)$ |
| 3 | $S_E = \{b_i, b_j\};\ 1 \leq i \leq m;\ 1 \leq j \leq m$ and $i \neq j$ | $r_e(c_n c_1\|S_E) = r_e(c_1 c_2\|S_E)$ |
| 4 | $S_E = \{x_i, x_j\};\ 1 \leq i \leq p;\ 1 \leq j \leq p$ and $i \neq j$ | $r_e(c_n c_1\|S_E) = r_e(c_1 c_2\|S_E)$ |
| 5 | $S_E = \{c_i, c_j\};\ 1 \leq i \leq n;\ 1 \leq j \leq n$ and $i \neq j$ | $r_e(a_l a_1\|S_E) = r_e(a_1 a_2\|S_E)$ |
| 6 | $S_E = \{a_i, w_s\};\ 1 \leq i \leq l,\ 1 \leq s \leq o$ | $r_e(c_n c_1\|S_E) = r_e(c_1 c_2\|S_E)$ |
| 7 | $S_E = \{a_i, b_j\};\ 1 \leq i \leq l,\ 1 \leq j \leq m$ | $r_e(c_n c_1\|S_E) = r_e(c_1 c_2\|S_E)$ |
| 8 | $S_E = \{a_i, x_z\};\ 1 \leq i \leq l,\ 1 \leq z \leq p$ | $r_e(c_n c_1\|S_E) = r_e(c_1 c_2\|S_E)$ |
| 10 | $S_E = \{w_s, b_j\};\ 1 \leq s \leq o,\ 1 \leq j \leq m$ | $r_e(c_n c_1\|S_E) = r_e(c_1 c_2\|S_E)$ |
| 11 | $S_E = \{w_s, x_z\};\ 1 \leq s \leq o,\ 1 \leq z \leq p$ | $r_e(c_n c_1\|S_E) = r_e(c_1 c_2\|S_E)$ |
| 12 | $S_E = \{w_s, c_k\};\ 1 \leq s \leq o,\ 1 \leq k \leq n$ | $r_e(a_l a_1\|S_E) = r_e(a_1 a_2\|S_E)$ |
| 13 | $S_E = \{b_j, x_z\};\ 1 \leq j \leq m,\ 1 \leq z \leq p$ | $r_e(c_n c_1\|S_E) = r_e(c_1 c_2\|S_E)$ |
| 14 | $S_E = \{b_j, c_k\};\ 1 \leq j \leq m,\ 1 \leq k \leq n$ | $r_e(a_l a_1\|S_E) = r_e(a_1 a_2\|S_E)$ |
| 15 | $S_E = \{x_z, c_k\};\ 1 \leq z \leq p,\ 1 \leq k \leq n$ | $r_e(a_l a_1\|S_E) = r_e(a_1 a_2\|S_E)$ |
| 16 | $S_E = \{a_i, c_k\};\ 1 \leq i \leq l,\ 1 \leq k \leq n$ | $r_e(b_1 b_2\|S_E) = r_e(b_m b_1\|S_E)$ |

From the above listed observations, we found that any set consisting of 2 vertices can never be a edge resolving set for $KP(C^3, P^2)$ when the middle cycle is even. Therefore, from this, we find that $KP(C^3, P^2) \geq 3$, which concludes the proof.

**Subcase 2: When $l$ is even and $n$ is odd.**
Suppose $l = 2d$, $m = 2e$ and $n = 2h + 1$, where $l, m, n \geq 6$; $o, p \geq 2$; and $d, e, h \geq 3$. Consider the set $S_E = \{a_3, b_3, c_3\}$. Then, the representations of each edge of $KP(C^3, P^2)$ with respect to $S_E$ are shown below from Table no. 36 to 42:

Table 36. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $i = 1$ | $1$ | $e + o + i - 2$ | $i + o + e + p + 3$ |
| $2 \leq i \leq 3$ | $0$ | $e + o + i - 2$ | $i + o + e + p + 3$ |
| $4 \leq i \leq d$ | $i - 3$ | $e + o + i - 2$ | $i + o + e + p + 3$ |
| $d + 1 \leq i \leq d + 2$ | $i - 3$ | $e + o + l - i - 1$ | $l - i + o + e + p + 4$ |
| $d + 3 \leq i \leq l$ | $l - i + 2$ | $e + o + l - i - 1$ | $l - i + o + e + p + 4$ |

Table 37. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $j = 1$ | $o + e + 3 - j$ | $1$ | $2 + p + j$ |
| $2 \leq j \leq 3$ | $o + e + 3 - j$ | $0$ | $2 + p + j$ |
| $4 \leq j \leq e$ | $o + e + 3 - j$ | $j - 3$ | $2 + p + j$ |
| $e + 1 \leq j \leq e + 2$ | $o + j - e + 2$ | $j - 3$ | $3 + p + m - j$ |
| $e + 3 \leq j \leq m$ | $o + j - e + 2$ | $m - j + 2$ | $3 + p + m - j$ |

From the above-shown edge metric codes, we have determined that each edge of $KP(C^3, P^2)$ has a unique metric. Thus, we conclude that $edim(KP(C^3, P^2)) \leq 3$. Further, we consider $edim(KP(C^3, P^2)) \geq 3$.

TABLE 38. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $k=1$ | $e+k+o+p+3$ | $p+k+2$ | $1$ |
| $2\leq k\leq 3$ | $e+k+o+p+3$ | $p+k+2$ | $0$ |
| $4\leq k\leq h+1$ | $e+k+o+p+3$ | $p+k+2$ | $k-3$ |
| $h+2\leq k\leq h+3$ | $n-k+o+e+p+4$ | $3+p+n-k$ | $k-3$ |
| $h+4\leq k\leq n$ | $n-k+o+e+p+4$ | $3+p+n-k$ | $n-k+2$ |

TABLE 39. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $1\leq f\leq o-1$ | $2+f$ | $e+o-2-f$ | $p+e+o-f+3$ |

TABLE 40. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $1\leq g\leq p-1$ | $o+e+3+g$ | $2+g$ | $p-g+2$ |

TABLE 41. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $E=a_1w_1$ | $2$ | $e+o-2$ | $3+p+e+o$ |
| $E=b_1x_1$ | $o+e+3$ | $2$ | $2+p$ |

TABLE 42. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $e-3$ | $3+p+e$ |
| $E=x_pc_1$ | $o+e+3+p$ | $2+p$ | $2$ |

By using the same process as in Subcase 1, a similar type of contradiction is obtained. Thus, we conclude that $edim(KP(C^3,P^2))=3$.

**Subcase 3: When $l$ is odd and $n$ is even.**
Suppose $l=2d+1$, $m=2e$ and $n=2h$, where $l,m,n\geq 6$; $o,p\geq 2$; and $d,e,h\geq 3$. Consider the set $S_E=\{a_3,b_3,c_3\}$. Then, the representations of each edge with respect to $S_E$ are shown below from Table no. 43 to 49:

TABLE 43. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $i=1$ | $1$ | $e+o+i-2$ | $i+o+e+p+3$ |
| $2\leq i\leq 3$ | $0$ | $e+o+i-2$ | $i+o+e+p+3$ |
| $4\leq i\leq d+1$ | $i-3$ | $e+o+i-2$ | $i+o+e+p+3$ |
| $d+2\leq i\leq d+3$ | $i-3$ | $e+o+l-i-1$ | $l-i+o+e+p+4$ |
| $d+4\leq i\leq l$ | $l-i+2$ | $e+o+l-i-1$ | $l-i+o+e+p+4$ |

From the above shown edge metric codes, we have determined that each edge of $KP(C^3,P^2)$ has a unique metric. Thus, we conclude that $edim(KP(C^3,P^2))\leq 3$. Further, we consider $edim(KP(C^3,P^2))\geq 3$.

TABLE 44. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $j=1$ | $o+e+3-j$ | $1$ | $2+p+j$ |
| $2 \leq j \leq 3$ | $o+e+3-j$ | $0$ | $2+p+j$ |
| $4 \leq j \leq e$ | $o+e+3-j$ | $j-3$ | $2+p+j$ |
| $e+1 \leq j \leq e+2$ | $o+j-e+2$ | $j-3$ | $3+p+m-j$ |
| $e+3 \leq j \leq m$ | $o+j-e+2$ | $m-j+2$ | $3+p+m-j$ |

TABLE 45. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $k=1$ | $e+k+o+p+3$ | $p+k+2$ | $1$ |
| $2 \leq k \leq 3$ | $e+k+o+p+3$ | $p+k+2$ | $0$ |
| $4 \leq k \leq h$ | $e+k+o+p+3$ | $p+k+2$ | $k-3$ |
| $h+1 \leq k \leq h+2$ | $n-k+o+e+p+4$ | $3+p+n-k$ | $k-3$ |
| $h+3 \leq k \leq n$ | $n-k+o+e+p+4$ | $3+p+n-k$ | $n-k+2$ |

TABLE 46. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $1 \leq f \leq o-1$ | $2+f$ | $e+o-2-f$ | $p+e+o-f+3$ |

TABLE 47. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $1 \leq g \leq p-1$ | $o+e+3+g$ | $2+g$ | $p-g+2$ |

TABLE 48. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $E=a_1w_1$ | $2$ | $e+o-2$ | $3+p+e+o$ |
| $E=b_1x_1$ | $o+e+3$ | $2$ | $2+p$ |

TABLE 49. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $e-3$ | $3+p+e$ |
| $E=x_pc_1$ | $o+e+3+p$ | $2+p$ | $2$ |

By using the same process as in Subcase 1, a similar type of contradiction is obtained. Thus, we conclude that $edim(KP(C^3, P^2)) = 3$.

**Subcase 4: When $l$ and $n$ is odd.**
Suppose $l = 2d+1$, $m = 2e$ and $n = 2h+1$, where $l, m, n \geq 6$; $o, p \geq 2$; and $d, e, h \geq 3$. Consider the set $S_E = \{a_3, b_3, c_3\}$. Then the representations of each edge of $KP(C^3, P^2)$ with respect to $S_E$ are shown below from Table no. 50 to 56:
From the -shownshown edge metric codes, we have determined that each edge of $KP(C^3, P^2)$ has a unique metric. Thus, we conclude that $edim(KP(C^3, P^2)) \leq 3$. Further, we consider $edim(KP(C^3, P^2)) \geq 3$.

TABLE 50. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $i=1$ | $1$ | $e+o+i-2$ | $i+o+e+p+3$ |
| $2\leq i\leq 3$ | $0$ | $e+o+i-2$ | $i+o+e+p+3$ |
| $4\leq i\leq d+1$ | $i-3$ | $e+o+i-2$ | $i+o+e+p+3$ |
| $d+2\leq i\leq d+3$ | $i-3$ | $e+o+l-i-1$ | $l-i+o+e+p+4$ |
| $d+4\leq i\leq l$ | $l-i+2$ | $e+o+l-i-1$ | $l-i+o+e+p+4$ |

TABLE 51. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $j=1$ | $o+e+3-j$ | $1$ | $2+p+j$ |
| $2\leq j\leq 3$ | $o+e+3-j$ | $0$ | $2+p+j$ |
| $4\leq j\leq e$ | $o+e+3-j$ | $j-3$ | $2+p+j$ |
| $e+1\leq j\leq e+2$ | $o+j-e+2$ | $j-3$ | $3+p+m-j$ |
| $e+3\leq j\leq m$ | $o+j-e+2$ | $m-j+2$ | $3+p+m-j$ |

TABLE 52. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $k=1$ | $e+k+o+p+3$ | $p+k+2$ | $1$ |
| $2\leq k\leq 3$ | $e+k+o+p+3$ | $p+k+2$ | $0$ |
| $4\leq k\leq h+1$ | $e+k+o+p+3$ | $p+k+2$ | $k-3$ |
| $h+2\leq k\leq h+3$ | $n-k+o+e+p+4$ | $3+p+n-k$ | $k-3$ |
| $h+4\leq k\leq n$ | $n-k+o+e+p+4$ | $3+p+n-k$ | $n-k+2$ |

TABLE 53. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $1\leq f\leq o-1$ | $2+f$ | $e+o-2-f$ | $p+e+o-f+3$ |

TABLE 54. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $1\leq g\leq p-1$ | $o+e+3+g$ | $2+g$ | $p-g+2$ |

TABLE 55. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $E=a_1w_1$ | $2$ | $e+o-2$ | $3+p+e+o$ |
| $E=b_1x_1$ | $o+e+3$ | $2$ | $2+p$ |

TABLE 56. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ |
|---|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $e-3$ | $3+p+e$ |
| $E=x_pc_1$ | $o+e+3+p$ | $2+p$ | $2$ |

By using the same process as in Subcase 1, a similar type of contradiction is obtained. Thus, we conclude that $edim(KP(C^3,P^2))=3$. $\square$

Based on the definition of an independent edge metric dimension and edge resolving set given in Theorem 3.1, we derive the following result.

**Theorem 3.2.** *The independent edge metric dimension of $KP(C^3, P^2)$ is,*

$$iedim(KP(C^3, P^2)) = \begin{cases} 2; & if\ m \quad is \quad odd, \\ 3; & if\ m \quad is \quad even. \end{cases}$$

## 4. Edge Metric Dimension of 4-Cycle Extended Kayak Paddle Graph

A 4-cycle extended kayak paddle is a graph made up of four cycles $(C_1, C_2, C_3,\ \&\ C_4)$ and three paths $(P_1, P_2\ \&\ P_3)$ as shown in Fig. 3. We denote it by $KP(C^4, P^3)$. It has vertices of degree 2 and 3. The cycles $C_1$, $C_2$, $C_3$, & $C_4$ consist of $l, m, n,$ & $u$ vertices, where $l, m, n,$ & $u$ are greater than or equal to 6. Also $P_1, P_2,$ & $P_3$ consists of $o$, $p$, & $r$ number of vertices, where $o$, $p$, and $r$ both are greater than or equal to 2. The vertex set and edge set of $KP(C^4, P^3)$ are denoted by $V(KP(C^4, P^3))$ and $E(KP(C^4, P^3))$, respectively. These sets are given as follows:

$$\begin{aligned} V(KP(C^4, P^3)) = &\{a_1, a_2, \dots, a_l, b_1, b_2, \dots, b_m, c_1, c_2, \dots, c_n, q_1, q_2, \dots, q_u, w_1, w_2, \\ &\dots, w_o, x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_r\} \end{aligned}$$

and

$$\begin{aligned} E(KP(l, m, n, o, p)) &= \{a_i a_{i+1} : 1 \le i \le l\} \cup \{b_j b_{j+1} : 1 \le j \le m\} \\ &\cup \{c_k c_{k+1} : 1 \le k \le n\} \cup \{q_t q_{t+1} : 1 \le t \le u\} \\ &\cup \{w_f w_{f+1} : 1 \le f \le o-1\} \cup \{x_g x_{g+1} : 1 \le g \le p-1\} \cup \{y_s y_{s+1} : 1 \le s \le r-1\} \\ &\cup \{a_1 w_1, w_o b_{e+1}\} \cup \{b_1 x_1, x_p c_{h+1}\} \cup \{c_1 y_1, y_r q_1\}, \end{aligned}$$

where

$$\begin{aligned} &A = \{a_i a_{i+1} : 1 \le i \le l\},\ B = \{b_j b_{j+1} : 1 \le j \le m\},\ C = \{c_k c_{k+1} : 1 \le k \le n\}, \\ &I = \{q_t q_{t+1} : 1 \le t \le u\},\ D = \{w_f w_{f+1} : 1 \le f \le o-1\}, \\ &F = \{x_g x_{g+1} : 1 \le g \le p-1\},\ J = \{y_s y_{s+1} : 1 \le s \le r-1\}, \\ &G = \{a_1 w_1, b_1 x_1, c_1 y_1\},\ \text{and } H = \{w_n b_{e+1}, x_n c_{e+1}, y_n q_1\}. \end{aligned}$$

Note that, $a_{l+1} = a_1$, $b_{m+1} = b_1$, $c_{n+1} = c_1$, and $q_{u+1} = q_1$. As a convention, we will label the vertices of $C_1$, $C_2$, and $C_3$ are in counter clockwise direction and $C_4$ in clockwise direction. Now, we will determine its minimum edge resolving set and the respective edge metric dimension. Then, we have the following result:

**Theorem 4.1.** *Let $KP(C^4, P^3)$ be a 4-cycle extended kayak paddle graph. Then,*

$$edim(KP(C^4, P^3)) \begin{cases} = 2; & m \text{ and } n \text{ are odd}, \\ = 3; & m \text{ is odd and } n \text{ is even}, \\ = 3; & m \text{ is even and } n \text{ is odd}, \\ \le 4; & m \text{ and } n \text{ are even}. \end{cases}$$

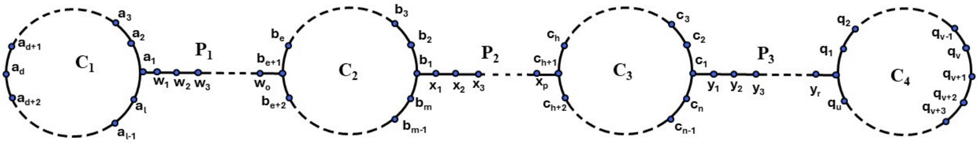

FIGURE 3. $KP(C^4, P^3)$.

*Proof.* In $KP(C^4, P^3)$, we consider the even-odd criteria based on the vertices present in the cycles of $KP(C^4, P^3)$. Thus, we have the following cases:

**Case 1: When $m$ and $n$ are odd.**
Then, we have the following subcases:

**Subcase 1: When $l$ and $u$ are odd.**
Suppose $l = 2d+1$, $m = 2e+1$, $n = 2h+1$ and $u = 2v+1$, where $l, m, n, u \geq 6$; $o, p, r \geq 2$; and $d, e, h, v \geq 3$. Consider the set $S_E = \{a_3, q_3\}$. Then, the representations of each edge of $KP(C^4, P^3)$ with respect to $S_E$ are shown below from Table no. 57 to 65.

TABLE 57. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $i = 1$ | $1$ | $r+h+p+e+o+i+4$ |
| $2 \leq i \leq 3$ | $0$ | $r+h+p+e+o+i+4$ |
| $4 \leq i \leq d+1$ | $i-3$ | $r+h+p+e+o+i+4$ |
| $d+2 \leq i \leq d+3$ | $i-3$ | $r+h+p+e+o+l-i+5$ |
| $d+4 \leq i \leq l$ | $l-i+2$ | $r+h+p+e+o+l-i+5$ |

TABLE 58. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \leq j \leq e$ | $3+o+e-j$ | $r+h+p+j+3$ |
| $e+1 \leq j \leq m$ | $2+o+j-e$ | $r+h+p+4+m-j$ |

TABLE 59. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \leq k \leq h$ | $4+o+e+p+h-k$ | $2+r+k$ |
| $h+1 \leq k \leq n$ | $3+o+e+p+k-h$ | $3+r+n-k$ |

From the above shown edge metric codes with respect to edge resolving set $S_E$, we found that $r_e(a|S_E) \neq r_e(b|S_E)$; $\forall\ a, b(a \neq b) \in E(KP(C^4, P^3))$. Thus, $S_E$ is a edge resolving set for $KP(C^4, P^3)$. Hence $edim(KP(C^4, P^3)) \leq 2$. By using proposition 1, we conclude that $edim(KP(C^4, P^3)) = 2$.

**Subcase 2: When $l$ and $u$ are odd.**
By applying the proof method from Theorem 1 and subcase 1 of case 1 of Theorem 3, we obtain $edim(KP(C^4, P^3)) = 2$.

TABLE 60. Edge Metric Codes for $I$

| $r_e(I\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $t=1$ | $o+e+p+h+r+4+t$ | 1 |
| $2 \le t \le 3$ | $o+e+p+h+r+4+t$ | 0 |
| $4 \le t \le v+1$ | $o+e+p+h+r+4+t$ | $t-3$ |
| $v+2 \le t \le v+3$ | $5+o+e+p+h+r+u-t$ | $t-3$ |
| $v+4 \le t \le u$ | $5+o+e+p+h+r+u-t$ | $u-t+2$ |

TABLE 61. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \le f \le o-1$ | $2+f$ | $r+h+p+e+o-f+4$ |

TABLE 62. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \le g \le p-1$ | $3+o+e+g$ | $r+h+p-g+3$ |

TABLE 63. Edge Metric Codes for $J$

| $r_e(J\|S_E)$ | $DF\ a_3$ | $DF\ c_3$ |
|---|---|---|
| $1 \le s \le r-1$ | $4+o+e+p+h+s$ | $r-s+2$ |

TABLE 64. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ q_3$ |
|---|---|---|
| $E=a_1w_1$ | 2 | $r+h+p+e+o+4$ |
| $E=b_1x_1$ | $o+e+3$ | $r+h+p+3$ |
| $E=c_1y_1$ | $o+e+p+h+4$ | $2+r$ |

TABLE 65. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ q_3$ |
|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $r+h+p+e+4$ |
| $E=x_pc_{h+1}$ | $o+e+3+p$ | $r+h+3$ |
| $E=y_rq_1$ | $o+e+p+h+4+r$ | 2 |

**Subcase 3: When $l$ is even and $u$ is odd.**
By applying the proof method from Theorem 1 and subcase 1 of case 1 of Theorem 3, we obtain $edim(KP(C^4, P^3)) = 2$.

**Subcase 4: When $l$ and $u$ are even.**
By applying the proof method from Theorem 1 and subcase 1 of case 1 of Theorem 3, we obtain $edim(KP(C^4, P^3)) = 2$.

**Case 2:** When $m$ is odd and $n$ is even.
By considering $S_E = \{a_3, b_3, q_3\}$ and applying the proof method from Theorem 1 and case 1 of Theorem 3, $edim(KP(C^4, P^3)) = 3$.

**Case 3:** When $m$ is even and $n$ is odd
By considering $S_E = \{a_3, c_3, q_3\}$ and applying the proof method from Theorem 1 and case 1 of Theorem 3, $edim(KP(C^4, P^3)) = 3$.

**Case 4: When $m$ and $n$ are even.**
Then, we have the following subcases:

**Subcase 1: When $l$ and $u$ are even.**
Suppose $l = 2d$, $m = 2e$, $n = 2h$ and $u = 2v$, where $l, m, n, u \geq 6$; $o, p, r \geq 2$; and $d, e, h, v \geq 3$. Consider the set $S_E = \{a_3, b_3, c_3, q_3\}$. Then, the representations of each edge of $KP(C^4, P^3)$ with respect to $S_E$ are shown below from Table no. 156 to 173:

TABLE 66. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $i = 1$ | $1$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $2 \leq i \leq 3$ | $0$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $4 \leq i \leq d$ | $i-3$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $d+1 \leq i \leq d+2$ | $i-3$ | $e+o+l-i-1$ | $h+p+e+o+l-i$ | $r+h+p+e+o+l-i+5$ |
| $d+3 \leq i \leq l$ | $l-i+2$ | $e+o+l-i-1$ | $h+p+e+o+l-i$ | $r+h+p+e+o+l-i+5$ |

TABLE 67. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $j = 1$ | $3+o+e-j$ | $1$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $2 \leq j \leq 3$ | $3+o+e-j$ | $0$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $4 \leq j \leq e$ | $3+o+e-j$ | $j-3$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $e+1 \leq j \leq e+2$ | $2+o+j-e$ | $j-3$ | $h+p+m-j-1$ | $4+r+h+p+m-j$ |
| $e+3 \leq j \leq m$ | $2+o+j-e$ | $m-j+2$ | $h+p+m-j-1$ | $4+r+h+p+m-j$ |

TABLE 68. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $k = 1$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $1$ | $2+r+k$ |
| $2 \leq k \leq 3$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $0$ | $2+r+k$ |
| $4 \leq k \leq h$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $k-3$ | $2+r+k$ |
| $h+1 \leq k \leq h+2$ | $3+o+e+p+k-h$ | $2+p+k-h$ | $k-3$ | $3+r+n-k$ |
| $h+3 \leq k \leq n$ | $3+o+e+p+k-h$ | $2+p+k-h$ | $n-k+2$ | $3+r+n-k$ |

TABLE 69. Edge Metric Codes for $I$

| $r_e(I\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $t = 1$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $1$ |
| $2 \leq t \leq 3$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $0$ |
| $4 \leq t \leq v$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $t-3$ |
| $v+1 \leq t \leq v+2$ | $5+o+e+p+h+r+u-t$ | $p+h+r+4+u-t$ | $3+r+u-t$ | $t-3$ |
| $v+3 \leq t \leq u$ | $5+o+e+p+h+r+u-t$ | $p+h+r+4+u-t$ | $3+r+u-t$ | $u-t+2$ |

TABLE 70. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1 \leq f \leq o-1$ | $2+f$ | $e-2+o-f$ | $h+p+e+o-f-1$ | $r+h+p+e+o-f+4$ |

TABLE 71. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1 \leq g \leq p-1$ | $o+e+g+3$ | $2+g$ | $h-2+p-g$ | $r+h+p-g+3$ |

TABLE 72. Edge Metric Codes for $J$

| $r_e(J\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1 \leq s \leq r-1$ | $o+e+p+h+4+s$ | $p+h+s+3$ | $2+s$ | $2+r-s$ |

TABLE 73. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $E=a_1w_1$ | $2$ | $e-2+o$ | $h+p+e+o-1$ | $r+h+p+e+o+4$ |
| $E=b_1x_1$ | $o+e+3$ | $2$ | $h-2+p$ | $r+h+p+3$ |
| $E=c_1y_1$ | $o+e+p+h+4$ | $p+h+3$ | $2$ | $2+r$ |

TABLE 74. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $e-2$ | $h+p+e-1$ | $r+h+p+e+4$ |
| $E=x_pc_{h+1}$ | $o+e+3+p$ | $2+p$ | $h-2$ | $r+h+3$ |
| $E=y_rq_1$ | $o+e+p+h+4+r$ | $p+h+3+r$ | $2+r$ | $2$ |

From the above shown edge metric codes, we have determined that each edge of $KP(C^4, P^3)$ has a unique metric.
Thus, we conclude that $edim(KP(C^4, P^3)) \leq 4$.

**Subcase 2: When $l$ is even and $u$ is odd.**

Suppose $l=2d$, $m=2e$, $n=2h$ and $u=2v+1$, where $l,m,n,u \geq 6$ and $o,p,r \geq 2$. Consider the set $S_E = \{a_3, b_3, c_3, q_3\}$. Then, the representations of each edge of $KP(C^4, P^3)$ with respect to $S_E$ are shown below from Table no. 174 to 182:

TABLE 75. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $i=1$ | $1$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $2 \leq i \leq 3$ | $0$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $4 \leq i \leq d$ | $i-3$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $d+1 \leq i \leq d+2$ | $i-3$ | $e+o+l-i-1$ | $h+p+e+o+l-i$ | $r+h+p+e+o+l-i+5$ |
| $d+3 \leq i \leq l$ | $l-i+2$ | $e+o+l-i-1$ | $h+p+e+o+l-i$ | $r+h+p+e+o+l-i+5$ |

TABLE 76. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $j=1$ | $3+o+e-j$ | $1$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $2 \leq j \leq 3$ | $3+o+e-j$ | $0$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $4 \leq j \leq e$ | $3+o+e-j$ | $j-3$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $e+1 \leq j \leq e+2$ | $2+o+j-e$ | $j-3$ | $h+p+m-j-1$ | $4+r+h+p+m-j$ |
| $e+3 \leq j \leq m$ | $2+o+j-e$ | $m-j+2$ | $h+p+m-j-1$ | $4+r+h+p+m-j$ |

From the above shown edge metric codes, we have determined that each edge of $KP(C^4, P^3)$ has a unique metric. Thus, we conclude that $edim(KP(C^4, P^3)) \leq 4$.

Table 77. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $k=1$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $1$ | $2+r+k$ |
| $2\leq k\leq 3$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $0$ | $2+r+k$ |
| $4\leq k\leq h$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $k-3$ | $2+r+k$ |
| $h+1\leq k\leq h+2$ | $3+o+e+p+k-h$ | $2+p+k-h$ | $k-3$ | $3+r+n-k$ |
| $h+3\leq k\leq n$ | $3+o+e+p+k-h$ | $2+p+k-h$ | $n-k+2$ | $3+r+n-k$ |

Table 78. Edge Metric Codes for $I$

| $r_e(I\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $t=1$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $1$ |
| $2\leq t\leq 3$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $0$ |
| $4\leq t\leq v+1$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $t-3$ |
| $v+2\leq t\leq v+3$ | $5+o+e+p+h+r+u-t$ | $p+h+r+4+u-t$ | $3+r+u-t$ | $t-3$ |
| $v+4\leq t\leq u$ | $5+o+e+p+h+r+u-t$ | $p+h+r+4+u-t$ | $3+r+u-t$ | $u-t+2$ |

Table 79. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1\leq f\leq o-1$ | $2+f$ | $e-2+o-f$ | $h+p+e+o-f-1$ | $r+h+p+e+o-f+4$ |

Table 80. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1\leq g\leq p-1$ | $o+e+g+3$ | $2+g$ | $h-2+p-g$ | $r+h+p-g+3$ |

Table 81. Edge Metric Codes for $J$

| $r_e(J\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1\leq s\leq r-1$ | $o+e+p+h+4+s$ | $p+h+s+3$ | $2+s$ | $2+r-s$ |

Table 82. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $E=a_1w_1$ | $2$ | $e-2+o$ | $h+p+e+o-1$ | $r+h+p+e+o+4$ |
| $E=b_1x_1$ | $o+e+3$ | $2$ | $h-2+p$ | $r+h+p+3$ |
| $E=c_1y_1$ | $o+e+p+h+4$ | $p+h+3$ | $2$ | $2+r$ |

Table 83. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $e-2$ | $h+p+e-1$ | $r+h+p+e+4$ |
| $E=x_pc_{h+1}$ | $o+e+3+p$ | $2+p$ | $h-2$ | $r+h+3$ |
| $E=y_rq_1$ | $o+e+p+h+4+r$ | $p+h+3+r$ | $2+r$ | $2$ |

**Subcase 3: When $l$ is odd and $u$ is even.**

Suppose $l=2d+1$, $m=2e$, $n=2h$ and $u=2v$, where $l,m,n,u\geq 6$ and $o,p,r\geq 2$. Consider the set $S_E=\{a_3,b_3,c_3,q_3\}$. Then, the representations of each edge of $KP(C^4,P^3)$ with respect to $S_E$ are shown below from Table no. 183 to 191:
From the above shown edge metric codes, we have determined that each edge of $KP(C^4,P^3)$ has a unique metric. Thus, we conclude that $edim(KP(C^4,P^3))\leq 4$.

**Subcase 4: When $l$ and $u$ are odd.**

TABLE 84. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $i=1$ | $1$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $2\le i\le 3$ | $0$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $4\le i\le d+1$ | $i-3$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $d+2\le i\le d+3$ | $i-3$ | $e+o+l-i-1$ | $h+p+e+o+l-i$ | $r+h+p+e+o+l-i+5$ |
| $d+4\le i\le l$ | $l-i+2$ | $e+o+l-i-1$ | $h+p+e+o+l-i$ | $r+h+p+e+o+l-i+5$ |

TABLE 85. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $j=1$ | $3+o+e-j$ | $1$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $2\le j\le 3$ | $3+o+e-j$ | $0$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $4\le j\le e$ | $3+o+e-j$ | $j-3$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $e+1\le j\le e+2$ | $2+o+j-e$ | $j-3$ | $h+p+m-j-1$ | $4+r+h+p+m-j$ |
| $e+3\le j\le m$ | $2+o+j-e$ | $m-j+2$ | $h+p+m-j-1$ | $4+r+h+p+m-j$ |

TABLE 86. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $k=1$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $1$ | $2+r+k$ |
| $2\le k\le 3$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $0$ | $2+r+k$ |
| $4\le k\le h$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $k-3$ | $2+r+k$ |
| $h+1\le k\le h+2$ | $3+o+e+p+k-h$ | $2+p+k-h$ | $k-3$ | $3+r+n-k$ |
| $h+3\le k\le n$ | $3+o+e+p+k-h$ | $2+p+k-h$ | $n-k+2$ | $3+r+n-k$ |

TABLE 87. Edge Metric Codes for $I$

| $r_e(I\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $t=1$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $1$ |
| $2\le t\le 3$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $0$ |
| $4\le t\le v$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $t-3$ |
| $v+1\le t\le v+2$ | $5+o+e+p+h+r+u-t$ | $p+h+r+4+u-t$ | $3+r+u-t$ | $t-3$ |
| $v+3\le t\le u$ | $5+o+e+p+h+r+u-t$ | $p+h+r+4+u-t$ | $3+r+u-t$ | $u-t+2$ |

TABLE 88. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1\le f\le o-1$ | $2+f$ | $e-2+o-f$ | $h+p+e+o-f-1$ | $r+h+p+e+o-f+4$ |

TABLE 89. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1\le g\le p-1$ | $o+e+g+3$ | $2+g$ | $h-2+p-g$ | $r+h+p-g+3$ |

TABLE 90. Edge Metric Codes for $J$

| $r_e(J\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1\le s\le r-1$ | $o+e+p+h+4+s$ | $p+h+s+3$ | $2+s$ | $2+r-s$ |

TABLE 91. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $E=a_1w_1$ | $2$ | $e-2+o$ | $h+p+e+o-1$ | $r+h+p+e+o+4$ |
| $E=b_1x_1$ | $o+e+3$ | $2$ | $h-2+p$ | $r+h+p+3$ |
| $E=c_1y_1$ | $o+e+p+h+4$ | $p+h+3$ | $2$ | $2+r$ |

Suppose $l=2d+1$, $m=2e$, $n=2h$ and $u=2v+1$, where $l,m,n,u\geq 6$ and $o,p,r\geq 2$. Consider the set $S_E=\{a_3,b_3,c_3,q_3\}$. Then, the representations of each edge with respect to $S_E$ are shown below from Table no. 192 to 200:

TABLE 92. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $E = w_o b_{e+1}$ | $2+o$ | $e-2$ | $h+p+e-1$ | $r+h+p+e+4$ |
| $E = x_p c_{h+1}$ | $o+e+3+p$ | $2+p$ | $h-2$ | $r+h+3$ |
| $E = y_r q_1$ | $o+e+p+h+4+r$ | $p+h+3+r$ | $2+r$ | $2$ |

TABLE 93. Edge Metric Codes for $A$

| $r_e(A\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $i=1$ | $1$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $2 \leq i \leq 3$ | $0$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $4 \leq i \leq d+1$ | $i-3$ | $e+o+i-2$ | $h+p+e+o+i-1$ | $r+h+p+e+o+i+4$ |
| $d+2 \leq i \leq d+3$ | $i-3$ | $e+o+l-i-1$ | $h+p+e+o+l-i$ | $r+h+p+e+o+l-i+5$ |
| $d+4 \leq i \leq l$ | $l-i+2$ | $e+o+l-i-1$ | $h+p+e+o+l-i$ | $r+h+p+e+o+l-i+5$ |

TABLE 94. Edge Metric Codes for $B$

| $r_e(B\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $j=1$ | $3+o+e-j$ | $1$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $2 \leq j \leq 3$ | $3+o+e-j$ | $0$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $4 \leq j \leq e$ | $3+o+e-j$ | $j-3$ | $h+p+j-2$ | $r+h+p+j+3$ |
| $e+1 \leq j \leq e+2$ | $2+o+j-e$ | $j-3$ | $h+p+m-j-1$ | $4+r+h+p+m-j$ |
| $e+3 \leq j \leq m$ | $2+o+j-e$ | $m-j+2$ | $h+p+m-j-1$ | $4+r+h+p+m-j$ |

TABLE 95. Edge Metric Codes for $C$

| $r_e(C\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $k=1$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $1$ | $2+r+k$ |
| $2 \leq k \leq 3$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $0$ | $2+r+k$ |
| $4 \leq k \leq h$ | $4+o+e+p+h-k$ | $3+p+h-k$ | $k-3$ | $2+r+k$ |
| $h+1 \leq k \leq h+2$ | $3+o+e+p+k-h$ | $2+p+k-h$ | $k-3$ | $3+r+n-k$ |
| $h+3 \leq k \leq n$ | $3+o+e+p+k-h$ | $2+p+k-h$ | $n-k+2$ | $3+r+n-k$ |

TABLE 96. Edge Metric Codes for $I$

| $r_e(I\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $t=1$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $1$ |
| $2 \leq t \leq 3$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $0$ |
| $4 \leq t \leq v+1$ | $o+e+p+h+r+4+t$ | $p+h+r+3+t$ | $2+r+t$ | $t-3$ |
| $v+2 \leq t \leq v+3$ | $5+o+e+p+h+r+u-t$ | $p+h+r+4+u-t$ | $3+r+u-t$ | $t-3$ |
| $v+4 \leq t \leq u$ | $5+o+e+p+h+r+u-t$ | $p+h+r+4+u-t$ | $3+r+u-t$ | $u-t+2$ |

TABLE 97. Edge Metric Codes for $D$

| $r_e(D\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1 \leq f \leq o-1$ | $2+f$ | $e-2+o-f$ | $h+p+e+o-f-1$ | $r+h+p+e+o-f+4$ |

TABLE 98. Edge Metric Codes for $F$

| $r_e(F\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1 \leq g \leq p-1$ | $o+e+g+3$ | $2+g$ | $h-2+p-g$ | $r+h+p-g+3$ |

From the above shown edge metric codes, we have determined that each edge of $KP(C^4, P^3)$ has a unique metric. Thus, we conclude that $edim(KP(C^4, P^3)) \leq 4$. $\square$

Table 99. Edge Metric Codes for $J$

| $r_e(J\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $1 \leq s \leq r-1$ | $o+e+p+h+4+s$ | $p+h+s+3$ | $2+s$ | $2+r-s$ |

Table 100. Edge Metric Codes for $G$

| $r_e(G\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $E=a_1w_1$ | $2$ | $e-2+o$ | $h+p+e+o-1$ | $r+h+p+e+o+4$ |
| $E=b_1x_1$ | $o+e+3$ | $2$ | $h-2+p$ | $r+h+p+3$ |
| $E=c_1y_1$ | $o+e+p+h+4$ | $p+h+3$ | $2$ | $2+r$ |

Table 101. Edge Metric Codes for $H$

| $r_e(H\|S_E)$ | $DF\ a_3$ | $DF\ b_3$ | $DF\ c_3$ | $DF\ q_3$ |
|---|---|---|---|---|
| $E=w_ob_{e+1}$ | $2+o$ | $e-2$ | $h+p+e-1$ | $r+h+p+e+4$ |
| $E=x_pc_{h+1}$ | $o+e+3+p$ | $2+p$ | $h-2$ | $r+h+3$ |
| $E=y_rq_1$ | $o+e+p+h+4+r$ | $p+h+3+r$ | $2+r$ | $2$ |

Based on the definition of an independent edge metric dimension and edge resolving set given in Theorem 3.2, we derive the following result.

**Theorem 4.2.** *The independent edge metric dimension of $KP(C^4, P^3)$ is,*

$$iedim(KP(C^4,P^3)) \begin{cases} = 2; & m \text{ and } n \text{ are odd}, \\ = 3; & m \text{ is odd and } n \text{ is even}, \\ = 3; & m \text{ is even and } n \text{ is odd}, \\ \leq 4; & m \text{ and } n \text{ are even}. \end{cases}$$

## 5. Conclusion

In this paper, we have studied some resolvability parameters like edge metric dimension and independent edge metric dimension for 3-cycle and 4-cycle extended kayak paddle graphs, and also derived that the resolvability parameter is bounded for these planar graphs. Furthermore, we have also studied the concept of independence in the respective minimum edge resolving set for 3-cycle and 4-cycle kayak paddle graphs. In the future, we will continue to explore the other variants of metric dimension for $KP(C^n, P^{n-1})$. For further directions, we posted the following open problems:

1. What is the lower bound for the edge metric dimension of the 4-cycle extended kayak paddle graph when the interior cycles are even.
2. What is the edge metric dimension of the $n$-cycle extended kayak paddle graph, where $n \geq 5$.

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