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Closure operators and Galois connections in categories of modules

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Abstract. In this paper we define a Galois connection between modules and rings. In case of finite generated modules over a principal ring we characterize the closed subsets with respect to associated closure operators.

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1. Prerequisites

In [1] are presented the next definitions and theorems

Definition 1.1. *Let S be a set. A function*

$$cl : \mathcal{P}(S) \to \mathcal{P}(S)$$

with the properties:
$(\forall X \subseteq S) \quad cl(X) \supseteq X$
$(\forall X, Y \subseteq S) \quad X \subseteq Y \Rightarrow cl(X) \subseteq cl(Y)$
$(\forall X \subseteq S) \quad cl(cl(X)) = cl(X)$
is called a **closure operator on** S*.*

If cl is a closure operator on S, the subsets $X \subseteq S$ satisfying $cl(X) = X$, equivalently, the subsets of the form $cl(Y)$ $(Y \subseteq S)$ are called **the closed subsets of** S **under** cl.

Definition 1.2. *Let S, T be sets. A pair of maps*

$$\lambda : \mathcal{P}(S) \to \mathcal{P}(T), \quad \mu : \mathcal{P}(T) \to \mathcal{P}(S)$$

with the properties
i) $(\forall A, A' \subseteq S) \quad A \subseteq A' \quad \Rightarrow \lambda(A) \supseteq \lambda(A')$
$(\forall B, B' \subseteq T) \quad B \subseteq B' \quad \Rightarrow \mu(B) \supseteq \mu(B')$
ii) $(\forall A \subseteq S) \quad (\mu \circ \lambda)(A) \supseteq A$
$(\forall B \subseteq T) \quad (\lambda \circ \mu)(B) \supseteq B$
is called a **Galois connection between the sets** S **and** T**.**

Theorem 1.1. *If $\lambda : \mathcal{P}(S) \to \mathcal{P}(T)$ and $\mu : \mathcal{P}(T) \to \mathcal{P}(S)$ is a Galois connection between the sets S and T, then:*
i) *$\mu \circ \lambda : \mathcal{P}(S) \to \mathcal{P}(S)$ and $\lambda \circ \mu : \mathcal{P}(T) \to \mathcal{P}(T)$ are closure operators on S and T respectively.*
ii) *The sets $\lambda(A)$ $(A \subseteq S)$ are precisely the closed subsets of T and the sets $\mu(B)$ $(B \subseteq T)$ are precisely the closed subsets of S with respect to these closure operators.*

iii) *The maps λ and μ, restricted to closed sets give an antiisomorphism between the complete lattices of closed subsets of S and T.*

If M is a module over the ring R and $x \in M$, then

$$Ann_R(x) := \{a \in R \,|\, ax = 0\}. \tag{1}$$

If R is a principal ring, then $Ann_R(x)$ is a principal ideal generated by an element denoted μ_x and called **the order of** x.

The next reprezentation theorem is proved in [3].

Theorem 1.2. *(of invariant factors) Let M be a finite generated module over the principal ring R. Then there are $m, n \in \mathbb{N}$, $m \leq n$ and $x_1, ..., x_n \in M$ so that*

$$M = Rx_1 \oplus ... \oplus Rx_m \oplus Rx_{m+1} \oplus ... \oplus Rx_n \tag{2}$$

where $\mu_{x_i} \sim d_i \neq 0$, $d_i \notin U(R)$, for $1 \leq i \leq m$, $d_1|d_2|...|d_m$ and $\mu_{x_j} = 0$ for $m < j \leq n$.

$d_1, ..., d_m$ *are called* **the invariant factors of** M.

2. A Galois connection in category of modules

Let R be a unitary commutative ring and M a $R-$module. We define two maps:

$$\lambda : \mathcal{P}(R) \to \mathcal{P}(M)$$

$$\lambda(A) := \{x \in M \,|\, Ann_R(x) \supseteq A\} \quad (A \subseteq R) \tag{3}$$

$$\mu : \mathcal{P}(M) \to \mathcal{P}(R)$$

$$\mu(L) := \{x \in R \,|\, (\forall x \in L)\, ax = 0\} = \bigcap_{x \in L} Ann_R(x) \quad (L \subseteq M). \tag{4}$$

Theorem 2.1. *The maps λ and μ define a Galois connection between the sets R and M.*

Proof. If $A, A' \in \mathcal{P}(R)$ and $A \subseteq A'$, then from (3) it results $\lambda(A) \supseteq \lambda(A')$. Analogous, if $L, L' \in \mathcal{P}(M)$ and $L \subseteq L'$, then from (4) it results $\mu(L) \supseteq \mu(L')$. Also, from (3) and (4) it results

$$(\forall A \subseteq R) \quad (\mu \circ \lambda)(A) \supseteq A$$

$$(\forall L \subseteq M) \quad (\lambda \circ \mu)(L) \supseteq L.$$

Hence, the pair of maps (λ, μ) defines a Galois connection between R and M. □

From Theorem 1.1 we deduce

Consequence 2.1. i) *$\mu \circ \lambda : \mathcal{P}(R) \to \mathcal{P}(R)$ and $\lambda \circ \mu : \mathcal{P}(M) \to \mathcal{P}(M)$ are closure operators on R and M respectively.*

ii) *The sets $\lambda(A)$ $(A \subseteq R)$ are precisely the closed subsets of M with respect to $\lambda \circ \mu$ and the sets $\mu(L)$ $(L \subseteq M)$ are precisely the closed subsets of R with respect to $\mu \circ \lambda$.*

iii) *The maps λ and μ restricted to the closed subsets give an antiisomorphism between the complete lattice of $\mu \circ \lambda$ closed subsets of R and the complete lattice of $\lambda \circ \mu$ closed subsets of M.*

3. About the lattice associated to a factorial ring

The next construction is usual. Let S be a nonempty set and ρ a preorder relation on S (ρ is transitive and reflexive). Let ρ^d be the dual relation of ρ defined by

$$(x,y) \in \rho^d \Leftrightarrow (y,x) \in \rho.$$

We note $\widehat{\rho} = \rho \cap \rho^d$. Hence

$$x\widehat{\rho}y \Leftrightarrow x\,\rho\,y \text{ and } x\,\widehat{\rho}\,y \Leftrightarrow x\,\rho\,y \text{ and } y\,\rho\,x.$$

$\widehat{\rho}$ is an equivalence relation on S and we note $\widehat{x} = \widehat{\rho}(x)$ the equivalence class of $x \in S$. Let $\widehat{S} = S/\widehat{\rho}$ be the factor set of S by $\widehat{\rho}$. We define a relation $\overline{\rho}$ on $\widehat{S}$ so

$$\widehat{x}\,\overline{\rho}\,\widehat{y} \Leftrightarrow x\,\rho\,y \quad (x,y \in S).$$

$\overline{\rho}$ is well defined. Moreover, $\overline{\rho}$ is an order relation on $\widehat{S}$.

We apply this construction in case when $S = R$ is an entire ring and $\rho = |$ is the divisibility relation on R

$$a|b \Leftrightarrow \exists c \in R \text{ so that } b = ac \quad (a,b \in R).$$

In this case $\rho^d = \vdots$,

$$a\vdots b \Leftrightarrow \exists c \in R \text{ so that } a = bc \quad (a,b \in R)$$

and $\widehat{\rho} = \sim$ is the divisibility associated relation on R

$$a \sim b \Leftrightarrow a|b \text{ and } b|a \quad (a,b \in R).$$

$\widehat{R} = R/\sim = \{\widehat{a} \,|\, a \in R\}$ is the set of equivalence classes with respect to $\widehat{\rho}$. The relation $\overline{\rho}$:

$$a\,\overline{\rho}\,b \Leftrightarrow a|b \quad (a,b \in R)$$

is an order relation on $\widehat{R}$.

Moreover, if R is a factorial ring, then $(\widehat{R}, \overline{\rho})$ is a lattice: $\widehat{a} \wedge \widehat{b} = \widehat{d}$ where d is the greatest common divisor of a and b, $\widehat{a} \vee \widehat{b} = \widehat{m}$ where m is the least common multiple of a and b.

The notions of greatest common divisor and least common multiple may be extended to infinite sets.

Let R be an entire ring and $X \subseteq R$. We say $d \in R$ is the greatest common divisor of X and we note $d \sim d(X)$ if:

i) $(\forall a \in X) \quad d|a$;

ii) $d' \in R$ and $(\forall a \in X) \quad d'|a \Rightarrow d'|d$

We say $m \in R$ is the least common multiple of X and we note $m \sim m(X)$ if:

i) $(\forall a \in X) \quad a|m$;

ii) $m' \in R$ and $(\forall a \in X) \quad a|m' \Rightarrow m|m'$

Theorem 3.1. *Let R be a factorial ring. Then $(\widehat{R}, \overline{\rho})$ is a complete lattice.*

Proof. $\widehat{1}$ is the least element of lattice $\widehat{R}$ and $\widehat{0}$ is the greatest element of lattice $\widehat{R}$. It is sufficient to show that every infinite subset of R has one greatest common divisor and one least common multiple.

Usually, we consider $(p_i)_{i \in I}$ a complete system of representants for $\widehat{\rho}$ equivalence classes of prime (irreducible) elements. If $a \in R \setminus \{0\}$, then a allows a canonical decomposition (unique):

$$a = u \prod_{i \in I} p_i^{\alpha_i}$$

where $u \in U(R)$, $\alpha_i \in \mathbb{N}$, $i \in I$ and almost all zero.

Let $X = \{a_j \mid j \in J\} \subseteq R$, where J is an infinite set. If $0 \in X$, then $d(X) \sim d(X \setminus \{0\})$ and $m(X) \sim 0$. Next, $0 \notin X$. Let

$$a_j = u_j \prod_{i \in I} p_i^{\alpha_{ij}}$$

be the canonical decomposition of elements a_j, $j \in J$.

$$d(X) \sim \prod_{i \in I} p_i^{\alpha_i},$$

where $\alpha_i = \min_{j \in J} \alpha_{ij}$, $i \in I$.

Clearly,

$$\wedge\{\widehat{x} \mid x \in X\} = \widehat{d(X)}$$

and $(\widehat{R}, \overline{\rho})$ is a lower semicomplete semilattice.

Let $I_0 := \{i \in I \mid \exists j \in J \text{ so that } \alpha_{ij} \neq 0\}$. If I_0 is infinite, then $m(X) \sim 0$. If I_0 is finite but there is $i_0 \in I_0$ such that $\{\alpha_{i_0 j} \mid j \in J\}$ is unbounded, then again $m(X) \sim 0$.

If I_0 is finite and all the sets $\{\alpha_{ij} \mid j \in J\}$, $i \in I_0$, are bounded, then

$$m(X) \sim \prod_{i \in I} p_i^{\beta_i}$$

where $\beta_i = \max\{\alpha_{ij} \mid j \in J\}$.

Clearly,

$$\vee\{\widehat{x} \mid x \in X\} = \widehat{m(X)}$$

and $(\widehat{R}, \overline{\rho})$ is an upper semicomplete semilattice.

Thus, $(\widehat{R}, \overline{\rho})$ is a complete lattice. □

Theorem 3.2. *If R is a factorial ring and X is an infinite subset of R, then there is a finite subset X' of X, so that*

$$d(X) \sim d(X').$$

Proof. We use the notices of Theorem 3.1 and we suppose $0 \notin X$.

$$X = \{a_j \mid j \in J\}, \quad a_j = u_j \prod_{i \in I} p_i^{\alpha_{ij}}.$$

Let $s \in J$, arbitrarily chosen, but fixed and

$$\{i_1, ..., i_r\} = \{i \in I \mid \alpha_{is} \neq 0\}, \quad \alpha_i = \min\{\alpha_{ij} \mid j \in J\}, \quad i \in I.$$

If $i \notin \{i_1, ..., i_r\}$, then $\alpha_i = 0$.

For $k \in \overline{1, r}$ let j_k chosen so that $\alpha_{i_k j_k} = \alpha_{i_k}$.

We take

$$X' = \{a_s, a_{j_1} ..., a_{j_r}\}.$$

Then

$$d(X') \sim \prod_{k=1}^{r} p_k^{\alpha_{i_k}} = \prod_{i \in I} p_i^{\alpha_i} \sim d(X).$$

□

Consequence 3.1. *Let R be a principal ring and $X = \{a_j \mid j \in J\}$ a nonempty subset of R. Then there are $c_j \in R$, $j \in J$, almost all zero, so that*

$$d(X) \sim \sum_{j \in J} c_j a_j.$$

Proof. If X is finite then the result is known. Since R is a principal ring it results that R is a factorial ring. If X is infinite subset of R, then, since Theorem 3.2 there is a finite subset J_0 of J so that

$$d(X) \sim d(X')$$

where $X' = \{a_j \,|\, j \in J_0\}$.

There are $c_j \in R$, $j \in J_0$ such that

$$d(X') \sim \sum_{j \in J_0} c_j a_j.$$

For $j \in J \setminus J_0$ we take $c_j = 0$. Then

$$d(X) \sim d(X') \sim \sum_{j \in J} c_j a_j.$$

□

4. Characterization of closed subsets with respect to closure operator $\lambda \circ \mu$

From Theorem 1.1 it results that the closet subsets of M with respect to closure operator $\lambda \circ \mu$ are precisely the sets $\lambda(A)$, $A \subseteq R$.

Lemma 4.1. *Let R be a principal ring and M a $R-$module. If $A \subseteq R$, then*

$$\lambda(A) = \lambda(\{d\}), \text{ where } d \sim d(A).$$

Proof. If $A = \emptyset$ or $A = \{0\}$, then $d(A) = 0$ and $\lambda(A) = \lambda(\{0\}) = M$. If $A \neq \emptyset$ and $A \neq \{0\}$, then there is a finite subset $A' = \{a_1, ..., a_n\} \subseteq A$ such that $d(A) \sim d(A') \sim d$ (Theorem 3.2).

There are $u_1, ..., u_n \in A$ such that

$$d = u_1 a_1 + ... + u_r a_r.$$

If $x \in \lambda(A)$, then $Ann_R(x) \supseteq A \supseteq A'$.

$$dx = u_1 a_1 x + ... + u_r a_r x = 0$$

Thus $x \in \lambda(\{d\})$.

Conversely, if $x \in \lambda(\{d\})$, then $Ann_R(x) \ni d$.

$$(\forall a \in A) \quad d|a \Rightarrow ax = 0$$

It results that $Ann_R(x) \supseteq A$ and $x \in \lambda(A)$. Therefore $\lambda(A) = \lambda(\{d\})$. □

For $x \in \mathbb{R}$, we note

$$x_+ = \begin{cases} x, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Theorem 4.1. *Let $M = Rx_1 \oplus ... \oplus Rx_m \oplus ... \oplus Rx_n$ be a finite generated module over the principal ring R and*

$$d_i = \prod_{j \in I} p_j^{\alpha_{ji}}, \quad i \in \overline{1, m}$$

the invariant factors of M. Let $d = \prod_{j \in I} p_j^{s_j} \in R \setminus \{0\}$. Then

$$\lambda(\{d\}) = Rt_1 x_1 \oplus ... \oplus Rt_m x_m$$

where

$$t_i = \prod_{j \in I} p_j^{\beta_{ji}}, \ \beta_{ji} = (\alpha_{ji} - s_j)_+, \quad j \in J, \quad i \in \overline{1,m}.$$

Proof. $d \cdot t_i = \prod_{j \in I} p_j^{s_j + \beta_{ji}}$.

Since $s_j + \beta_{ji} = s_j + (\alpha_{ji} - s_j)_+ \geq \alpha_{ji}$, it results $d_i | dt_i$ and $dt_i x_i = 0$, $i \in \overline{1,m}$. Hence

$$\lambda(\{d\}) \supseteq Rt_1 x_1 \oplus ... \oplus Rt_m x_m.$$

Conversely, let $x \in \lambda(\{d\})$.

$$x = a_1 x_1 + ... + a_m x_m + a_{m+1} x_{m+1} + ... + a_n x_n$$

and $0 = dx = da_1 x_1 + ... + da_m x_m + da_{m+1} x_{m+1} + ... + da_n x_n$.

It follows that $da_i x_i = 0$, for $i \in \overline{1,n}$.

For $i > m$, $\mu_{x_i} = 0 \Rightarrow da_i = 0 \Rightarrow a_i = 0$.

For $i \leq m \Rightarrow d_i | da_i$.

If $a_i = 0$, then $a_i x_i \in Rt_i x_i$.

If $a_i \neq 0$, $a_i = u_i \prod_{j \in I} p_j^{\gamma_{ji}}$, then $da_i = u_i \prod_{j \in I} p_j^{s_j + \gamma_{ji}}$.

$$d_i | da_i \Rightarrow \alpha_{ji} \leq s_j + \gamma_{ji}.$$

If $\alpha_{ji} \geq s_j$, then $\beta_{ji} = \alpha_{ji} - s_j \leq \gamma_{ji}$.

If $\alpha_{ji} < s_j$, then $\beta_{ji} = 0 \leq \gamma_{ji}$.

Hence, $t_i | a_i$ and $a_i x_i \in Rt_i x_i$. Finally,

$$x \in Rt_1 x_1 + ... + Rt_m x_m$$

and $\lambda(\{d\}) = Rt_1 x_1 + ... + Rt_m x_m$. □

Theorem 4.2. *Let $M = Rx_1 \oplus ... \oplus Rx_m \oplus ... \oplus Rx_n$ be a finite generated module over the principal ring R and*

$$d_i = \prod_{j \in I} p_j^{\alpha_{ji}}, \quad i \in \overline{1,m}$$

the invariant factors of M. Let $(s_j)_{j \in I} \in \mathbb{N}^I$ where s_j are almost all zero. Let

$$t_i = \prod_{j \in I} p_j^{\beta_{ji}}, \ \beta_{ji} = (\alpha_{ji} - s_j)_+, \quad j \in I, \quad i \in \overline{1,m}.$$

Then

$$L = Rt_1 x_1 \oplus ... \oplus Rt_m x_m \tag{5}$$

is a closed subset of M with respect to $\lambda \circ \mu$.

Proof. Let $d = \prod_{j \in I} p_j^{s_j}$. Repeating the calculation from Theorem 4.1 we obtain

$$\lambda(\{d\}) = L$$

Hence, L is a closed subset of M with respect to $\lambda \circ \mu$. □

Consequence 4.1. *Let M be a finite generated module over the principal ring R. Then the closed subsets of M with respect to closure operator $\lambda \circ \mu$ are precisely M and the submodules of the form (5) from Theorem 4.2.*

5. Characterization of closed subsets with respect to closure operator $\mu \circ \lambda$

From Theorem 1.1 it results that the closed subsets of R with respect to closure operator $\mu \circ \lambda$ are precisely the sets $\mu(L)$, $L \subseteq M$.

We consider a finite generated module M over the principal ring R:

$$M = Rx_1 \oplus ... \oplus Rx_m \oplus ... \oplus Rx_n$$

where $\mu_{x_i} \sim d_i$, $i \in \overline{1, m}$ are the invariant factors of M and $\mu_{x_i} = 0$, $i \in \overline{m+1, n}$.

$$t(M) = Rx_1 \oplus ... \oplus Rx_m$$

is the torsion submodule of M.

$$P = Rx_{m+1} \oplus ... \oplus Rx_n$$

is a free module.

If $x \in M$, then there are $y \in t(M)$ and $z \in P$ such that $x = y + z$.

Let $L \subseteq M$. If there is $x = y + z \in L$ with $z \in P \setminus \{0\}$ then

$$(\forall a \in R)\, ax = 0 \Rightarrow ay = az = 0 \Rightarrow a = 0.$$

Hence, $\mu(L) = \{0\}$.

if $L \subseteq t(M)$ then $(\forall x \in L) d_m \cdot x = 0$ and

$$\mu(L) = \cap_{x \in L} Ann_R(x) \supseteq (d_m).$$

Because $\mu(L)$ is an ideal and R is a principal ring, there is $d \in R$ so that $\mu(L) = (d)$ and $d | d_m$.

Conversely, let $d \in R$ so that $d | d_m$. There is $t \in R$ so that $d_m = dt$. $\mu_{tx_m} \sim d$.

For $L = \{tx_m\}$,

$$\mu(L) = Ann_R(tx_m) = (\mu_{tx_m}) = (d).$$

We proved:

Theorem 5.1. *Let M be a finite generated module over the principal ring R. Then the closed subsets of R with respect to closure operator $\mu \circ \lambda$ are precisely the ideals $\{0\}$ and (d) with $d | d_m$, where d_m is the last invariant factor of M.*

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