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Study of some Anisotropic Parabolic Problems with Degenerate Coercivity

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Abstract. The aim of this paper is the study of the anisotropic parabolic problems described by

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{N} D^i a_i(x,t,u,\nabla u) = f(x,t,u) & \text{in} \quad Q_T = \Omega \times (0,T), \\ u = 0 & \text{on } \Sigma_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^{\mathbb{N}}$ with $N \geq 2$. We prove the existence of renormalized solutions result for (1). We point out that the function $f(x,t,u)$ satisfies only some growth conditions with respect to u, and the initial data u_0 belongs to $L^1(\Omega)$.

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$. Fan et al. have studied in [16] the nonlinear elliptic Dirichlet problem of the form

$$\begin{cases} -\Delta_{p(\cdot)} u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

here, the nonlinear term $f(x,u)$ verifying some growth condition with respect to u. They proved the existence of weak solutions for (1.1) in variable exponent Sobolev spaces. In [1], Ahmedatt et al. have studied the existence of renormalized solutions for the following quasilinear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + |u|^{s(\cdot)-1}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the Leray-Lions operator $Au = -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u)$ verifying some degenerated coercivity condition, we refer the reader also to [2]. In [10], Boccardo et al. have

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studied the existence of solutions for the parabolic problem defined by

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u + \alpha_0 |u|^{s-1} u = f & \text{in } Q_T = \Omega \times (0,T), \\ u = 0 & \text{on } \Sigma_T = \partial\Omega \times (0,T), \\ u(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where f belongs to $L^1(Q_T)$ with α_0 is a strictly positive constant. They also concluded some regularity results. Blanchard el al. in [9] have considered the quasilinear parabolic problem given by

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u) + \Phi(u)) = f - \operatorname{div}(g) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where the datum $f \in L^1(Q_T)$, the initial data u_0 belongs to $L^1(\Omega)$, with $g \in (L^{p'}(Q_T))^N$ and $\Phi(\cdot)$ is a continuous function. They have established the existence and uniqueness of renormalized solution. In the case where the function a doesn't depend on the solution u, and $\Phi = g = 0$, the authors have proved in [7] the existence and uniqueness of renormalized solution for the quasilinear problem (1.4), we refer the reader also to [8]. In the framework of Sobolev spaces with variable exponents, we refer the reader to [3], [4], [5], and [11].

In the anisotropic parabolic spaces, Chrif et al. have established in [12] the existence of entropy solutions for the quasilinear parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{N} D^i a_i(x,t,\nabla u) + g(x,t,u,\nabla u) + d(x,t)|u|^{p_0-2}u = f - div(\Phi(u)) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

where the lower order term $g(x,t,s,\xi)$ satisfies the sign and some growth condition, with $f(x,t) \in L^1(Q_T)$, and the initial data $u_0 \in L^1(\Omega)$, we refer the reader also to [13, 14] for more details.

In this paper, we study the degenerated quasilinear parabolic problem

$$\begin{cases} u_t - \displaystyle\sum_{i=1}^{N} D^i(a_i(x,t,u,\nabla u)) = f(x,t,u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.6}$$

in anisotropic parabolic Sobolev spaces, where $f(x,t,s)$ verifies some growth condition with respect to s and the initial data u_0 belongs to $L^1(\Omega)$. We are interested on establishing the existence of renormalized solutions for (1.6). However, there are some difficulties connected to our problem, such as the lack of coercivity due to the degenerate coercivity condition, which requires the use of the penalized term $\dfrac{1}{n}|u|^{p_0-2}u$.

This paper is structured as follows : In the section 2 we introduce some definitions and properties concerning the anisotropic Sobolev spaces, then we recall some essentials lemmas. The Section 3 is devoted to presenting the assumptions on the functions

$a_i(x,t,s,\xi)$ and $f(x,t,s)$ under which our problem has at least one renormalized solution. The Section 5, will be devoted to proved our mains result.

2. Notations and preliminaries

This section is devoted to introduce some definitions and basic properties concerning the anisotropic parabolic Sobolev spaces.
Let Ω be an open bounded domain in $I\!R^N$ $(N \geq 2)$ with a Lipschitz boundary $\partial\Omega$.
Let $p_1, p_2, \dots, p_N$ be N real exponents, such that $1 < p_i < \infty$ for $i = 1, \dots, N$.
We set $\overrightarrow{p} = (p_1, \dots, p_N)$, with

$$\underline{p} = \min\{p_1, p_2, \dots, p_N\} \qquad \text{and} \qquad p_0 = \max\{p_1, p_2, \dots, p_N\}.$$

Moreover, we denote

$$D^0 u = u \qquad \text{and} \qquad D^i u = \frac{\partial u}{\partial x_i} \qquad \text{for} \qquad i = 1, \dots, N.$$

The anisotropic Sobolev space $W^{1,\overrightarrow{p}}(\Omega)$ is defined by

$$W^{1,\overrightarrow{p}}(\Omega) = \left\{ u \in L^{p_0}(\Omega) \quad \text{such that} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \dots, N \right\},$$

this space is equipped with the norm

$$\|u\|_{1,\overrightarrow{p}} = \sum_{i=0}^{N} \|D^i u\|_{p_i}. \tag{2.1}$$

We set $W_0^{1,\overrightarrow{p}}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}}(\Omega)$ for the norm (2.1).
The Sobolev spaces $W^{1,\overrightarrow{p}}(\Omega)$ and $W_0^{1,\overrightarrow{p}}(\Omega)$ are separable and reflexive Banach spaces.

Proposition 2.1. (see [17, 22].) Let $u \in W_0^{1,\vec{p}}(\Omega)$, we have
(i) Poincaré's inequality : there exists a constant $C_p > 0$, such that

$$\|u\|_{L^{p_i}(\Omega)} \leq C_p \, \|D^i u\|_{L^{p_i}(\Omega)} \qquad \text{for any} \quad i = 1, \dots, N.$$

(ii) Sobolev's inequality : there exists a constant $C_s > 0$, such that

$$\|u\|_{L^q(\Omega)} \leq \frac{C_s}{N} \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & \text{if} \quad \overline{p} < N, \\ q \in [1, +\infty[& \text{if} \quad \overline{p} \geq N. \end{cases}$$

Lemma 2.1. *Let Ω be a bounded open set in $I\!R^N$ with a Lipschitz boundary. Then, the following embedding are compact.*

- *if* $\overline{p} < N$, *then* $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ *for any* $r \in [1, \underline{p}^*[$, *where* $\dfrac{1}{\underline{p}^*} = \dfrac{1}{\underline{p}} - \dfrac{1}{N}$.
- *if* $\overline{p} = N$, *then* $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ *for any* $r \in [1, +\infty[$,
- *if* $\overline{p} > N$, *then* $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$.

The proof is based on the continuous embedding of $W_0^{1,\overrightarrow{p}}(\Omega)$ into $W_0^{1,\underline{p}}(\Omega)$, and the compact embedding theorem for Sobolev spaces.

Definition 2.1. The dual of the anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ is denoted by $W^{-1,\overrightarrow{p'}}(\Omega)$, where $\overrightarrow{p'} = (p'_1, p'_2, \dots, p'_N)$ giving by :

$$\begin{aligned} rW^{-1,\overrightarrow{p'}}(\Omega) &= \Big\{F = F_0 - \sum_{i=1}^{N} D^i F_i \text{ such that } F_0 \in L^{p'_0}(\Omega) \\ &\qquad \text{and} \quad F_i \in L^{p'_i}(\Omega) \quad \text{for} \quad i = 1, 2, \dots, N\Big\}. \end{aligned}$$

Moreover, for all $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ we have

$$\langle F, u\rangle = \sum_{i=0}^{N} \int_\Omega F_i D^i u \, dx.$$

The norm on the dual space is defined by

$$\begin{aligned} lr\|F\|_{-1,\overrightarrow{p'}} &= \inf\left\{ \sum_{i=0}^{N} \|F_i\|_{p'_i} \text{ with } F = F_0 - \sum_{i=1}^{N} D^i F_i,\right. \\ &\qquad \left. \text{where} \quad F_0 \in L^{p'_0}(\Omega) \text{ and } F_i \in L^{p'_i}(\Omega)\right\}. \end{aligned}$$

Let $T > 0$, we put $Q_T = \Omega \times (0, T)$ and $\sum_T$ the surface $\partial\Omega \times (0, T)$. We introduce the anisotropic parabolic space $L^{\overrightarrow{p}}(0, T; W^{1,\overrightarrow{p}}(\Omega))$ by

$$L^{\overrightarrow{p}}(0, T; W^{1,\overrightarrow{p}}(\Omega)) = \left\{ u \text{ measurable function } / \sum_{i=0}^{N} \int_0^T \|D^i u\|^{p_i}_{L^{p_i}(\Omega)} \, dt < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{\overrightarrow{p}}(0,T;W^{1,\overrightarrow{p}}(\Omega))} = \sum_{i=0}^{N} \left\|D^i u\right\|_{L^{p_i}(Q_T)}.$$

The functional space $L^{\overrightarrow{p}}(0, T; W_0^{1,\overrightarrow{p}}(\Omega))$ is defined by

$$L^{\overrightarrow{p}}(0, T; W_0^{1,\overrightarrow{p}}(\Omega)) = \left\{ u \in L^{\overrightarrow{p}}(0, T; W^{1,\overrightarrow{p}}(\Omega)) / u = 0 \text{ on } \partial\Omega \times [0, T] \right\}.$$

Note that $L^{\overrightarrow{p}}(0, T; W^{1,\overrightarrow{p}}(\Omega))$ and $L^{\overrightarrow{p}}(0, T; W_0^{1,\overrightarrow{p}}(\Omega))$ are separable and reflexive Banach spaces.

Definition 2.2. The dual space of $L^{\overrightarrow{p}}(0, T; W_0^{1,\overrightarrow{p}}(\Omega))$ is defined as follows

$$L^{\overrightarrow{p'}}(0, T; W^{-1,\overrightarrow{p'}}(\Omega)) = \left\{ F = f_0 - \sum_{i=1}^{N} D^i f_i, \text{ with } f_i \in L^{p'_i}(Q_T) \right\}.$$

We define a norm on the dual space by

$$\|F\|_{L^{\overrightarrow{p'}}(0,T;W^{-1,\overrightarrow{p'}}(\Omega))} = \inf\left\{ \sum_{i=0}^{N} \|f_i\|_{L^{p'_i}(Q_T)} / \quad F = f_0 - \sum_{i=1}^{N} D^i f_i \text{ with } f_i \in L^{p'_i}(Q_T) \right\}.$$

The duality of the spaces $L^{\overrightarrow{p}}(0,T;W_0^{1,\overrightarrow{p}}(\Omega))$ and $L^{\overrightarrow{p}'}(0,T;W^{-1,\overrightarrow{p}'}(\Omega))$ is given by the relation

$$\int_0^T \langle F, v\rangle dt = \sum_{i=0}^{N}\int_{Q_T} f_i D^i v \, dxdt \quad \text{for all} \quad v\in L^{\overrightarrow{p}}(0,T;W_0^{1,\overrightarrow{p}}(\Omega)).$$

Lemma 2.2 (see [21].). *Let B_0, B and B_1 be a Banach spaces with $B_0\subset B\subset B_1$. Let us set*

$$Y=\big\{u\ :\ u\in L^{p_0}(0,T;B_0) \quad \textit{and} \quad u'\in L^{p_1}(0,T;B_1)\big\}$$

where $p_0>1$ and $p_1>1$ are reals numbers.
Assuming that the embedding $B_0\hookrightarrow\hookrightarrow B$ is compact, then

$$Y\hookrightarrow\hookrightarrow L^{p_0}(0,T;B)$$

and this embedding is compact.

Remark 2.1. Let $\underline{p}>\dfrac{2N}{N+2}$, we set

$$B_0=W_0^{1,\overrightarrow{p}}(\Omega), \qquad B=L^2(\Omega) \qquad \text{and} \qquad B_1=W^{-1,\overrightarrow{p}'}(\Omega),$$

with $p_0=\underline{p}$ and $p_1=p_0'$. In view of the Lemma 2.2 we obtain

$$\big\{u\ :\ u\in L^{\overrightarrow{p}}(0,T;W_0^{1,\overrightarrow{p}}(\Omega)) \quad \text{and} \quad u'\in L^{\overrightarrow{p}'}(0,T;W^{-1,\overrightarrow{p}'}(\Omega))\big\}\subseteq Y\hookrightarrow\hookrightarrow L^1(Q_T). \tag{2.2}$$

Moreover, in view of [5], we have

$$\big\{u\ :\ u\in L^{\overrightarrow{p}}(0,T;W_0^{1,\overrightarrow{p}}(\Omega)) \quad \text{and} \quad u'\in L^{\overrightarrow{p}'}(0,T;W^{-1,\overrightarrow{p}'}(\Omega))\big\}\subseteq C([0,T];L^1(\Omega)). \tag{2.3}$$

Now, we recall some helpful lemmas for the proof of our main result.

Lemma 2.3 (see [18].). *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u\in L^1(\Omega)$ such that*
- **(i):** *$u_n\to u$ a.e. in Ω,*
- **(ii):** *$u_n\geq 0$ and $u\geq 0$ a.e. in Ω,*
- **(iii):** $\displaystyle\int_\Omega u_n\,dx\to\int_\Omega u\,dx$,

then u_n converges to u strongly in $L^1(\Omega)$.

Lemma 2.4. *Let $1<p<\infty$, we assume that $g(x)\in L^p(\Omega)$ and the sequence $(g_n)_n$ is uniformly bounded in $L^p(\Omega)$ with $\|g_n\|_p\leq C$.*
If $g_n(x)\to g(x)$ a.e. in Ω, then $g_n(x)\rightharpoonup g(x)$ weakly in $L^p(\Omega)$.

Definition 2.3. For $k>0$, we define the truncation function $T_k(\cdot):\mathbb{R}\longmapsto\mathbb{R}$ by

$$T_k(s)=\max\{-k,\min(s,k)\}.$$

Proposition 2.5 (see [13]). *Let $\mu>0$, we define the time mollification of a function u that belongs to $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ by*

$$u_\mu(x,t)=\mu\int_{-\infty}^{t}\bar{u}(x,s)\exp(\mu(s-t))ds, \qquad \textit{where } \bar{u}(x,s)=u(x,s)\chi_{(0,T)}(s).$$

Thus, we have

(i) *If* $u \in L^q(Q_T)$*, then* u_μ *is measurable in* Q_T*. Moreover,* $(u_\mu)_t = \mu(u - u_\mu)$ *and*

$$\int_{Q_T} |u_\mu|^q \, dx\, dt \leq \int_{Q_T} |u|^q \, dx\, dt.$$

(ii) *If* $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$*, thus* $u_\mu \to u$ *strongly in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *as* $\mu \to +\infty$*.*

(iii) *If* $u_n \to u$ *strongly in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$*, then* $(u_n)_\mu \to u_\mu$ *strongly in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$*.*

3. Essential Assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, with Lipschitz boundary $\partial\Omega$.
We consider the operator $Au \;:\; L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \mapsto L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$ defined by

$$Au = -\sum_{i=1}^{N} D^i a_i(x,t,u,\nabla u), \tag{3.1}$$

such that $a_i(x,t,s,\xi) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a Carathéodory function that verifies the following conditions

$$|a_i(x,t,s,\xi)| \leq \beta(K_i(x,t) + |s|^{p_i-1} + |\xi_i|^{p_i-1}), \tag{3.2}$$

where $K_i(.,.)$ is a nonnegative function that belonging to $L^{p_i'}(Q_T)$,

$$(a_i(x,t,s,\xi) - a_i(x,t,s,\xi^*))(\xi_i - \xi_i^*) > 0 \quad \text{for any} \quad \xi_i \neq \xi_i^*, \tag{3.3}$$

and

$$a_i(x,t,s,\xi)\xi_i \geq b(|s|)|\xi_i|^{p_i}, \tag{3.4}$$

such that $b(|\cdot|) : \mathbb{R} \longmapsto \mathbb{R}^+$ is a decreasing positive function, that verifies

$$b(|s|) \geq \frac{b_0}{(1+|s|)^\lambda} \qquad \text{with} \quad 0 \leq \lambda < \underline{p} - 1, \tag{3.5}$$

where β and b_0 are two strictly nonnegatives constants.

The Carathéodory function $f(x,t,s)$ having only the growth condition

$$|f(x,t,s)| \leq f_0(x,t) + c(x,t)|s|^{q_0}, \tag{3.6}$$

where $f_0(.,.) \in L^1(Q_T)$ and $0 < q_0 \leq \underline{p} - \lambda - 1$, the positive measurable function $c(.,.)$ belongs to $L^m(Q_T)$ with $\dfrac{\underline{p} - \lambda - 1}{\underline{p} - \lambda - 1 - q_0} < m$.

Lemma 3.1. *Assuming that the conditions* (3.2) − (3.4) *hold true. Let* $(u_n)_n$ *be a sequence in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *with* $(u_n)_t \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$*, such that* $u_n \rightharpoonup u$ *weakly in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *and*

$$\lim_{n\to\infty} \Big(\sum_{i=1}^{N} \int_{Q_T} (a_i(x,t,T_k(u_n),\nabla u_n) - a_i(x,t,T_k(u_n),\nabla u))(D^i u_n - D^i u)\, dx\, dt$$
$$+ \int_{Q_T} (|u_n|^{p_0-2} u_n - |u|^{p_0-2} u)(u_n - u)\, dx\, dt \Big) = 0,$$

then $u_n \to u$ *strongly in* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *for a subsequence.*

The proof of lemma 3.1 follows the same technique used in [13].

4. Main result

Definition 4.1. A measurable function u is a renormalized solution of the anisotropic quasilinear parabolic problem (1.6) if

$$u \in C(0,T;L^1(\Omega)),\ T_k(u) \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$$

for any $k > 0$, and $f(x,t,u) \in L^1(Q_T)$, with

$$\sum_{i=1}^{N} \int_{\{h<|u|\leq h+1\}} a_i(x,t,u,\nabla u) D^i u \, dx \, dt \to 0 \qquad \text{as} \quad h \longrightarrow \infty,$$

and u verifies the following equality

$$\int_{Q_T} \frac{\partial u}{\partial t} S'(u)\nu \ dx \ dt + \sum_{i=1}^{N} \int_{Q_T} a_i(x,t,u,\nabla u)(S''(u)D^i u\nu + S'(u)D^i\nu) \, dx \, dt$$
$$= \int_{Q_T} f(x,t,u)S'(u)\nu \ dx \, dt.$$

for any $\nu \in D(Q_T)$ such that $\nu(.,T) = 0$, and $S'(\cdot) \in C_0^1(\mathbb{R})$.

Theorem 4.1. *We assume that* (3.2) − (3.6) *hold true. Then, there exists at least one renormalized solution u for the quasilinear anisotropic parabolic problem* (1.6).

Proof of Theorem 4.1

Step 1: Approximate problems.
We define the approximate problem as follows

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \displaystyle\sum_{i=1}^{N} D^i a_i(x,t,T_n(u_n),\nabla u_n) + \frac{1}{n}|u_n|^{p_0-2}u_n = f_n(x,t,T_n(u_n)) & \text{in } Q_T, \\ u_n(x,t) = 0 & \text{on } \Sigma_T, \\ u_n(x,0) = u_{0,n} & \text{in } \Omega. \end{cases} \tag{4.1}$$

with $u_{0,n} = T_n(u_0(x))$ and $f_n(x,t,T_n(s)) = T_n(f(x,t,T_n(s)))$.
We consider the following operators :
A_n and $F_n : L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \longmapsto L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$, defined by

$$\int_0^T \langle A_n u, v\rangle \ dt = \sum_{i=1}^{N} \int_{Q_T} a_i(x,t,T_n(u),\nabla u)D^i v \, dx \, dt + \frac{1}{n}\int_{Q_T} |u|^{p_0-2}uv \, dx \, dt, \tag{4.2}$$

and

$$\int_0^T \langle F_n u, v\rangle \ dt = \int_{Q_T} f_n(x,t,T_n(u))v \, dx \, dt, \tag{4.3}$$

for every $u, v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$.

Lemma 4.2. *We define the operator* $B_n = A_n - F_n$. B_n *is bounded and pseudo-monotone. Moreover,* B_n *is coercive in the following sense:*

$$\frac{\displaystyle\int_0^T \langle B_n u, u\rangle\ dt}{\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}} \longrightarrow \infty \quad as \quad \|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \longrightarrow \infty,$$

for any $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$.

For the proof of Lemma 4.2, see Appendix.
Thanks to Lemma 4.2 (cf. [19]), the approximate problems (4.1) has at least one weak solution $u_n \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, i.e

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, v\rangle\ dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n) D^i v\, dxdt + \frac{1}{n}\int_{Q_T} |u_n|^{p_0-2} u_n v\, dxdt$$
$$= \int_{Q_T} f_n(x,t,T_n(u_n)) v\, dxdt, \quad \text{for any} \quad v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)).$$

Step 2: A priori estimates.
Let $k > 0$ and $1 < \theta < \underline{p} - \lambda$, we consider the following functions

$$\varphi_k(s) = \int_0^s T_k(\tau)\, d\tau = \begin{cases} \dfrac{s^2}{2} & \text{if} \quad |s| \le k, \\ k|s| - \dfrac{k^2}{2} & \text{if} \quad |s| > k, \end{cases}$$

and

$$B(s) = \int_0^s \frac{1}{(1+|\tau|)^\theta} d\tau \qquad \text{then} \qquad 0 < B(\infty) < \infty.$$

By choosing $T_k(u_n)e^{B(|u_n|)}$ as a test function in (4.1), we obtain

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n)e^{B(|u_n|)}\rangle\ dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n) D^i T_k(u_n) e^{B(|u_n|)}\, dx\, dt$$
$$+ \sum_{i=1}^N \int_{Q_T} \frac{a_i(x,t,T_n(u_n),\nabla u_n) D^i u_n}{(1+|u_n|)^\theta} |T_k(u_n)| e^{B(|u_n|)}\, dx\, dt$$
$$+ \frac{1}{n}\int_{Q_T} |u_n|^{p_0-2} u_n T_k(u_n) e^{B(|u_n|)}\, dx\, dt$$
$$= \int_{Q_T} f_n(x,t,T_n(u_n)) T_k(u_n) e^{B(|u_n|)}\, dx\, dt,$$

Thanks to (3.4) and (3.6), we obtain

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n)e^{B(|u_n|)}\rangle\ dt + \frac{b_0}{(1+k)^\lambda} \sum_{i=1}^N \int_{Q_T} |D^i T_k(u_n)|^{p_i}\, dx\, dt$$
$$+ b_0 \sum_{i=1}^N \int_{\{|u_n|>k\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+\theta}} |T_k(u_n)|\, dx\, dt + \frac{1}{n}\int_{Q_T} |u_n|^{p_0-1}|T_k(u_n)|\, dt\, dx$$
$$\le k e^{B(\infty)} \|f_0\|_{L^1(Q_T)} + \int_{Q_T} |c(x,t)||T_n(u_n)|^{q_0} |T_k(u_n)| e^{B(|u_n|)}\, dx\, dt, \tag{4.4}$$

For the first term on the left-hand side of (4.4), we have

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n)e^{B(|u_n|)}\rangle \, dt = \int_\Omega \int_0^T \frac{\partial u_n}{\partial t} T_k(u_n)e^{B(|u_n|)} \, dt \, dx \\
&= \int_\Omega \int_0^T \frac{d}{dt} \int_0^{u_n} T_k(s)e^{B(|s|)} \, ds \, dt \, dx \\
&= \int_\Omega \int_0^{u_n(T)} T_k(s)e^{B(|s|)} \, ds \, dx - \int_\Omega \int_0^{u_n(0)} T_k(s)e^{B(|s|)} \, ds \, dx \\
&\geq \int_\Omega \int_0^{u_n(T)} T_k(s) \, ds \, dx - e^{B(\infty)} \int_\Omega \int_0^{u_n(0)} T_k(s) \, ds \, dx \\
&= \int_\Omega \varphi_k(u_n(T)) \, dx - e^{B(\infty)} \int_\Omega \varphi_k(u_{0,n}) \, dx,
\end{aligned} \tag{4.5}$$

Moreover, for any $t \in (0, T)$ we have $u(\cdot, t) \in W_0^{1,\vec{p}}(\Omega)$. In view of Poincaré's inequality, we establish that

$$\begin{aligned}
&\int_{\{|u_n|\geq k\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}} |T_k(u_n)| \, dx \geq k \int_{\{|u_n|\geq k\}} \frac{|D^i u_n|^{\underline{p}}}{(1+|u_n|)^{\theta+\lambda}} \, dx - k\,|\Omega| \\
&\geq k \int_\Omega \left| D^i \int_{|T_k(u_n)|}^{|u_n|} \frac{1}{(1+s)^{\frac{\theta+\lambda}{\underline{p}}}} ds \right|^{\underline{p}} dx - k\,|\Omega| \\
&\geq \frac{k}{C^{\underline{p}}} \int_\Omega \left| \int_{|T_k(u_n)|}^{|u_n|} \frac{1}{(1+s)^{\frac{\theta+\lambda}{\underline{p}}}} ds \right|^{\underline{p}} dx - k\,|\Omega| \\
&\geq \frac{k}{C^{\underline{p}}} \int_\Omega \frac{(|u_n| - |T_k(u_n)|)^{\underline{p}}}{(1+|u_n|)^{\theta+\lambda}} \, dx - k\,|\Omega| \\
&\geq \frac{k}{2^{\underline{p}-1}C^{\underline{p}}} \int_\Omega \frac{|u_n|^{\underline{p}}}{(1+|u_n|)^{\theta+\lambda}} \, dx - \frac{k}{C^{\underline{p}}} \int_\Omega \frac{|T_k(u_n)|^{\underline{p}}}{(1+|u_n|)^{\theta+\lambda}} \, dx - k\,|\Omega| \\
&\geq kC_0 \int_{\{|u_n|>1\}} |u_n|^{\underline{p}-\lambda-\theta} \, dx - kC_1 \int_\Omega \frac{|T_k(u_n)|^{\underline{p}}}{(1+|u_n|)^{\theta+\lambda}} \, dx - k\,|\Omega|,
\end{aligned} \tag{4.6}$$

Then, we integrate between $(0, T)$ we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{|u_n|\geq k\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}} |T_k(u_n)| \, dx \, dt \\
&\qquad \geq kC_2 \int_{\{|u_n|>1\}} |u_n|^{\underline{p}-\lambda-\theta} \, dx \, dt - kC_3 \int_{Q_T} \frac{|T_k(u_n)|^{\underline{p}}}{(1+|u_n|)^{\theta+\lambda}} \, dx \, dt - kC_4,
\end{aligned} \tag{4.7}$$

For the third term on the right-hand side of (4.4), by using Young's inequality we obtain

$$\begin{aligned}
&\int_{Q_T} |c(x,t)||T_n(u_n)|^{q_0} |T_k(u_n)| e^{B(|u_n|)} \, dx \, dt \\
&\leq \frac{C_2}{2} \int_{Q_T} |u_n|^{\underline{p}-\lambda-\theta} |T_k(u_n)| \, dx \, dt + C_5 \int_{Q_T} |c(x,t)|^{\frac{\underline{p}-\lambda-\theta}{\underline{p}-\lambda-\theta-q_0}} |T_k(u_n)| \, dx \, dt \\
&\leq kC_6 + \frac{kC_2}{2} \int_{\{|u_n|>1\}} |u_n|^{\underline{p}-\lambda-\theta} \, dx \, dt + C_5 \int_{Q_T} |c(x,t)|^{\frac{\underline{p}-\lambda-\theta}{\underline{p}-\lambda-\theta-q_0}} |T_k(u_n)| \, dx \, dt,
\end{aligned} \tag{4.8}$$

By combining (4.4) and (4.5) – (4.8) we conclude that

$$\begin{aligned}&\int_{\Omega} \varphi_k(u_n(T))\ dx + \frac{b_0}{(1+k)^{\lambda}} \sum_{i=1}^{N} \int_{Q_T} |D^i T_k(u_n)|^{p_i}\ dx\, dt \\ &+ \frac{kC_2}{2} \int_{\{|u_n|>1\}} |u_n|^{\underline{p}-\lambda-\theta}\ dxdt + \frac{1}{n} \int_{Q_T} |u_n|^{p_0-1} |T_k(u_n)|\, dt\, dx \\ &\leq C_7 \left\{ k + (1+k)^{\underline{p}-(\theta+\lambda)+1} \right\},\end{aligned} \tag{4.9}$$

On the other hand, we have

$$\int_{Q_T} |T_k(u_n)|^{p_0}\, dt\, dx \leq k^{p_0} \text{meas}(Q_T), \tag{4.10}$$

Using (4.9) and (4.10), we deduce that : there exists a nonnegative constant $C(k)$ independently on and n.

$$\|T_k(u_n)\|^{\underline{p}}_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \leq C_7(k). \tag{4.11}$$

Hence, the sequence $(T_k(u_n))_n$ is uniformly bounded in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, which implies the existence of a subsequence still denoted $(T_k(u_n))_{n\in\mathbb{N}}$ and a measurable function $v_k \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k \text{ weakly in } L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)), \\ T_k(u_n) \to v_k \text{ strongly in } L^{\underline{p}}(Q_T) \text{ and a.e in } Q_T. \end{cases} \tag{4.12}$$

Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure.
Indeed, by using Hölder's inequality, Poincaré's inequality and thanks to (4.9), we have

$$\begin{aligned} k^{\underline{p}}\text{meas}(\{|u_n| > k\}) &= \int_{\{|u_n|>k\}} |T_k(u_n)|^{\underline{p}}\ dx\, dt \\ &\leq \int_{Q_T} |T_k(u_n)|^{\underline{p}}\ dx\, dt = \|T_k(u_n)\|^{\underline{p}}_{L^{\underline{p}}(Q_T)} \\ &\leq C_{\underline{p}}^{\underline{p}} \|\nabla T_k(u_n)\|^{\underline{p}}_{L^{\underline{p}}(Q_T)} \\ &\leq C_8 \sum_{i=1}^{N} \int_{Q_T} |D^i T_k(u_n)|^{p_i}\ dxdt + C_9 \text{meas}(Q_T) \\ &\leq C_{10} \left((1+k)^{\lambda+1} + (1+k)^{\underline{p}-\theta+1} \right) + C_9 \text{meas}(Q_T). \end{aligned}$$

It follows that

$$\text{meas}(\{|u_n| > k\}) \leq \frac{C_{10} \left((1+k)^{\lambda+1} + (1+k)^{\underline{p}-\theta+1} \right)}{k^{\underline{p}}} + \frac{C_9 \text{meas}(Q_T)}{k^{\underline{p}}} \longrightarrow 0 \ \text{ as } \ k \to \infty. \tag{4.13}$$

For all $\delta > 0$, we have

$$\begin{aligned} \text{meas}\left\{|u_n - u_m| > \delta\right\} \leq\ & \text{meas}\left\{|u_n| > k\right\} + \text{meas}\left\{|u_m| > k\right\} \\ & + \text{meas}\left\{|T_k(u_n) - T_k(u_m)| > \delta\right\}. \end{aligned}$$

Let $\varepsilon > 0$, using (4.13) we can choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\left\{|u_n| > k\right\} \leq \frac{\varepsilon}{3} \quad \text{ and } \quad \text{meas}\left\{|u_m| > k\right\} \leq \frac{\varepsilon}{3}. \tag{4.14}$$

Moreover, thanks to (4.12) we have $T_k(u_n)$ converges to v_k strongly in $L^{\underline{p}}(Q_T)$ and a.e. in Q_T. Thus, $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, it follows that : for all $k>0$ and $\delta,\varepsilon>0$, there exists $n_0=n_0(k,\delta,\varepsilon)$ such that

$$\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}\leq\frac{\varepsilon}{3}\quad\text{ for all } m,n\geq n_0(k,\delta,\varepsilon). \tag{4.15}$$

By combining (4.13) $-$ (4.15), we conclude that

$$\text{for all } \delta,\varepsilon>0, \exists\, n_0=n_0(\delta,\varepsilon) \text{ such that } \operatorname{meas}\{|u_n-u_m|>\delta\}\leq\varepsilon,\ \forall\, n,m\geq n_0(\delta,\varepsilon).$$

Therefore, the sequence $(u_n)_n$ is a Cauchy sequence in measure, then there exists a subsequence still denoted $(u_n)_n$ and a measurable function u, such that $u_n \longrightarrow u$ almost everywhere in Q_T. As result, we have

$$T_k(u_n)\rightharpoonup T_k(u) \text{ weakly in } L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad\text{and}\quad \text{a.e in}\quad \Omega. \tag{4.16}$$

Furthermore, in view of (4.16) and Lebesgue's dominated convergence theorem we have

$$T_k(u_n)\longrightarrow T_k(u) \text{ strongly in } L^{p_i}(Q_T) \quad\text{ for any }\quad i\in\{1,\cdots,N\}. \tag{4.17}$$

Moreover, in view of (4.9) we show that

$$\int_{Q_T}|u_n|^{\underline{p}-\lambda-\theta}\,dxdt\leq C(k). \tag{4.18}$$

Step 3 : The equi-integrability of $(|u_n|^{\sigma})_n$.
In this step, we will show that

$$|u_n|^{\sigma}\longrightarrow|u|^{\sigma}\quad\text{strongly in}\quad L^1(Q_T)\quad\text{for any}\quad 0<\sigma<\underline{p}-\lambda-1. \tag{4.19}$$

Let $1<\theta<\underline{p}-\lambda$ and $0<\delta<\underline{p}-\lambda-\theta$, in view of (4.16) we have

$$|u_n|^{\underline{p}-\lambda-\theta-\delta}\longrightarrow|u|^{\underline{p}-\lambda-\theta-\delta}\quad\text{a.e in}\quad Q_T.$$

Thanks to Vitali's theorem, for proving (4.19) it is enough to prove the uniformly equi-integrability of the sequence $\left(|u_n|^{\underline{p}-\lambda-\theta-\delta}\right)_n$.
Let $h>0$, and let E be any measurable subset in Q_T, we have

$$\int_E|u_n|^{\underline{p}-\lambda-\theta-\delta}\,dx\,dt\leq\int_E|T_h(u_n)|^{\underline{p}-\lambda-\theta-\delta}\,dx\,dt+\int_{\{|u_n|>h\}}|u_n|^{\underline{p}-\lambda-\theta-\delta}\,dx\,dt. \tag{4.20}$$

Having in mind (4.18), we obtain

$$h^{\delta}\int_{\{|u_n|>h\}}|u_n|^{\underline{p}-\lambda-\theta-\delta}\,dx\,dt\leq\int_{\{|u_n|>h\}}|u_n|^{\underline{p}-\lambda-\theta}\,dx\,dt\leq\int_{\Omega}|u_n|^{\underline{p}-\lambda-\theta}\,dx\,dt\leq C_{11},$$

where C_{11} is a nonnegative constant independent on n, it follows that

$$\lim_{h\to\infty}\int_{\{|u_n|>h\}}|u_n|^{\underline{p}-\lambda-\theta-\delta}\,dx\,dt\leq\frac{C_{11}}{h^{\delta}}=0.$$

Therefore, for every $\epsilon>0$ there exists $h_0(\epsilon)>0$ such that

$$\int_{\{|u_n|>h\}}|u_n|^{\underline{p}-\lambda-\theta-\delta}\,dx\,dt\leq\frac{\epsilon}{2}\quad\text{ for any } h\geq h_0(\epsilon). \tag{4.21}$$

On the other hand : there exists $\eta(\epsilon, h) > 0$, such that

$$\int_E |T_h\,(u_n)|^{\underline{p}-\lambda-\theta-\delta}\ dx\ dt \leq \frac{\epsilon}{2} \quad \text{for any} \quad E \subset Q_T \quad \text{with} \quad \text{meas}(E) \leq \eta(\epsilon, h). \tag{4.22}$$

We combine the results (4.20) and (4.21) – (4.22), we conclude that : for every $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ small enough such that

$$\int_E |u_n|^{\underline{p}-\lambda-\theta-\delta}\ dx\ dt \leq \epsilon \quad \text{for any} \quad E \subset Q_T \quad \text{with} \quad \text{meas}(E) \leq \eta(\epsilon).$$

Thus, the equi-integrability of the seqence $\Big(|u_n|^{\underline{p}-\lambda-\theta-\delta}\Big)_n$ is concluded for any $1 < \theta < \underline{p} - \lambda$ and any $0 < \delta < \underline{p} - \lambda - \theta$. Consequently, we deduce (4.19).

Step 4 : Equi-integrability of the sequence $\left(\frac{1}{n}|u_n|^{p_0-2}u_n\right)_n$ **.**

By taking $(T_{h+1}(u_n) - T_h(u_n))$ as a test function for the approximate problem (4.1), we have

$$\begin{aligned}&\int_0^T \langle \frac{\partial u_n}{\partial t}, (T_{h+1}(u_n) - T_h(u_n))\rangle\ dt\\ &\quad + \sum_{i=1}^N \int_{Q_T} a_i(x, t, T_n(u_n), \nabla u_n) D^i(T_{h+1}(u_n) - T_h(u_n))\ dx\ dt\\ &\quad + \frac{1}{n}\int_{Q_T} |u_n|^{p_0-2}u_n(T_{h+1}(u_n) - T_h(u_n))\ dt\ dx\\ &= \int_{Q_T} f_n(x, t, T_n(u_n))(T_{h+1}(u_n) - T_h(u_n))\ dx\ dt.\end{aligned}$$

In view of (3.6), and since $(T_{h+1}(u_n) - T_h(u_n))$ has the same sign as u_n, then

$$\begin{aligned}&\int_0^T \langle \frac{\partial u_n}{\partial t}, (T_{h+1}(u_n) - T_h(u_n))\rangle\ dt\\ &\quad + \sum_{i=1}^N \int_{\{h<|u_n|\leq h+1\}} a_i(x, t, T_n(u_n), \nabla u_n) D^i u_n\ dx\ dt\\ &\quad + \frac{1}{n}\int_{Q_T} |u_n|^{p_0-1}|T_{h+1}(u_n) - T_h(u_n)|\ dt\ dx\\ &\leq \int_{Q_T} |f_0||T_{h+1}(u_n) - T_h(u_n)|\ dxdt\\ &\quad + \int_{Q_T} |c(x,t)||T_n(u_n)|^{q_0}|T_{h+1}(u_n) - T_h(u_n)|\ dxdt,\end{aligned} \tag{4.23}$$

We have

$$\begin{aligned}&\int_0^T \langle \frac{\partial u_n}{\partial t}, (T_{h+1}(u_n) - T_h(u_n))\rangle\ dt\\ &= \int_\Omega \int_0^T \frac{d}{dt}\int_0^{u_n} (T_{h+1}(s) - T_h(s))\ ds\ dt\ dx\\ &= \int_\Omega \int_0^{u_n(T)} (T_{h+1}(s) - T_h(s))\ ds\ dx - \int_\Omega \int_0^{u_n(0)} (T_{h+1}(s) - T_h(s))\ ds\ dx\\ &= \int_\Omega \varphi_{h+1}(u_n(T)) - \varphi_h(u_n(T))\ dx - \int_\Omega \varphi_{h+1}(u_{0,n}) - \varphi_h(u_{0,n})\ dx,\end{aligned} \tag{4.24}$$

Moreover, we have

$$\begin{aligned}\int_\Omega \varphi_{h+1}(u_n(T)) - \varphi_h(u_n(T))\,dx &= \frac{1}{2}\int_{\{h\leq|u_n(T)|\leq h+1\}} (|u_n(T)|-h)^2\,dx\\ &+\int_{\{h+1\leq|u_n(T)|\}} |u_n(T)|-h-\frac{1}{2}\,dx \geq 0.\end{aligned} \tag{4.25}$$

For the third term on the right-hand side of (4.23), by using Young's inequality we get

$$\begin{aligned}&\int_{Q_T} |c(x,t)||T_n(u_n)|^{q_0}|T_{h+1}(u_n)-T_h(u_n)|\,dx\,dt\\ &\leq \int_{Q_T} |u_n|^{\underline{p}-\lambda-\theta}|T_{h+1}(u_n)-T_h(u_n)|\,dx\,dt\\ &\quad +C_7\int_{\{h\leq|u_n|\}} |c(x,t)|^{\frac{\underline{p}-\lambda-\theta}{\underline{p}-\lambda-\theta-q_0}}|T_{h+1}(u_n)-T_h(u_n)|\,dx\,dt,\end{aligned} \tag{4.26}$$

By combining (4.23) and (4.24) – (4.26), we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_{\{h\leq|u_n|\leq h+1\}} a_i(x,t,T_n(u_n),\nabla u_n)D^iu_n\,dx\,dt\\ &\quad+\frac{1}{n}\int_{Q_T}|u_n|^{p_0-1}|T_{h+1}(u_n)-T_h(u_n)|\,dx\,dt\\ &\leq\int_{\{h\leq|u_n|\}}|f_0|\,dx\,dt+\int_\Omega\varphi_{h+1}(u_{0,n})-\varphi_h(u_{0,n})\,dx\\ &\quad+\int_{\{h\leq|u_n|\}}|u_n|^{\underline{p}-\lambda-\theta}\,dx\,dt+C_7\int_{\{h\leq|u_n|\}}|c(x,t)|^{\frac{\underline{p}-\lambda-\theta}{\underline{p}-\lambda-\theta-q_0}}\,dx\,dt,\end{aligned} \tag{4.27}$$

Since $\operatorname{meas}(\{h\leq|u_n|\})\to 0$ as $h\to\infty$, and $f_0(x,t),|c(x,t)|^{\frac{\underline{p}-\lambda-\theta}{\underline{p}-\lambda-\theta-q_0}}\in L^1(Q_T)$, we obtain

$$\int_{\{h\leq|u_n|\}}|f_0|\,dx\,dt+C_7\int_{\{h\leq|u_n|\}}|c(x,t)|^{\frac{\underline{p}-\lambda-\theta}{\underline{p}-\lambda-\theta-q_0}}\,dx\,dt\longrightarrow 0\quad\text{as}\quad n,h\longrightarrow\infty \tag{4.28}$$

Furthermore, by (4.19) we get

$$\int_{\{h\leq|u_n|\}}|u_n|^{\underline{p}-\lambda-\theta}\,dx\,dt\longrightarrow 0\qquad\text{as}\quad n,h\longrightarrow\infty. \tag{4.29}$$

For the second term on right-hand side of (4.27), since $u_0\in L^1(\Omega)$, it follows that

$$\begin{aligned}\int_\Omega\varphi_{h+1}(u_{0,n})-\varphi_h(u_{0,n})\,dx &= \frac{1}{2}\int_{\{h\leq|u_{0,n}|\leq h+1\}}(|u_{0,n}|-h)^2\,dx\\ &+\int_{\{h+1\leq|u_{0,n}|\}}|u_{0,n}|-h-\frac{1}{2}\,dx\longrightarrow 0,\end{aligned} \tag{4.30}$$

as $n,h\longrightarrow\infty$. In view of (4.27) – (4.30), we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^N\int_{\{h\leq|u_n|<h+1\}}a_i(x,t,T_n(u_n),\nabla u_n)D^iu_n\,dx\,dt=0, \tag{4.31}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{n}\int_{\{h+1\leq|u_n|\}}|u_n|^{p_0-1}\,dx\,dt=0. \tag{4.32}$$

Now, we will prove that the sequence $(\frac{1}{n}|u_n|^{p_0-2}u_n)_n \to 0$ strongly in $L^1(Q_T)$.
Firstly, from (4.16), we have $\frac{1}{n}|u_n|^{p_0-2}u_n \longrightarrow 0$ a.e in Q_T.
Now, we show the equi-integrability of the sequence $(\frac{1}{n}|u_n|^{p_0-2}u_n)_n$.
Thanks to (4.32) : for any $\epsilon > 0$ there exists a positive constant $h(\epsilon) > 1$ such that

$$\frac{1}{n}\int_{\{h(\epsilon)\le|u_n|\}} |u_n|^{p_0-1}\,dx\,dt \le \frac{\epsilon}{2}. \tag{4.33}$$

Let E be a measurable subset in Q_T, we have

$$\frac{1}{n}\int_E |u_n|^{p_0-1}\,dx\,dt \quad \le \frac{1}{n}\int_E |T_{h(\epsilon)}(u_n)|^{p_0-1}\,dx\,dt + \frac{1}{n}\int_{\{h(\epsilon)\le|u_n|\}} |u_n|^{p_0-1}\,dx\,dt, \tag{4.34}$$

On the other hand, we have : for any $\epsilon > 0$, there exists $\zeta(\epsilon) > 0$ such that

$$\frac{1}{n}\int_E |T_{h(\epsilon)}(u_n)|^{p_0-1}\,dx\,dt \le \frac{\epsilon}{2} \qquad \text{for any} \quad E\subset\Omega \quad \text{with} \quad \text{meas}(E)\le\zeta(\epsilon), \tag{4.35}$$

Therefore, by combining (4.33) and (4.34)-(4.35), we conclude that

$$\forall\epsilon>0, \quad \exists\zeta(\epsilon)>0 \quad \text{such that} \quad \frac{1}{n}\int_{Q_T} |u_n|^{p_0-1}\,dx\,dt \le \epsilon \quad \text{for} \quad \text{meas}(E)\le\zeta(\epsilon), \tag{4.36}$$

Thus, the sequence $(\frac{1}{n}|u_n|^{p_0-1})_n$ is uniformly equi-integrable over Ω. In view of Vitali's theorem, we deduce that

$$\frac{1}{n}|u_n|^{p_0-1} \longrightarrow 0 \quad \text{strongly in} \quad L^1(Q_T), \tag{4.37}$$

Furthermore, in view of (3.6) and Young's inequality we have

$$|f_n(x,t,T_n(u_n))| \le |f_0(x,t)| + |c(x,t)|^{\frac{\underline{p}-\lambda-\theta}{\underline{p}-\lambda-\theta-q_0}} + |u_n|^{\underline{p}-\lambda-\theta}, \tag{4.38}$$

Thanks to (4.19), we deduce that $(f_n(x,t,T_n(u_n)))_n$ is uniformly equi-integrable and since $f_n(x,t,T_n(u_n)) \longrightarrow f(x,t,u)$ a.e in Q_T, we conclude that

$$f_n(x,t,T_n(u_n)) \longrightarrow f(x,t,u) \quad \text{strongly in} \quad L^1(Q_T), \tag{4.39}$$

Step 5 : Strong convergence of truncation.
Let $\varepsilon_i(n)$ be some real numbers functions, that verifies $\varepsilon_i(n)\to 0$ as $n\to\infty$, (resp. $\varepsilon_i(n,h), \varepsilon_i(n,\mu)$ and $\varepsilon_i(n,\mu,h)$).
For $h>0$, we consider the increasing function $S_h(\cdot)\in C^2(\mathbb{R})$, where

$$\begin{cases} S_h(r) = r \quad \text{for} \quad |r|\le h, \\ \text{supp}(S_h') \subset [-(h+1), h+1]\,, \\ \text{supp}(S_h'') \subset [-(h+1), -h]\cup[h, h+1]\,. \end{cases}$$

Let $h \geq k > 0$, we choose $(T_k(u_n) - (T_k(u))_\mu)S'_h(u_n)$ as a test function in (4.1), we have

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, (T_k(u_n) - (T_k(u))_\mu)S'_h(u_n)\rangle \, dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)(D^i T_k(u_n) - D^i(T_k(u))_\mu)S'_h(u_n)\, dx\, dt \\
&\quad + \sum_{i=1}^N \int_{\{h \leq |u| \leq h+1\}} a_i(x,t,T_n(u_n),\nabla u_n) D^i u_n S''_h(u_n)(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt \\
&\quad + \frac{1}{n}\int_{Q_T} |u_n|^{p_0-2}u_n(T_k(u_n) - (T_k(u))_\mu)S'_h(u_n)\, dx\, dt \\
&= \int_{Q_T} f_n(x,t,T_n(u_n))(T_k(u_n) - (T_k(u))_\mu)S'_h(u_n)\, dx\, dt.
\end{aligned}$$

Having in mind that $S'_h(r) = 1$ on $\{|u_n| \leq k\}$, we have

$$\begin{aligned}
&\int_0^T \langle \frac{\partial u_n}{\partial t}, (T_k(u_n) - (T_k(u))_\mu)S'_h(u_n)\rangle \, dt \\
&\quad + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u_n)) D^i(T_k(u_n) - (T_k(u))_\mu)\, dx\, dt \\
&\quad - \sum_{i=1}^N \int_{\{k < |u_n| \leq h+1\}} a_i(x,t,T_n(u_n),\nabla u_n) D^i(T_k(u))_\mu S'_h(u_n)\, dx\, dt \\
&\leq \|S'_h(\cdot)\|_{L^\infty(\mathbb{R})} \int_{Q_T} |f_n(x,t,T_n(u_n))||T_k(u_n) - (T_k(u))_\mu|\, dx\, dt \\
&\quad + \|S'_h(\cdot)\|_{L^\infty(\mathbb{R})} \frac{1}{n}\int_{Q_T} |u_n|^{p_0-1}|T_k(u_n) - (T_k(u))_\mu|\, dx\, dt \\
&\quad + 2k\|S''_h(\cdot)\|_{L^\infty(\mathbb{R})} \sum_{i=1}^N \int_{\{h \leq |u_n| \leq h+1\}} a_i(x,t,T_n(u_n),\nabla u_n) D^i u_n\, dx\, dt,
\end{aligned} \tag{4.40}$$

Thanks to (4.39) we have $f_n(x,t,T_n(u_n)) \longrightarrow f(x,t,u)$ strongly in $L^1(Q_T)$, and since $(T_k(u_n) - (T_k(u))_\mu) \rightharpoonup 0$ weak$-*$ in $L^\infty(Q_T)$, we obtain

$$\varepsilon_1(n,\mu) = \int_{Q_T} |f_n(x,t,T_n(u_n))||T_k(u_n) - (T_k(u))_\mu|\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n,\mu \longrightarrow \infty. \tag{4.41}$$

Similarly, in view of (4.37) we get

$$\varepsilon_2(n,\mu) = \frac{1}{n}\int_{Q_T} |u_n|^{p_0-1}|T_k(u_n) - (T_k(u))_\mu|\, dx\, dt \longrightarrow 0, \quad \text{as} \quad n,\mu \longrightarrow \infty. \tag{4.42}$$

Moreover, thanks to (4.31) we have

$$\varepsilon_3(n,h) = 2k\|S''_h(\cdot)\|_{L^\infty(\mathbb{R})} \sum_{i=1}^N \int_{\{h \leq |u_n| \leq h+1\}} a_i(x,t,T_n(u_n),\nabla u_n) D^i u_n\, dx\, dt \longrightarrow 0, \tag{4.43}$$

as $n, h \to \infty$.

Now, we study the behavior of first term on the left-hand side of (4.40), we use the same argument as in ([11, 15]) we have

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, (T_k(u_n) - (T_k(u))_\mu) S'_h(u_n)\rangle \, dt \geq \varepsilon_5(n) \tag{4.44}$$

In view of (3.2), the sequence $\big(a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))\big)_n$ is uniformly bounded in $L^{p'_i}(Q_T)$, then there exists a measurable function $\vartheta_i \in L^{p'_i}(Q_T)$ such that $a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n)) \rightharpoonup \vartheta_i$ weakly in $L^{p'_i}(Q_T)$ as n goes to infinity, it follows that

$$\begin{aligned}\varepsilon_6(n,\mu) &= \sum_{i=1}^N \int_{\{k<|u_n|\leq h+1\}} a_i(x,t,T_n(u_n),\nabla u_n) D^i(T_k(u))_\mu S'_h(u_n)\, dx\, dt \\ &\leq \|S'_h(\cdot)\|_{L^\infty(\mathbb{R})} \sum_{i=1}^N \int_{\{k<|u_n|\leq h+1\}} |a_i(x,t,T_n(u_n),\nabla u_n)|\, |D^i(T_k(u))_\mu|\, dx\, dt \\ &\longrightarrow \|S'_h(\cdot)\|_{L^\infty(\mathbb{R})} \sum_{i=1}^N \int_{\{k<|u|\leq h+1\}} |\vartheta_i|\, |D^i(T_k(u))_\mu|\, dx\, dt \to 0 \text{ as } \mu \to \infty.\end{aligned} \tag{4.45}$$

By combining (4.40) and (4.41) – (4.45), we obtain

$$\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u_n))(D^i T_k(u_n) - D^i(T_k(u))_\mu)\, dx\, dt \leq \varepsilon_7(n,\mu,h).$$

It follows that

$$\begin{aligned}&\sum_{i=1}^N \int_{Q_T} (a_i(x,t,T_k(u_n),\nabla T_k(u_n)) - a_i(x,t,T_k(u_n),\nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u)) dxdt \\ &\leq \varepsilon_7(n,\mu,h) - \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u_n))(D^i T_k(u) - D^i(T_k(u))_\mu)\, dxdt \\ &\quad - \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u))(D^i T_k(u_n) - D^i T_k(u))\, dxdt.\end{aligned} \tag{4.46}$$

We have $(a_i(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ is uniformly bounded sequence in $L^{p'_i}(Q_T)$, then there exists a measurable function ϱ_i such that $a_i(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varrho_i$ weakly in $L^{p'_i}(Q_T)$, we obtain

$$\begin{aligned}\varepsilon_8(n,\mu) &= \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u_n))(D^i T_k(u) - D^i(T_k(u))_\mu)\, dxdt \\ &\longrightarrow \sum_{i=1}^N \int_{Q_T} \varrho_i (D^i T_k(u) - D^i(T_k(u))_\mu)\, dxdt \quad \text{as} \quad n \to \infty \\ &\longrightarrow 0 \quad \text{as} \quad \mu \to \infty.\end{aligned} \tag{4.47}$$

Thanks to (4.17), we have $a_i(x,t,T_k(u_n),\nabla T_k(u))$ converges to $a_i(x,t,T_k(u),\nabla T_k(u))$ strongly in $L^{p'_i}(Q_T)$, and since $D^i T_k(u_n) \rightharpoonup D^i T_k(u))$ weakly in $L^{p_i}(Q_T)$, we get

$$\varepsilon_9(n) = \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u))(D^i T_k(u_n) - D^i T_k(u))\, dx\, dt \to 0 \text{ as } n \to \infty. \tag{4.48}$$

Thanks to (4.17), we have $T_k(u_n) \to T_k(u)$ strongly in $L^{p_0}(\Omega)$. By combining (4.46) and (4.47) – (4.48), then by letting n tends to infinity we deduce that

$$\begin{aligned}\sum_{i=1}^{N}\int_{Q_T}&(a_i(x,t,T_k(u_n),\nabla T_k(u_n)) - a_i(x,t,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n) - D^iT_k(u))\,dxdt\\ &+\int_{Q_T}\left(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u)\right)(T_k(u_n) - T_k(u))\,\,dxdt \to 0.\end{aligned} \tag{4.49}$$

In view of Lemma 3.1, we obtain

$$\begin{cases} T_k(u_n) \longrightarrow T_k(u) & \text{strongly in} \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)),\\ D^iu_n \to D^iu & \text{a.e in} \quad Q_T.\end{cases} \tag{4.50}$$

It follows that $a_i(x,t,T_n(u_n),\nabla u_n) \longrightarrow a_i(x,t,u,\nabla u)$ almost everywhere in Q_T, and thanks to (3.2) we conclude that

$$a_i(x,t,T_n(u_n),\nabla u_n) \rightharpoonup a_i(x,t,u,\nabla u) \qquad \text{weakly in} \quad L^{p_i}(Q_T) \quad \text{for} \quad i=1,\dots,N. \tag{4.51}$$

Moreover, by using (4.31), (4.50) and Fatou's Lemma we show that

$$\lim_{h\to\infty}\sum_{i=1}^{N}\int_{\{h\le|u|<h+1\}} a_i(x,t,u,\nabla u)D^iu\,dxdt = 0. \tag{4.52}$$

Step 6 : The convergence of the sequence $(u_n)_n$ in $C(0,T;L^1(\Omega))$.
Let $0 < s \le T$, by taking $T_1(u_n - (T_h(u))_\mu)\cdot\chi_{[0,s]}(t)$ as a test function for the approximate problem (4.1), we obtain

$$\begin{aligned}&\int_0^s\int_\Omega \frac{\partial u_n}{\partial t}T_1(u_n - (T_h(u))_\mu)\,dxdt\\ &+\sum_{i=1}^{N}\int_0^s\int_\Omega a_i(x,t,T_n(u_n),\nabla u_n)D^iT_1(u_n - (T_h(u))_\mu)\,dx\,dt\\ &+\frac{1}{n}\int_0^s\int_\Omega |u_n|^{p_0-2}\,u_nT_1(u_n - (T_h(u))_\mu)\,dx\,dt\\ &=\int_0^s\int_\Omega f_n(x,t,T_n(u_n))T_1(u_n - (T_h(u))_\mu)\,dx\,dt.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_0^s\int_\Omega \frac{\partial(u_n - (T_h(u))_\mu)}{\partial t}T_1(u_n - (T_h(u))_\mu)\,dxdt\\ &+\int_0^s\int_\Omega \frac{\partial(T_h(u))_\mu}{\partial t}T_1(u_n - (T_h(u))_\mu)\,dxdt\\ &+\sum_{i=1}^{N}\int_0^s\int_\Omega a_i(x,t,T_n(u_n),\nabla u_n)D^iT_1(u_n - (T_h(u))_\mu)\,dxdt\\ &+\frac{1}{n}\int_0^s\int_\Omega |u_n|^{p_0-2}\,u_nT_1(u_n - (T_h(u))_\mu)\,dx\,dt\\ &=\int_0^s\int_\Omega f_n(x,t,T_n(u_n))T_1(u_n - (T_h(u))_\mu)\,dx\,dt.\end{aligned} \tag{4.53}$$

Considering the first and second terms on the left-hand side of (4.53), one has

$$\begin{aligned}&\int_\Omega\int_0^s \frac{\partial(u_n-(T_h(u))_\mu)}{\partial t}T_1(u_n-(T_h(u))_\mu)\;dt\,dx\\&=\int_\Omega\int_0^s \frac{\partial\varphi_1(u_n-(T_h(u))_\mu)}{\partial t}\;dt\,dx\\&=\int_\Omega \varphi_1(u_n(s)-(T_h(u(s)))_\mu)\;dx-\int_\Omega\varphi_1(u_{0,n}-(T_h(u_0))_\mu)\;dx,\end{aligned}\tag{4.54}$$

and

$$\begin{aligned}&\int_\Omega\int_0^s \frac{\partial(T_h(u))_\mu}{\partial t}T_1(u_n-(T_h(u))_\mu)\;dt\,dx\\&=\mu\int_\Omega\int_0^s (T_h(u)-(T_h(u))_\mu)T_1(u_n-(T_h(u))_\mu)\;dt\,dx\\&\longrightarrow\mu\int_\Omega\int_0^s (T_h(u)-(T_h(u))_\mu)T_1(u-(T_h(u))_\mu)\;dt\,dx\geq 0\quad\text{as}\quad n\to\infty.\end{aligned}\tag{4.55}$$

For the third term on the left-hand side of (4.53), thanks to (4.51) we have $a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))$ converges to $a_i(x,t,T_{h+1}(u),\nabla T_{h+1}(u))$ weakly in $L^{p_i'}(Q_T)$, and since $D^iT_1(u_n-(T_h(u))_\mu)\to D^iT_1(u-(T_h(u)))$ strongly in $L^{p_i}(Q_T)$ as $n,\mu\to\infty$, in view of (4.31) we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_0^s\int_\Omega a_i(x,t,T_n(u_n),\nabla u_n)D^iT_1(u_n-(T_h(u))_\mu)\;dx\,dt\\&=\sum_{i=1}^N\int_0^s\int_\Omega a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))D^iT_1(u_n-(T_h(u))_\mu)\;dx\,dt\\&\longrightarrow\sum_{i=1}^N\int_0^s\int_\Omega a_i(x,t,T_{h+1}(u),\nabla T_{h+1}(u))D^iT_1(u-T_h(u))\;dx\,dt\quad\text{as}\quad n,\mu\to\infty\\&=\sum_{i=1}^N\int_{\{h<|u|\leq h+1\}} a_i(x,t,u,\nabla u)D^iu\;dx\,dt\longrightarrow 0\quad\text{as}\quad h\to\infty.\end{aligned}\tag{4.56}$$

Moreover, thanks to (4.37) and the fact that $T_1(u_n-(T_h(u))_\mu)\rightharpoonup 0$ weak$-^*$ in $L^\infty(Q_T)$, we obtain

$$\frac{1}{n}\int_0^s\int_\Omega |u_n|^{p_0-2}u_nT_1(u_n-(T_h(u))_\mu)\;dx\,dt\longrightarrow 0\text{ as }n,\mu,h\longrightarrow\infty,\tag{4.57}$$

Similarly, in view of (4.39) we have

$$\int_0^s\int_\Omega f_n(x,t,T_n(u_n))T_1(u_n-(T_h(u))_\mu)\;dx\,dt\longrightarrow 0\quad\text{as}\quad n,\mu,h\longrightarrow\infty.\tag{4.58}$$

By combining (4.53) and (4.54) – (4.58), we obtain

$$\begin{aligned}\int_\Omega\varphi_1(u_n(s)-(T_h(u(s)))_\mu)\;dx\;&\leq\int_\Omega\varphi_1(u_{0,n}-(T_h(u_0))_\mu)\;dx+\varepsilon_{10}(n,\mu,h)\\&\longrightarrow\int_\Omega\varphi_1(u_0-T_h(u_0))\;dx+\varepsilon_{10}(n,\mu,h)\text{ as }n,\mu\to\infty.\end{aligned}\tag{4.59}$$

Having in mind that u_0 belongs to $L^1(\Omega)$, it follows that

$$\int_\Omega \varphi_1(u_n(s) - (T_h(u(s)))_\mu)\, dx \longrightarrow 0 \quad \text{as} \quad n, \mu, h \to \infty. \tag{4.60}$$

Thanks to the fact that

$$\begin{aligned} &\int_{\{|u_n(s)-(T_h(u(s)))_\mu|\leqslant 1\}} |u_n(s) - (T_h(u(s)))_\mu|^2 \; dx \\ &\quad + \int_{\{|u_n(s)-(T_h(u(s)))_\mu|>1\}} |u_n(s) - (T_h(u(s)))_\mu| \; dx \\ &\quad \leqslant 2 \int_\Omega \varphi_1(u_n(s) - (T_h(u(s)))_\mu)\; dx. \end{aligned} \tag{4.61}$$

and

$$\begin{aligned} \int_\Omega |u_n(s) - (T_h(u(s)))_\mu| \; dx &= \int_{\{|u_n(s)-(T_h(u(s)))_\mu|\leqslant 1\}} |u_n(s) - (T_h(u(s)))_\mu| \; dx \\ &\quad + \int_{\{|u_n(s)-(T_h(u(s)))_\mu|>1\}} |u_n(s) - (T_h(u(s)))_\mu| \; dx \\ &\leq C_0 \Big(\int_{\{|u_n(s)-(T_h(u(s)))_\mu|\leq 1\}} |u_n(s) - (T_h(u(s)))_\mu|^2 \; dx \Big)^{\frac{1}{2}} \\ &\quad + \int_{\{|u_n(s)-(T_h(u(s)))_\mu|>1\}} |u_n(s) - (T_h(u(s)))_\mu| \; dx \longrightarrow 0 \quad \text{as} \quad n, \mu, h \to \infty. \end{aligned} \tag{4.62}$$

Thus, we show that

$$\int_\Omega |u_n(s) - u_m(s)|\, dx \leq \int_\Omega |u_n(s) - (T_h(u(s)))_\mu|\, dx + \int_\Omega |u_m(s) - (T_h(u(s)))_\mu|\, dx \longrightarrow 0,$$

as $n, m, \mu, h \longrightarrow \infty$.

As result, $(u_n)_n$ is a Cauchy sequence in $C([0,T]; L^1(\Omega))$, which implies that $u \in C([0,T]; L^1(\Omega))$. Therefore, we get

$$u_n(s) \longrightarrow u(s) \quad \text{strongly in} \quad L^1(\Omega) \quad \text{for any} \quad 0 \leqslant s \leqslant T. \tag{4.63}$$

Step 7 : Passage to the limit.

Let $\nu \in D(Q_T)$, such that $\nu(.,T) = 0$, and $S(.) \in C_c^\infty(\mathbb{R})$ with $supp(S'(.)) \subset [-M, M]$ where $M > 0$. Using $S'(u_n)\nu$ as a test function for the approximate problem (4.1), we obtain

$$\begin{aligned} &\int_0^T \langle \frac{\partial u_n}{\partial t}, S'(u_n)\nu\rangle \; dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n), \nabla u_n)(D^i u_n S''(u_n)\nu + D^i \nu S'(u_n)) \; dx\; dt \\ &\quad + \frac{1}{n} \int_{Q_T} |u_n|^{p_0-2} u_n S'(u_n)\nu \; dx\; dt = \int_{Q_T} f_n(x,t,T_n(u_n)) S'(u_n)\nu \; dx\; dt. \end{aligned} \tag{4.64}$$

For the first term on the left-hand side of (4.64), we have

$$\begin{aligned} \int_0^T \left\langle \frac{\partial u_n}{\partial t}, S'(u_n)\nu \right\rangle \; dt \quad &= \int_{Q_T} \frac{\partial S(u_n)}{\partial t} \nu \; dxdt \\ &= -\int_{Q_T} S(u_n) \frac{\partial \nu}{\partial t} \; dxdt - \int_\Omega S(u_{n,0})\nu(x,0) \; dx, \end{aligned} \tag{4.65}$$

Since $S(u_n) \rightharpoonup S(u)$ weak-* in $L^\infty(Q_T)$ and $\dfrac{\partial \nu}{\partial t} \in L^1(Q_T)$, it follows that

$$\int_{Q_T} S(u_n)\frac{\partial \nu}{\partial t}\,dxdt \to \int_{Q_T} S(u)\frac{\partial \nu}{\partial t}\,dxdt, \tag{4.66}$$

and in view of (4.63), we have

$$\int_\Omega S(u_{n,0})\nu(x,0)\,dx \to \int_\Omega S(u_0)\nu(x,0)\,dx. \tag{4.67}$$

From (4.66) and (4.67) we deduce that

$$\begin{aligned}\lim_{n\to\infty}\int_{Q_T}\frac{\partial S(u_n)}{\partial t}\nu\,dxdt &= -\int_{Q_T} S(u)\frac{\partial \nu}{\partial t}\,dxdt - \int_\Omega S(u_0)\nu(x,0)\,dx\\ &= \int_{Q_T}\frac{\partial S(u)}{\partial t}\nu\,dx\,dt.\end{aligned} \tag{4.68}$$

Having in mind (3.2) and (4.50), it yields $a_i(x,t,T_M(u_n),\nabla T_M(u_n))$ converges to $a_i(x,t,T_M(u),\nabla T_M(u))$ weakly in $L^{p_i'}(Q_T)$. Moreover, we have

$$D^iT_M(u_n)S''(u_n)\nu + S'(u_n)D^i\nu \longrightarrow D^iT_M(u)S''(u)\nu + S'(u)D^i\nu \text{ strongly in } L^{p_i}(Q_T).$$

Then, it follows that

$$\begin{aligned}&\lim_{n\to\infty}\int_{Q_T} a_i(x,t,T_M(u_n),\nabla T_M(u_n))(D^iT_M(u_n)S''(u_n)\nu + S'(u_n)D^i\nu)\,dx\,dt\\ &= \int_{Q_T} a_i(x,t,T_M(u),\nabla T_M(u))(D^iT_M(u)S''(u)\nu + S'(u)D^i\nu)\,dx\,dt\\ &= \int_{Q_T} a_i(x,t,u,\nabla u)(D^iuS''(u)\nu + S'(u)D^i\nu)\,dx\,dt,\end{aligned} \tag{4.69}$$

Moreover, in view of (4.37) we have $\dfrac{1}{n}|u_n|^{p_0-2}u_n \to 0$ strongly in $L^1(Q_T)$ and since $S'(u_n)\nu$ converges to $S'(u)\nu$ weak$-*$ in $L^\infty(Q_T)$, we obtain

$$\frac{1}{n}\int_{Q_T}|u_n|^{p_0-2}u_nS'(u_n)\nu\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n\to\infty, \tag{4.70}$$

similarly, in view of (4.39) we have

$$\int_{Q_T} f_n(x,t,T_n(u_n))S'(u_n)\nu\,dxdt \longrightarrow \int_{Q_T} f(x,t,u)S'(u)\nu dxdt \quad \text{as} \quad n\to\infty. \tag{4.71}$$

By combining (4.64) and (4.68) – (4.71), we obtain

$$\begin{aligned}&\int_0^T\langle\frac{\partial S(u)}{\partial t},\nu\rangle\,dt + \sum_{i=1}^N\int_{Q_T} a_i(x,t,u,\nabla u)(D^iuS''(u)\nu + D^i\nu S'(u))\,dx\,dt\\ &\quad= \int_{Q_T} f(x,t,u)S'(u)\nu\,dx\,dt,\end{aligned} \tag{4.72}$$

which conclude the proof of the Theorem 4.1.

5. Appendix

In view of growth conditions (3.2), (3.6) and the Hölder's inequality, we establish the boundedness of the operator B_n in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$.

Lemma 5.1. *The bounded operator B_n is pseudo-monotone. Moreover, B_n is coercive in the following sense :*

$$\frac{\displaystyle\int_0^T \langle B_n u, u\rangle \ dt}{\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}} \longrightarrow \infty \quad as \quad \|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \longrightarrow \infty,$$

for any $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$.

Proof of Lemma 5.1. In view of (3.4), we have

$$\begin{aligned}
&\int_0^T \langle B_n u, u\rangle \ dt \\
&= \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u),\nabla u) D^i u \, dxdt + \frac{1}{n}\int_{Q_T} |u|^{p_0} \, dxdt \\
&\quad - \int_{Q_T} f_n(x,t,T_n(u_n)) u \, dx\, dt \\
&\geq \sum_{i=1}^N \int_{Q_T} b(|T_n(u)|)|D^i u|^{p_i} \ dx\, dt + \frac{1}{n}\int_{Q_T} |u|^{p_0} \ dx\, dt - n\int_{Q_T} |u| \ dx\, dt \\
&\geq \frac{b_0}{(1+n)^\lambda} \sum_{i=1}^N \int_{Q_T} |D^i u|^{p_i} \ dx\, dt + \frac{1}{n}\int_{Q_T} |u|^{p_0} \ dx\, dt - nC_0\|u\|_{L^{p_0}(Q_T)} \\
&\geq \min\left\{\frac{b_0}{(1+n)^\lambda}, \frac{1}{n}\right\} \sum_{i=0}^N \int_{Q_T} |D^i u|^{p_i} \ dx\, dt - nC_0\|u\|_{L^{p_0}(Q_T)} \\
&\geq \min\left\{\frac{b_0}{(1+n)^\lambda}, \frac{1}{n}\right\} \|u\|^{\underline{p}}_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} - C_1(N+1) - nC_0\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))},
\end{aligned} \tag{5.1}$$

It follows that

$$\frac{\int_0^T \langle B_n u, u\rangle \ dt}{\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}} \geq \frac{\min\left\{\frac{b_0}{(1+n)^\lambda}, \frac{1}{n}\right\} \|u\|^{\underline{p}}_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}}{\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}} - \frac{C_1(N+1) + nC_0\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}}{\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}}.$$

Then, the coercivity of the operator B_n is concluded.
It remains to establish that B_n is a pseudo-monotone operator. Indeed, we consider the sequence $(u_k)_{k\in\mathbb{N}}$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, such that

$$\begin{cases}
u_k \rightharpoonup u \text{ weakly in } L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)), \\
B_n u_k \rightharpoonup \chi_n \ \text{ weakly in } \ L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)), \\
\limsup\limits_{k\to\infty} \displaystyle\int_0^T \langle B_n u_k, u_k\rangle \ dt \leq \int_0^T \langle \chi_n, u\rangle \ dt.
\end{cases} \tag{5.2}$$

The objective is to show that

$$B_n u = \chi_n \qquad \text{and} \qquad \langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \quad \text{as} \quad k \to \infty.$$

In view of Remark 2.1, we have u_k converges to u strongly in $L^1(Q_T)$ and $u_k \to u$ a.e in Q_T, for a subsequence still denoted $(u_k)_{k\in\mathbb{N}^*}$, then thanks to (3.2), we have $(a_i(x,t,T_n(u_k),\nabla u_k))_k$ is bounded sequence in $L^{p_i'}(Q_T)$, then there exists a measurable function $\psi_{i,n} \in L^{p_i'}(Q_T)$ where

$$a_i(x,t,T_n(u_k),\nabla u_k) \rightharpoonup \psi_{i,n} \quad \text{weakly in} \quad L^{p_i'}(Q_T) \quad \text{for} i = 1,\dots,N. \tag{5.3}$$

Moreover, we have $f_n(x,t,u_k)$ converges to $f_n(x,t,u)$ almost everywhere in Q_T and $|f_n(x,t,u_k)| \le n$. In view of Lebesgue's dominated convergence theorem we get

$$f_n(x,t,u_k) \longrightarrow f_n(x,t,u) \qquad \text{strongly in} \quad L^{p_0'}(Q_T), \tag{5.4}$$

also, we have

$$\frac{1}{n}|u_k|^{p_0-2}u_k \rightharpoonup \frac{1}{n}|u|^{p_0-2}u \qquad \text{weakly in} \quad L^{p_0'}(Q_T). \tag{5.5}$$

Thanks to (5.2), we have

$$\begin{aligned}
&\int_0^T \langle \chi_n, v\rangle \, dt = \lim_{k\to\infty} \int_0^T \langle B_n u_k, v\rangle \, dt \\
&= \lim_{k\to\infty} \Big(\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k) D^i v \, dx + \frac{1}{n}\int_{Q_T} |u_k|^{p_0-2}u_k v \, dx\, dt \\
&\qquad\qquad - \int_{Q_T} f_n(x,t,u_k) v \, dx\, dt \Big) \\
&= \sum_{i=1}^N \int_{Q_T} \psi_{i,n} D^i v \, dx\, dt + \frac{1}{n}\int_{Q_T} |u|^{p_0-2}uv \, dx\, dt - \int_{Q_T} f_n(x,t,u) v \, dx\, dt,
\end{aligned}$$

for any $v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$. Thus, according to (5.2) we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \int_0^T \langle B_n u_k, u_k\rangle \, dt \\
&= \limsup_{k\to\infty} \Big(\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k) D^i u_k \, dx\, dt + \frac{1}{n}\int_{Q_T} |u_k|^{p_0-2}u_k u_k \, dx\, dt \\
&\qquad\qquad - \int_{Q_T} f_n(x,t,u_k) u_k \, dx\, dt \Big) \\
&\le \sum_{i=1}^N \int_{Q_T} \psi_{i,n} D^i u \, dx\, dt + \frac{1}{n}\int_{Q_T} |u|^{p_0} \, dx\, dt - \int_{Q_T} f_n(x,t,u) u \, dx\, dt.
\end{aligned}$$

In the light of (5.3) − (5.5), we conclude that

$$\begin{aligned}
&\limsup_{k\to\infty} \Big(\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k) D^i u_k \, dx\, dt + \frac{1}{n}\int_{Q_T} |u_k|^{p_0} \, dx\, dt \Big) \\
&\le \sum_{i=1}^N \int_{Q_T} \psi_{i,n} D^i u \, dx\, dt + \frac{1}{n}\int_{Q_T} |u|^{p_0} \, dx\, dt.
\end{aligned} \tag{5.6}$$

On the one hand, thanks to (3.3) we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_{Q_T}(a_i(x,t,T_n(u_k),\nabla u_k)-a_i(x,t,T_n(u_k),\nabla u))(D^iu_k-D^iu)\;dx\,dt\\&\quad+\frac{1}{n}\int_{Q_T}(|u_k|^{p_0-2}u_k-|u|^{p_0-2}u)(u_k-u)\;dx\,dt\geq 0,\end{aligned}$$

it follows that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu\;dx\,dt\\&\quad+\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u)(D^iu_k-D^iu)\;dx\,dt\\&\quad+\frac{1}{n}\int_{Q_T}|u_k|^{p_0-2}u_ku\;dx\,dt+\frac{1}{n}\int_{Q_T}|u|^{p_0-2}u(u_k-u)\;dx\,dt\\&\leq\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu_k\;dx\,dt+\frac{1}{n}\int_{Q_T}|u_k|^{p_0}\;dx\,dt.\end{aligned}$$

In view of Lebesgue's dominated convergence theorem, we have $T_n(u_k)$ converges to $T_n(u)$ strongly in $L^{p_i}(Q_T)$, then $a_i(x,t,T_n(u_k),\nabla u)$ converges to $a_i(x,t,T_n(u),\nabla u)$ strongly in $L^{p_i'}(Q_T)$, and thanks to (5.3) and (5.5), we get

$$\begin{aligned}&\sum_{i=1}^{N}\int_{Q_T}\psi_{i,n}D^iu\;dx\,dt+\int_{Q_T}\frac{1}{n}|u|^{p_0}\;dx\,dt\\&\leq\liminf_{k\to\infty}\Big(\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu_k\;dx\,dt+\int_{Q_T}\frac{1}{n}|u_k|^{p_0}\;dx\,dt\Big),\end{aligned}\tag{5.7}$$

Thanks to (5.6), we conclude that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{Q_T}\psi_{i,n}D^iu\;dx\,dt+\frac{1}{n}\int_{Q_T}|u|^{p_0}\;dx\,dt\\&=\lim_{k\to\infty}\Big(\sum_{i=1}^{N}\int_{Q_T}a_i(x,t,T_n(u_k),\nabla u_k)D^iu_k\;dx\,dt+\frac{1}{n}\int_{Q_T}|u_k|^{p_0}\;dx\,dt\Big),\end{aligned}\tag{5.8}$$

Having in mind (5.4), we have

$$\lim_{k\to\infty}\int_{Q_T}f_n(x,t,u_k)u_k\;dx\,dt=\int_{Q_T}f_n(x,t,u)u\;dx\,dt.\tag{5.9}$$

We conclude that $\lim\limits_{k\to\infty}\int_0^T\langle B_nu_k,u_k\rangle\;dt=\int_0^T\langle\chi_n,u\rangle\;dt.$
Moreover, thanks to (5.8), we have

$$\begin{aligned}&\lim_{k\to\infty}\Big(\sum_{i=1}^{N}\int_{Q_T}(a_i(x,t,T_n(u_k),\nabla u_k)-a_i(x,t,T_n(u_k),\nabla u))(D^iu_k-D^iu)\;dx\,dt\\&\quad+\int_{Q_T}(\frac{1}{n}|u_k|^{p_0-2}u_k-\frac{1}{n}|u|^{p_0-2}u)(u_k-u)\;dx\,dt\Big)=0,\end{aligned}$$

Thanks to Lemma 3.1, we deduce that

$$u_k \longrightarrow u \quad \text{strongly in} \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(Q_T)) \quad \text{and} \quad D^i u_k \longrightarrow D^i u \quad \text{a.e in} \quad Q_T.$$

As a consequence we have

$$a_i(x,t,T_n(u_k),\nabla u_k) \rightharpoonup a_i(x,t,T_n(u),\nabla u) \quad \text{weakly in} \quad L^{p_i'}(Q_T), \quad \text{for} \quad i=1,\dots,N.$$

Finally, by using (5.3), (5.4) and (5.5) we conclude that $B_n u = \chi_n$. which conclude the proof of the Lemma 5.1.

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