Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 52(2), 2025, Pages 485–507, DOI: 10.52846/ami.v52i2.2002
ISSN: 1223-6934

Doubly-measure Pseudo S-asymptotically Bloch Type Periodicity and Applications to some Stochastic Integrodifferential Equations

Mamadou Moustapha Mbaye, Amadou Diop, and Bipan Hazarika

Abstract. In the present work, for a separable complex Hilbert space $\mathbb{V}$, we introduce the concept of doubly-measure pseudo S-asymptotically Bloch-type periodicity to define the space of $(\nu, \mu)-$pseudo S-asymptotically Bloch-type periodic (or $(\omega, k)-$periodic) stochastic process with values in the complex Banach space of all strongly-measurable, p-integrable $\mathbb{V}$-valued random variables. We first looked into some completeness, composition and convolution theorems for such stochastic processes. Second, the existence and uniqueness in the p^{th}-mean $(\nu, \mu)-$pseudo S-asymptotically Bloch-type periodic ((ν, μ)-PSABP, in short) mild solutions of some stochastic integrodifferential equations is formally investigated. In conclusion, we provide examples to support our findings.

2020 *Mathematics Subject Classification.* 30D45, 34C25, 60H15, 60G22.
Key words and phrases. Stochastic processes, stochastic evolution equations, Fractional Brownian motion, Pseudo S-asymptotically Bloch-type periodic functions.

The theory of (ω, k)-periodic (or Bloch-type periodic) phenomena has its origins in the publication [4] by F. Bloch, investigating the crystalline solid's conductivity. Bloch-type periodic functions have been formally examined by N'guérékata and Hasler in [31], and thus, ω-periodicity and ω-anti-periodicity concepts become particular cases of this notion. To characterize the impact of perturbations on (ω, k)–periodic functions, some quasi-(ω, k)-periodicity concepts are presented in some publications. For example, the asymptotically Bloch periodic function has been studied with its applications in [31, 32, 17], while notions of (pseudo) S-asymptotically (ω, k)–periodic functions have been investigated with its applications in [14, 15, 16]. The previous mentioned quasi-Bloch periodicity's notions can be viewed as generalization of classical asymptotically ω-periodic and (pseudo) S-asymptotically ω-periodic functions in the deterministic case which have been examined in several researches, see [12, 2, 11, 35, 8, 9, 28, 29, 24, 33, 34, 42, 40, 46].

On the other hand, with the aid of measure theory, Blot et al. [6] introduced the notions μ-ergodic function and provided some fundamental properties of μ-pseudo-almost periodic (μ-PAP) functions which encloses the classical concepts of PAP functions due to Zhang [47, 48, 49] and weighted PAP functions introduced by Diagana [25, 26] as particular cases. From then on, many papers have been devoted to the study of μ-PAP function from many ways [27]. Due to the fact that most real life phenomena are basically stochastic rather than deterministic, a tremendous interest in generalizing certain classical deterministic concepts to stochastic one has been noted

Received August 10, 2024. Accepted August 9, 2025.

in the literature. For instance, in [23] the authors introduced and studied the concept of p^{th}-mean μ-PAP processes and obtained sufficient condition for the existence of p^{th}-mean μ-PAP mild solutions to some class of non-linear stochastic evolution equations. The authors of [7] formulated the notion of (ν, μ)-pseudo almost periodic processes and gave some results for well-posedness of (ν, μ)-pseudo almost periodic mild solutions in p^{th}-mean sense for a class of non-linear stochastic evolution systems.

However, we mentioned that the researches which focus on the notion of S-asymptotically ω-periodicity for stochastic processes and related application on stochastic evolution systems is rather well furnished (see [21, 50, 51, 45]). Moreover, up to now, no work has been reported yet regarding the concept of (ν, μ)-PSABP in the p^{th}-mean sense for stochastic processes, which mainly motivates this present study. This issue is interesting and new and, hence, the question even if there exists a (ν, μ)-PSABP mild solution in p^{th}-mean sense is still untreated for stochastic evolutions systems.

In this work there are three fundamental goals, described as follows:

(1) Firstly, we introduce a new concept of p^{th}-mean (ν, μ)-pseudo S-asymptotically Bloch-type periodic for stochastic processes and establish some composition, completeness, and convolution theorems for such stochastic processes.

(2) Secondly, we investigate the existence and uniqueness of p^{th}-mean (ν, μ)-pseudo S-asymptotically Bloch-type periodic mild solutions to the following stochastic evolution equations:

$$\begin{cases} dz(\tau) = \left[Az(\tau) + \alpha \displaystyle\int_{-\infty}^{\tau} \frac{(\tau-u)^{\mathbf{b}-1}}{\Gamma(\mathbf{b})} e^{-\mathbf{a}(\tau-u)} Az(u)du + g(\tau, z(\tau)) \right] d\tau \\ \qquad\qquad +f(\tau, z(\tau))dW(\tau), \quad \tau \in \mathbb{R}, \end{cases} \tag{1}$$

where A is a closed linear operator generator of an uniformly stable and strongly continuous family operators $\{\mathcal{R}(\tau)\}_{\tau \geq 0} \subset \mathcal{B}(\mathbb{V})$ on a separable complex Hilbert space $\mathbb{V}$, $\alpha \neq 0$, $\mathbf{a} > 0$, $\mathbf{b} \geq 1$, z, g, f are $\mathbb{V}$-valued stochastic processes, $\Gamma(\cdot)$ is the Gamma function. Here $(W(\tau))_{\tau\in\mathbb{R}}$ represents a two-sided and standard one-dimensional Brownian motion on $\mathbb{V}$. If $f(\tau, z(\tau)) \equiv 0$, then problem (1)degrades to the following deterministic semilinear integro-differential equation

$$\frac{d}{d\tau} z(\tau) = Az(\tau) + \int_{-\infty}^{\tau} \xi(\tau-u)Az(u)du + g(\tau, z(\tau)), \quad \tau \in \mathbb{R}. \tag{2}$$

where $\xi(\tau) = \dfrac{\alpha(\tau)^{\mathbf{b}-1}}{\Gamma(\mathbf{b})} e^{-\mathbf{a}\tau}$. In [35], Lizama and N'Guérékata provided sufficient conditions for the existence and uniqueness of bounded solutions, such as (asymptotically) ω-periodic solutions, S-asymptotically ω-periodic solutions, (asymptotically, pseudo) almost periodic and (asymptotically, pseudo) almost automorphic solutions to problem (1) when g is bounded continuous with certain recurrence. For bounded solutions to some occurence of problem (2) with some specific kernels $\xi(\cdot)$, we refer to [10, 19, 36] and its references to address this issue.

(3) Thirdly, we give also the existence and uniqueness of p^{th}-mean (ν, μ)-pseudo S-asymptotically Bloch-type periodic mild solutions to the following class of

stochastic fractional evolution equations:

$$\begin{cases} \partial_\tau^\alpha z(\tau) = Az(\tau) + \alpha \displaystyle\int_{-\infty}^{\tau} \frac{(\tau-u)^{\mathbf{b}-1}}{\Gamma(\mathbf{b})} e^{-\mathbf{a}(\tau-u)} Az(u)du + g(\tau, z(\tau)) \\ \qquad + f(\tau, z(\tau)\dfrac{dW(\tau)}{d\tau}, \quad \tau \in \mathbb{R}, \end{cases} \tag{3}$$

where ∂_τ^α denotes the Weyl fractional derivative of order $\alpha > 0$, A generate a α-resolvent family $\{\mathcal{R}_\alpha(\tau)\}_{\tau\geq 0}$ on a Hilbert space $\mathbb{V}$. Noted that when $f(\tau, z(\tau)) \equiv 0$, then problem (3) is reduced to the following deterministic semilinear integro-differential equation

$$\partial_\tau^\alpha z(\tau) = Az(\tau) + \int_{-\infty}^{\tau} \xi(\tau-u)Az(u)du + g(\tau, z(\tau)), \quad \tau \in \mathbb{R}, \tag{4}$$

where $\xi(\tau) = \dfrac{\alpha(\tau)^{\mathbf{b}-1}}{\Gamma(\mathbf{b})} e^{-\mathbf{a}\tau}$. With some specific kernels $\xi(\cdot)$, bounded solutions to problem (4) is first explored in [44], in which the existence and uniqueness of (asymptotically) ω-periodic solutions, S-asymptotically ω-periodic solutions, (asymptotically, pseudo) almost periodic and (asymptotically, pseudo) almost automorphic solutions are studied when g is a bounded continuous function with certain condition. Some existence results of weighted pseudo almost automorphic solutions to problem (3) when g is Stepanov-like weighted pseudo almost automorphic are established in [13], and the existence and uniqueness of weighted pseudo antiperiodic solutions to problem (3)) when g is Stepanov-like weighted pseudo antiperiodic is accomplished in[3]. Oueama-Guengai and N'Guérékata [41] studied the existence and uniqueness of S-asymptotically ω-periodic and (ω, k)-Bloch periodic solutions to problem (3) when g is a bounded continuous function satisfying additional conditions.

Let us mention that some special forms of problems (1) and (3) have also been investigated but to the best of our knowledge, no work has been published on the existence and uniqueness of p^{th}-mean (ν, μ)-pseudo S-asymptotically Bloch-type periodic mild solutions of the above problems and we propose extending the recent results on Bloch-type periodic (or $(\omega, k)-$periodic) stochastic process developed in [21, 22, 39]. Additionally, the present work can be considered as a continuation of [41, 44, 31] in the stochastic setting when it comes to Bloch- periodic process. Note that, problems of types (1) and (3) usually arises in the models of viscoelastic materials or memory materials where stochastic effects need to be considered (see for instance [37]). The obtained outcomes show that for each p^{th}-mean (ν, μ)-pseudo S-asymptotically Bloch-type periodic input, the output is still a bounded and continuous mild solutions to the reference equation, which is also p^{th}-mean (ν, μ)-pseudo S-asymptotically Bloch-type periodic.

This paper is organized as follows: Section 1 is concerned with some basic definitions, lemmas, notations, and mainly focused on properties of p^{th}-mean (ν, μ)-pseudo S-asymptotically Bloch-type periodicity. Section 2 is concerned to applications to some stochastic evolution equations in Hilbert spaces. To end this work, we give some illustrations in Section 3.

1. Preliminaries

Suppose that $(\mho, \mathcal{F}, \mathbb{P})$ represents a probability space, $(\mathbb{V}, \|\cdot\|)$ is a complex separable Hilbert space and $\big(\mathbb{L}^p(\mho, \mathbb{V}), \|\cdot\|_{\mathbb{L}^p}\big)$ $(p \geq 2)$ is the complex Banach space of all strongly-measurable, p-integrable $\mathbb{V}$-valued random variables, equipped with the norm

$$\|\psi\|_{\mathbb{L}^p} = (\mathsf{E}\|\psi\|^p)^{1/p},\ \psi \in \mathbb{L}^p(\mho, \mathbb{V}),$$

where $\mathsf{E}(\cdot)$ is the expectation defined by $(\mathsf{E}\|\psi\|)^p = \displaystyle\int_{\mho} \|\psi\|^p d\mathbb{P}$. For each $\tau \in \mathbb{R}$, $\mathcal{F}_\tau$ is the σ-field generated by the random variables $\big\{W(u), u \leq \tau\big\}$ and the $\mathbb{P}$-null sets.

Definition 1.1. A stochastic process $\psi : \mathbb{R} \to \mathbb{L}^p(\mho, \mathbb{V})$ is referred to as stochastically bounded and continuous process if there exist $\zeta > 0$ such that

$$\mathsf{E}\|\psi(\tau)\|^p = \int_{\mho} \|\psi(\tau)\|^p d\mathbb{P} < \zeta, \quad \forall \tau \in \mathbb{R}$$

and

$$\lim_{u\to v} \mathsf{E}\|\psi(u) - \psi(v)\|^p = 0 \quad \text{for all} \quad v \in \mathbb{R}.$$

We will use the notation $(\mathcal{BC}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V})), \|\cdot\|_\infty)$ (for $p \geq 2$) represents the complex Banach space of all stochastically bounded and continuous processes ψ : $\mathbb{R} \to \mathbb{L}^p(\mho, \mathbb{V})$ such that $\|\psi\|_\infty = \left(\sup\limits_{s\in\mathbb{R}} \mathsf{E}\|\psi(s)\|^p\right)^{1/p} < \infty$.
The following particular Burkholder-Davis-Gundy type.

Lemma 1.1. [?] *Let $q > 0$ and $\psi : [0, q] \times \mho \to \mathbb{L}^p(\mho, \mathbb{V})$ be an $\mathcal{F}_\tau$-adapted measurable stochastic process such that*

$$\int_0^q \mathsf{E}\|\psi(\varrho)\|^2 d\varrho < \infty \ \ a.\,s.$$

Then $\forall\, p \geq 1$, $\exists\, C_p > 0$ such that

$$\mathsf{E} \sup_{\tau\in[0,q]} \left\| \int_0^\tau \psi(\varrho) dW(\varrho) \right\|^p \leq C_p \mathsf{E} \left(\int_0^q \|\psi(\varrho)\|^2 d\varrho \right)^{p/2}.$$

1.1. p^{th}-mean (ν, μ)-pseudo S-asymptotically Bloch-type periodic process.

We use the symbol $\mathcal{B}$ to represent the Lebesgue σ-field of $\mathbb{R}$, and the set of all positive measures μ on $\mathcal{B}$ such that $\mu(\mathbb{R}) = +\infty$ and $\mu([q_1, q_2]) < +\infty$ for all $q_1, q_2, \in \mathbb{R}$ $(q_1 \leq q_2)$ will be denoted by $\mathcal{M}$.

Definition 1.2. Let $\mu \in \mathcal{M}$ and $p \geq 2$. A stochastic process $\psi \in \mathcal{BC}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$ is referred as p^{th}-mean $(\nu, \mu)-$speudo-S-asymptotically Bloch-type periodic (or (ω, k) periodic) if for given $\omega \in \mathbb{R}, k \in \mathbb{R}$,

$$\lim_{m\to+\infty} \frac{1}{\nu([-m, m])} \int_{-m}^{m} \mathbb{E}\|\psi(s + \omega) - e^{ik\omega}\psi(s)\|^p d\mu(s) = 0.$$

The set of all the p^{th}-mean (ν,μ)-pseudo-S-asymptotically (ω,k)-periodic stochastic processes is denoted by $\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$ and

$$\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}\times\mathbb{L}^p(\mho,\mathbb{V}),\mathbb{L}^p(\mho,\mathbb{V})) = \Big\{h(\cdot,z)\in\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V})) \mid \text{ for any } z\in\mathbb{L}^p(\mho,\mathbb{V})\Big\}.$$

Now, we can show the following basic properties.

Lemma 1.2. *Let $p\geq 2$ and $z_1,z_2,z\in\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$. The subsequent results hold:*

(a): *$z_1+z_2\in\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$, and $az\in\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$ for each $a\in\mathbb{C}$.*

(b): *Assume that $\limsup_{m\to+\infty}\dfrac{\nu([-m,m])}{\mu([-m,m])}=l<\infty$; then $\left(\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V})),\|\cdot\|_\infty\right)$ is a Banach space.*

Proof. (a) Using Definition 1.2, we have

$$\begin{aligned}
&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|(z_1+z_2)(\tau+\omega)-e^{ik\omega}(z_1+z_2)(\tau)\|^p d\mu(\tau)\\
&\qquad\leq\frac{2^{p-1}}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z_1(\tau+\omega)-e^{ik\omega}z_1(\tau)\|^p d\mu(\tau)\\
&\qquad\quad+\frac{2^{p-1}}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z_2(\tau+\omega)-e^{ik\omega}z_2(\tau)\|^p d\mu(\tau)\\
&\qquad\longrightarrow 0 \text{ as } m\to\infty
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|az(\tau+\omega)-e^{ik\omega}az(\tau)\|^p d\mu(\tau)\\
&=\frac{|a|^p}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)\\
&\longrightarrow 0 \text{ as } m\to\infty.
\end{aligned}$$

Thus $z_1+z_2,\ a\,z\in\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$.

(b) Let $\{z_n\}_n\subseteq\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$ such that $\lim_{n\to\infty}\|z_n-z\|_\infty=0$ as $n\to\infty$. We have

$$\begin{aligned}
&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)\\
=&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z(\tau+\omega)-z_n(\tau+\omega)+z_n(\tau+\omega)-e^{ik\omega}z_n(\tau)+e^{ik\omega}z_n(\tau)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)\\
\leq&\frac{3^{p-1}}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z(\tau+\omega)-z_n(\tau+\omega)\|^p d\mu(\tau)+\frac{3^{p-1}}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z_n(\tau+\omega)-e^{ik\omega}z_n(\tau)\|^p d\mu(\tau)\\
&+\frac{3^{p-1}}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|e^{ik\omega}z_n(\tau)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)
\end{aligned}$$

$$\leq 3^{p-1}\frac{\mu([-m,m])}{\nu([-m,m])}\sup_{\tau\in\mathbb{R}}\left(\mathsf{E}\|z(\tau)-z_n(\tau)\|^p\right)+\frac{3^{p-1}}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z_n(\tau+\omega)-e^{ik\omega}z_n(\tau)\|^p d\mu(\tau)$$
$$+\,3^{p-1}\frac{\mu([-m,m])}{\nu([-m,m])}\sup_{\tau\in\mathbb{R}}\left(\mathsf{E}\|z_n(\tau)-z(\tau)\|^p\right).$$

It follows that,

$$\limsup_{m\to+\infty}\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)$$
$$\leq 3^{p-1}\,l\,\|z-z_n\|_\infty^p+3^{p-1}\limsup_{m\to+\infty}\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z_n(\tau+\omega)-e^{ik\omega}z_n(\tau)\|^p d\mu(\tau)$$
$$+\,3^{p-1}\,l\,\|z_n-z\|_\infty^p.$$

Since $\lim_{n\to\infty}\|z_n-z\|^p=0$ and

$$\lim_{m\to+\infty}\frac{3^{p-1}}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z_n(\tau+\omega)-e^{ik\omega}z_n(\tau)\|^p d\mu(\tau)=0,$$

we deduce that

$$\lim_{m\to+\infty}\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)=0.$$

This implies that the space $\mathcal{SABP}_{\omega,k}^{\nu,\mu}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$ is a closed sub-space of $\mathcal{BC}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$, so it is a Banach space equipped with the sup-norm. □

The following lemma offers a characterization of processes belonging to the class of p^{th}-mean $(\nu,\mu)-$speudo-S-asymptotically Bloch-type periodic.

Lemma 1.3. *Let $\nu,\mu\in\mathcal{M}$ such that $\limsup_{m\to+\infty}\frac{\mu([-m,m])}{\nu([-m,m])}=l<\infty$, Y be a bounded interval (possibly $Y=\emptyset$) and $z\in\mathcal{BC}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$, then the following assertions are equivalent:*

(1) $z\in\mathcal{SABP}_{\omega,k}^{\nu,\mu}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$;

(2) $\lim_{m\to\infty}\frac{1}{\nu([-m,m]\backslash Y)}\int_{[-m,m]\backslash Y}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)=0$;

(3) *For each $\epsilon>0$,*

$$\lim_{m\to+\infty}\frac{\mu(\mathcal{Q}_{m,\epsilon}(z))}{\nu([-m,m]\backslash Y)}=0,\tag{5}$$

where $\mathcal{Q}_{m,\epsilon}(z)=\{\tau\in[-m,m]\backslash Y:\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p\geq\epsilon\}$.

Proof. **Claim 1 :** Let prove that (1) $\Longleftrightarrow$ (2).
Denote by $A=\nu(Y)$ and $B=\int_Y\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)$ and $C=\mu(Y)$. Thanks to the boundedness of the interval Y and $z\in\mathcal{BC}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$, we deduce that A, B and C are finite. Let $m>0$ be such that $Y\subset[-m,m]$ and $\nu([-m,m]\backslash Y)>0$.

Then, we have

$$\begin{aligned}&\frac{1}{\nu([-m,m]\backslash Y)}\int_{[-m,m]\backslash Y}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)\\&=\frac{1}{\nu([-m,m])-A}\left(\int_{[-m,m]}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)-B\right)\\&=\frac{\nu([-m,m])}{\nu([-m,m])-A}\left(\frac{1}{\nu([-m,m])}\int_{[-m,m]}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)-\frac{B}{\nu([-m,m])}\right).\end{aligned} \tag{6}$$

Since $\nu(\mathbb{R})=+\infty$, we derive that

$$\frac{1}{\nu([-m,m])}\int_{[-m,m]}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)=0.$$

Claim 2 : We prove that (2) $\Longrightarrow$ (3).
Suppose that (2) holds. For given $\epsilon>0$, we have

$$\begin{aligned}&\frac{1}{\nu([-m,m]\backslash Y)}\int_{\mu([-m,m]\backslash Y)}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)\\&\geq\frac{1}{\nu([-m,m]\backslash Y)}\int_{\mathcal{Q}_{m,\epsilon}(z)}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)\\&\geq\epsilon\frac{\mu(\mathcal{Q}_{m,\epsilon}(z))}{\nu([-m,m]\backslash Y)}\geq 0.\end{aligned}$$

Consequently, for m large enough, we get (3).
Claim 3 : We prove that (3) $\Longrightarrow$ (2)
Suppose that (3) hold. Let $z\in\mathcal{BC}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$. There exists a positive number $\zeta>0$ such that $\mathsf{E}\|z(\tau)\|^p\leq\zeta$ for $\tau\in\mathbb{R}$. Since $\lim_{m\to+\infty}\dfrac{\mu(\mathcal{Q}_{m,\epsilon}(z))}{\nu([-m,m]\backslash Y)}=0$, we derive that for any $\epsilon>0$, there exists $\zeta>0$ such that for $m\geq\zeta$,

$$\frac{\mu(\mathcal{Q}_{m,\epsilon}(z))}{\nu([-m,m]\backslash Y)}\leq\frac{\epsilon}{2^p\zeta+1}.$$

We have

$$\begin{aligned}&\frac{1}{\nu([-m,m]\backslash Y)}\int_{-m}^{m}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p\,d\mu(\tau)\\&=\frac{1}{\nu([-m,m]\backslash Y)}\int_{\mathcal{Q}_{m,\epsilon}(z)}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p\,d\mu(\tau)\\&\quad+\frac{1}{\nu([-m,m]\backslash Y)}\int_{([-m,m]\backslash Y)\backslash\mathcal{Q}_{m,\epsilon}(z)}\mathsf{E}\|z(\tau+\omega)-e^{ik\omega}z(\tau)\|^p d\mu(\tau)\\&\leq\frac{1}{\nu([-m,m]\backslash Y)}\int_{\mathcal{Q}_{m,\epsilon}(z)}2^{p-1}\left(\mathsf{E}\|z(\tau+\omega)\|^p+\mathsf{E}\|z(\tau)\|^p\right)\,d\mu(\tau)\\&\quad+\frac{1}{\nu([-m,m]\backslash Y)}\int_{([-m,m]\backslash Y)\backslash\mathcal{Q}_{m,\epsilon}(z)}\epsilon\,d\mu(\tau)\\&\leq 2^p\zeta\frac{\mu(\mathcal{Q}_{m,\epsilon}(z))}{\nu([-m,m]\backslash Y)}+\frac{\epsilon}{\nu([-m,m]\backslash Y)}\int_{[-m,m]\backslash Y}d\mu(\tau)\end{aligned}$$

$$\leq \frac{2^p\zeta}{2^p\zeta+1}\epsilon + \frac{\epsilon\mu([-m,m]\backslash Y)}{\nu([-m,m]\backslash Y)} \leq \epsilon + \frac{\mu([-m,m])-\mu(Y)}{\nu([-m,m])-\nu(Y)}\epsilon$$
$$\leq \epsilon + \frac{\mu([-m,m])}{\nu([-m,m])} \times \frac{1-\frac{\mu(Y)}{\mu([-m,m])}}{1-\frac{\nu(Y)}{\nu([-m,m])}}\epsilon.$$

By the fact that $\mu(\mathbb{R}) = \nu(\mathbb{R}) = +\infty$ and $\limsup_{m\to+\infty} \frac{\mu([-m,m])}{\nu([-m,m])} = l < \infty$, it follows that

$$\limsup_{m\to+\infty} \frac{1}{\nu([-m,m]\backslash Y)} \int_{-m}^{m} \mathsf{E}\|z(\tau+\omega) - e^{ik\omega}z(\tau)\|^p \, d\mu(\tau) \leq (1+l)\epsilon.$$

This implies that (2) holds. □

Definition 1.3 ([5]). Let μ_1 and $\mu_2 \in \mathcal{M}$. μ_1 is referred to be equivalent to μ_2 ($\mu_1 \sim \mu_2$) If there exist constants α and $\beta > 0$ and a bounded interval Y (eventually $Y = \emptyset$) such that

$$\alpha\mu_1(K) \leq \mu_2(K) \leq \beta\mu_1(K)$$

for $K \in \mathcal{B}$ verifying $K \cap Y = \emptyset$.

Theorem 1.4. *Let μ_1, μ_2, ν_1 and $\nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then*

$$\mathcal{SABP}^{\nu_1,\mu_1}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V})) = \mathcal{SABP}^{\nu_2,\mu_2}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V})).$$

Proof. Since $\mu_1 \sim \mu_2$, $\nu_1 \sim \nu_2$ and $\mathcal{B}$ the Lebesgue σ-field $\mathbb{R}$, there exist $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} > 0$, such that

$$\alpha\,\mu_1 \leq \mu_2 \leq \beta\,\mu_1 \quad \text{and} \quad \widetilde{\alpha}\,\nu_1 \leq \nu_2 \leq \widetilde{\beta}\,\nu_1.$$

We obtain for m sufficiently large

$$\frac{\alpha}{\widetilde{\beta}} \frac{\mu_1\Big\{\tau \in [-m,m]\backslash Y : \mathsf{E}\|z(\tau+\omega) - e^{ik\omega}z(\tau)\|^p \geq \epsilon\Big\}}{\nu_1\{\tau \in [-m,m]\backslash Y\}}$$
$$\leq \frac{\mu_2\Big\{\tau \in [-m,m]\backslash Y : \mathsf{E}\|z(\tau+\omega) - e^{ik\omega}z(\tau)\|^p \geq \epsilon\Big\}}{\nu_2\{\tau \in [-m,m]\backslash Y\}}$$
$$\leq \frac{\beta}{\widetilde{\alpha}} \frac{\mu_1\Big\{\tau \in [-m,m]\backslash Y : \mathsf{E}\|z(\tau+\omega) - e^{ik\omega}z(\tau)\|^p \geq \epsilon\Big\}}{\nu_1\{\tau \in [-m,m]\backslash Y\}}.$$

By using Lemma 1.3, we deduce that

$$\mathcal{SABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}), \mu_1) = \mathcal{SABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}), \mu_2)$$

□

For $\mu \in \mathcal{M}$ and $y \in \mathbb{R}$, we let μ_y the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_y(X) = \mu(\{x + y : x \in X\}) \text{ for } X \in \mathcal{B}.$$

Now, from $\mu \in \mathcal{M}$, we come up with the subsequent assumption:

($\mathbf{H^0}$) For all $y \in \mathbb{R}$, there exists a bounded interval Y and $q > 0$ satisfying

$$\mu_y(X) \leq q\,\mu(X) \text{ when } X \in \mathcal{B} \text{ satisfies } X \cap Y = \emptyset.$$

And, we recall the following useful result:

Lemma 1.5 ([5]). *Let $\mu \in \mathcal{M}$.*
(i): *μ fulfills $(\mathbf{H^0})$ if and only if $\mu \sim \mu_y$ for all $y \in \mathbb{R}$.*
(ii): *If condition $(\mathbf{H^0})$ hold, then for all $\sigma > 0$,*

$$\limsup_{m\to+\infty} \frac{\mu([-m-\sigma, m+\sigma])}{\mu([-m,m])} < +\infty.$$

Lemma 1.6. *Let $\nu, \mu \in \mathcal{M}$ and $f \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$. If μ, ν fulfills $(\mathbf{H^0})$ then $f_y \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$ for each $y \in \mathbb{R}$, where $f_y(\tau) := f(\tau + y)$ for each $\tau \in \mathbb{R}$.*

Proof. Let $f \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$ and $y \in \mathbb{R}$. It gives that

$$\begin{aligned}
0 &\leq \frac{1}{\nu([-m,m])} \int_{[-m,m]} \mathsf{E}\|f(\tau+y+\omega) - e^{ik\omega} f(\tau+y)\|^p d\mu(\tau) \\
&= \frac{\nu([-m-|y|, m+|y|])}{\nu([-m,m])} \cdot \left(\frac{1}{\nu([-m-|y|, m+|y|])} \int_{[-m,m]} \mathsf{E}\|f(\tau+y+\omega) - e^{ik\omega} f(\tau+y)\|^p d\mu(\tau) \right) \\
&= \frac{\nu([-m-|y|, m+|y|])}{\nu([-m,m])} \cdot \left(\frac{1}{\nu([-m-|y|, m+|y|])} \int_{[-m+y, m+y]} \mathsf{E}\|f(\tau+\omega) - e^{ik\omega} f(\tau)\|^p d\mu_{-y}(\tau) \right) \\
&\leq \frac{\nu([-m-|y|, m+|y|])}{\nu([-m,m])} \cdot \left(\frac{q}{\nu([-m-|y|, m+|y|])} \int_{[-m-|y|, m+|y|]} \mathsf{E}\|f(\tau+\omega) - e^{ik\omega} f(\tau)\|^p d\mu(\tau) \right),
\end{aligned}$$

where $q > 0$ is a constant, ensuring the equivalence between μ and μ_y. Thanks to Lemma 1.5-(ii), we derive that $f_y \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$ for all $y \in \mathbb{R}$. $\square$

Now, present some compositions results and convolutions theorems for p^{th}-mean (ν, μ)-pseudo S-asymptotically (ω, k)-periodic stochastic processes. Let $\nu, \mu \in \mathcal{M}$, $h \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^p(\mho, \mathbb{V}), \mathbb{L}^p(\mho, \mathbb{V}))$ and consider the following assumptions:

(H0): For all $(s, \psi) \in \mathbb{R} \times \mathbb{L}^p(\mho, \mathbb{V})$,

$$\lim_{m\to\infty} \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\|h(s+\omega, \psi) - e^{ik\omega} h(s, e^{-ik\omega}\psi)\|^p d\mu(s) = 0 \tag{7}$$

uniformly on any bounded set of $\mathbb{L}^p(\mho, \mathbb{V})$.

(H1): There exists a number $L > 0$ such that for any $z_1, z_2 \in \mathbb{L}^p(\mho, \mathbb{V})$,

$$\mathsf{E}\|h(\tau, z_1) - h(\tau, z_2)\|^p \leq L.\, \mathsf{E}\|z_1 - z_2\|^p,$$

uniformly for all $\tau \in \mathbb{R}$.

(H2): For any $\epsilon > 0$ and any bounded subset $B \subset \mathbb{L}^p(\mho, \mathbb{V})$, there exist $T_{\epsilon,B} \in \mathbb{R}$ and $\delta_{\epsilon,B} > 0$ such that

$$\mathsf{E}\|h(\tau, z_1) - h(\tau, z_2)\|^p \leq \epsilon$$

for all $z_1, z_2 \in B$ with $\mathsf{E}\|z_1 - z_2\|^p \leq \delta_{\epsilon,B}$ and $\tau \geq T_{\epsilon,B}$.

We now state the following composition results.

Theorem 1.7. *Let $\mu, \nu \in \mathcal{M}$. If $h \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^p(\mho, \mathbb{V}), \mathbb{L}^p(\mho, \mathbb{V}))$ satisfies $(\mathbf{H0})-(\mathbf{H1})$, then $h(\cdot, z(\cdot)) \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$ for every $z \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$.*

Proof. Since $z \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$, for each $\tau \in \mathbb{R}$, we have

$$\lim_{m\to+\infty} \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\|z(\tau+\omega) - e^{ik\omega} z(\tau)\|^p d\mu(\tau) = 0.$$

We have

$$\begin{aligned}
0 \leq & \frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|h(\tau+\omega, z(\tau+\omega)) - e^{ik\omega}h(\tau, z(\tau))\|^p d\mu(\tau)\\
\leq & \frac{2^{p-1}}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|h(\tau+\omega, z(\tau+\omega)) - e^{ik\omega}h(\tau, e^{-ik\omega}z(\tau+\omega))\|^p d\mu(\tau)\\
& + \frac{2^{p-1}}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|e^{ik\omega}h(\tau, e^{-ik\omega}z(\tau+\omega)) - e^{ik\omega}h(\tau, z(\tau))\|^p d\mu(\tau)\\
\leq & \frac{2^{p-1}}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|h(\tau+\omega, z(\tau+\omega)) - e^{ik\omega}h(\tau, e^{-ik\omega}z(\tau+\omega))\|^p d\mu(\tau)\\
& + \frac{2^{p-1}L}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|e^{-ik\omega}z(\tau+\omega)) - z(\tau)\|^p d\mu(\tau)\\
& \to 0 \text{ as } m \to +\infty.
\end{aligned}$$

Thus,

$$\lim_{m\to+\infty} \frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|h(\tau+\omega, z(\tau+\omega)) - e^{ik\omega}h(\tau, z(\tau))\|^p d\mu(\tau) = 0,$$

i.e., $h(\cdot, z(\cdot)) \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$. □

Theorem 1.8. *Let $\nu, \mu \in \mathcal{M}$ such that $\limsup_{m\to+\infty} \frac{\mu([-m,m])}{\nu([-m,m])} = l < \infty$. If $h \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^p(\mho, \mathbb{V}), \mathbb{L}^p(\mho, \mathbb{V}))$ satisfies* (**H0**) *and* (**H2**)*, then $h(\cdot, z(\cdot)) \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$ for every $z \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$.*

Proof. By $z \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$, Lemma 1.3 and assumption (**H0**), for any $\epsilon > 0$, there exists $q_\epsilon > 0$ such that for each $m \geq q_\epsilon$,

$$\frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|h(\tau+\omega, z(\tau+\omega)) - e^{ik\omega}h(s, e^{-ik\omega}z(\tau+\omega))\|^p \mu(\tau) < \frac{\epsilon}{2^{p-1}}$$
$$\text{and} \quad \frac{\mu(\mathcal{Q}_{m,\epsilon}(z))}{\nu([-m,m])} \leq \frac{\epsilon}{2^{2p-1}\|h\|^p_\infty + 1},$$

where $\mathcal{Q}_{m,\epsilon}(z) = \{\tau \in [-m,m] : \mathsf{E}\|z(\tau+\omega) - e^{ik\omega}z(\tau)\|^p \geq \epsilon\}$. Thanks to condition (**H2**), we have that for any $\epsilon > 0$, there exists $\delta_{\epsilon,B} := \epsilon$ and $T_{\epsilon,B} := q_\epsilon$ such that

$$\mathsf{E}\|h(\tau, e^{-ik\omega}z(\tau+\omega)) - h(\tau, z(\tau))\|^p \leq \frac{\epsilon}{2^{p-1}}$$

whenever $\mathsf{E}\|z(\tau+\omega) - e^{ik\omega}z(\tau)\|^p \leq \epsilon$ and $|\tau| \geq q_\epsilon$

$$\begin{aligned}
\frac{1}{\nu([-m,m])}&\int_{-m}^{m} \mathsf{E}\|h(\tau+\omega, z(\tau+\omega)) - e^{ik\omega}h(\tau, z(\tau))\|^p \, d\mu(\tau)\\
&\leq \frac{2^{p-1}}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|h(\tau, e^{-ik\omega}z(\tau+\omega)) - e^{ik\omega}h(\tau, e^{-ik\omega}z(\tau+\omega))\|^p d\mu(\tau)\\
&+ \frac{2^{p-1}}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|e^{ik\omega}h(s, e^{-ik\omega}z(\tau+\omega)) - e^{ik\omega}h(\tau, z(\tau))\|^p d\mu(\tau)
\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon + \frac{2^{p-1}}{\nu([-m,m])} \int_{[-m,m]\setminus \mathcal{Q}_{m,\epsilon}(z)} \mathsf{E}\|h(\tau, e^{-ik\omega} z(\tau+\omega)) - h(\tau, z(\tau))\|^p d\mu(\tau) \\
&\quad + \frac{2^{p-1}}{\nu([-m,m])} \int_{\mathcal{Q}_{m,\epsilon}(z)} \mathsf{E}\|h(\tau, e^{-ik\omega} z(\tau+\omega)) - h(\tau, z(\tau))\|^p d\mu(\tau) \\
&\leq \epsilon + \frac{\epsilon}{\nu([-m,m])} \int_{[-m,m]\setminus \mathcal{Q}_{m,\epsilon}(z)} d\mu(\tau) + \frac{2^{p-1}}{\nu([-m,m])} \int_{\mathcal{Q}_{m,\epsilon}(z)} 2^p \|h\|_\infty^p d\mu(\tau) \\
&\leq \epsilon + \frac{\epsilon}{\nu([-m,m])} \int_{-m}^{m} d\mu(\tau) + \frac{2^{2p-1}\|h\|_\infty^p}{\nu([-m,m])} \int_{\mathcal{Q}_{m,\epsilon}(z)} d\mu(\tau) \\
&\leq \epsilon + \epsilon \frac{\mu([-m,m])}{\nu[-m,m]} + 2^{2p-1}\|h\|_\infty^p \frac{\mu(\mathcal{Q}_{m,\epsilon}(z))}{\nu[-m,m]} \\
&\leq \epsilon + \epsilon\, l + \frac{2^{2p-1}\|h\|_\infty^p}{2^{2p-1}\|h\|_\infty^p + 1}\epsilon \leq (2+\, l)\epsilon
\end{aligned}
$$

for each $m \geq q_\epsilon$. Thus

$$
\lim_{m\to\infty} \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\|h(\tau+\omega, z(\tau+\omega)) - e^{ik\omega} h(\tau, z(\tau))\|^p \, d\mu(\tau) = 0.
$$

This completes the proof. □

Lemma 1.9. *Let $\nu, \mu \in \mathcal{M}$ satisfy $(\mathbf{H^0})$. If $\{\mathcal{K}(\tau)\}_{\tau\geq 0} \subset \mathcal{B}(\mathbb{V})$ is uniformly* 1*-integrable and strongly continuous family of operators, and $X \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$, then*

$$
\Phi(\tau) = \int_{-\infty}^{\tau} \mathcal{K}(\tau - s) X(s) ds \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V})).
$$

Proof. Since the operator family $\{\mathcal{K}(\tau)\}_{\tau\geq 0}$ is uniformly 1-integrable, there exists $M > 0$ such that $\int_0^\infty \|\mathcal{K}(\tau)\| dt \leq M$. Let $X \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$, then for any $m > 0$ we get

$$
\begin{aligned}
&\frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\|\Phi(\tau+\omega) - e^{ik\omega}\Phi(\tau)\|^p d\mu(\tau) \\
=& \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\left\| \int_{-\infty}^{\tau+\omega} \mathcal{K}(\tau+\omega-s)X(s)ds - e^{ik\omega} \int_{-\infty}^{\tau} \mathcal{K}(\tau-s)X(s)ds \right\|^p d\mu(\tau) \\
=& \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\left\| \int_{-\infty}^{\tau} \mathcal{K}(\tau-s)X(s+\omega)ds - e^{ik\omega} \int_{-\infty}^{\tau} \mathcal{K}(\tau-s)X(s)ds \right\|^p d\mu(\tau) \\
=& \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\left\| \int_{-\infty}^{\tau} \mathcal{K}(\tau-s)[X(s+\omega) - e^{ik\omega}X(s)]ds \right\|^p d\mu(\tau) \\
=& \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\left(\left\| \int_{-\infty}^{\tau} \mathcal{K}(\tau-s)[X(s+\omega) - e^{ik\omega}X(s)]ds \right\| \right)^p d\mu(\tau) \\
\leq& \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\left(\int_{-\infty}^{\tau} \|\mathcal{K}(\tau-s)\| \left\| X(s+\omega) - e^{ik\omega}X(s) \right\| ds \right)^p d\mu(\tau)
\end{aligned}
$$

$$\leq \frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\left(\int_{-\infty}^{\tau} \|\mathcal{K}(\tau-s)\|^{\frac{p-1}{p}} \|\mathcal{K}(\tau-s)\|^{\frac{1}{p}} \left\|X(s+\omega)-e^{ik\omega}X(s)\right\| ds\right)^{p} d\mu(\tau)$$

$$\leq \frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\left(\left[\int_{-\infty}^{\tau} \left(\|\mathcal{K}(\tau-s)\|^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}} ds\right]^{\frac{p-1}{p}} \times \left[\int_{-\infty}^{\tau} \left(\|\mathcal{K}(\tau-s)\|^{\frac{1}{p}} \left\|X(s+\omega)-e^{ik\omega}X(s)\right\|\right)^{p} ds\right]^{\frac{1}{p}}\right)^{p} d\mu(\tau)$$

$$\leq \frac{1}{\nu([-m,m])}\int_{-m}^{m} \left(\left[\int_{-\infty}^{\tau} \|\mathcal{K}(\tau-s)\| ds\right]^{p-1} \times \left[\int_{-\infty}^{\tau} \|\mathcal{K}(\tau-s)\| \mathsf{E}\left\|X(s+\omega)-e^{ik\omega}X(s)\right\|^{p} ds\right]\right) d\mu(\tau).$$

From the Fubini theorem, it follows that

$$\leq \frac{M^{p-1}}{\nu([-m,m])}\int_{-m}^{m} \left[\int_{0}^{\infty} \|\mathcal{K}(s)\| \; \mathsf{E}\|X(\tau-s+\omega)-e^{ik\omega}X(\tau-s)\|^{p} ds\right] d\mu(\tau)$$
$$= M^{p-1}\int_{0}^{\infty} \|\mathcal{K}(s)\| \left[\frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|X(\tau-s+\omega)-e^{ik\omega}X(\tau-s)\|^{p} d\mu(\tau)\right] ds$$
$$= M^{p-1}\int_{0}^{\infty} \|\mathcal{K}(s)\| \left[\frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|X_{-s}(\tau+\omega)-e^{ik\omega}X_{-s}(\tau)\|^{p} d\mu(\tau)\right] ds.$$

Since $X \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$, thanks to Lemma 1.6, we know that for any $s \in \mathbb{R}$, we have

$$\lim_{m\to\infty} \frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|X_{-s}(\tau+\omega)-e^{ik\omega}X_{-s}(\tau)\|^{p} d\mu(\tau) = 0.$$

Then Lebesgue dominated convergence theorem yield that

$$\int_{0}^{\infty} \|\mathcal{K}(s)\| \left[\frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|X_{-s}(\tau+\omega)-e^{ik\omega}X_{-s}(\tau)\|^{p} d\mu(\tau)\right] ds \to 0$$

as $m \to +\infty$. Therefore

$$\lim_{m\to+\infty} M^{p-1}\int_{0}^{\infty} \|\mathcal{K}(s)\| \left[\frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|X_{-s}(\tau+\omega)-e^{ik\omega}X_{-s}(\tau)\|^{p} d\mu(\tau)\right] ds = 0.$$

It proves that $\Phi \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$. □

Lemma 1.10. *Let $\nu,\mu \in \mathcal{M}$ satisfy $(\mathbf{H^0})$. If $\{\mathcal{K}(\tau)\}_{\tau\geq 0} \subset \mathcal{B}(\mathbb{V})$ is uniformly 2−integrable and strongly continuous family of operators.*
Suppose that $g \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$, then

$$\Psi(\tau) = \int_{-\infty}^{\tau} \mathcal{K}(\tau-s)g(s)dW(s) \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))\,.$$

Proof. We have

$$\begin{aligned}&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|\Psi(\tau+\omega)-e^{ik\omega}\Psi(\tau)\|^{p}d\mu(\tau)\\=&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\left\|\int_{-\infty}^{\tau+\omega}\mathcal{K}(\tau+\omega-s)g(s)dW(s)-e^{ik\omega}\int_{-\infty}^{\tau}\mathcal{K}(\tau-s)g(s)dW(s)\right\|^{p}d\mu(\tau)\\=&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\left\|\int_{-\infty}^{\tau}\mathcal{K}(\tau-s)g(s+\omega)dW(s+\omega)-e^{ik\omega}\int_{-\infty}^{\tau}\mathcal{K}(\tau-s)g(s)dW(s)\right\|^{p}d\mu(\tau).\end{aligned}$$

Let $\widetilde{W}(s)=W(s+\omega)-W(\omega)$. We know that $\widetilde{W}$ is a Brownian motion and has the same distribution as W. Then we obtain

$$\begin{aligned}&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|\Psi(\tau+\omega)-e^{ik\omega}\Psi(\tau)\|^{p}d\mu(\tau)\\=&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\left\|\int_{-\infty}^{\tau}\mathcal{K}(\tau-s)g(s+\omega)d\widetilde{W}(s)-e^{ik\omega}\int_{-\infty}^{\tau}\mathcal{K}(\tau-s)g(s)d\widetilde{W}(s)\right\|^{p}d\mu(\tau)\\=&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\left\|\int_{-\infty}^{\tau}\mathcal{K}(\tau-s)\left[g(s+\omega)-e^{ik\omega}g(s)\right]d\widetilde{W}(s)\right\|^{p}d\mu(\tau).\end{aligned}$$

Since $g\in\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^{p}(\mho,\mathbb{V}))$ and for all $\tau\in\mathbb{R}$, $g(\tau)\in\mathbb{L}^{p}(\mho,\mathbb{V})\subset L^{2}(\Omega,\mathbb{V})$ then

$$\begin{aligned}&\int_{-\infty}^{\tau}\mathsf{E}\left\|\mathcal{K}(\tau-s)\left[g(s+\omega)-e^{ik\omega}g(s)\right]\right\|^{2}ds\\&\qquad\le 2\sup_{s\in\mathbb{R}}\mathsf{E}\|g(s)\|^{2}\int_{0}^{\infty}\|\mathcal{K}(s)\|^{2}ds<\infty\quad\text{for all}\quad\tau\in\mathbb{R}.\end{aligned}$$

Then by Lemma 1.1, there exists $C_p>0$ such that

$$\begin{aligned}&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|\Psi(\tau+\omega)-e^{ik\omega}\Psi(\tau)\|^{p}d\mu(\tau)\\\le& C_p\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\left(\int_{-\infty}^{\tau}\left\|\mathcal{K}(\tau-s)\left[g(s+\omega)-e^{ik\omega}g(s)\right]\right\|^{2}ds\right)^{p/2}d\mu(\tau).\end{aligned}$$

Therefore, using Hölder's inequality and the Fubini's theorem, we obtain that

$$\begin{aligned}&\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|\Psi(\tau+\omega)-e^{ik\omega}\Psi(\tau)\|^{p}d\mu(\tau)\\\le& C_p\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\Bigg[\left(\int_{-\infty}^{\tau}\left(\|\mathcal{K}(\tau-s)\|^{2.\frac{p-2}{p}}\right)^{\frac{p}{p-2}}ds\right)^{\frac{p-2}{p}}\\&\qquad\times\left(\int_{-\infty}^{\tau}\left(\|\mathcal{K}(\tau-s)\|^{\frac{4}{p}}\|g(s+\omega)-e^{ik\omega}g(s)\|^{2}\right)^{\frac{p}{2}}ds\right)^{\frac{2}{p}}\Bigg]^{p/2}d\mu(\tau)\\\le& C_p\left(\int_{-\infty}^{\tau}\|\mathcal{K}(\tau-s)\|^{2}ds\right)^{\frac{p-2}{2}}\\&\quad\times\int_{0}^{\infty}\|\mathcal{K}(s)\|^{2}\left[\frac{1}{\nu([-m,m])}\int_{-m}^{m}\mathsf{E}\|g(\tau-s+\omega)-e^{ik\omega}g(\tau-s)\|^{p}d\mu(\tau)\right]ds.\end{aligned}$$

Since $g \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$ and $\{\mathcal{K}(\tau)\}_{\tau\geq 0}$ is uniformly 2−integrable, therefore using Lemma 1.6 and Lebesgue dominated converge theorem, we obtain that

$$\lim_{m\to+\infty} \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\|\Psi(\tau+\omega) - e^{ik\omega}\Psi(\tau)\|^p d\mu(\tau) = 0. \tag{8}$$

which proves that $\Psi \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$. □

2. Existence of p^{th}-mean (ν, μ)-speudo S-asymptotically Bloch-type periodic mild solutions

Firstly, we discuss the existence and uniqueness of (ν, μ)-pseudo S-asymptotically (ω, k)-periodic mild solution for problem(1). We state the following conditions:

(A1) A generates an immediately norm continuous C_0-semigroup on a Hilbert space $\mathbb{V}$.

(A2) $\mathrm{Re}\Big((-\alpha)^{1/\mathbf{b}}) - \mathbf{a}\Big) < 0$ and

$$\sup\Big\{\mathrm{Re}(\lambda), \lambda \in \mathbb{C} \,:\, \lambda(\lambda+\mathbf{a})^{\mathbf{b}}((\lambda+\mathbf{a})^{\mathbf{b}}+\alpha)^{-1} \in \sigma(A)\Big\} < 0.$$

Let us consider the following linear Cauchy problem

$$z'(\tau) = Az(\tau) + \alpha \int_{-\infty}^{\tau} \frac{(\tau-s)^{\mathbf{b}-1}}{\Gamma(\mathbf{b})} e^{-\mathbf{a}(\tau-s)} Az(s)ds + g(\tau), \quad \tau \in \mathbb{R}. \tag{9}$$

Under conditions (A1)-(A2), from [18, Proposition 3.1], we know that there exists a uniformly stable and strongly continuous family operators $\{\mathcal{R}(\tau)\}_{\tau\geq 0} \subset \mathcal{B}(\mathbb{V})$, i.e., there exist constants $d_1, d_2 > 0$ such that for all $\tau > 0$,

$$\|\mathcal{R}(\tau)\| \leq d_1 e^{-d_2\tau}. \tag{10}$$

From [18, Theorem 3.2], the mild solution of problem (9) can be expressed by

$$z(\tau) = \int_{-\infty}^{\tau} \mathcal{R}(\tau-s)g(s)ds.$$

Motivated by [18], we present the concept of mild solution for the stochastic integrodifferential equation (1).

Definition 2.1. An $\mathcal{G}_\tau$-progressively measurable process $\{z(\tau)\}_{\tau\in\mathbb{R}}$ is referred to as a mild solution of problem (1) if for $\tau \in \mathbb{R}$, $z(\tau)$ $\mathbb{P}$-almost surely satisfies the following integral equation

$$z(\tau) = \int_{-\infty}^{\tau} \mathcal{R}(\tau-s)g(s,z(s))ds + \int_{-\infty}^{\tau} \mathcal{R}(\tau-s)f(s,z(s))dW(s). \tag{11}$$

Theorem 2.1. *Let $\mu, \nu \in \mathcal{M}$ satisfy $(\mathbf{H^0})$ and $\limsup_{m\to+\infty} \frac{\nu([-m,m])}{\mu([-m,m])} < \infty$. Suppose that (A1)-(A2) hold and $g, f \in \mathcal{BC}(\mathbb{R}\times\mathbb{L}^p(\mho,\mathbb{V}), \mathbb{L}^p(\mho,\mathbb{V}))$, such that for all $(\tau, x) \in \mathbb{R}\times\mathbb{L}^p(\mho,\mathbb{V})$,*

$$\lim_{m\to\infty} \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\|f(\tau+\omega,x) - e^{ik\omega}f(\tau, e^{-ik\omega}x)\|^p d\mu(\tau) = 0 \text{ and}$$

$$\lim_{m\to\infty} \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E}\|g(\tau+\omega,x) - e^{ik\omega}g(\tau, e^{-ik\omega}x)\|^p d\mu(\tau) = 0.$$

uniformly on any bounded set of $\mathbb{L}^p(\mho,\mathbb{V})$. *In addition, assume that there exist constants* $L, L' > 0$ *such that for any* $v_1, v_2 \in \mathbb{L}^p(\mho,\mathbb{V})$,

$$\begin{aligned} \mathsf{E}\|g(\tau,v_1)-g(\tau,v_2)\|^p &\le L\,\mathsf{E}\|v_1-v_2\|^p, \\ \mathsf{E}\|f(\tau,v_1)-f(\tau,v_2)\|^p &\le L'\,\mathsf{E}\|v_1-v_2\|^p, \end{aligned}$$

uniformly for all $\tau \in \mathbb{R}$. *Then problem* (1) *admit a unique mild solution* $z \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$ *in p-th mean sense on* $\mathbb{R}$, *if*

$$2^{p-1}\left[\frac{(d_1)^p}{(d_2)^p}L + \frac{C_p(d_1)^p}{(2d_2)^{p/2}}L'\right] < 1 \;\; \text{for } p > 2$$

and

$$\left[\frac{2(d_1)^2}{(d_2)^2}L + \frac{(d_1)^2}{(d_2)}L'\right] < 1 \;\; \text{for } p = 2.$$

Proof. Define the operator $\mathcal{S} : \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V})) \to \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$ by

$$(\mathcal{S}z)(\tau) = \int_{-\infty}^{\tau} \mathcal{R}(\tau-s)g(s,z(s))ds + \int_{-\infty}^{\tau} \mathcal{R}(\tau-s)f(s,z(s))dW(s), \tag{12}$$

where $\{\mathcal{R}(\tau)\}_{\tau\ge 0}$ satisfies the relation (10).

From Theorem 1.7, for each $z \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$, the stochastic processes $\tau \longmapsto f(\tau,z(\tau))$, $\tau \longmapsto g(\tau,z(\tau))$ belongs to $\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$. It follows from Lemmas 1.9, 1.10 and 1.2-(a) that $\mathcal{S}z \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$.

For all $v_1, v_2 \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$, we have

$$\begin{aligned} \mathsf{E}\|(\mathcal{S}v_1)(\tau)-(\mathcal{S}v_2)(\tau)\|^p \le& 2^{p-1}\mathsf{E}\|\int_{-\infty}^{\tau}\mathcal{R}(\tau-s)[g(s,v_1(s))-g(s,v_2(s))]ds\|^p \\ &+ 2^{p-1}\mathsf{E}\|\int_{-\infty}^{\tau}\mathcal{R}(\tau-s)[f(s,v_1(s))-f(s,v_2(s))]dW(s)\|^p \\ =& 2^{p-1}(J_1+J_2), \end{aligned}$$

where, for $p > 2$

$$\begin{aligned} J_1 =& \mathsf{E}\|\int_{-\infty}^{\tau}\mathcal{R}(\tau-s)[g(s,v_1(s))-g(s,v_2(s))]ds\|^p \\ \le& \left(\int_{-\infty}^{\tau}\|\mathcal{R}(\tau-s)\|ds\right)^{p-1}\int_{-\infty}^{\tau}\|\mathcal{R}(\tau-s)\|\,\mathsf{E}\|g(s,v_1(s))-g(s,v_2(s))\|^p ds \\ \le& \frac{(d_1)^p}{(d_2)^{p-1}}\,L\int_{-\infty}^{\tau} e^{-d_2(\tau-s)}\,\mathsf{E}\|v_1(s)-v_2(s)\|^p ds \\ \le& \frac{(d_1)^p}{(d_2)^{p-1}}\,L\int_{-\infty}^{\tau} e^{-d_2(\tau-s)}\,\mathsf{E}\|v_1(s)-v_2(s)\|^p ds \\ \le& \frac{(d_1)^p}{(d_2)^p}\,L\sup_{\tau\in\mathbb{R}}\mathsf{E}\|v_1(\tau)-v_2(\tau)\|^p, \end{aligned}$$

and

$$\begin{aligned}
J_2 =&\mathsf{E}\| \int_{-\infty}^{\tau} \mathcal{R}(\tau-s)[f(s,v_1(s))-f(s,v_2(s))]dW(s)\|^p\\
\leq& C_p\left(\int_{-\infty}^{\tau} \mathsf{E}\left\|\mathcal{R}(\tau-s)\left[f(s,v_1(s))-f(s,v_2(s))\right]\right\|^2 ds\right)^{p/2}\\
\leq& C_p\left(\int_{-\infty}^{\tau} \|\mathcal{R}(\tau-s)\|^2 ds\right)^{\frac{p-2}{2}} \int_{-\infty}^{\tau} \|\mathcal{R}(\tau-s)\|^2\, \mathsf{E}\|f(s,v_1(s))-f(s,v_2(s))\|^p ds\\
\leq& C_p(d_1)^{p-2}\left(\int_{-\infty}^{\tau} e^{-2d_2(\tau-s)}ds\right)^{\frac{p-2}{2}} (d_1)^2\int_{-\infty}^{\tau} e^{-2d_2(\tau-s)}\, \mathsf{E}\|f(s,v_1(s))-f(s,v_2(s))\|^p ds\\
\leq& \frac{C_p(d_1)^p}{(2d_2)^{p/2}}\, L' \sup_{\tau\in\mathbb{R}} \mathsf{E}\|v_1(\tau)-v_2(\tau)\|^p.
\end{aligned}$$

Therefore,

$$\|\mathcal{S}v_1-\mathcal{S}v_2\|_\infty^p \leq 2^{p-1}\left[\frac{(d_1)^p}{(d_2)^p}L+\frac{C_p(d_1)^p}{(2d_2)^{p/2}}L'\right]\|v_1-v_2\|_\infty^2.$$

In the case $p=2$, from Ito's isometry identity and by similar calculations, we obtain that

$$\begin{aligned}
&\mathsf{E}\|(\mathcal{S}v_1)(\tau)-(\mathcal{S}v_2)(\tau)\|^2\\
\leq& 2\mathsf{E}\| \int_{-\infty}^{\tau} \mathcal{R}(\tau-s)[g(s,v_1(s))-g(s,v_2(s))]ds\|^2\\
&+2\mathsf{E}\| \int_{-\infty}^{\tau} \mathcal{R}(\tau-s)[f(s,v_1(s))-f(s,v_2(s))]dW(s)\|^2\\
\leq& 2(d_1)^2\left(\int_{-\infty}^{\tau} e^{-d_2(\tau-s)}ds\right)\int_{-\infty}^{\tau} e^{-d_2(\tau-s)}\, \mathsf{E}\|g(s,v_1(s))-g(s,v_2(s))\|^2 ds\\
&+2(d_1)^2\int_{-\infty}^{\tau} e^{-2d_2(\tau-s)}\, \mathsf{E}\|g(s,v_1(s))-g(s,v_2(s))\|^2 ds\\
\leq& \frac{2(d_1)^2}{(d_2)^2}L\sup_{\tau\in\mathbb{R}}\mathsf{E}\|v_1(\tau)-v_2(\tau)\|^2+\frac{(d_1)^2}{(d_2)}L'\sup_{\tau\in\mathbb{R}}\mathsf{E}\|v_1(\tau)-v_2(\tau)\|^2\\
\leq& \left[\frac{2(d_1)^2}{(d_2)^2}L+\frac{(d_1)^2}{(d_2)}L'\right]\|v_1-v_2\|_\infty^2.
\end{aligned}$$

Then $\mathcal{S}$ is a contraction mapping in the Banach space $\mathcal{SABP}_{\omega,k}^{\nu,\mu}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$ owing to the condition

$$2^{p-1}\left[\frac{(d_1)^p}{(d_2)^p}L+\frac{C_p(d_1)^p}{(2d_2)^{p/2}}L'\right]<1 \text{ for } p>2$$

and

$$\left[\frac{2(d_1)^2}{(d_2)^2}L+\frac{(d_1)^2}{(d_2)}L'\right]<1 \text{ for } p=2.$$

Thus there exists a unique $z\in\mathcal{SABP}_{\omega,k}^{\nu,\mu}(\mathbb{R},\mathbb{L}^p(\mho,\mathbb{V}))$ such that $\mathcal{S}z=z$ via Banach fixed point theorem. □

Now, we discuss the existence and uniqueness of (ν,μ)-pseudo S-asymptotically (ω,k)-periodic mild solution for Eq.(3). To prove the existence of mild solutions of

Eq.(3), we need to recall some facts about Weyl fractional integral and derivative of order $\alpha > 0$, and α-resolvent operators that will be used to develop the main results of this section. For more details on properties α-resolvent operators, one can take reference to [44]. Now, suppose that $\mathbb{X}$ is a Banach space. For a given function $h : \mathbb{R} \to \mathbb{X}$, the Weyl fractional integral of order $\alpha > 0$ is defined by

$$\partial_\tau^{-\alpha} h(\tau) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\tau} (\tau - s)^{\alpha-1} h(s) ds, \qquad \tau \in \mathbb{R},$$

when this integral is convergent. The Weyl fractional derivative ∂_τ^α of order α is defined by

$$\partial_\tau^\alpha h(\tau) = \frac{d^n}{d\tau^n} \partial_\tau^{-(n-\alpha)} h(\tau), \quad \tau \in \mathbb{R},$$

where $n = [\alpha] + 1$, and the notation $[\alpha]$ represents the integer part of α. Now, Let A be a closed and linear operator with domain $\mathcal{D}(A)$ defined on a Banach space X, and $\alpha > 0$. For a given kernel $b(\cdot) \in L^1_{loc}(\mathbb{R}_+)$, it is said that A is the generator of a α-resolvent family if there exists $\xi > 0$ and a strongly continuous family $\mathcal{R}_\alpha : \mathbb{R}_+ \to \mathcal{B}(\mathbb{X})$ such that

$$\left\{ \frac{\lambda^\alpha}{1 + \hat{b}(\lambda)} : \mathrm{Re}(\lambda) > \xi \right\} \subseteq \rho(A)$$

and for all $y \in \mathbb{X}$,

$$(\lambda^\alpha - (1+\hat{b}(\lambda))A)^{-1} y = \frac{1}{1 + \hat{b}(\lambda)} \left(\frac{\lambda^\alpha}{1 + \hat{b}(\lambda)} - A \right)^{-1} y = \int_0^\infty e^{-\lambda t} \mathcal{R}_\alpha(\tau) y d\tau, \ \mathrm{Re}\lambda > \xi.$$

$\{\mathcal{R}_\alpha(\tau)\}_{\tau \geq 0}$ is called the α-resolvent family generated by the operator A.

Motivated by Ponce [44], we present the concept of mild solutions for Eq.(3).

Definition 2.2. An $\mathcal{F}_\tau$-progressively measurable process $\{z(\tau)\}_{\tau \in \mathbb{R}}$ is referred to as a mild solution of problem (1) if it satisfies the following stochastic integral equation

$$z(\tau) = \int_{-\infty}^{\tau} \mathcal{R}_\alpha(\tau - s) g(s, z(s)) ds + \int_{-\infty}^{\tau} \mathcal{R}_\alpha(\tau - s) f(s, z(s)) dW(s)$$

for all $\tau \in \mathbb{R}$, where $\{\mathcal{R}_\alpha(\tau)\}_{\tau \geq 0}$ the resolvent family generated by the operator A.

Theorem 2.2. *Let $\mu, \nu \in \mathcal{M}$ satisfy $(\mathbf{H^0})$ and $\limsup_{m \to +\infty} \frac{\nu([-m,m])}{\mu([-m,m])} < \infty$. Suppose that (A1)-(A2) hold. Suppose that $\{\mathcal{R}_\alpha(\tau)\}_{\tau \geq 0} \subset \mathcal{B}(\mathbb{V})$ such that $\mathcal{R}_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ and $g, f \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^p(\mho, \mathbb{V}), \mathbb{L}^p(\mho, \mathbb{V}))$, such that for all $(\tau, x) \in \mathbb{R} \times \mathbb{L}^p(\mho, \mathbb{V})$,*

$$\lim_{m \to \infty} \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E} \| f(\tau + \omega, x) - e^{ik\omega} f(\tau, e^{-ik\omega} x) \|^p d\mu(\tau) = 0$$
$$\lim_{m \to \infty} \frac{1}{\nu([-m,m])} \int_{-m}^{m} \mathsf{E} \| g(\tau + \omega, x) - e^{ik\omega} g(\tau, e^{-ik\omega} x) \|^p d\mu(\tau) = 0,$$

uniformly on any bounded set of $\mathbb{L}^p(\mho, \mathbb{V})$. In addition, assume that there exist constants $C_p, L, L' > 0$ such that for any $v_1, v_2 \in \mathbb{L}^p(\mho, \mathbb{V})$,

$$\mathsf{E} \| g(\tau, v_1) - g(\tau, v_2) \|^p \leq L\, \mathsf{E} \| v_1 - v_2 \|^p,$$
$$\mathsf{E} \| f(\tau, v_1) - f(\tau, v_2) \|^p \leq L'\, \mathsf{E} \| v_1 - v_2 \|^p,$$

uniformly for all $\tau \in \mathbb{R}$. *Then equation* (3) *has a unique mild solution* $z \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$ *in* p^{th}*-mean sense on* $\mathbb{R}$, *if*

$$2^{p-1}\left[L\left(\|\mathcal{R}_\alpha\|_{L^1}\right)^p + C_p L'\left(\|\mathcal{R}_\alpha\|_{L^2}\right)^p\right] < 1 \ \ \textit{for}\, p > 2,$$

and

$$2\left(\|\mathcal{R}_\alpha\|^2_{L^1} L + L'\|\mathcal{R}_\alpha\|^2_{L^2}\right) < 1 \ \ \textit{for}\, p = 2.$$

Proof. Define the operator $\mathcal{S} : \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V})) \to \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$ by

$$(\mathcal{S}z)(\tau) = \int_{-\infty}^{\tau} \mathcal{R}_\alpha(\tau - s) g(s, z(s)) ds + \int_{-\infty}^{\tau} \mathcal{R}_\alpha(\tau - s) f(s, z(s)) dW(s), \tag{13}$$

where $\{\mathcal{R}(\tau)\}_{\tau \geq 0}$ satisfies the relation (10).

From Theorem 1.7, for each $z \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$, the stochastic processes $\varsigma \longmapsto f(\varsigma, z(\varsigma))$, $\varsigma \longmapsto g(\varsigma, z(\varsigma))$ belongs to $\mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$. It follows from Lemmas 1.9, 1.10 and 1.2-(a) that $\mathcal{S}z \in \mathcal{SABP}^{\nu,\mu}_{\omega,k}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$. Let $v_1, v_2 \in \mathbb{L}^p(\mho, \mathbb{V})$. For $p > 2$, we have by Lemma 1.1 and Hölder's inequality

$$\begin{aligned}
&\mathsf{E}\|(\mathcal{S}v_1)(\tau) - (\mathcal{S}v_2)(\tau)\|^p \\
\leq& 2^{p-1}\mathsf{E}\left\|\int_{-\infty}^{\tau} \mathcal{R}_\alpha(\tau - s)[g(s, v_1(s)) - g(s, v_2(s))]ds\right\|^p \\
&+ 2^{p-1}\mathsf{E}\left\|\int_{-\infty}^{\tau} \mathcal{R}_\alpha(\tau - s)[f(s, v_1(s)) - f(s, v_2(s))]dW(s)\right\|^p \\
\leq& 2^{p-1}\left(\int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\| ds\right)^{p-1} \int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\|\, \mathsf{E}\|g(s, v_1(s)) - g(s, v_2(s))\|^p ds \\
&+ 2^{p-1}C_p\left(\int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\|^2 ds\right)^{\frac{p-2}{2}} \times \int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\|^2\, \mathsf{E}\|f(s, v_1(s)) - f(s, v_2(s))\|^p ds \\
\leq& 2^{p-1}\left(\int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\| ds\right)^{p-1} \int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\|\, \mathsf{E}\|g(s, v_1(s)) - g(s, v_2(s))\|^p ds \\
&+ 2^{p-1}C_p\left(\int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\|^2 ds\right)^{\frac{p-2}{2}} \int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\|^2\, \mathsf{E}\|f(s, v_1(s)) - f(s, v_2(s))\|^p ds \\
\leq& 2^{p-1}L\|v_1 - v_2\|^p_\infty \left(\int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\| ds\right)^{p-1} \int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\|\, ds \\
&+ 2^{p-1}C_p L'\|v_1 - v_2\|^p_\infty \left(\int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\|^2 ds\right)^{\frac{p-2}{2}} \int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\|^2 ds \\
\leq& 2^{p-1}L\|v_1 - v_2\|^p_\infty \left(\int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\| ds\right)^{p} + 2^{p-1}C_p L'\|v_1 - v_2\|^p_\infty \left(\int_{-\infty}^{\tau} \|\mathcal{R}_\alpha(\tau - s)\|^2 ds\right)^{\frac{p}{2}} \\
\leq& 2^{p-1}L\|v_1 - v_2\|^p_\infty \left(\|\mathcal{R}_\alpha\|_{L^1}\right)^p + 2^{p-1}C_p L'\|v_1 - v_2\|^p_\infty \left(\|\mathcal{R}_\alpha\|_{L^2}\right)^p \\
\leq& 2^{p-1}\left[L\left(\|\mathcal{R}_\alpha\|_{L^1}\right)^p + C_p L'\left(\|\mathcal{R}_\alpha\|_{L^2}\right)^p\right]\|v_1 - v_2\|^p_\infty.
\end{aligned}$$

In the case $p = 2$, from Ito's isometry identity and by similar calculations, we obtain that

$$\begin{aligned}\mathsf{E}\|(\mathcal{S}v_1)(\tau) - (\mathcal{S}v_2)(\tau)\|^2 \le & 2\mathsf{E}\|\int_{-\infty}^{\tau} \mathcal{R}(\tau - s)[g(s, v_1(s)) - g(s, v_2(s))]ds\|^2 \\ & + 2\mathsf{E}\|\int_{-\infty}^{\tau} \mathcal{R}(\tau - s)[f(s, v_1(s)) - f(s, v_2(s))]dW(s)\|^2 \\ \le & 2\left[L\|\mathcal{R}_\alpha\|_{L^1}^2 + L'\|\mathcal{R}_\alpha\|_{L^2}^2\right]\|v_1 - v_2\|_\infty^2.\end{aligned}$$

Since

$$2^{p-1}\left[L\left(\|\mathcal{R}_\alpha\|_{L^1}\right)^p + C_p L'\left(\|\mathcal{R}_\alpha\|_{L^2}\right)^p\right] < 1, \quad \text{for } p > 2$$

and

$$2\left(\|\mathcal{R}_\alpha\|_{L^1}^2 L + L'\|\mathcal{R}_\alpha\|_{L^2}^2\right) < 1, \quad \text{for } p = 2,$$

we deduce that $\mathcal{S}$ is a contraction mapping in the Banach space $\mathcal{SABP}_{\omega,k}^{\nu,\mu}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$. Consequently, there exists a unique $z \in \mathcal{SABP}_{\omega,k}^{\nu,\mu}(\mathbb{R}, \mathbb{L}^p(\mho, \mathbb{V}))$ such that $\mathcal{S}z = z$ via Banach fixed point theorem. □

3. Examples

To apply our theoretical results, we take the measures μ and ν, whose Radon-Nikodym derivative are given respectively

$$\rho(\tau) = \begin{cases} e^\tau & \text{for } \tau \le 0, \\ 1 & \text{for } \tau > 0 \end{cases}$$

and

$$\widetilde{\rho}(\tau) = |\tau|, \quad \tau \in \mathbb{R}.$$

That is,

$$\mu(Y) = \int_Y \rho(\tau)d\tau \quad \text{and} \quad \nu(Y) = \int_Y \widetilde{\rho}(\tau)d\tau \qquad \text{for} \quad Y \in \mathcal{B},$$

where $d\tau$ denotes the Lebesgue measure on $\mathbb{R}$ and $\mathcal{B}$ the Lebesgue σ-field $\mathbb{R}$. From [6], we know that ν and μ satisfies $(\mathbf{H^0})$ and simple computation yield that

$$\lim \sup_{m \to +\infty} \frac{\mu([-m, m])}{\nu([-m, m])} = 0 < \infty.$$

Let $f(\tau, z)(s) = M_1(\tau, z)(s) + \widetilde{M_1}(\tau, z)(s)$, and $g(\tau, z)(s) = M_2(\tau, z)(s) + \widetilde{M_2}(\tau, z)(s)$, where

$$M_1(\tau, z)(s) = \gamma(\tau)\sigma_1(z(s)), \quad M_2(\tau, z)(s) = \gamma(\tau)\sigma_2(z(s))$$

$$\widetilde{M_1}(\tau, z)(s) = \gamma(\tau)\arctan(\tau)\cos(z(s)) \text{ and } \widetilde{M_2}(\tau, z)(s) = \gamma(\tau)\arctan(\tau)\sin(z(s)).$$

We suppose that $\gamma(\tau)$ is continuous, bounded and w-periodic and for $\sigma_i\,(i = 1, 2)$ verifies the conditions

$$\sigma_i(e^{ikw}x) = e^{ikw}\sigma_i(x), \text{ and } \quad \mathsf{E}\|\sigma_i(u) - \sigma_i(v)\|_{\mathbb{V}}^2 \le L_i\,\mathsf{E}\|u - v\|_{\mathbb{V}}^2, \qquad i = 1, 2.$$

Since for $(i = 1, 2)$, we obtain

$$M_i(\tau + w, z)(s) = \gamma(\tau + w)\sigma_i(z(s)) = \gamma(\tau)e^{ikw}\sigma_i(e^{-ikw}z(s)) = e^{ikw}M_i(\tau, e^{-ikw}z)(s),$$

then we get following estimation:

$$
\begin{aligned}
0 \leq & \frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|f(\tau+\omega,z)-e^{ik\omega}f(\tau,e^{-ik\omega}z)\|_{\mathbb{V}}^2 d\mu(\tau)\\
=&\frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|\gamma(\tau+\omega)M_1(\tau+\omega,z)+\widetilde{M_1}(\tau+\omega,z)\\
&\qquad - e^{ik\omega}\Big(\gamma(\tau)M_1(\tau,e^{-ik\omega}z)+\widetilde{M_1}(\tau,e^{-ik\omega}z)\Big)\|_{\mathbb{V}}^2\, d\mu(\tau)\\
\leq&\frac{\|\gamma\|_\infty^2}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|\widetilde{M_1}(\tau+w,z)-e^{ik\omega}\widetilde{M_1}(\tau,e^{-ik\omega}z)\|_{\mathbb{V}}^2\, d\mu(\tau)\\
\leq&\frac{\|\gamma\|_\infty^2}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|\arctan(\tau+\omega)\cos(z)-\arctan(\tau)e^{ik\omega}\cos(e^{-ik\omega}z)\|_{\mathbb{V}}^2\, d\mu(\tau)\\
\leq&\frac{2\|\gamma\|_\infty^2}{\nu([-m,m])}\int_{-m}^{m} \Big(|\arctan(\tau+\omega)|^2+|\arctan(\tau)|^2\Big)\, d\mu(\tau)\\
\leq&\frac{2\|\gamma\|_\infty^2}{\nu([-m,m])}\int_{-m}^{m} \frac{\pi^2}{2}\, d\mu(\tau)=\pi^2\|\gamma\|_\infty^2\frac{\mu([-m,m])}{\nu([-m,m])}\to 0 \text{ as } r\to\infty.
\end{aligned}
$$

Similarly we have

$$
\lim_{m\to+\infty}\frac{1}{\nu([-m,m])}\int_{-m}^{m} \mathsf{E}\|g(\tau+\omega,z)-e^{ik\omega}g(\tau,e^{-ik\omega}z)\|_{\mathbb{V}}^2 d\mu(\tau)=0.
$$

Let $u,v\in\mathbb{L}^2(\Omega,\mathbb{V})$ and $\tau\in\mathbb{R}$, then we have following estimation:

$$
\begin{aligned}
\mathsf{E}\|f(\tau,u)-f(\tau,v)\|_{\mathbb{V}}^2 &\leq 2\|\gamma\|_\infty^2\Bigg(\mathsf{E}\|\sigma_1(u)-\sigma_1(v)\|_{\mathbb{V}}^2+\mathsf{E}\|\cos u-\cos v\|_{\mathbb{V}}^2\Bigg)\\
&\leq 2\|\gamma\|_\infty^2\Bigg(L_1\mathsf{E}\|u-v\|_{\mathbb{V}}^2+\mathsf{E}\|u-v\|_{\mathbb{V}}^2\Bigg)\\
&\leq 2\|\gamma\|_\infty^2\Bigg(L_1+1\Bigg)\mathsf{E}\|u-v\|_{\mathbb{V}}^2.
\end{aligned}
$$

Similarly, we obtain

$$
\mathsf{E}\|g(\tau,u)-g(\tau,v)\|_{\mathbb{V}}^2\leq 2\|\gamma\|_\infty^2\Bigg(L_2+1\Bigg)\mathsf{E}\|u-v\|_{\mathbb{V}}^2.
$$

Example 3.1.

Consider the stochastic partial stochastic integro-differential equations as follows:

$$
\begin{cases}
dy(\tau,x) &= \Bigg[\dfrac{\partial^2 y(\tau,x)}{\partial x^2}-\dfrac{1}{2}\displaystyle\int_{-\infty}^{\tau}\dfrac{\tau-s}{\Gamma(2)}e^{-(\tau-s)}\dfrac{\partial^2 y(s,x)}{\partial x^2}ds+f(\tau,y(\tau,x))\Bigg]dt\\
&\quad +g(\tau,y(\tau,x))dW(\tau)),\quad (\tau,x)\in\mathbb{R}\times(0,\pi),\\
y(0,\tau) &= y(\pi,\tau)=0,
\end{cases}
\tag{14}
$$

where $W(\tau)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},P,\mathcal{F}_\tau)$ with $\mathcal{F}_\tau=\sigma\{W(w)-W(v)|w,v\leq\tau\}$. Let $\mathbb{V}=L^2[0,\pi]$ and define $A:=\dfrac{\partial^2}{\partial x^2}$, with domain

$$
\mathcal{D}(A)=\{y\in\mathbb{V}: y,y' \text{ are absolutely continuous}, y''\in\mathbb{V},\ y(0)=y(\pi)=0\}.
$$

The problem (14) can be written into the form (1) with $y(\tau)(x) = y(\tau, x)$, $\alpha = 1/2$, $\mathbf{a} = 1$ and $\nu = 2$. From [18, Example 4.6], there exists a uniformly exponentially stable family of operators $\{\mathcal{R}(\tau)\}_{\tau\geq 0}$ such that $\|\mathcal{R}(\tau)\| \leq Ce^{-\delta\tau}$. Therefore, by Theorem 2.1, the problem (14) has a unique square-mean (ν, μ)-pseudo S-asymptotically Bloch-type periodic mild solution on $\mathbb{R}$ if only $\|\gamma\|^2_\infty < \left((\frac{2L_1}{\delta} + L_2)\frac{C^2}{\delta}\right)^{-1}$.

Example 3.2.

Let $\mathbb{V} = L^2[0, \pi]$, $1 < \alpha < 2$, $\rho > 0$ and consider the problem

$$\begin{cases} \partial_t^\alpha y(\tau, x) = & -\rho\, y(\tau, x) - \dfrac{\rho^2}{4}\displaystyle\int_{-\infty}^{\tau} \frac{(\tau - s)^{\alpha-1}}{\Gamma(\alpha)} y(\tau, x) ds \\ & +g(\tau, y(\tau, x)) + f(\tau, y(\tau, x))\dfrac{\partial W(\tau)}{\partial \tau}, \quad (\tau, x) \in \mathbb{R} \times (0, \pi), \\ & u(0, \tau) = u(\pi, \tau) = 0. \end{cases} \tag{15}$$

Putting $y(\tau)(x) = y(\tau, x)$, $\alpha = \mathbf{a}$ and $A = \rho I$, I is the identity operator on the Hilbert space $\mathbb{V}$, the problem (15) can be written into the form (3). It follows from [[44], Example 4.17] that A generates a α-resolvent family $\{\mathcal{R}_\alpha(\tau)\}_{\tau\geq 0}$ with its Laplace transform satisfying

$$\hat{\mathcal{R}}_\alpha(\lambda) = \frac{\lambda^\alpha}{(\lambda^\alpha + \rho/2)^2} = \frac{\lambda^{\alpha-\rho/2}}{(\lambda^\alpha + \rho/2)} \cdot \frac{\lambda^{\alpha-\rho/2}}{(\lambda^\alpha + \rho/2)}$$

and

$$\mathcal{R}_\alpha(\tau) = (r * r)(\tau) \text{ where } r(\tau) = \tau^{\frac{\alpha}{2}} E_{\alpha,\alpha/2}\Big(- \frac{\rho}{2}\tau^\alpha\Big)$$

and $E_{\alpha,\alpha/2}(\cdot)$ is the Mittag-Leffter function (see [1]). From [[43], Theorem 4.12], $\mathcal{R}_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Therefore, by Theorem 2.2, the problem (15) has a unique square-mean $(\nu, \mu)-$pseudo S-asymptotically Bloch-type periodic mild solution on $\mathbb{R}$ if only $\|\gamma\|_\infty$ is small enough.

References

[1] S. Abbas, M. Benchohra, G.M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[2] B. De Andrade, C. Cuevas, S-asymptotically ω-periodic and asymptotically ω-periodic solutions to semilinear Cauchy problems with non-dense domain, *Nonl. Anal.* **72** (2010), no. 6, 3190–3208.

[3] E. Alvarez, C. Lizama, R. Ponce, Weighted pseudo antiperiodic solutions for fractional integro-differential equations in Banach spaces, *Appl. Math. Comput.* **259** (2015), 164–172.

[4] F. Bloch, Überdie quanten mechanik der elektronen in kristall gittern, *Z. Phys.* **52** (1929), 555–600.

[5] J. Blot, P. Cieutat, K. Ezzinbi, Measure theory and pseudo almost automorphic function: New developments and applications, *Nonlinear Analysis* **75** (2012), 2426–2447.

[6] J. Blot, P. Cieutat, K. Ezzinbi, New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications, *Applicable Analysis* **92** (2013), no. 3, 493-526.

[7] N. Belmabrouk, M. Damak, M. Miraoui, Measure pseudo almost periodic solution for a class of nonlinear delayed stochastic evolution equations driven by Brownian motion, *Filomat* **35** (2021), no. 2, 515–534.

[8] D. Brindle, G.M. N'Guérékata, S-asymptotically ω-periodic mild solutions to fractional differential equations, *Electron. J. Differ. Equ.* **30** (2020), 1–12.

[9] D. Brindle, G.M. N'Guérékata, S-asymptotically τ-periodic integrodifferential equations, *PanAmer. Math. J.* **29** (2019), no. 2, 63–74.

[10] Y.T. Bian, Y.K. Chang, J.J. Nieto, Weighted asymptotic behavior of solutions to a semilinear integro-differential equation in Banach spaces, *Electron. J. Diff. Equ.* **91** (2014), 1–16.

[11] C. Cuevas, J.C. De Souza, Existence of S-asymptotically ω-periodic solutions for fractional order functional integro-differential equations with infinite delay, *J. Nonl. Anal.* **72** (2010), no. 3, 1683–1689.

[12] A. Caicedo, C. Cuevas, H.R. Henríquez, Asymptotic periodicity for a class of partial integrodifferential equations, *Int. Sch. Res. Notices* **2011** (2011), Article ID 537890. https://doi.org/10.5402/2011/537890

[13] Y.K. Chang, M.J. Zhang, R. Ponce, Weighted pseudo almost automorphic solutions to a semilinear fractional differential equation with Stepanov-like weighted pseudo almost automorphic nonlinear term, *Appl. Math. Comput.* **257** (2015), 158–168.

[14] Y.K. Chang, Y. Wei, S-asymptotically Bloch-type periodic solutions to some semi-linear evolution equations in Banach spaces, *Acta Math. Sci. Ser. B* **41** (2021), 413–425

[15] Y.K. Chang, Y. Wei, Pseudo S-asymptotically Bloch-type periodicity with applications to some evolution equations, *Z. Anal. Anwend.* **40** (2021), 33–50.

[16] Y.K. Chang, J. Zhao, Weighted pseudo asymptotically Bloch periodic solutions to nonlocal Cauchy problems of integrodifferential equations in Banach spaces, *Int. J. Nonlinear Sci. Numer. Simul.* 24 (2021), no. 2, 581–598. https://doi.org/10.1515/ijnsns-2021-0251

[17] Y.K. Chang, Y. Wei, Pseudo S-asymptotically Bloch-type periodicity with applications to some evolution equations, *Zeitschrift für Analysis und ihre Anwendungen* **40** (2021), no. 1, 33–50.

[18] Y.K. Chang, R. Ponce, Uniform exponential stability and its applications to bounded solutions of integro-differential equations in Banach spaces, *J. Integral Equ. Appl.* **30** (2018), 347–369.

[19] Y.K. Chang, X.Y. Wei, G. M. N'Guérékata, Some new results on bounded solutions to a semilinear integro-differential equation in Banach spaces, *J. Integral Equ. Appl.* **27** (2015), 153–178.

[20] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge, 1992.

[21] A. Diop, M.M. Mbaye, G.M. N'Guérékata, Y.K. Chang, On square-mean S-asymptotically Bloch-type periodicity of some stochastic evolution equations, *Analele Universităţii Oradea Fasc. Matematica* **30** (2023), no. 2, 83–112.

[22] A. Diop, M.M. Mbaye, Y.K. Chang, G.M. N'Guérékata, Measure Pseudo-S-asymptotically Bloch-Type Periodicity of Some Semilinear Stochastic Integrodifferential Equations, *Journal of Theoretical Probability* **37** (2024), no. 3, 1–24.

[23] M.A. Diop, K. Ezzinbi, M.M. Mbaye, Existence and global attractiveness of a pseudo almost periodic solution in p-th mean sense for stochastic evolution equation driven by a fractional Brownian motion, *Stochastics* **87** (2015), no. 6, 1061–1093.

[24] J.P.C. Dos Santos, H.R. Henríquez, Existence of S-asymptotically ω-periodic solutions to abstract integro-differential equations, *Appl. Math. Comput.* **256** (2015), 109–118.

[25] T. Diagana, Weighted pseudo-almost periodic functions and applications, *C. R. Math. Acad. Sci. Paris* **343**(10) (2006), no. 10, 643–646.

[26] T. Diagana, Weighted pseudo-almost periodic solutions to some differential equations, *Nonlinear Anal. Theory Methods Appl.* **68** (2008), no. 8, 2250–2260.

[27] W.S. Du, M. Kostić, M. Pinto, Almost Periodic Functions and Their Applications: A Survey of Results and Perspectives, *J. Math.* **2021** (2021), Article ID 5536018.

[28] H. Henríquez, M. Pierri, P. Tàboas, Existence of S-asymptotically ω-periodic solutions for abstract neutral equations, *Bull. Austral. Math. Soc.* **78** (2008), no. 3, 365–382.

[29] H. Henríquez, M. Pierri, P. Tàboas, On S-asymptotically ω-periodic functions on Banach spaces and applications, *J. Math. Anal. Appl.* **343** (2008), no. 2, 1119–1130.

[30] Y. Hino, S. Murakami, T. Naito, *Functional-differential equations with infinite delay* Lecture notes in mathematics **1473**, Springer-Verlag, Berlin, 1991.

[31] M.F. Hasler, G.M. N'Guérékata, Bloch-periodic functions and some applications, *Nonlinear Stud.* **21** (2014), 21–30.

[32] M. Kostić, D. Velinov, Asymptotically Bloch-periodic solutions of abstract fractional nonlinear differential inclusions with piecewise constant argument, *Funct. Anal. Approx. Comput.* **9** (2017), 27–36.

[33] Q. Li, L. Liu, M. Wei, Existence of positive S-asymptotically periodic solutions of the fractional evolution equations in ordered Banach spaces, *Nonlinear Anal. Model. Control* **26** (2021), no. 5, 928–946.

[34] Q. Li, L. Liu, M. Wei, S-asymptotically Periodic Solutions for Time-Space Fractional Evolution Equation, *Mediterr. J. Math.* **18** (2021), no. 4, 1–21.

[35] C. Lizama, G.M. N'Guérékata, Bounded mild solutions for semilinear integro-differential equations in Banach spaces, *Integr. Equat. Oper. Th.* **68** (2010), no. 2, 207–227.

[36] C. Lizama, R. Ponce, Bounded solutions to a class of semilinear integrodifferential equations in Banach spaces, *Nonlin. Anal.* **74** (2011), 3397– 3406.

[37] M.A. Meyers, K.K. Chawla, *Mechanical Behavior of Materials*, Cambridge University Press, Cambridge, 2009.

[38] X. Mao, *Stochastic differential equations and their applications*, Horwood Publishing Ltd., Chichester, 1997.

[39] M.M. Mbaye, A. Diop, M. Dieye, Weighted pseudo S-asymptotically Bloch type periodic solutions for a class of mean field stochastic fractional evolution equations, *Malaya Journal of Matemati k* **11** (2024), no. 4, 378–402.

[40] S. Nicola, M. Pierri, A note on S-asymptotically ω-periodic functions, *Nonl. Anal.* **10** (2009), no. 5, 2937–2938.

[41] E.R. Oueama-Guengai, G.M. N'Guérékata, S-asymptotically ω-periodic mild solutions to some fractional integro-differential equations with infinite delay, *Lib. Math. (NS)* **38** (2018), no. 2, 111–124.

[42] M. Pierri, D. O'Regan, S-asymptotically ω-periodic solutions for abstract neutral differential equations, *Electron. J. Diff. Equ.* **2015** (2015), Article no. 210.

[43] R. Ponce, Asymptotic behavior of mild solutions to fractional Cauchy problems in Banach spaces, *Appl. Math. Lett.* **105** (2020), 1–9.

[44] R. Ponce, Bounded mild solutions to fractional integro-differential equations in Banach spaces, **Semigroup Forum 87** (2013), 377–392.

[45] L. Qiang, X. Wu, Existence and asymptotic behavior of square-mean S-asymptotically periodic solutions for fractional stochastic evolution equation with delay, Fract. Calc. Appl. Anal. 26 (2023), 718–750.

[46] X. Shu, F. Xu, Y. Shi, S-asymptotically ω-positive periodic solutions for a class of neutral fractional differential equations, *Appl. Math. Comput.* **270** (2015), 768–776.

[47] C. Zhang, Integration of vector-valued pseudo almost periodic functions, *Proc. Amer. Math. Soc.* **121** (1994), no. 1, 167–174.

[48] C. Zhang, Pseudo almost-periodic solutions of some Differential Equations, *J. Math. Anal. Appl.* **181** (1994), no. 1, 62-76.

[49] C. Zhang, Pseudo almost periodic solutions of some differential equations II, *J. Math. Anal. Appl.* 192 (1995), no. 2, 543–561.

[50] S. Zhao, M. Song, S-asymptotically ω-periodic solutions in distribution for a class of Stochastic fractional functional differential equations, arXiv : 1609.01453v1 [math.DS](2016).

[51] S. Zhao, M. Song, Square-mean S-asymptotically ω-periodic solutions for a Stochastic fractional evolution equation driven by Levy noise with piecewise constant argument, arXiv : 1609.01444v1 [math.DS] (2016).

(Mamadou Moustapha Mbaye) Département de Mathématiques, Faculté des Sciences et Technique, Université Cheikh Anta Diop, BP-5005, Dakar-Fann, Senegal
E-mail address: mamadoumoustapha3.mbaye@ucad.edu.sn

(Amadou Diop) Classes préparatoires aux grandes Écoles (CPGE), Thiés, Senegal. Laboratory of Numerical Analysis and Computer Science, Gaston Berger University, BP 234, Saint-Louis, Senegal
E-mail address: diop.amadou@ugb.edu.sn; amadou0712.diop@gmail.com

(Bipan Hazarika) Department of Mathematics, Gauhati University, Guwahati 781014, Assam, India
E-mail address: bh_rgu@yahoo.co.in; bh_gu@gauhati.ac.in