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Existence of Solutions of an Infinite System of Nonlinear Integral Equations in a New WC–Banach Algebra

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Abstract. In this paper we investigate some new examples of WC–Banach algebras and we use them as a framework to prove the existence of solutions of an infinite system of nonlinear integral quadratic equations. The analysis is based on a Krasnoselskii fixed point theorem type under weak topology setting. An illustrative example is provided to clarify the obtained result.

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1. Introduction

Over the past few decades, fixed-point theory has emerged as an active area of research with a broad range of applications in various fields. The theory itself represents a beautiful fusion of pure and applied analysis, topology, and geometry. On the one hand, fixed-point theory has significant applications in diverse domains such as physics, engineering, and game theory. On the other hand, it provides powerful tools for solving infinite systems of nonlinear integral equations, including quadratic ones (see [2, 3, 10, 11, 12]).

In this paper, we investigate an infinite system of nonlinear integral equations of the Volterra-Hammerstein type. In [2], Banas, Krichen, and Mefteh demonstrated that such a system has a solution in the WC-Banach algebra $\mathcal{C}([0,1], c_0)$, which consists of all functions mapping the interval $[0,1]$ into the sequence space c_0 and are continuous on $[0,1]$. Building on their work, we extend this investigation by applying a fixed-point theorem in the context of the weak topology. Our main contribution is to establish a stronger result, proving that an infinite system of Volterra-Hammerstein integral equations has at least one solution in the WC-Banach algebra $\mathcal{C}([0,1], l_1)$, which consists of all functions defined and continuous on the interval $[0,1]$ with values in the sequence space l_1. Notably, each such solution also belongs to the WC-Banach algebra $\mathcal{C}([0,1], c_0)$, as discussed in [2]).

It is worth mentioning that [2] was the first study to explore the solvability of infinite systems in the weak topology. This paper continues and extends the research presented in [2], as well as related works such as [10] and [12]. Finally, we provide an example to illustrate how our results can be applied to the theory of nonlinear integral equations.

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2. Preliminaries and main results

A non-associative Banach algebra over a field $\mathbb{K}$, ($\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$) is a Banach space X with the norm $\|\cdot\|$ endowed with an inner operation of multiplication $x \cdot y$ of elements $x, y \in X$ which is bilinear, and such that

$$\|x \cdot y\| \leq \|x\|\|y\|,$$

for $x, y \in X$. Here, the binary multiplication operation is not assumed to be associative. Examples include Lie algebras, Jordan algebras, the octonions, and three-dimensional Euclidean space equipped with the cross product operation.

Now, assume that X is a non-associative Banach algebra with the zero element θ. If (x_n) is a sequence in X, then we write $x_n \to x$ to denote that (x_n) converges strongly to $x \in X$ (with respect to the norm topology), and we write $x_n \rightharpoonup x$ to denote that (x_n) converges weakly to $x \in X$ (with respect to the weak topology).

For two arbitrary subsets U, V of X, we define the product $U \cdot V$ in the following way:

$$U \cdot V = \{u \cdot v \ : u \in U, \ v \in V\}.$$

Recall from [1] that the product of compacts sets U and V in X is compact. Moreover, it was shown that the product of a compact set and a weakly compact set is weakly compact. However, the product of two weakly compact sets does not need to be weakly compact. For more details, the reader may refer to (see [1, 4, 8]).

Definition 2.1. [1] A non-associative Banach algebra X is said weakly compact (in short, WC), if the product $U \cdot V$ of two arbitrary weakly compact sets U and V in X is weakly compact.

An natural interesting question is to give some characterizations of these Banach algebras, with the WC property. A few results in this direction were established in [1], [5], and [8]. In 2010, Ben Amar, Chouayekh, and Jeribi [5] introduced a class of Banach algebra satisfying the following sequential condition:

$$(\mathcal{P}) \qquad \begin{cases} \text{For any sequences } (x_n) \text{ and } (y_n) \text{ of } X \text{ such that } x_n \rightharpoonup x \\ \text{and } y_n \rightharpoonup y, \text{ then } x_n \cdot y_n \rightharpoonup x \cdot y; \text{ where } X \text{ is a Banach algebra,} \end{cases}$$

and they proved some fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators. In 2015, Jeribi and Krichen [8] asked the question of whether there exists a WC–Banach algebra in which the property $(\mathcal{P})$ fails and this question remained unanswered until 2019. Banas and Olszowy [4] provided a positive answer and proved that a Banach algebra X satisfies condition $(\mathcal{P})$ if and only if, X is a WC–Banach algebra.

Theorem 2.1. [4] A Banach algebra X satisfies condition $(\mathcal{P})$ if and only if X is a WC–Banach algebra.

In previous years, several researchers in functional analysis did not find other WC–Banach algebras that were sufficiently well-known, in this paper, we will provide several new examples of WC–Banach algebras.

Example 2.1. (i) Clearly, every finite dimensional Banach algebra is a WC–Banach algebra.

(ii) Let X be a commutative Banach algebra with Dunford-Pettis Property [8]. Then, X is a WC–Banach algebra (see [1]).

(iii) If X is a WC–Banach algebra then the set $\mathcal{C}(\mathcal{K}, X)$ (here, $\mathcal{K}$ is a compact Hausdorff space) of all continuous functions from $\mathcal{K}$ to X is also a WC–Banach algebra. The proof is based on Dobrakov's Theorem [6].

(iv) Let X be a Banach algebra with the Schur property (see [13]). Then, X is a WC–Banach algebra.

In the following example, we will present a new example of a non-associative WC–Banach algebra.

Example 2.2. Let $X = \ell^1$ the classical Banach sequence space, consists of all real sequences whose series is absolutely convergent,

$$l_1 = \left\{ x = (x_n) \in \mathbb{R}^\infty : \sum_{n=1}^{+\infty} |x_n| < \infty \right\},$$

with the standard norm,

$$\|x\|_{l_1} = \sum_{n=1}^{+\infty} |x_n|,$$

for $x = (x_n) \in l_1$, and with the multiplication law defined by

$$x \cdot y = (x_n) \cdot (y_n) = \left(\sum_{n=1}^{\infty} x_n y_n, 0, 0, 0, \cdots \right),$$

for all $x = (x_n)$ and $y = (y_n)$ in l_1. One can easily show that this law is non-associative and X with this inner operation of multiplication forms a Banach algebra.

$$\begin{aligned} \|x \cdot y\|_{l_1} &= \left| \sum_{n=1}^{+\infty} x_n y_n \right| \\ &\leq \sum_{n=1}^{+\infty} |x_n y_n| \\ &\leq \|x\|_{l_1} \sum_{n=1}^{+\infty} |y_n| \\ &\leq \|x\|_{l_1} \|y\|_{l_1}. \end{aligned}$$

Conversely, if $\mathcal{C}(\mathcal{K}, X)$ is WC–Banach algebra, then this property can be lifted to X. Indeed, the following proposition provides the converse of property (iii) in Example 2.1.

Proposition 2.1. Assume that $\mathcal{K}$ is a Hausdorff compact space and X is a Banach algebra such that $\mathcal{C}(\mathcal{K}, X)$ is WC–Banach algebra. Then, X is a WC–Banach algebra.

Proof. We reason by contradiction. We suppose that X is not a WC–Banach algebra. Then, by Theorem 2.1 there exist a two sequences (x_n) and (y_n) of X with $x_n \rightharpoonup x \in X$ and $y_n \rightharpoonup y \in X$ such that

$$x_n \cdot y_n \not\rightharpoonup x \cdot y.$$

Now, take the sequences (f_n) and (g_n) in $\mathcal{C}(\mathcal{K}, X)$ defined by $f_n(t) = x_n$ and $g_n(t) = y_n$ for all $t \in \mathcal{K}$, then we have $f_n(t) \rightharpoonup x$ and $g_n(t) \rightharpoonup y$, for all $t \in \mathcal{K}$.
Put $f(t) = x$ and $g(t) = y$ for all $t \in \mathcal{K}$. By Dobrakov's Theorem [6] we have

$$f_n \rightharpoonup f \text{ and } g_n \rightharpoonup g \text{ in } \mathcal{C}(\mathcal{K}, X).$$

Since $\mathcal{C}(\mathcal{K}, X)$ is a WC–Banach algebra, then

$$f_n \cdot g_n \rightharpoonup f \cdot g.$$

So, we have

$$f_n(t) \cdot g_n(t) \rightharpoonup f(t) \cdot g(t) = x \cdot y.$$

Thus,

$$x_n \cdot y_n \rightharpoonup x \cdot y,$$

which contradicts our supposition.
We conclude that:

$\mathcal{C}(\mathcal{K}, X)$ is a WC–Banach algebra if and only if X is a WC–Banach algebra. □

Now, we will present a new example of an associative WC–Banach algebra.

Example 2.3. Let the classical Banach sequence space l_1, and let us consider the multiplication of elements in l_1, defined in the following way :

$$x \cdot y = (x_n) \cdot (y_n) = \left(\sum_{n=1}^{\infty} x_n \sum_{n=1}^{\infty} y_n, 0, 0, 0, \cdots \right), \tag{2.1}$$

for all $x = (x_n)$ and $y = (y_n)$ in l_1.
It is easy to verify that this law is associative and

$$\begin{aligned} \|x \cdot y\|_{l_1} &= \left| \sum_{n=1}^{+\infty} x_n \sum_{n=1}^{+\infty} y_n \right| \\ &= \left| \sum_{n=1}^{+\infty} x_n \right| \left| \sum_{n=1}^{+\infty} y_n \right| \\ &= \|x\|_{l_1} \|y\|_{l_1}. \end{aligned}$$

Thus, l_1 with the above-defined inner operation of multiplication forms an associative Banach algebra.

Example 2.4. Assume that $\mathcal{K}$ is a Hausdorff compact space and l_1 is the Banach sequence algebra with the above-defined inner operation (2.1), then the set of all continuous functions from $\mathcal{K}$ to l_1 is also a WC–Banach algebra.

Proof. According to Example 2.1, it suffices to show that l_1 is the WC–Banach algebra. Let (x_k) and (y_k) two sequences of l_1 such that $x_k \rightharpoonup x \in l_1$ and $y_k \rightharpoonup y \in l_1$, since l_1 is a Banach algebra with Schur property, then $x_k \to x$ and $y_k \to y$. On the other hand, by continuity of the application $(x, y) = ((x_n), (y_n)) \mapsto x \cdot y$ on $l_1 \times l_1$, which implies $x_k \cdot y_k \to x \cdot y$, then we have $x_k \cdot y_k \rightharpoonup x \cdot y$. Thus, the proof is complete. □

We conclude this section by recalling the following fixed point theorem and the well-known conditions (H_1) and (H_2) (see [9]).

Definition 2.2. We will say that the operator A satisfies the condition (H_1) if for any sequence $(x_n) \subset D(A)$ which is weakly convergent in X, the sequence (Ax_n) has a strongly convergent subsequence in X.
We say that the operator A satisfies the condition (H_2) if for each weakly convergent sequence $(x_n) \subset D(A)$ the sequence (Ax_n) contains a weakly convergent subsequence in X.

It is worth noting that the conditions (H_1) and (H_2) have been studied in references [1, 2, 9]. For properties of operators that satisfy these conditions, we refer to monograph [8] and review article [9].

Theorem 2.2. Let X be a WC–Banach algebra and let S be a nonempty, bounded, closed and convex subset of X. Further, let be given three operators A, B, C such that $A, C \colon X \to X$ and $B \colon S \to X$, which satisfy the following conditions:

(i) The operators A, C satisfy the condition (H_2) and are Lipschitzian with Lipschitz constants α and β respectively.

(ii) A is a regular operator.

(iii) The operator B is continuous on S, satisfies the condition (H_1) and the set BS is relatively weakly compact.

(iv) For each $y \in S$ the following implication holds

$$x = AxBy + Cx \Rightarrow x \in S.$$

(v) $L\alpha + \beta < 1$, where $L = \|BS\|$.

Under the above assumptions the operator equation $x = AxBx + Cx$ has at least one solution in the set S.

Subsequently, we will apply the fixed point theorem in the next section.

3. Applications to infinite systems of integral equations in $\mathcal{C}([0,1], l_1)$

In this section we consider the following infinite system of nonlinear quadratic integral equations of the Volterra-Hammerstein type

$$x_n(t) = c_n(t, x_n(t)) + a_n(x_n(t)) \int_0^1 k_n(t,s) f_n(s, x_n(s), x_{n+1}(s), \dots) ds, \qquad (3.1)$$

where $n = 1, 2, \dots$ and $t \in I = [0, 1]$.

In [2], Banas, Krichen, and Mefteh proved that system (3.1) has at least one solution in the space $\mathcal{C}(I, c_0)$, consisting of all functions acting from the interval I into the Banach sequence space c_0, which are continuous on I, where c_0 is the space consisting of all sequences converging to zero. Our aim is to show that the infinite system of integral equations (3.1) has a solution $t \mapsto x(t) = (x_n(t))$ in the space $\mathcal{C}_1 = \mathcal{C}(I, l_1)$, consisting of all continuous functions acting from the interval I into the Banach real sequence space l_1.
In fact, every function of $\mathcal{C}_1$ can be regarded as a function sequence

$$x(t) = (x_n(t)) = (x_1(t), x_2(t), \dots),$$

for $t \in I$, where for any fixed $t \in I$ the sequence $(x_n(t))$ is a real sequence being an element of the space l_1.
Obviously, the norm in the space $\mathcal{C}_1$ has the following norm :

$$\|x\|_{\mathcal{C}_1} = \sup_{t\in I} \|x_n(t)\|_{l_1} = \sup_{t\in I} \sum_{n=1}^{+\infty} |x_n(t)|.$$

Let us mention that $\mathcal{C}_1$ forms a WC–Banach algebra (cf. Example 2.4).

Now, we formulate assumptions under which we will investigate infinite system (3.1).

(i) The function $a_n\colon \mathbb{R} \to (0, +\infty)$ is equibounded on $\mathbb{R}$, i.e., there exists a constant M_1 such that

$$|a_n(x)| \leq M_1,$$

for all $x \in \mathbb{R}$, and $n = 1, 2, \ldots$. Moreover, there exists a bounded function $\gamma\colon I \to \mathbb{R}_+$ with bound Γ such that

$$|a_n(x_n(t)) - a_n(y_n(t))| \leq \gamma(t)|x_n(t) - y_n(t)|,$$

for all $x = (x_n)$ and $y = (y_n)$ in $\mathcal{C}_1$, $t \in I$, and $n = 1, 2, \ldots$.

(ii) The function $c_n\colon I \times \mathbb{R} \to \mathbb{R}$ is equibounded on $I \times \mathbb{R}$, i.e., there exists a constant M_2 such that

$$|c_n(t, s)| \leq M_2,$$

for all $t \in I$, $s \in \mathbb{R}$, and $n = 1, 2, \ldots$. Moreover, there exists a bounded function $\lambda\colon I \to \mathbb{R}_+$ with bound Λ such that for all $r, s \in \mathbb{R}$ and $t \in I$,

$$|c_n(t, r) - c_n(t, s)| \leq \lambda(t)|r - s|,$$

for all r, s in $\mathbb{R}$, $t \in I$, and $n = 1, 2, \ldots$.

(iii) The sequences $(a_n(x_n(t)))$ and $(c_n(t, x_n(t)))$ are in l_1 for all $x = (x_n) \in \mathcal{C}_1$ and $t \in I$. Moreover for every $x = (x_n) \in \mathcal{C}_1$ the functions sequence $(a_n(x_n(\cdot)))$ and $(c_n(\cdot, x_n(\cdot)))$ are continuous on I.

(iv) The functions $k_n(\cdot, \cdot) := k_n$ are continuous on I^2 for all $n = 1, 2, \ldots$. Moreover, the functions $t \mapsto k_n(t, s)$ are equicontinuous on I uniformly with respect to $s \in I$, i.e., the following condition is satisfied

$$\forall \varepsilon > 0, \exists \delta > 0, \forall n \in \mathbb{N}^*, \forall s \in I, \forall t_1, t_2 \in I,$$

we have

$$|t_2 - t_1| < \delta \Rightarrow |k_n(t_2, s) - k_n(t_1, s)| < \varepsilon.$$

(v) There exists a constant $K_1 > 0$ such that

$$\sum_{n=1}^{+\infty} \int_0^1 |k_n(t, s)| ds \leq K_1,$$

for all $t \in I$.

(vi) The function sequence $(k_n(\cdot, \cdot))$ is equibounded on I^2, i.e., there exists a constant K_2 such that

$$|k_n(t, s)| \leq K_2,$$

for all $t, s \in I$, and $n = 1, 2, \ldots$.

(vii) The function f_n acts from the set $I \times \mathbb{R}^\infty$ into $\mathbb{R}$ for any $n = 1, 2, \ldots$. Moreover, we assume that there exist two functions sequence (p_n) and (q_n) with positive terms defined and continuous on I and the following inequality is satisfied

$$|f_n(t, x_n, x_{n+1}, \dots) \leq p_n(t) + q_n(t) \sum_{i=n}^{n+j} |x_i(s)|,$$

for all $t \in I$, $x = (x_n) \in l_1$, $n = 1, 2, \dots$, and a fixed $j \in \mathbb{N}$.
Assume that the functions series $\sum_{n\geq 1} p_n$ and $\sum_{n\geq 1} q_n$ are uniformly convergent on I.

Remark 3.1. From the above formulated assumption we deduce that $P < \infty$ and $Q < \infty$ where the constants P and Q are defined by the equalities :

$$P = \sum_{n=1}^{+\infty} \int_0^1 p_n(s)ds, \text{ and } Q = \sum_{n=1}^{+\infty} \int_0^1 q_n(s)ds.$$

Our further assumptions are as follows.

$(viii)$ The family of function $\{f_n\}_{n\in\mathbb{N}^*}$ is uniformly equicontinuous on the set $I \times l_1$. This means that for every $\varepsilon > 0$ there exists $\delta > 0$, such that for any $n \in \mathbb{N}^*$, $t \in I$, and for all $x = (x_n)$, $y = (y_n) \in l_1$ with $\|x - y\|_{l_1} < \delta$ we have that

$$|f_n(t, x_n, x_{n+1}, \dots) - f_n(t, y_n, y_{n+1} \dots)| < \varepsilon.$$

(ix) The following inequality holds

$$MK_2Q < 1,$$

where $M = \max\{M_1, M_2\}$. For further purposes we define the number r_0 by putting

$$r_0 = M\frac{K_2P + 1}{1 - MK_2Q}. \tag{3.2}$$

(x) The following inequality is satisfied

$$\Gamma K_2 \frac{P + MQ}{1 - MK_2Q} + \Lambda < 1.$$

Theorem 3.1. Under assumptions $(i) - (x)$ the infinite system of integral equations (3.1) has at least one solution $x(t) = (x_n(t))$ in the space $\mathcal{C}_1 = \mathcal{C}(I, l_1)$.

Proof. In order to prove our theorem we will apply the result contained in Theorem 2.2. To this end let us define on the space $\mathcal{C}_1$ three operators A, B, C by putting:

$$\begin{cases} (Ax)(t) = (a_n(x_n(t))) = (a_1(x_1(t)), a_2(x_2(t)), \ldots), \\ (Bx)(t) = \displaystyle\int_0^1 k_n(t,s) f_n(s, x_n(s), x_{n+1}(s), \dots) ds, \\ (Cx)(t) = (c_n(t, x_n(t))) = (c_1(t, x_1(t)), c_2(t, x_2(t)), \ldots), \end{cases}$$

for an arbitrary element $x = (x_n) \in \mathcal{C}_1$ and for $t \in I = [0, 1]$. We show that these operators satisfy the assumptions of Theorem 2.2.
It can be easily seen that the infinite system of integral equations (3.1) is equivalent to the equation $x = Ax \cdot Bx + Cx$. To prove our statement, it is sufficient to show that the operators A, B, and C satisfy the hypothesis of Theorem 2.2.

We start with investigations concerning the operator A. Initially, let us observe that assumption (i) ensures that A is regular. According to assumption (iii), it is easy to check that $Ax(t) \in l_1$ for all $t \in I$, and we have that $Ax \in \mathcal{C}_1$ for all $x = (x_n) \in \mathcal{C}_1$, indeed, let (t_m) be a sequence of I such that $t_m \to t$ in I. Then,

$$\|Ax(t_m) - Ax(t)\|_{l_1} = \|a_n(x_n(t_m)) - a_n(x_n(t))\|_{l_1} \to 0,$$

as $m \to +\infty$. Hence, the operator A is well defined.
Next, we check that A is Lipschitzian. Let $x, y \in \mathcal{C}_1$ and $t \in I$,

$$\begin{aligned}
\|Ax(t) - Ay(t)\|_{l_1} &= \sum_{n=1}^{+\infty} |a_n(x_n(t)) - a_n(y_n(t))| \\
&\leq \gamma(t) \sum_{n=1}^{+\infty} |x_n(t) - y_n(t)| \\
&\leq \gamma(t)\|x(t) - y(t)\|_{l_1} \\
&\leq \Gamma\|x(t) - y(t)\|_{l_1}.
\end{aligned}$$

Taking the supremum over $t \in I$, we obtain that A is Γ Lipschitzian.
In order to verify that the operator A meets the condition (H_2). So, let $(x_n) \subset \mathcal{C}_1$ which is weakly convergent to a function $x \in C_1$. So, we denote

$$x_n(t) = (x_1^n(t), x_2^n(t), x_3^n(t), \dots),$$

for $n = 1, 2, \dots$ and for an arbitrary $t \in I$ and if we denote $x(t) = (x_1(t), x_2(t), x_3(t), \dots)$, then, by the Dobrakov theorem [6] we infer that $x_1^n(t) \rightharpoonup x_1(t)$, $x_2^n(t) \rightharpoonup x_2(t)$, ..., $x_k^n(t) \rightharpoonup x_k(t), \dots$ for any $t \in I$, if $n \to \infty$. Since l_1 is a Banach algebra with the Schur property, we deduce that: $x_1^n(t) \to x_1(t), x_2^n(t) \to x_2(t), \dots, x_k^n(t) \to x_k(t), \dots$ for any $t \in I$, if $n \to \infty$.
Now, let us consider the sequence (Ax_n) i.e.,

$$\begin{aligned}
(Ax_n) &= (A(x_1^n, x_2^n, x_3^n, \dots)) = (a_1(x_1^n), a_2(x_2^n), a_3(x_3^n), \dots) \\
&= (a_k(x_k^n)),
\end{aligned}$$

for $k = 1, 2, \dots$.
Then, for an arbitrarily fixed $t \in I$ we obtain:

$$\begin{aligned}
((Ax_n)(t)) &= (a_1(x_1^n(t)), a_2(x_2^n(t)), a_3(x_3^n(t)), \dots) \\
&= (a_k(x_k^n(t))),
\end{aligned}$$

for $k = 1, 2, \dots$.
Given our assumptions, we have that $x_k^n(t) \to x_k(t)$ as $n \to \infty$ $(k = 1, 2, \dots)$, this implies that $a_k(x_k^n(t)) \to a_k(x_k(t))$ $(k = 1, 2, \dots)$, which is a simple consequence of the continuity of each function $a_k (k = 1, 2, \dots)$ on the set $\mathbb{R}$. But this means that the sequence (Ax_n) is strongly convergent in the space $\mathcal{C}_1$. Thus the operator A satisfies the condition (H_1), in particular, it satisfies the condition (H_2).
Similarly, we can establish that the operator C is Λ-Lipschitzian. Furthermore, C maps the space $\mathcal{C}_1$ to itself and fulfills condition (H_2).
Hereafter, we will consider the operator B. To accomplish this, let us define the set $S = B(\theta, r_0)$, where r_0 is a number described by equality (3.2). Initially, we start by showing that B maps the set S into the space $\mathcal{C}_1$. Thus, let us take an arbitrary

function $x = (x_n) \in S$. Fix arbitrarily a number $t \in I$. Next, considering the assumptions (vi) and (vii) and Remark 3.1, we obtain:

$$\begin{aligned}
\sum_{n=1}^{+\infty} |(Bx)(t)| &= \sum_{n=1}^{+\infty} \left| \int_0^1 k_n(t,s) f_n(s, x_n(s), x_{n+1}(s), \dots) ds \right| \\
&\leq \sum_{n=1}^{+\infty} \int_0^1 |k_n(t,s) f_n(s, x_n(s), x_{n+1}(s), \dots)| \, ds \\
&\leq \sum_{n=1}^{+\infty} \int_0^1 |k_n(t,s)| \left\{ p_n(s) + q_n(s) \sum_{i=n}^{n+j} |x_i(s)| \right\} ds \\
&\leq K_2 \sum_{n=1}^{+\infty} \int_0^1 \left\{ p_n(s) + q_n(s) \sum_{i=n}^{n+j} |x_i(s)| \right\} ds \\
&\leq K_2 (P + Q \|x\|_{\mathcal{C}_1}) \\
&< +\infty.
\end{aligned}$$

Thus, $(Bx)(t) \in l_1$ for all $t \in I$.
We have also the following inequality,

$$\|Bx\|_{\mathcal{C}_1} \leq K_2 (P + Q r_0) \tag{3.3}$$

Next, we show that Bx is continuous for all $x \in S$. To this end, fix $\varepsilon > 0$ and choose a number δ according to assumption (iv). Next, take $t_2, t_1 \in I$ such that $|t_2 - t_1| < \delta$ and using assumptions (iv) and (vii), we obtain

$$\begin{aligned}
\|(Bx)(t_2) - (Bx)(t_1)\|_{l_1} &= \sum_{n=1}^{+\infty} \left| \int_0^1 \{k_n(t_2,s) - k_n(t_1,s)\} f_n(s, x_n(s), x_{n+1}(s), \dots) ds \right| \\
&\leq \varepsilon \sum_{n=1}^{+\infty} \int_0^1 \left\{ p_n(s) + q_n(s) \sum_{i=n}^{n+j} |x_i(s)| \right\} ds \\
&\leq \varepsilon (P + Q \|x\| |_{\mathcal{C}_1}) \\
&\leq \varepsilon (P + Q r_0).
\end{aligned}$$

We deduce that Bx is continuous for all $x \in S$. Furthermore, we prove that the operator B is continuous on the set S.
In order to prove this fact, we fix arbitrarily an $\varepsilon > 0$ and we choose a number $\delta > 0$ according to assumption $(viii)$. Next we take $x, y \in S$ such that $\|x - y\|_{\mathcal{C}_1} < \delta$. So, for any $t \in I$ we have:

$$\sup_{t \in I} \|x(t) - y(t)\|_{l_1} < \delta.$$

Equivalently, we can express this as:

$$\sup_{t \in I} \sum_{n=1}^{+\infty} \|x(t) - y(t)\|_{l_1} < \delta.$$

Therefore, by synthesizing the aforementioned estimates, we derive the following inequalities:

$$\|Bx - By\|_{\mathcal{C}_1} = \sup_{t\in I} \|(Bx)(t) - (By)(t)\|_{l_1}$$
$$\leq \sup_{t\in I} \sum_{n=1}^{+\infty} |(Bx)(t) - (By)(t)|$$
$$\leq \sup_{t\in I} \sum_{n=1}^{+\infty} \left| \int_0^1 k_n(t,s) \left\{ f_n(s, x_n(s), x_{n+1}(s), \dots) - f_n(s, y_n(s), y_{n+1}(s), \dots) \right\} \right| ds$$
$$\leq \sup_{t\in I} \sum_{n=1}^{+\infty} \int_0^1 |k_n(t,s)||f_n(s, x_n(s), x_{n+1}(s), \dots) - f_n(s, y_n(s), y_{n+1}(s), \dots)|ds.$$

Therefore, considering assumptions (v) and $(viii)$, we derive

$$\begin{aligned}\|Bx - By\|_{\mathcal{C}_1} &\leq \varepsilon \sup_{t\in I} \sum_{n=1}^{+\infty} \int_0^1 |k_n(t,s)|ds \\ &\leq \varepsilon K_1.\end{aligned}$$

This establishes that the operator B is continuous (and even uniformly continuous) on the set S.

Next, we will prove that the operator B fulfills the condition (H_1) on the set S. Note that estimate (3.3) shows that functions in the set BS are equibounded on the interval I. Based on the above analysis, we can further establish that the set BS is equicontinuous on I. Now, we can apply Arzela-Ascoli's theorem (see [7]) to get that BS is relatively compact. In particular, The operator B satisfies the condition (H_1).
In our next step we show that assumption (iv) of Theorem 2.2 is satisfied . To this end let us fix arbitrarily $y \in S = B(\theta, r_0)$. Next, assume that an element $x \in \mathcal{C}_1$ satisfies the equality

$$x = AxBy + Cx.$$

This yields

$$\|x\|_{\mathcal{C}_1} \leq \|Ax\|_{\mathcal{C}_1}\|By\|_{\mathcal{C}_1} + \|Cx\|_{\mathcal{C}_1}.$$

Then,

$$\begin{aligned}\|x\|_{\mathcal{C}_1} &\leq M_1\|By\|_{\mathcal{C}_1} + M_2 \\ &\leq M_1K_2(P + Qr_0) + M_2 = M(K_2(P + Qr_0) + 1)\end{aligned}$$

Therefore, taking into account assumption (ix), we derive that $\|x\|_{\mathcal{C}_1} \leq r_0$. This establishes $x \in S$ and demonstrates the satisfaction of assumption (iv) of Theorem 2.2. Finally, let us notice that in view of equality (3.2) and estimate (3.3) we have:

$$\begin{aligned}L = \|BS\| &\leq K_2(P + Qr_0) \\ &\leq K_2 \frac{P + MQ}{1 - MK_2Q}.\end{aligned}$$

By linking the above inequality with assumption (x), we observe that assumption (v) of Theorem 2.2 is satisfied. Thus, the proof is complete. □

4. An example

This section is dedicated to provide an example that illustrates the applicability of the result stated in Theorem 3.1.

Consider the following infinite system of integral equations of the form

$$x_n(t) = \frac{t\sin(x_n(t))}{Se^t} + \left[\frac{1}{2n^2} + \frac{T}{T'}\frac{\ln(1+x_n^2(t))}{1+\ln(1+x_n^2(t))}\right] \times \int_0^1 \frac{e^{-(n+t)\frac{s}{2}}}{n}\left\{\frac{\sin(ns)s^n}{3n!} + \frac{\arctan(s)}{2n^2}\sum_{k=1}^{3}\frac{1}{k}\frac{x_{n+k-1}(s)}{x_{n+k-1}^2(s)+k}\right\}ds, \quad (4.1)$$

where $t \in [0,1]$, $T, T' > 0$, $S > 1$ and $n = 1, \ldots$.

Put $M_1 = \frac{1}{S}$, $M_2 = \frac{1}{2} + \frac{T}{T'}$, and $M = \max\{M_1, M_2\}$.
In the rest of this paper, we will assume that

$$M < \frac{48}{\pi^3 - 2\pi^2\ln(2)},$$

and

$$\frac{T}{T'}\frac{\frac{e-1}{3}M\frac{\pi^3-2\pi^2\ln(2)}{48}}{1 - M\frac{\pi^3-2\pi^2\ln(2)}{48}} + \frac{1}{S} < 1.$$

Note that the infinite system (4.1) is a special case of the infinite system of quadratic integral equations of Volterra-Hammerstein (3.1) when we set:

$$a_n(x_n) = \frac{1}{2n^2} + \frac{T}{T'}\frac{\ln(1+x_n^2)}{1+\ln(1+x_n^2)},$$

$$c_n(t, x_n) = \frac{t\sin(x_n)}{Se^t},$$

$$k_n(t,s) = \frac{e^{-(n+t)\frac{s}{2}}}{n}, \text{ and}$$

$$f(t, x_n, x_{n+1}, \ldots) = \frac{\sin(nt)t^n}{3n!} + \frac{\arctan(t)}{2n^2}\sum_{k=1}^{3}\frac{1}{k}\frac{x_{n+k-1}(t)}{x_{n+k-1}^2(t)+k}$$ for $t \in [0,1]$ and $n \geq 1$.

Now, we show that infinite system of integral equations (4.1) has a solution in the Banach algebra $\mathcal{C}_1 = \mathcal{C}([0,1], l_1)$ by showing that all the conditions of Theorem 3.1 are satisfied.
Take the sequence of functions (a_n) defined on the set $\mathbb{R}$ by

$$a_n(x) = \frac{1}{2n^2} + \frac{T}{T'}\frac{\ln(1+x^2)}{1+\ln(1+x^2)}.$$

Note that $a_n(x) \geq \frac{1}{2n^2} > 0$ for $n = 1, 2, \ldots$ which implies that a_n acts from $\mathbb{R}$ into $(0, +\infty)$. Moreover, for arbitrary $x \in \mathbb{R}$ and for a fixed natural number $n \in \mathbb{N}^*$ we have:

$$|a_n(x)| = \left|\frac{1}{2n^2} + \frac{T}{T'}\frac{\ln(1+x^2)}{1+\ln(1+x^2)}\right| \leq \frac{1}{2} + \frac{T}{T'} := M_1.$$

Further, for arbitrary $x, y \in \mathbb{R}$ and for a fixed $n \in \mathbb{N}^*$ we obtain :

$$\begin{aligned} |a_n(x) - a_n(y)| &= \left| \frac{1}{2n^2} + \frac{T}{T'} \frac{\ln(1+x^2)}{1+\ln(1+x^2)} - \frac{1}{2n^2} - \frac{T}{T'} \frac{\ln(1+y^2)}{1+\ln(1+y^2)} \right| \\ &\leq \frac{T}{T'} \left| \frac{\ln(1+x^2)(1+\ln(1+y^2)) - \ln(1+y^2)(1+\ln(1+x^2))}{(1+\ln(1+x^2))(1+\ln(1+y^2))} \right| \\ &\leq \frac{T}{T'} |\ln(1+x^2) - \ln(1+y^2)| \\ &\leq \frac{T}{T'} |x-y|. \end{aligned}$$

Which shows that the function sequence (a_n) is Lipschitzian with constant $\frac{T}{T'}$. The above inequality shows that assumption (i) is satisfied with $\Gamma = \frac{T}{T'}$.
Assume that $x = (x_n) \in \mathcal{C}_1$, and $t \in I$, we have

$$\begin{aligned} |a_n(x_n(t))| = \left| \frac{1}{2n^2} + \frac{T}{T'} \frac{\ln(1+x_n^2(t))}{1+\ln(1+x_n^2(t))} \right| &= \frac{1}{2n^2} + \frac{T}{T'} \frac{\ln(1+x_n^2(t))}{1+\ln(1+x_n^2(t))} \\ &\leq \frac{1}{2n^2} + \frac{T}{T'} \ln(1+x_n^2(t)) \\ &\leq \frac{1}{2n^2} + \frac{T}{T'} x_n^2(t), \end{aligned}$$

since $x = (x_n) \in \mathcal{C}_1$, it follows that for all $t \in [0,1]$, $\frac{1}{2n^2} + \frac{T}{T'} x_n^2(t) \in l_1$.
We also have the function $t \mapsto a_n(x_n(t))$ is continuous on $[0,1]$. Similarly, we define the function sequence (c_n) on $[0,1] \times \mathbb{R}$ by:

$$c_n(t,s) = \frac{t \sin(s)}{Se^t},$$

where $S > 1$. For arbitrary $r, s \in \mathbb{R}$ and $t \in [0,1]$ we have

$$\begin{aligned} |c_n(t,r) - c_n(t,s)| &= \left| \frac{t\sin(r)}{Se^t} - \frac{t\sin(s)}{Se^t} \right| \\ &\leq \frac{1}{S} |\sin(r) - \sin(s)| \\ &\leq \frac{1}{S} |r-s|. \end{aligned}$$

So, the function sequence (c_n) is Lipschitzian with constant $\frac{1}{S}$ and assumption (ii) is satisfied with $\Lambda = \frac{1}{S}$.
Obviously, the sequence $(c_n(t, x_n(t)))$ belongs to l_1 for all $x = (x_n) \in \mathcal{C}_1$ and $t \in I$, and $t \mapsto c_n(t, x_n(t)))$ is continuous on $[0,1]$. Moreover, for all $t \in [0,1]$, $s \in \mathbb{R}$, and $n \geq 1$, we have

$$\begin{aligned} |c_n(t, x_n(t))| &= \frac{t|\sin(s)|}{Se^t} \\ &\leq \frac{1}{S} := M_2. \end{aligned}$$

Then, the function sequence $(c_n(t,s))$ is equibounded on $I \times \mathbb{R}$, for $n \geq 1$. Thus, hypotheses (ii) and (iii) of Theorem 3.1 are satisfied.
Further, let us observe that the functions k_n are continuous on I^2 for $n \geq 1$. Next, fix a natural number $n \in \mathbb{N}^*$ and $s \in I$. Then, for arbitrary $t_1, t_2 \in I$ we obtain

$$\begin{aligned} |k_n(t_2,s) - k_n(t_1,s)| &= \left| \frac{e^{-(n+t_2)\frac{s}{2}}}{n} - \frac{e^{-(n+t_1)\frac{s}{2}}}{n} \right| \\ &\leq \left| e^{-t_2\frac{s}{2}} - e^{-t_1\frac{s}{2}} \right| \\ &\leq \frac{s}{2} |t_2 - t_1| . \end{aligned}$$

Thus, the function sequence $(k_n(t,s))$ satisfies assumption (iv).
Next, for arbitrarily fixed $n \in \mathbb{N}^*$ and $t, s \in [0,1]$ we get

$$|k_n(t,s)| = \left| \frac{e^{-(n+t)\frac{s}{2}}}{n} \right| \leq 1 := K_2.$$

Hence, we infer that assumption (vi) is satisfied with the constant $K_2 = 1$.
Furthermore, observe that

$$\begin{aligned} \sum_{n=1}^{+\infty} \int_0^1 |k_n(t,s)| ds &= \sum_{n=1}^{+\infty} \int_0^1 \frac{e^{-(n+t)\frac{s}{2}}}{n} ds \\ &= \sum_{n=1}^{+\infty} \frac{2}{n(n+t)} \left(1 - e^{-(n+t)\frac{1}{2}} \right) \\ &\leq 2 \sum_{n=1}^{+\infty} \frac{1}{n^2} \\ &\leq \frac{\pi^2}{3}. \end{aligned}$$

Hence, we infer that assumption (v) is satisfied with the constant $K_1 = \dfrac{\pi^2}{3}$.
Furthermore, we show that the function $f_n = f_n(t, x_n, x_{n+1}, \dots)$, where $n = 1, \dots$ verifies assumption $(viii)$. Assume that $t \in [0,1]$ and $x = (x_n) \in l_1$, we obtain

$$\begin{aligned} |f_n(t, x_n, x_{n+1}, \dots)| &= \left| \frac{\sin(nt)t^n}{3n!} + \frac{\arctan(t)}{2n^2} \sum_{k=1}^{3} \frac{1}{k} \frac{x_{n+k-1}(t)}{x_{n+k-1}^2(t) + k} \right| \\ &\leq \frac{t^n}{3n!} + \frac{\arctan(t)}{2n^2} \frac{|x_n|}{x_n^2 + 1} + \frac{\arctan(t)}{4n^2} \frac{|x_{n+1}|}{x_{n+1}^2 + 2} + \frac{\arctan(t)}{6n^2} \frac{|x_{n+2}|}{x_{n+2}^2 + 3} \\ &\leq \frac{t^n}{3n!} + \frac{\arctan(t)}{2n^2} \left[|x_n| + |x_{n+1}| + |x_{n+2}| \right] \\ &\leq \frac{t^n}{3n!} + \frac{\arctan(t)}{2n^2} \sum_{i=n}^{n+2} |x_i|. \end{aligned}$$

Hence, we see that the function f_n satisfies the inequality from assumption (vii) with the functions sequences (p_n) and (q_n) defined as follows:

$$p_n(t) = \frac{t^n}{3n!}, \quad q_n(t) = \frac{\arctan(t)}{2n^2},$$

for all $t \in [0,1]$ and $n \geq 1$.
It is easy to verify that the function series $\sum p_n$ and $\sum q_n$ are normally convergent, and therefore uniformly convergent on $[0,1]$.
Further, let $x = (x_n), y = (y_n) \in l_1$ and for $t \in [0,1]$, we obtain

$$\begin{aligned} |f_n(t, x_n, x_{n+1}, \dots) - f_n(t, y_n, y_{n+1}, \dots)| &\leq \frac{\arctan(t)}{2n^2} \left(\sum_{k=n}^{n+2} |x_k| - \sum_{k=n}^{n+2} |y_k| \right) \\ &\leq \frac{\arctan(t)}{2n^2} \left(\sum_{k=n}^{n+2} |x_k - y_k| \right) \\ &\leq \frac{\pi}{4} \|x - y\|_{l_1}. \end{aligned}$$

Thus, the function $f_n = f_n(t, x_n, x_{n+1}, \dots)$ satisfies a Lipschitz condition with a constant $\frac{\pi}{4}$ on the set $[0,1] \times l_1$. Consequently, the family of functions $\{f_n\}_n$ is uniformly equicontinuous on the set $[0,1] \times l_1$.
Moreover, we have

$$\begin{aligned} Q &= \sum_{n=1}^{+\infty} \int_0^1 q_n(s) ds = \sum_{n=1}^{+\infty} \int_0^1 \frac{\arctan(s)}{2n^2} ds \\ &= \sum_{n=1}^{+\infty} \frac{1}{2n^2} \int_0^1 \arctan(s) ds \\ &= \frac{\pi^3 - 2\pi^2 \ln(2)}{48}, \end{aligned}$$

and

$$M = \max\{M_1, M_2\} = \max\left\{ \frac{1}{S}, \frac{1}{2} + \frac{T}{T'} \right\}.$$

Based on the accepted assumptions stated above, we derive:

$$\begin{aligned} MK_2Q &= M \frac{\pi^3 - 2\pi^2 \ln(2)}{48} \\ &< 1. \end{aligned}$$

Thus, we observe that assumption (ix) is satisfied.
Finally, we are going to verify assumption (x). Indeed, considering all the constants Γ, Λ, K_1, K_2, M, Q, and

$$\begin{aligned} P &= \sum_{n=1}^{+\infty} \int_0^1 p_n(s) ds = \sum_{n=1}^{+\infty} \int_0^1 \frac{s^n}{3n!} ds \\ &= \frac{1}{3} \int_0^1 \left(\sum_{n=1}^{+\infty} \frac{s^n}{n!} \right) ds \\ &= \frac{e-1}{3}. \end{aligned}$$

Then, we see that the mentioned inequality has the form

$$\frac{T}{T'} \frac{\frac{e-1}{3} M \frac{\pi^3 - 2\pi^2 \ln(2)}{48}}{1 - M \frac{\pi^3 - 2\pi^2 \ln(2)}{48}} + \frac{1}{S} < 1.$$

It is easy to verify that the above inequality has a positive solution for appropriate values of the parameters T, T', and S. For example, with $T=4$, $T'=3$, and $S=6$ we obtain

$$\frac{4}{3}\frac{\frac{e-1}{3}\frac{11}{6}\frac{\pi^3-2\pi^2\ln(2)}{48}}{1-\frac{11}{6}\frac{\pi^3-2\pi^2\ln(2)}{48}}+\frac{1}{6}\approx 0,8<1.$$

Thus, by applying Theorem 3.1, we conclude that the infinite system of integral equations (4.1) has a solution in the Banach algebra $\mathcal{C}_1$.

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