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The Generalized Power Fractional Derivative Operators with Respect to Another Function in the Kernel

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Abstract. In this work, a new notion of the fractional derivatives with non-singular kernels the so-called power Caputo and Power Riemann-Liouville (R-L) operators associated with another function in the kernel are presented. The new defined operators are the generalization of different operators found in the literature. Some basic properties and formulas of the new operators are discussed. Additionally, novel formulas and properties of fractional derivatives and integrals in the Power Caputo and Power R-L senses are presented in this study.

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1. Introduction

The fractional derivative is a generalization of integer order classical derivative. Despite fractional calculus's impressive 325-year history, many questions remain unanswered from both a theoretical and practical standpoint. Abel solves the tautochrone problem using fractional calculus [1]. Researchers' attention is drawn to the use of fractional calculus in differential and integral equations by this work [2]. Many mathematical models have been created in recent years using the topic of fractional calculus with boundary conditions to address a variety of real-world problems. The fractional order derivatives correspond to physical representations of diverse phenomena found in a range of disciplines, including as dynamic systems, physics, biology, and mechanics [3]. One of the main topics of study for fractional order differential equations is the existence theory of results, which is something that analysts are closely monitoring. It is quite difficult to get a precise solution for a differential equation of fractional order.

On other side, the modeling of memory effects has changed recently, as evidenced by a thorough review of the fractional calculus literature. These modifications include taking into account the exponential outcome when using the Caputo–Fabrizio derivative [4], Atangana-Baleanu and Al-Refai operators [5, 6], the modified Atangana-Baleanu operators [7], and the new generalized fractional operator of Hattaf [8, 9].

2. Preliminaries

Let's review the key terminologies.

The definitions of beta and gamma functions from [10] are as follows.

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Definition 2.1. The widely recognized Gamma function, which is defined by

$$\Gamma(\delta)=\int_0^{\infty}\theta^{\delta-1}e^{-\theta}d\theta,\quad \mathcal{R}(\delta)>0.$$

Definition 2.2. The well-known Beta function is described as follows:

$$B(u,v)=\int_0^{1}\theta^{u-1}(1-\theta)^{v-1}d\theta,$$

where $\mathcal{R}(u)>0,\ \mathcal{R}(v)>0$.

And also, its relation with Gamma function is given by

$$B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

Definition 2.3. Lotfi *et al.* [11] have recently presented the power M-L function in the following manner:

$$^{p}E_{\sigma,\omega}\left(\theta\right)=\sum_{j=0}^{\infty}\frac{\left(\theta\ln p\right)^{j}}{\Gamma\left(\sigma j+\omega\right)} \tag{1}$$

where $\theta\in\mathbb{C},$ p $>1,\Re\left(\sigma\right)>0$ and $\Re\left(\omega\right)>0$.

Remark 2.1. Special cases of power M-L function are:
(1) If we take $\sigma=\omega=1$and $p=e$, we get

$$^{e}E_{1,1}\left(\theta\right)=\sum_{j=0}^{\infty}\frac{\left(\theta\right)^{j}}{\Gamma\left(j+1\right)}=e^{\theta}.$$

(2) If we take $\omega=1$and $p=e$,we get

$$^{p}E_{\sigma,1}\left(\theta\right)=\sum_{j=0}^{\infty}\frac{\left(\theta\right)^{j}}{\Gamma\left(\sigma j+1\right)}.$$

(3) If we take $p=e$,we get

$$^{e}E_{\sigma,\omega}\left(\theta\right)=\sum_{j=0}^{\infty}\frac{\left(\theta\right)^{j}}{\Gamma\left(\sigma j+\omega\right)}.$$

Definition 2.4. [5] If $\psi'\in H^{1}\left(0,T\right)$, then the A-B fractional operator in Caputo sense of order $0<\delta<1$ is defined as:

$$_{ABC}D_0^{\delta}\psi\left(\theta\right)=\frac{K\left(\delta\right)}{1-\delta}\int_0^{\theta}E_{\delta}\left(-\varepsilon_{\delta}\left(\theta-\vartheta\right)^{\delta}\right)\psi'\left(\vartheta\right)d\vartheta,\ \vartheta\geq 0.$$

And its associated integral operator is provided by

$$_{ABC}\mathrm{I}_0^{\delta}\psi\left(\theta\right)=\frac{1-\delta}{K\left(\delta\right)}\psi\left(\theta\right)+\frac{K\left(\delta\right)}{1-\delta}\int_0^{\theta}\left(\theta-\vartheta\right)^{\delta}\psi\left(\vartheta\right)d\vartheta,\ \vartheta\geq 0. \tag{2}$$

Definition 2.5. [5] Definition of A-B fractional operator of order $0<\delta<1$ associated with weighted function, for a given $\psi'\in L\left(0,T\right)$:

$$_{ABC}D_0^{\delta}\psi\left(\theta\right)=\frac{K\left(\delta\right)}{1-\delta}\int_0^{\theta}E_{\delta}\left(-\varepsilon_{\delta}\left(\theta-\vartheta\right)^{\delta}\right)\varpi\left(\vartheta\right)\psi'\left(\vartheta\right)d\vartheta,\ \vartheta\geq 0,$$

and its associated integral operator is given by

$$_{ABC}\mathrm{I}_0^{\delta}\psi\left(\theta\right)=\frac{1-\delta}{K\left(\delta\right)}\psi\left(\theta\right)+\frac{K\left(\delta\right)}{1-\delta}\frac{1}{\varpi\left(\vartheta\right)}\int_0^{\vartheta}\left(\theta-\vartheta\right)^{\delta}\varpi\left(\vartheta\right)\psi\left(\vartheta\right)d\vartheta,\quad \vartheta\geq 0.$$

The normalized function $K(\lambda)$, has property where $K\left(0\right)=K\left(1\right)=1$.

Definition 2.6. [8] With regard to the weight function $\varpi(\theta)$, the new generalized fractional derivative of order δ in the Caputo sense is defined as

$$^{C}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi\left(\theta\right)=\frac{K\left(\delta\right)}{1-\delta}\frac{1}{\varpi\left(\theta\right)}\int_a^{y}E_{\sigma}\left[-\varepsilon_{\delta}\left(\theta-\vartheta\right)^{\upsilon}\right]\frac{d}{d\vartheta}\left(\varpi\psi\right)\left(\vartheta\right)d\vartheta,\tag{3}$$

where $\varpi\in C^1\left(r,s\right),\varpi>0$ on $[r,s]$, $K\left(\vartheta\right)$ is normalization function follow $K\left(0\right)=K\left(1\right)=1$. $\varepsilon_{\delta}=\frac{\delta}{1-\delta}$ and $E_{\sigma}\left(\theta\right)=\sum_{j=0}^{\infty}\frac{\theta^j}{\Gamma(\sigma j+1)}$ is the M-L function of parameter σ.

It is important to keep in consideration that the definition given above refers to certain cases that can be found in the literature.

Remark 2.2. In (3),

(1) When $\varpi\left(\theta\right)=1$ and $\sigma=\delta=1$, then we get definition defined by [5]

$$^{C}D_{a,\theta,1}^{\delta,1,1}\psi\left(\theta\right)=\frac{K\left(\delta\right)}{1-\delta}\int_a^{\theta}\exp\left[-\varepsilon_{\delta}\left(\theta-\vartheta\right)\right]\psi'\left(\vartheta\right)d\vartheta.$$

(2) When $\varpi\left(\theta\right)=1$ and $\sigma=\upsilon=\delta$, then we get definition defined by [5]

$$^{C}D_{a,\theta,1}^{\delta,\delta,\delta}\psi\left(\theta\right)=\frac{K\left(\delta\right)}{1-\delta}\int_a^{y}E_{\delta}\left[-\varepsilon_{\delta}\left(-\vartheta\right)^{\delta}\right]\psi'\left(\vartheta\right)d\vartheta.$$

(3) When $\sigma=\upsilon=\delta$, we get definition defined in [6]

$$^{C}D_{a,\theta,\varpi}^{\delta,\delta,\delta}\psi\left(\theta\right)=\frac{K\left(\delta\right)}{1-\delta}\frac{1}{\varpi\left(\theta\right)}\int_a^{\theta}E_{\delta}\left[-\varepsilon_{\delta}\left(\theta-\vartheta\right)^{\delta}\right]\frac{d}{d\vartheta}\left(\varpi\psi\right)\left(\vartheta\right)d\vartheta.$$

We develop a family of power fractional operators (PFOs) that extend the existing generalized fractional operators, based on a generalized power M-L function and their numerous implications, our novel mathematical idea enables us to expand and unify the fractional literature. The power parameter p is use in our paper. Many of the cited results can now be generalized and unified thanks to the power fractional calculus that is currently being introduced. This makes it possible for scientists, engineers, and researchers to choose the right fractional derivative in relation to the phenomenon they are studying in a natural way because our new definitions include the parameter p. In the numerical simulation phase, the effect of the parameter p on a system is demonstrated. Here choosing the appropriate value for p is essential for describing actual data using the selected model and for describing the existing trajectories, and to accurately forecast the asymptotic behavior in subsequent times. For more information, refer to our section on resolving power fractional differential equations (PFDEs). Additionally, the inverse power fractional integral operator (PFIO) of the defined power fractional derivative is constructed by the application of the convolution theorem and the Laplace transform. In conclusion, we assert that our PFOs have great possibility of progress of computational modeling across a number of domains and within the area of mathematics itself.

The structure of the paper is as follow: with the power M-L Function serving as it kernels, the new generalized power fractional derivative operator (Caputo's definition) is the focus of Section 3. The definition of R-L and the properties of the new power fractional derivative in the Caputo sense are given in Section 4. The Laplace transform of the new power fractional Derivative is shown in Section 5. The new Generalized power fractional integral allied to the new Derivative is shown in section 6. Section 7 presents some novel Calculus and Characteristics. The conclusion is presented in Section 8.

3. The New Generalized Power Fractional Derivative Operator

Inside this part, we defined a new power fractional derivative operator by using the power M-L function as it kernel.

Definition 3.1. Suppose that the Sobolev spaces $K^1(r,s)$ of order one is defined as:

$$K^1(r,s) \equiv \left\{u \in L^2(r,s), u'^2(r,s)\right\}$$

and consider $\delta \in [0,1), \sigma, \upsilon > 0$and $\psi \in K^1(r,s)$. The new generalized fractional derivative of order δ of power Caputo sense of the function $\psi(\theta)$ with respect to the weight function

$$ {}^{pC}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\psi(\theta) = \frac{K(\delta)}{1-\delta}\frac{1}{\varpi(\theta)}\int_a^\theta {}^{p}E_{\sigma,1}\left[-\varepsilon_\delta\left(\chi(\theta)-\chi(\vartheta)\right)^{\upsilon}\right]\frac{d}{d\vartheta}(\varpi\psi)(\vartheta)\,d\vartheta \qquad (4)$$

where $\varpi \in C^1(a,b), \varpi > 0$ on $[a,b]$, $K(\delta)$ is a normalization function obeying

$$K(0) = K(1) = 1,\ \varepsilon_\delta = \frac{\delta}{1-\delta} \text{ and } {}^{p}E_{\sigma,1}(\theta) = \sum_{u=0}^{\infty}\frac{(\theta\ln\psi)^u}{\Gamma(\sigma u+1)}$$

where $p > 1$ is the power M-L function of parameter σ.

The special cases of (4) are:

(1) If we take $\chi(\theta) = \theta$ in (4), then we get the special case of [12].
(2) If we take $\chi(\theta) = \theta$, $\varpi(\theta) = 1$, $\sigma = \upsilon = 1$ and $p = e$, we get the power Caputo–Fabrizio fractional derivative [4].
(3) When $\chi(\theta) = \theta$, $\varpi(\theta) = 1$, σ=υ=δ and $p = e$, we get the A–B fractional derivative [5].
(4) When $\chi(\theta) = \theta$ and $p = e$, we obtain fractional derivative that recently defined in [8].

Similarly, one can get many existing operators by applying certain conditions on $\chi(\theta) = \theta$.

4. Properties of new power fractional derivative of Caputo sense

For all scalars c_1, c_2, the power fractional derivative in the Caputo meaning is a linear operator and functions $\psi, \varphi \in K^1(r,s)$.

(1) ${}^{pC}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}(c_1\psi(\theta) + c_2\varphi(\theta)) = c_1{}^{pC}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\psi(\theta) + c_2{}^{pC}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\varphi(\theta)$.
(2) ${}^{pC}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,1}\psi(\theta) = 0$, for all constant function $\psi(\theta) = m$.

(3) ${}^{pC}_{\chi}D^{0,\sigma,\upsilon}_{a,\theta,\varpi}\psi\left(\theta\right)= \frac{K(0)}{1-0}\frac{1}{\varpi(\theta)}\int_a^\theta {}^{e}E_{\sigma,1}\left[-\varepsilon_0\left(\chi(\theta)-\chi(\vartheta)\right)^{\upsilon}\right]\frac{d}{d\vartheta}\left(\varpi\psi\right)(\vartheta)\,d\vartheta$, where $K\left(0\right)=1$, $\varepsilon_{\delta}=\frac{\delta}{1-\delta}$,

$$ {}^{pC}_{\chi}D^{0,\sigma,\upsilon}_{a,\theta,\varpi}\psi\left(\theta\right)=\frac{1}{\varpi\left(\theta\right)}\left(\varpi\left(\theta\right)\psi\left(\theta\right)-\varpi\left(a\right)\psi\left(a\right)\right). $$

Definition 4.1. Let $\delta\in[0,1),\sigma,\upsilon>0$ and $\psi\in K^{1}\left(r,s\right)$. The new power fractional derivative of order δ of R-L sense of the function $\psi\left(\theta\right)$ with regard to weight function $\varpi\left(\theta\right)$is provided by

$$ {}^{pRL}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\psi\left(\theta\right)=\frac{1}{\varpi\left(\theta\right)}\frac{1}{\chi'(\theta)}\frac{d}{d\theta}\int_a^\theta {}^{p}E_{\delta,1}\left[-\varepsilon_0\left(\chi(\theta)-\chi(\vartheta)\right)^{\upsilon}\right]\chi'(\vartheta)\varpi\left(\vartheta\right)\psi\left(\vartheta\right)d\vartheta, a<\theta. $$

If $\delta=0$ and $p=e$, then we have

$$ {}^{pRL}_{\chi}D^{0,\sigma,\upsilon}_{a,\theta,\varpi}\psi\left(\theta\right)=\frac{1}{\varpi\left(\theta\right)}\frac{1}{\chi'(\theta)}\frac{d}{d\theta}\int_a^\theta {}^{e}E_{0,1}\left[-\varepsilon_0\left(\chi(\theta)-\chi(\vartheta)\right)^{\upsilon}\right]\chi'(\vartheta)\varpi\psi\left(\vartheta\right)d\vartheta, $$

$$ {}^{pRL}_{\chi}D^{0,\sigma,\upsilon}_{a,\theta,\varpi}\psi\left(\theta\right)=\frac{1}{\varpi\left(\theta\right)}\frac{1}{\chi'(\theta)}\frac{d}{d\theta}\int_a^\theta {}^{e}E_{0,1}\left(0\right)\chi'(\vartheta)\varpi\psi\left(\vartheta\right)d\vartheta $$

Moreover, for all scalars c_1,c_2, the new power fractional derivative is a linear operator in the sense of R-L and functions $\psi,\varphi\in K^{1}\left(r,s\right)$, then we have

$$ \begin{aligned} &{}^{pRL}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\left(c_1\psi\left(\theta\right)+c_2\varphi\left(\theta\right)\right)\\ &=\frac{K\left(\delta\right)}{1-\delta}\frac{1}{\varpi\left(\theta\right)}\frac{1}{\chi'(\theta)}\frac{d}{d\theta}\int_a^\theta {}^{p}E_{\sigma,1}\left[-\varepsilon_\delta\left(\chi(\theta)-\chi(\vartheta)\right)^{\upsilon}\right]\chi'(\vartheta)\varpi\left(\vartheta\right)\left(c_1\psi\left(\theta\right)+c_2\varphi\left(\theta\right)\right)d\vartheta\\ &=c_1{}^{pRL}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\psi\left(\theta\right)+c_2{}^{pRL}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\varphi\left(\theta\right). \end{aligned} $$

Theorem 4.1. *Assume that $\varpi\psi$ is an analytical function. Then we have*

$$ {}^{pRL}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\psi\left(\theta\right)={}^{pC}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\psi\left(\theta\right)+\frac{K\left(\delta\right)}{1-\delta}\frac{1}{\varpi\left(\theta\right)}\frac{1}{\chi'(\theta)}{}^{p}E_{\sigma,1}\left[-\varepsilon_\delta\left(\chi(\theta)-\chi(a)\right)^{\upsilon}\right]\varpi\psi\left(a\right). \tag{5} $$

Proof. As $\varpi\psi$ is an analytic function then

$$ \left(\varpi\psi\right)\left(\vartheta\right)=\sum_{i=0}^{\infty}\frac{\left(\varpi\psi\right)^{i}\left(\theta\right)}{i!}\left(\chi(\vartheta)-\chi(\theta)\right)^{i} $$

and

$$ \begin{aligned} &{}^{pRL}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\psi\left(\theta\right)\\ &=\frac{K\left(\delta\right)}{1-\delta}\frac{1}{\varpi\left(\theta\right)}\frac{1}{\chi'(\theta)}\frac{d}{d\theta}\int_a^\theta\sum_{u=0}^{\infty}\frac{\left(-\varepsilon_\delta\left(\chi(\theta)-\chi(\vartheta)\right)^{\upsilon}\ln p\right)^{u}}{\Gamma\left(\sigma u+1\right)}\sum_{i=0}^{\infty}\frac{\left(\varpi\psi\right)^{i}\left(\theta\right)}{i!}\left(\chi(\vartheta)-\chi(\theta)\right)^{i}d\vartheta\\ &=\frac{K\left(\delta\right)}{1-\delta}\frac{1}{\varpi\left(\theta\right)}\frac{1}{\chi'(\theta)}\frac{d}{d\theta}\sum_{i=0}^{\infty}\sum_{u=0}^{\infty}\frac{\left(-1\right)^{i}\left(-\varepsilon_\delta\ln p\right)^{u}}{\Gamma\left(\sigma u+1\right)}\frac{\left(\varpi\psi\right)^{i}\left(\theta\right)}{i!}\int_a^\theta\chi'(\vartheta)\left(\chi(\theta)-\chi(\vartheta)\right)^{i+\upsilon k}d\vartheta\\ &=\frac{K\left(\delta\right)}{1-\delta}\frac{1}{\varpi\left(\theta\right)}\frac{1}{\chi'(\theta)}\left\{\begin{array}{l}\sum_{i=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^{i}(-\varepsilon_\delta\ln p)^{u}}{\Gamma(\sigma u+1)}\frac{(\varpi\psi)^{i+1}(\theta)}{i!}\frac{(\chi(\theta)-\chi(a))^{i+\upsilon u+1}}{(i+\upsilon u+1)}\\ +\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i}(-\varepsilon_\delta\ln p)^{u}}{\Gamma(\sigma u+1)}\frac{(\varpi\psi)^{i}(\theta)}{i!}\frac{(\chi(\theta)-\chi(a))^{\upsilon u+i}}{(i+\upsilon u+1)}\end{array}\right\} \end{aligned} $$

$$= \frac{K(\delta)}{1-\delta}\frac{1}{\varpi(\theta)}\frac{1}{\chi'(\theta)}$$
$$\times\left\{\begin{array}{l} \sum_{i=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^i(-\varepsilon_\delta \ln p)^u}{\Gamma(\sigma u+1)}\frac{(\varpi\psi)^{i+1}(\theta)}{i!}\int_a^\theta (\chi(\theta)-\chi(\vartheta))^{\upsilon u+i}\,d\vartheta \\ +\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}(\varpi\psi)^i(\theta)\times(\chi(\theta)-\chi(a))^i\sum_{u=0}^{\infty}\frac{(-\varepsilon_\delta \ln p)^u}{\Gamma(\sigma u+1)}(\chi(\theta)-\chi(a))^{\upsilon u}\end{array}\right\}$$
$$= {}^{pC}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\psi(\theta) + \frac{K(\delta)}{1-\delta}\frac{1}{\varpi(\theta)}\frac{1}{\chi'(\theta)}{}^{p}E_{\sigma,1}\left[-\varepsilon_\delta(\theta-\vartheta)^\upsilon\right]\varpi\psi(a).$$

This brings the proof to the conclusion. □

5. Transformation of New Power Fractional Derivative via Laplace

This part involves the calculation of the Laplace transform of the power fractional derivative of the R-L and Caputo types.

Proposition 5.1. *Let $\delta \in (0,1]$ $\sigma, \upsilon > 0$ and $\psi \in K^1(r,s)$, then the generalized fractional derivative in Caputo and R-L sense can be represented in series form as:*

$${}^{pC}_{\chi}D^{\delta,\sigma,\upsilon}_{0,\theta,\varpi}\psi(\theta) = \frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)}(-\varepsilon_\delta \ln p)^u\left({}^{RL}_{\chi}I^{\upsilon u+1}_{a,\theta,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)\right)(\theta)$$

and

$${}^{pRL}_{\chi}D^{\delta,\sigma,\upsilon}_{a,\theta,\varpi}\psi(\theta) = \frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)}(-\varepsilon_\delta \ln p)^u\left({}^{RL}_{\chi}I^{\upsilon u-1}_{a,\theta,\varpi}\psi\right)(\theta),$$

where $\left({}^{RL}_{\chi}I^{\upsilon u-1}_{a,\theta,\varpi}\psi\right)(\theta)$ is the generalized weighted R-L fractional integral defined by [13].

Proof. Since

$${}^{p}E_{\sigma,1}\left(-\varepsilon_\delta(\chi(\theta)-\chi(a))^\upsilon\right) = \sum_{u=0}^{\infty}\frac{\left(-\varepsilon_\delta(\chi(\theta)-\chi(a))^\upsilon \ln p\right)^u}{\Gamma(\sigma u+1)},$$

the above series is uniformly convergent.

Therefore,

$${}^{pC}_{\chi}D^{\delta,\sigma,\upsilon}_{0,\theta,\varpi}\psi(\theta)$$
$$= \frac{K(\delta)}{1-\delta}\frac{1}{\varpi(\theta)}\int_a^\theta {}^{p}E_{\sigma,1}\left(-\varepsilon_\delta(\chi(\theta)-\chi(\vartheta))^\upsilon\right)(\varpi\psi)'(\vartheta)d\vartheta$$
$$= \frac{K(\delta)}{1-\delta}\frac{1}{\varpi(\theta)}\int_a^\theta \chi'(\vartheta)\sum_{u=0}^{\infty}\frac{\left(-\varepsilon_\delta(\chi(\theta)-\chi(\vartheta))^\upsilon \ln p\right)^u}{\Gamma(\sigma u+1)}\left(\frac{(\varpi\psi)'(\vartheta)}{\chi'(\vartheta)}\right)d\vartheta$$
$$= \frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}\frac{(-\varepsilon_\delta \ln p)^u\,\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)}$$
$$\times\frac{1}{\varpi(\theta)\Gamma(\upsilon u+1)}\int_a^\theta \chi'(\vartheta)(\chi(\theta)-\chi(\vartheta))^{\upsilon u}\varpi(\vartheta)\left(\frac{(\varpi\psi)'(\vartheta)}{(\varpi\chi')(\vartheta)}\right)d\vartheta$$
$$= \frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)}(-\varepsilon_\delta \ln p)^u\left({}^{RL}_{\chi}I^{\upsilon u+1}_{a,\theta,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)\right)(\theta).$$

Similarly, one can get

$$ {}_{\chi}^{pRL}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta)=\frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)}(-\varepsilon_{\delta}\ln p)^{u}\left({}_{\chi}^{RL}I_{a,\theta,\varpi}^{\upsilon u-1}\psi\right)(\theta). $$

□

With the aid of Proposition 5.1, we prove the following:

Theorem 5.2. *The Laplace transform of* ${}_{\chi}^{pC}D_{0,\theta,\varpi}^{\delta,\sigma,\upsilon}$ *is given by*

$$ \begin{aligned} &L_{\chi}^{\varpi}\left\{{}_{\chi}^{pC}D_{0,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta)\right\}(s)\\ &=\frac{K(\delta)\left[sL_{\chi}^{\varpi}\{\psi(\theta)\}(s)-w(a)\psi(a)\right]}{(1-\delta)s}\sum_{u=0}^{\infty}\left(\frac{-\varepsilon_{\delta}\ln p}{s^{\upsilon}}\right)^{u}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)}, \end{aligned} \tag{6} $$

where $L_{\chi}^{\varpi}\{\psi(\theta)\}(s)$ *is the generalized weighted Laplace defined by Jarad et al.* [13].

Proof. By utilizing Proposition 5.1, we have

$$ \begin{aligned} &L_{\chi}^{\varpi}\left\{{}_{\chi}^{pC}D_{0,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta)\right\}(s)\\ &=\frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)}(-\varepsilon_{\delta}\ln p)^{u}L_{\chi}^{\varpi}\left\{\left({}_{\chi}^{RL}I_{a,\theta,\varpi}^{\upsilon u+1}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)\right)(\theta)\right\}(s)\\ &=\frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)s^{\upsilon u+1}}(-\varepsilon_{\delta}\ln p)^{u}L_{\chi}^{\varpi}\left\{D_{\varpi,\chi}^{1}\psi(\theta)\right\}(s), \end{aligned} $$

where $\left(D_{\varpi,\chi}^{1}\chi\right)(\theta)=\frac{(\varpi\psi)'}{(\varpi\chi')}(\theta)$, see [13]. Thus, we get

$$ \begin{aligned} &L_{\chi}^{\varpi}\left\{{}_{\chi}^{pC}D_{0,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta)\right\}(s)\\ &=\frac{K(\delta)\left[sL_{\chi}^{\varpi}\{\psi(\theta)\}(s)-w(a)\psi(a)\right]}{(1-\delta)s}\sum_{u=0}^{\infty}\left(\frac{-\varepsilon_{\delta}\ln p}{s^{\upsilon}}\right)^{u}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)}. \end{aligned} $$

□

Lemma 5.3. *If we take* $\sigma=\upsilon$ *and* $a=0$ *in Theorem* 5.2, *then we have*

$$ L_{\chi}^{\varpi}\left\{{}_{\chi}^{pC}D_{0,\theta,\varpi}^{\delta,\sigma,\sigma}\psi(\theta)\right\}(s)=\frac{K(\delta)}{1-\delta}\frac{s^{\sigma}L_{\chi}^{\varpi}\{\varpi(\theta)\psi(\theta)\}(s)-s^{\sigma-1}\varpi(0)\psi(0)}{s^{\sigma}+\varepsilon_{\delta}\ln p}. \tag{7} $$

Theorem 5.4. *The Laplace transform of* ${}_{\chi}^{pRL}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}$ *is given by*

$$ L_{\chi}^{\varpi}\left\{{}_{\chi}^{pRL}D_{0,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta)\right\}(s)=\frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}\left(\frac{-\varepsilon_{\delta}\ln p}{s^{\upsilon}}\right)^{u}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)}L_{\chi}^{\varpi}\{\psi(\theta)\}(s). \tag{8} $$

Proof. Similarly, by Proposition 5.1, we have

$$ \begin{aligned} L_{\chi}^{\varpi}\left\{{}_{\chi}^{pRL}D_{0,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta)\right\}(s)&=\frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)s^{u\upsilon}}(-\varepsilon_{\delta}\ln p)^{u}L_{\chi}^{\varpi}\{\psi(\theta)\}(s)\\ &=\frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}\left(\frac{(-\varepsilon_{\delta}\ln p)}{s^{\upsilon}}\right)^{u}\frac{\Gamma(\upsilon u+1)}{\Gamma(\sigma u+1)}\times L_{\chi}^{\varpi}\{\psi(\theta)\}(s). \end{aligned} $$

This completes the proof. □

Lemma 5.5. *If we take $\sigma = \upsilon$ in Theorem 5.4, then we have*

$$L_{\chi}^{\varpi}\left\{ {}_{\chi}^{pRL}D_{0,\theta,\varpi}^{\delta,\sigma,\sigma}\psi\left(\theta\right)\right\}(s) = \frac{K\left(\delta\right)}{1-\delta}\frac{s^{\sigma}L_{\chi}^{\varpi}\left\{\psi\left(\theta\right)\right\}(s)}{s^{\sigma}+\varepsilon_{\delta}\ln p}. \quad (9)$$

Remark 5.1. When $\chi(\theta) = \theta$, $\varpi\left(\theta\right) = 1$, $\sigma = \upsilon = \delta$ and $p = e$, then the Laplace transform of the A-B fractional derivatives is obtained in the manner described by Caputo and R-L in [6].

Lemma 5.6. *The Laplace transform of ${}^{p}E_{\sigma,1}\left(-\varepsilon_{\lambda}\left(\chi(\theta)-\chi(a)\right)^{\upsilon}\right)$ is given by*

$$\begin{aligned} &L_{\chi}^{\varpi}\left\{\varpi^{-1}(\theta){}^{p}E_{\sigma,1}\left(-\varepsilon_{\delta}\left(\chi(\theta)-\chi(a)\right)^{\upsilon}\right)\right\}(s) \\ &= \frac{1}{s}\sum_{u=0}^{\infty}\left(\frac{-\varepsilon_{\delta}\left(\chi(\theta)-\chi(a)\right)^{\upsilon}\ln p}{s^{\upsilon}}\right)^{u}\frac{\Gamma\left(\upsilon u+1\right)}{\Gamma\left(\sigma u+1\right)} \end{aligned} \quad (10)$$

Proof. As we know that

$${}^{p}E_{\sigma,1}\left(-\varepsilon_{\delta}\left(\chi(\theta)-\chi(a)\right)^{\upsilon}\right) = \sum_{u=0}^{\infty}\frac{\left(-\varepsilon_{\delta}\left(\chi(\theta)-\chi(a)\right)^{\upsilon}\ln p\right)^{u}}{\Gamma\left(\sigma u+1\right)},$$

$$\begin{aligned} &L_{\chi}^{\varpi}\left\{\omega^{-1}(u){}^{p}E_{\sigma,1}\left(-\varepsilon_{\delta}\left(\chi(\theta)-\chi(a)\right)^{\upsilon}\right)\right\}(s) \\ &= L_{\chi}^{\varpi}\left\{\sum_{u=0}^{\infty}\frac{\left(-\varepsilon_{\delta}\left(\chi(\theta)-\chi(a)\right)^{\upsilon}\ln p\right)^{u}}{\Gamma\left(\sigma u+1\right)}\right\}(s) \\ &= \sum_{u=0}^{\infty}\frac{\left(-\varepsilon_{\delta}\chi(\theta)^{\upsilon}\ln p\right)^{u}}{\Gamma\left(\sigma u+1\right)}L_{\chi}^{\varpi}\left\{\left(\chi(\theta)-\chi(a)\right)^{\upsilon u}\right\}(s) \\ &= \frac{1}{s}\sum_{u=0}^{\infty}\frac{\left(-\varepsilon_{\delta}\ln p\right)^{u}}{\left(s^{\upsilon}\right)^{u}}\frac{\Gamma\left(\upsilon u+1\right)}{\Gamma\left(\sigma u+1\right)}. \end{aligned}$$

□

Theorem 5.7. *Let $\delta \in (0,1]$ $\sigma, \upsilon > 0$ and $\psi \in K^{1}\left(r,s\right)$, then the relation between the modified Caputo and R-L fractional derivatives is given by*

$${}_{\chi}^{pC}D_{0,\theta,\varpi}^{\delta,\sigma,\sigma}\psi\left(\theta\right) = {}^{pRL}D_{0,\theta,\varpi}^{\delta,\sigma,\sigma}\psi\left(\theta\right) - \frac{K\left(\delta\right)}{1-\delta}\frac{{}^{p}E_{\sigma,1}\left(-\varepsilon_{\delta}\left(\chi(\theta)-\chi(a)\right)^{\upsilon}\right)}{\varpi(\theta)}\left(\varpi\psi\right)(a).$$

Proof. With the aid of Proposition 5.1 and Lemma 5.6, we have

$$\begin{aligned} &L_{\chi}^{\varpi}\left\{{}_{\chi}^{pC}D_{0,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi\left(\theta\right)\right\}(s) \\ &=\frac{K\left(\delta\right)}{\left(1-\delta\right)}\sum_{u=0}^{\infty}\left(-\varepsilon_{\delta}\ln p\right)^{u}\frac{\Gamma\left(\upsilon u+1\right)}{\Gamma\left(\sigma u+1\right)s^{\upsilon u}}L_{\chi}^{\varpi}\left\{\psi\left(\theta\right)\right\}(s) \\ &\quad-\frac{K\left(\delta\right)}{\left(1-\delta\right)}\sum_{u=0}^{\infty}\left(-\varepsilon_{\delta}\ln p\right)^{u}\frac{\Gamma\left(\upsilon u+1\right)}{\Gamma\left(\sigma u+1\right)s^{\upsilon u+1}}w\left(a\right)\psi\left(a\right) \\ &= L_{\chi}^{\varpi}\left\{{}^{pRL}D_{0,\theta,\varpi}^{\delta,\sigma,\sigma}\psi\left(\theta\right)\right\}(s) - \frac{K\left(\delta\right)}{1-\delta}L_{\chi}^{\varpi}\left\{\omega^{-1}(u){}^{p}E_{\sigma,1}\left(-\varepsilon_{\delta}\left(\chi(\theta)-\chi(a)\right)^{\upsilon}\right)\right\}(s). \end{aligned}$$

The required result is obtained by applying the inverse Laplace transform. □

6. The New Generalized Power Fractional Integral Allied to the New Derivative

The definition of the new generalized power fractional integral allied with the new generalized derivative is established in this part.

Theorem 6.1.

$$ {}_{\chi}^{pRL}D_{o,\theta,\varpi}^{\delta,\sigma,\sigma}\kappa\left(\theta\right)=\psi\left(\theta\right) $$

has a unique solution given by

$$ \kappa\left(\theta\right)=\frac{1-\delta}{K\left(\delta\right)}\psi\left(\theta\right)+\frac{\lambda\ln p}{K\left(\delta\right)\Gamma\left(\sigma\right)}\frac{1}{\varpi\left(\theta\right)}\int_{0}^{\theta}\left(\chi(\theta)-\chi(\vartheta)\right)^{\sigma-1}\chi'(\vartheta)\varpi\psi\left(\vartheta\right)d\vartheta. \quad (11) $$

Proof. We have

$$ {}_{\chi}^{pRL}D_{o,\theta,\varpi}^{\delta,\sigma,\sigma}\kappa\left(\theta\right)=\psi\left(\theta\right). $$

By using weighted Laplace transform and applying Lemma 5.6, we find

$$ \begin{aligned} L_{\chi}^{\varpi}\left\{\kappa\left(\theta\right)\right\}\left(s\right) &= \frac{1-\delta}{K\left(\delta\right)}L_{\chi}^{\varpi}\left\{\psi\left(\theta\right)\right\}\left(s\right)+\frac{1-\delta}{K\left(\delta\right)}\frac{\varepsilon_{\delta}\ln p}{s^{\sigma}}L_{\chi}^{\varpi}\left\{\psi\left(\theta\right)\right\}\left(s\right) \\ &= \frac{1-\delta}{K\left(\delta\right)}L_{\chi}^{\varpi}\left\{\psi\left(\theta\right)\right\}\left(s\right)+\frac{1-\delta}{K\left(\delta\right)}\frac{\varepsilon_{\delta}\ln p}{\Gamma\left(\sigma\right)}L_{\chi}^{\varpi}\left\{\chi(\theta)^{\sigma-1}*\psi\left(\theta\right)\right\}\left(s\right). \end{aligned} $$

The passage to inverse weighted Laplace leads to

$$ \begin{aligned} \kappa\left(\theta\right) &= \frac{1-\delta}{K\left(\delta\right)}\psi\left(\theta\right)+\frac{1-\delta}{K\left(\delta\right)}\frac{\varepsilon_{\delta}\ln p}{\Gamma\left(\sigma\right)}\left(\chi(\theta)^{\sigma-1}*\psi\left(\theta\right)\right) \\ \kappa\left(\theta\right) &= \frac{1-\delta}{K\left(\delta\right)}\psi\left(\theta\right)+\frac{\delta}{K\left(\delta\right)}\frac{\ln p}{\Gamma\left(\sigma\right)}\frac{1}{\varpi\left(\theta\right)}\int_{0}^{\theta}\left(\chi(\theta)-\chi(\vartheta)\right)^{\sigma-1}\chi'(\vartheta)\varpi\left(\vartheta\right)\psi\left(\vartheta\right)d\vartheta, \end{aligned} $$

which completes the proof. □

Definition 6.1. When $\upsilon=\sigma$, the generalized fractional integral that corresponds to the new fractional derivative is defined as

$$ {}_{\chi}^{pRL}I_{a,\theta,\varpi}^{\delta,\sigma,\sigma}\psi\left(\theta\right)=\frac{1-\delta}{K\left(\delta\right)}\psi\left(\theta\right)+\frac{\delta\ln p}{K\left(\delta\right)\Gamma\left(\sigma\right)}\frac{1}{\varpi\left(\theta\right)}\int_{a}^{\theta}\left(\chi(\theta)-\chi(\vartheta)\right)^{\sigma-1}\chi'(\vartheta)\varpi\left(\vartheta\right)\psi\left(\vartheta\right)d\vartheta. \quad (12) $$

This generalized fractional integral coincides with the A–B fractional integral when, $\chi(\theta)=\theta$, $\varpi\left(\theta\right)=1$, $p=e$ and $\upsilon=\sigma=\delta$, and with the weighted A–B fractional integral defined by Al Refai [12] when $\chi(\theta)=\theta$, $\delta=\sigma=\upsilon$ and $p=e$. Also, we recover the original function when $\delta=0$ and the ordinary integral when $\delta=1$.

Theorem 6.2. *Let u be a continuously differentiable function and ψ be a continuous function. The new generalized fractional derivative of the function ψ, for any constant c, is defined as*

$$ \psi\left(\theta\right)=\frac{1}{\varpi\left(\theta\right)}\int_{c}^{\varpi(\theta)u(\theta)}\psi\left(\vartheta\right)d\vartheta \quad (13) $$

Satisfies the following property:

$$\begin{aligned}
&{}_{\chi}^{pC}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta) \\
&= \psi(\varpi(\theta)u(\theta))\,{}_{\chi}^{pC}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}u(\theta) - \frac{K(\delta)}{1-\delta}\frac{v(a)}{\varpi(\theta)}\,{}^{p}E_{\sigma,1}\left[-\varepsilon_{\delta}(\chi(\theta)-\chi(a))^{\upsilon}\right] \\
&\quad - \frac{\delta\upsilon K(\delta)}{(1-\delta)^{2}}\frac{1}{\varpi(\theta)}\int_{a}^{\theta}(\chi(\theta)-\chi(\tau))^{\upsilon-1}\,{}^{p}E_{\sigma,\sigma+1}^{2}\left[-\varepsilon_{\delta}(\chi(t)-\chi(\tau))\,\upsilon\right]v(\tau)\,d\tau,
\end{aligned}$$

where

$$v(\tau) = \psi(\varpi(\theta)u(\theta))(\varpi(\theta)u(\theta)-\varpi(\tau)u(\tau)) + \int_{\varpi(\theta)u(\theta)}^{\varpi(\tau)u(\tau)}\psi(\vartheta)\,d\vartheta. \tag{14}$$

Furthermore, we have the following inequalities:

(1) *If ψ is increasing function, then*

$${}_{\chi}^{pC}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta) \le \psi(\varpi(\theta)u(\theta))\,{}_{\chi}^{pC}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}u(\theta). \tag{15}$$

(2) *If ψ is decreasing function, then*

$${}_{\chi}^{pC}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta) \ge \psi(\varpi(\theta)u(\theta))\,{}_{\chi}^{pC}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}u(\theta). \tag{16}$$

Proof. By using equation (4), we have

$${}_{\chi}^{pC}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta) = \frac{K(\delta)}{1-\delta}\frac{1}{\varpi(\theta)}\int_{a}^{\theta}{}^{p}E_{\sigma,1}\left[-\varepsilon_{\delta}(\chi(\theta)-\chi(\tau))^{\upsilon}\right]\psi(w(\tau)u(\tau))(wu)'(\tau)\,d\tau. \tag{17}$$

Consider

$$h(\theta) = {}_{\chi}^{pC}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}\psi(\theta) - \psi(\varpi(\theta)u(\theta))\,{}_{\chi}^{pC}D_{a,\theta,\varpi}^{\delta,\sigma,\upsilon}u(\theta). \tag{18}$$

Then,

$$\begin{aligned}
h(\theta) = \frac{K(\delta)}{1-\delta}\frac{1}{\varpi(\theta)} \times \int_{a}^{\theta}&{}^{p}E_{\sigma,1}\left[-\varepsilon_{\delta}(\chi(\theta)-\chi(\tau))^{\upsilon}\right]\psi(\varpi(\tau)u(\tau)) \\
&- \psi(\varpi(\theta)u(\theta))(\varpi u)'(\tau)\,d\tau. \qquad (19)
\end{aligned}$$

Obviously, $v'(\tau) = (\psi(\varpi(\tau)u(\tau)-\varpi(\theta)u(\theta)))(\varpi u)'(\tau)$ and $v(\theta)=0$. Integrating by parts the last integral, we find

$$\begin{aligned}
h(\theta) &= \frac{K(\delta)}{1-\delta}\frac{1}{\varpi(\theta)}\,{}^{p}E_{\sigma,1}\left[-\varepsilon_{\delta}(\chi(\theta)-\chi(\tau))^{\upsilon}\right]v(\tau)\Big|_{a}^{1} \qquad (20)\\
&= \frac{\delta\upsilon K(\delta)}{(1-\delta)^{2}}\frac{1}{\varpi(\theta)}\int_{a}^{\theta}(\chi(\theta)-\chi(\tau))^{\upsilon-1}\,{}^{p}E_{\sigma,\sigma+1}^{2}\left[-\varepsilon_{\delta}(\chi(\theta)-\chi(\tau))^{\upsilon}\right]v(\tau)\,d\tau.
\end{aligned}$$

Since $\lim\limits_{\tau\to\theta}\left[{}^{p}E_{\sigma,1}\left[-\varepsilon_{\delta}(\chi(\theta)-\chi(\tau))^{\upsilon}\right]v(\tau)\right]=0$, we have

$$\begin{aligned}
h(\theta) &= \frac{K(\delta)}{1-\delta}\frac{v(a)}{\varpi(\theta)}\,{}^{p}E_{\sigma,1}\left[-\varepsilon_{\delta}(\chi(\theta)-\chi(\tau))^{\upsilon}\right] \qquad (21)\\
&\quad -\frac{\delta\upsilon K(\delta)}{(1-\delta)^{2}}\frac{1}{\varpi(\theta)}\int_{a}^{\theta}(\chi(\theta)-\chi(\tau))^{\upsilon-1}\,{}^{p}E_{\sigma,\sigma+1}^{2}\left[-\varepsilon_{\delta}(\chi(\theta)-\chi(\tau))\,\upsilon\right]v(\tau)\,d\tau.
\end{aligned}$$

Now, we consider the following function:

$$\phi_{c,\psi}(\tau) = \psi(c)(c-\tau) + \int_{c}^{\tau}\psi(\vartheta)\,d\vartheta. \tag{22}$$

Obviously $\phi'_{c,\psi}(\tau) = \psi(\tau) - \psi(c)$. If ψ is an increasing function, then the function $\phi_{c,\psi}(\tau)$ is decreasing on the interval $(-\infty, c]$ and increasing on $[c, +\infty)$ with $\phi_{c,\psi}(c) = 0$. Hence, $\phi_{c,\psi}(\tau)$ has the global minimum at $\tau = c$. So,

$$\phi_{c,\psi}(\tau) \geq 0, \text{ for all } (c,\tau) \in IR^2 \tag{23}$$

Since $\upsilon(\tau) = \phi_{\varpi(\theta)u(\theta),\psi}(\varpi(\tau)u(\tau))$, we have $\upsilon(\tau) \geq 0$ for all $\tau \in IR$. This proves 1. Likewise, we can simply demonstrate 2. □

7. Novel Calculus and Characteristics

Let we denote ${}^{pC}_{\chi}D^{\delta,\sigma,\sigma}_{a,\theta,\varpi}$ by ${}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}$. Based on the above results, we can define generalized fractional integral operator and some basic formulas as follows:

Definition 7.1. The generalized fractional integral operator corresponding to ${}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}$ is defined by

$${}^{pRL}_{\chi}I^{\delta,\sigma}_{a,\varpi}\psi(\theta) = \frac{1-\delta}{N(\delta)}\psi(\theta) + \frac{\delta \ln p}{K(\delta)}{}^{RL}_{\chi}I^{\sigma}_{a,\varpi}\psi(\theta), \tag{24}$$

where ${}^{RL}_{\chi}I^{\sigma}_{a,\varpi}$ is the generalized weighted power R-L fractional integral of order σ defined by

$${}^{RL}_{\chi}I^{\sigma}_{a,\varpi}\psi(\theta) = \frac{1}{\Gamma(\sigma)}\frac{1}{\varpi(\theta)}\int_a^\theta (\chi(\theta) - \chi(\vartheta))^{\sigma-1}\chi'(\vartheta)\varpi(\vartheta)\psi(\vartheta)\,d\vartheta. \tag{25}$$

Remark 7.1. As a unique instance of (24), the Atangana-Baleanu fractional integral operator [6] can be considered as follows: $\varpi(\theta) = 1$, $p = e$ and $\sigma = \delta$.

Theorem 7.1. *Assume that* $\delta, \upsilon \in [0,1)$, $\sigma, \xi > 0$, *and* $\psi \in K^1(r,s)$. *Then*
(1) ${}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}\left({}^{pC}_{\chi}D^{\upsilon,\xi}_{a,\varpi}\psi\right)(\theta) = {}^{pC}_{\chi}D^{\upsilon,\xi}_{a,\varpi}\left({}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}\psi\right)(\theta)$.
(2) ${}^{pRL}_{\chi}I^{\delta,\sigma}_{a,\varpi}\left({}^{pRL}_{\chi}I^{\upsilon,\xi}_{a,\varpi}\psi\right)(\theta) = {}^{pRL}_{\chi}I^{\upsilon,\xi}_{a,\varpi}\left({}^{pRL}_{\chi}I^{\delta,\sigma}_{a,\varpi}\psi\right)(\theta)$.

This indicates that derivatives and generalized fractional integrals are commutative operators.

Proof. First, we demonstrate (1) in Theorem 7.1. Using Proposition 5.1, we obtain

$$\begin{aligned}
{}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}\left({}^{pC}_{\chi}D^{\upsilon,\xi}_{a,\varpi}\psi\right)(\theta) &= \frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}(-\varepsilon_\delta \ln p)^u\, {}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left(\frac{\left(\varpi D^{\upsilon,\xi}_{a,\varpi}\psi\right)'}{\varpi\chi'}\right)(\theta) \\
&= \frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}(-\varepsilon_\delta \ln p)^u\, {}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left(\frac{K(\upsilon)}{1-\upsilon}\sum_{i=0}^{\infty}(-\varepsilon_\upsilon \ln p)^i\, {}^{RL}_{\chi}I^{\xi i}_{a,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)\right)(\theta) \\
&= \frac{K(\delta)}{1-\delta}\frac{K(\upsilon)}{1-\upsilon}\sum_{u=0}^{\infty}\sum_{i=0}^{\infty}(-\varepsilon_\delta \ln p)^u(-\varepsilon_\upsilon \ln p)^i\, {}^{RL}_{\chi}I^{\sigma u+\xi i+1}_{a,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)(\theta).
\end{aligned}$$

Due to the symmetry of the last equation in δ and υ and also in $\sigma$and ξ. To prove (2), we have

$${}^{pRL}_{\chi}I^{\delta,\sigma}_{a,\varpi}\left({}^{pRL}_{\chi}I^{\upsilon,\xi}_{a,\varpi}\psi\right)(\theta) = \frac{1-\delta}{K(\delta)}\left({}^{RL}_{\chi}I^{\upsilon,\xi}_{a,\varpi}\psi\right)(\theta) + \frac{\delta \ln p}{K(\delta)}{}^{RL}_{\chi}I^{\sigma}_{a,\varpi}\left({}^{pRL}_{\chi}I^{\upsilon,\xi}_{a,\varpi}\psi\right)(\theta)$$

$$\begin{aligned}
&=\frac{1-\delta}{K(\delta)}\left(\frac{1-\upsilon}{K(\upsilon)}\psi(\theta)+\frac{\upsilon\ln p}{K(\upsilon)}{}^{RL}_{\chi}I^{\xi}_{a,\varpi}\psi(\theta)\right)\\
&+\frac{\delta\ln p}{K(\delta)}{}^{RL}_{\chi}I^{\sigma}_{a,\varpi}\left(\frac{1-\upsilon}{K(\upsilon)}\psi(\theta)+\frac{\upsilon\ln p}{K(\upsilon)}{}^{RL}_{\chi}I^{\xi}_{a,\varpi}\psi(\theta)\right)\\
&=\frac{(1-\delta)(1-\upsilon)}{N(\delta)N(\upsilon)}\psi(\theta)+\left(\frac{\upsilon(1-\delta)\ln p}{K(\delta)K(\upsilon)}{}^{RL}_{\chi}I^{\xi}_{a,\varpi}\psi(\theta)\right)\\
&+\frac{\delta(1-\upsilon)\ln p}{K(\delta)K(\upsilon)}{}^{RL}_{\chi}I^{\sigma}_{a,\varpi}\psi(\theta)+\left(\frac{\delta\upsilon(\ln p)^2}{K(\delta)K(\upsilon)}{}^{RL}_{\chi}I^{\xi}_{a,\varpi}\psi(\theta)\right).
\end{aligned}$$

This completes the proof. □

Theorem 7.2. *Suppose* $\delta\in[0,1)$, $\sigma>0$, *and* $\psi\in K^1(r,s)$. *Then,*

(1) ${}^{pRL}_{\chi}I^{\delta,\sigma}_{a,\varpi}\left({}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}\psi\right)(\theta)=\psi(\theta)-\left(\left(\varpi(a)\psi(a)/\varpi(\theta)\right)\right).$

(2) ${}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}\left({}^{pRL}_{\chi}I^{\delta,\sigma}_{a,\varpi}\psi\right)(\theta)=\psi(\theta)-\left(\left(\varpi(a)\psi(a)/\varpi(\theta)\right)\right).$

Proof. In alignment with equation 4 and Definition 6.1, we have

$$\begin{aligned}
&{}^{pRL}_{\chi}I^{\delta,\sigma}_{a,\varpi}\left({}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}\psi\right)(\theta)\\
&=\frac{1-\delta}{K(\delta)}\left({}^{pC}_{\chi}D^{\delta,\upsilon}_{a,\varpi}\psi\right)(\theta)+\frac{\delta\ln p}{K(\delta)}{}^{RL}_{\chi}I^{\sigma}_{a,\varpi}\left({}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}\psi\right)(\theta)\\
&=\frac{1-\delta}{K(\delta)}\left[\frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}(-\varepsilon_\delta\ln p)^u\,{}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)(\theta)\right]\\
&\quad+\frac{\delta\ln p}{K(\delta)}{}^{RL}_{\chi}I^{\sigma}_{a,\varpi}\left[\frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}(-\varepsilon_\delta\ln p)^u\,{}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)(\theta)\right]\\
&=\sum_{u=0}^{\infty}(-\varepsilon_\delta\ln p)^u\,{}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)(\theta)-\sum_{u=0}^{\infty}(-\varepsilon_\delta\ln p)^{u+1}\,{}^{RL}_{\chi}I^{\sigma(u+1)+1}_{a,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)(\theta).
\end{aligned}$$

Hence

$$\begin{aligned}
&{}^{pRL}_{\chi}I^{\delta,\sigma}_{a,\varpi}\left({}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}\psi\right)(\theta),\\
&=\sum_{u=0}^{\infty}(-\varepsilon_\delta\ln p)^u\,{}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)(\theta)-\sum_{u=1}^{\infty}(-\varepsilon_\delta\ln p)^u\,{}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)(\theta)\\
&={}^{RL}_{\chi}I^{1}_{a,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)(\theta)\\
&=\frac{1}{\varpi(\theta)}\int_a^\theta(\varpi\psi)'(\vartheta)\,d\vartheta\\
&=\frac{1}{\varpi(\theta)}\left(\varpi(\theta)\psi(\theta)-\varpi(a)\psi(a)\right).
\end{aligned}$$

Now, we will prove 2. With the aid of equation 4, we have

$${}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}\left({}^{pRL}_{\chi}I^{\delta,\sigma}_{a,\varpi}\psi\right)(\theta)=\frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}(-\varepsilon_\delta\ln p)^u\,{}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left(\frac{\left(\varpi\,{}^{pRL}_{\chi}I^{\lambda,\sigma}_{a,\varpi}\psi\right)'}{\varpi\chi'}\right)(\theta)$$

$$= \frac{K(\delta)}{1-\delta}\sum_{u=0}^{\infty}(-\varepsilon_\delta \ln p)^u \, {}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left[\frac{1-\delta}{K(\delta)}\frac{(\varpi\psi)'}{\varpi\chi'}+\frac{\delta\ln p}{K(\delta)}\frac{\left(\varpi{}^{RL}_{\chi}I^{\sigma}_{a,\varpi}\psi\right)'}{\varpi\chi'}\right](\theta)$$

$$= \sum_{u=0}^{\infty}(-\varepsilon_\delta \ln p)^u \, {}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left(\frac{(\varpi\psi)'}{\varpi\chi'}\right)(\theta)$$

$$+ \varepsilon_\delta \ln p\sum_{u=0}^{\infty}(-\varepsilon_\delta \ln p)^u \, {}^{RL}_{\chi}I^{\sigma u+1}_{a,\varpi}\left(\frac{\left(\varpi{}^{pRL}_{\chi}I^{\sigma}_{a,\varpi}\psi\right)'}{\varpi\chi'}\right)(\theta)$$

$$= \sum_{u=0}^{\infty}(-\varepsilon_\delta \ln p)^u \left[{}^{RL}_{\chi}I^{\sigma u}_{a,\varpi}\psi(\theta)-(\varpi\psi)(a)\,{}^{RL}_{\chi}I^{\sigma u}_{a,\varpi}\left(\frac{1}{\varpi\chi'}\right)(\theta)\right]$$

$$+\varepsilon_\delta \ln p\sum_{u=0}^{\infty}(-\varepsilon_\delta \ln p)^u \left[{}^{RL}_{\chi}I^{\sigma(u+1)}_{a,\varpi}\psi(\theta)-(\varpi\psi)(a)\,{}^{RL}_{\chi}I^{\sigma(u+1)}_{a,\varpi}\left(\frac{1}{\varpi\chi'}\right)(\theta)\right].$$

Thus,

$$\begin{aligned}
&{}^{pC}_{\chi}D^{\delta,\sigma}_{a,\varpi}\left({}^{pRL}_{\chi}I^{\delta,\sigma}_{a,\varpi}\psi\right)(\theta)\\
&=\sum_{u=0}^{\infty}(-\varepsilon_\delta \ln p)^u \left[{}^{RL}_{\chi}I^{\sigma u}_{a,w}\psi(\theta)-(\varpi\psi)(a)\,{}^{RL}_{\chi}I^{\sigma u}_{a,\varpi}\left(\frac{1}{\varpi\chi'}\right)(\theta)\right]\\
&\quad-\sum_{u=1}^{\infty}(-\varepsilon_\delta \ln p)^u \left[{}^{RL}_{\chi}I^{\sigma u}_{a,\varpi}\psi(\theta)-(\varpi\psi)(a)\,{}^{RL}_{\chi}I^{\sigma u}_{a,\varpi}\left(\frac{1}{\varpi\chi'}\right)(\theta)\right]\\
&={}^{RL}_{\chi}I^{0}_{a,\varpi}\psi(\theta)-(\varpi\psi)(a)\,{}^{RL}_{\chi}I^{0}_{a,\varpi}\left(\frac{1}{\varpi\chi'}\right)(\theta)\\
&=\psi(\theta)-\frac{(\varpi\psi)(a)}{\varpi(\theta)}.
\end{aligned}$$

This completes the proof. □

8. Conclusion

Using the Caputo Fabrizo fractional derivative, A-B fractional derivative, and R-L fractional derivative, we have presented a novel concept of power fractional derivative associated with generalized function in the kernel in this study. We generate distinct results using the power M-L function notion. We achieved the intended results by applying new modified power fractional derivative formulas and attributes. One can restore the known operators by applying certain conditions on $\chi(\theta)$. If we take $\chi(\theta) = 1$ and $p = e$, then we get the work done by [8, 9].

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