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On Integral Extension of Ankeny and Rivlin-Type Inequality

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Abstract. A well-known theorem due to Ankeny and Rivlin states that if $p(z)$ is a polynomial of degree n having no zero in $|z|<1$, then

$$\max_{|z|=R\geq 1}|p(z)|\leq\left(\frac{R^n+1}{2}\right)\max_{|z|=1}|p(z)|.$$

In this paper, we obtain an extension as well as an improvement of this inequality to the integral setting.

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1. Introduction

Let $p(z)$ be a polynomial of degree n over the set $\mathbb{C}$ of complex numbers, and for each real $q>0$, we define the integral mean of $p(z)$ on the unit circle $|z|=1$ by

$$\|p\|_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}.$$

If we let $q\to\infty$ in the above equality and make use of the well-known fact from the analysis [15] that

$$\lim_{q\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}=\max_{|z|=1}|p(z)|,$$

we can suitably denote

$$\|p\|_\infty=\max_{|z|=1}|p(z)|.$$

Also, we denote $M(p,R)=\max\limits_{|z|=R}|p(z)|$.

Serge Bernstein [4] established a relation that relates an estimate of the size of the derivative of a polynomial to that of the polynomial itself in the uniform norm on the unit circle in the complex plane which states that if $p(z)$ is a polynomial of degree n, then

$$\|p'\|_\infty\leq n\|p\|_\infty. \tag{1}$$

Inequality (1) is best possible and equality holds only for polynomials of the form $p(z)=\alpha z^n$, $\alpha\neq 0$ being a complex number.

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In 1945, S. Bernstein observed the following result, which in fact is a simple consequence of the maximum modulus principle (see [13]) also known as the Bernstein's inequality. It states that if $p(z)$ is a polynomial of degree n, then for $R \geq 1$,

$$M(p, R) \leq R^n \|p\|_\infty, \tag{2}$$

Equality in (2) holds for $p(z) = \alpha z^n$, $\alpha \neq 0$ being a complex number.

It was proved by Bernstein himself that inequality (1) can be obtained from inequality (2). However, it was not known if inequality (2) can also be obtained from inequality (1), and this has been shown by Govil et al. [7]. Thus both the inequalities (1) and (2) are equivalent in the sense that anyone can be obtained from the other.

Inequalities (1) and (2) can be obtained by letting $q \to \infty$ in the inequalities

$$\|p'\|_q \leq n\|p\|_q, \; q > 0, \tag{3}$$

and

$$\|p(Rz)\|_q \leq R^n \|p\|_q, \; q > 0, \tag{4}$$

respectively. Inequality (3) was proved by Zygmund [22] for $q \geq 1$, and by Arestov [2] for $0 < q < 1$, while inequality (4) was proved by Hardy [8].

Ankeny and Rivlin [1] considered a class of polynomials $p(z)$ of degree n having no zero in $|z| < 1$, and obtained a refinement of inequality (2) that for $R \geq 1$,

$$M(p, R) \leq \frac{R^n + 1}{2} \|p\|_\infty. \tag{5}$$

The result is best possible with equality only for polynomials $p(z) = \lambda + \mu z^n$, $|\lambda| = |\mu|$.

As an improvement of inequality (5), Aziz and Dawood [3] proved under the same hypothesis that

$$M(p, R) \leq \frac{R^n + 1}{2} \|p\|_\infty - \frac{R^n - 1}{2} m, \tag{6}$$

where $m = \min_{|z|=1} |p(z)|$.

Boas and Rahman [5] proved the integral extension of inequality (5) for $q \geq 1$, while Rahman and Schmeisser [14] extended it for $q > 0$. They established that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for $R \geq 1$,

$$\|p(Rz)\|_q \leq \frac{\|1 + R^n z^n\|_q}{\|1 + z^n\|_q} \|p\|_q. \tag{7}$$

The literature has witnessed significant attention towards extending Zygmund-type inequalities to their integral counterparts (see [17], [21], [18], [9], [16], [19], [11], [12], [10]), while extending Ankeny and Rivlin-type inequalities to their integral counterparts has a unique trajectory. Boas and Rahman pioneered the introduction of integral analog of inequality (5) way back in 1962, and it took about 26 years for Rahman and Schmeisser to provide an extension of the same. Since then, there has been a noticeable absence of generalizations into integral means for this particular type of inequalities.

This gap in the research landscape sparks curiosity and emphasizes the need to delve into potential extensions in integral mean versions of the Ankeny and Rivlin-type inequalities. In this context, we have successfully extended inequality (6) due to Aziz and Dawood [3] to an integral setting which also gives an improved form of inequality (7).

2. Lemmas

Definition 2.1. For $\gamma = (\gamma_0, ..., \gamma_n) \in \mathbb{C}^{n+1}$ and $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, we define

$$\Lambda_\gamma p(z) = \sum_{\nu=0}^{n} \gamma_\nu a_\nu z^\nu.$$

The operator Λ_γ is said to be admissible if it preserves one of the following properties:
(i) $p(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$,
(ii) $p(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

Lemma 2.1. *Let $\phi(x) = \psi(\log x)$, where ψ is a convex non-decreasing function on $\mathbb{R}$. Then for all polynomials $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree n, and each admissible operator Λ_γ,*

$$\int_0^{2\pi} \phi\left(|\Lambda_\gamma p(e^{i\theta})|\right) d\theta \leq \int_0^{2\pi} \phi\left(c(\gamma, n)|p(e^{i\theta})|\right) d\theta,$$

where $c(\gamma, n) = \max(|\gamma_0|, |\gamma_n|)$.

In particular, the lemma applies with $\phi : x \mapsto x^q$ for every $q > 0$, and with $\phi : x \mapsto \log x$ as well. Therefore, we have

$$\|\Lambda_\gamma p\|_q \leq c(\gamma, n)\|p\|_q \text{ for } q > 0. \tag{8}$$

The above lemma is due to Arestov [2].

Lemma 2.2. *For some fixed $a, b \in \mathbb{C}$, and some $\lambda \in \mathbb{C}$, with appropriate choice of the argument of λ, $|a + \lambda b|$ can be made equals either $|a| + |\lambda||b|$ or $||a| - |\lambda||b||$.*

Proof. Suppose $a = |a|e^{i\theta_1}$, $b = |b|e^{i\theta_2}$, and $\lambda = |\lambda|e^{i\theta}$, then

$$\begin{aligned} |a + \lambda b| &= \left||a|e^{i\theta_1} + |\lambda||b|e^{i(\theta+\theta_2)}\right| \\ &= \left||a| + |\lambda||b|e^{i(\theta+\theta_2-\theta_1)}\right|. \end{aligned}$$

On choosing θ as $-(\theta_2 - \theta_1)$ or $\pi - (\theta_2 - \theta_1)$, the result follows readily. □

3. Main result

In this paper, we obtain the following interesting result concerning integral mean setting. In fact, we prove

Theorem 3.1. *Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n having no zero in $|z| < 1$, then for $R \geq 1$, for every complex number λ with $|\lambda| < 1$, and for each $q > 0$,*

$$\|p(Rz) - \lambda m\|_q \leq \frac{\|1 + R^n z^n\|_q}{\|1 + z^n\|_q}\|p - \lambda m\|_q, \tag{9}$$

where $m = \min_{|z|=1} |p(z)|$.

Proof. Since the polynomial $p(z)$ has no zero in $|z|<1$, by Rouche's theorem, for every complex number λ with $|\lambda|<1$, the polynomial $G(z)=p(z)-\lambda m$ has no zero in $|z|<1$.

For $R\geq 1$ and $\gamma\in\mathbb{R}$, the polynomial

$$\sum_{\nu=0}^{n}\binom{n}{\nu}\left(R^{\nu}+e^{i\gamma}R^{n-\nu}\right)z^{\nu}$$

has all its zeros on the unit circle [13, (see problem 26, p.108)]. Since $G(z)$ does not vanish in $|z|<1$, by Szegö's convolution theorem [20] the same is true for

$$\begin{aligned}\Lambda G(z): &= (1+e^{i\gamma}R^{n})(a_{o}-\lambda m)+(R+e^{i\gamma}R^{n-1})a_{1}z+...+(R^{n}+e^{i\gamma})a_{n}z^{n}\\ &= G(Rz)+e^{i\gamma}R^{n}G\left(\frac{z}{R}\right).\end{aligned}$$

Therefore, Λ is an admissible operator. Applying (8) of Lemma 2.1, we obtain

$$\int_{0}^{2\pi}\left|G(Re^{i\theta})+e^{i\gamma}R^{n}G\left(\frac{e^{i\theta}}{R}\right)\right|^{q}d\theta\leq|1+R^{n}e^{i\gamma}|^{q}\int_{0}^{2\pi}|G(e^{i\theta})|^{q}d\theta. \tag{10}$$

Since

$$f(z):=\frac{z^{n}\overline{G(\frac{1}{\bar z})}}{G(z)}$$

is holomorphic for $|z|\leq 1$ with $|f(z)|=1$ on the unit circle, it follows from the maximum modulus principle that $|f((\frac{1}{R})e^{i\theta})|\leq 1$ for $\frac{1}{R}<1$, and so

$$\left|\frac{R^{n}G(\frac{e^{i\theta}}{R})}{G(Re^{i\theta})}\right|\geq 1\ (R\geq 1). \tag{11}$$

Now, integrating (10) with respect to γ on $[0,2\pi]$, and using (11), we obtain

$$\int_{0}^{2\pi}\int_{0}^{2\pi}|G(Re^{i\theta})|^{q}|1+e^{i\gamma}R(\theta)|^{q}d\theta d\gamma\leq\int_{0}^{2\pi}|1+R^{n}e^{i\gamma}|^{q}d\gamma\int_{0}^{2\pi}|G(e^{i\theta})|^{q}d\theta, \tag{12}$$

where $R(\theta)=\dfrac{R^{n}G(\frac{e^{i\theta}}{R})}{G(Re^{i\theta})}$.

It is known [6, Theorem 2] that if $G(z)\neq 0$ for $|z|<1$, then $|R(\theta)|\geq 1$, and therefore by a theorem of Hardy [8],

$$\int_{0}^{2\pi}|1+e^{i\gamma}R(\theta)|^{q}d\gamma\geq\int_{0}^{2\pi}|1+e^{i\gamma}|^{q}d\gamma$$

for all $\theta\in[0,2\pi]$. Using this in (12), we have

$$\int_{0}^{2\pi}|G(Re^{i\theta})|^{q}d\theta\leq\frac{\int_{0}^{2\pi}|1+R^{n}e^{i\gamma}|^{q}d\gamma}{\int_{0}^{2\pi}|1+e^{i\gamma}|^{q}d\gamma}\int_{0}^{2\pi}|G(e^{i\theta})|^{q}d\theta,$$

i.e.

$$\left\{\int_{0}^{2\pi}|p(Re^{i\theta})-\lambda m|^{q}d\theta\right\}^{\frac{1}{q}}\leq\frac{\left\{\int_{0}^{2\pi}|1+R^{n}e^{i\gamma}|^{q}d\gamma\right\}^{\frac{1}{q}}}{\left\{\int_{0}^{2\pi}|1+e^{i\gamma}|^{q}d\gamma\right\}^{\frac{1}{q}}}\left\{\int_{0}^{2\pi}|p(e^{i\theta})-\lambda m|^{q}d\theta\right\}^{\frac{1}{q}}.$$

This completes the proof of the theorem. □

Remark 3.1. Letting $q \to \infty$ on both sides of inequality (9) of Theorem 3.1, we obtain

$$\max_{|z|=1} |p(Rz) - \lambda m| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z) - \lambda m|, \tag{13}$$

from which we can further deduce inequality (6) as discussed below:

Let z_0 on $|z| = 1$ be such that $\max_{|z|=1} |p(Rz)| = |p(Rz_0)|$, then

$$\begin{aligned} \max_{|z|=1} |p(Rz) - \lambda m| &\geq |p(Rz_0) - \lambda m| \\ &\geq ||p(Rz_0)| - |\lambda| m|. \end{aligned} \tag{14}$$

By the minimum modulus principle, we know $|p(z)| \geq m$ for all $|z| \geq 1$.

Then, for $R \geq 1$ and $|z| = 1$, since $|Rz| = R \geq 1$, we have

$$|p(Rz)| \geq m > |\lambda| m. \tag{15}$$

Using (15) to (14), we get

$$\max_{|z|=1} |p(Rz) - \lambda m| \geq |p(Rz_0)| - |\lambda| m. \tag{16}$$

Combining (16) and (13), we have

$$\max_{|z|=1} |p(Rz)| - |\lambda| m \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z) - \lambda m|. \tag{17}$$

Again, let z_1 on $|z| = 1$ be such that

$$\max_{|z|=1} |p(z) - \lambda m| = |p(z_1) - \lambda m|. \tag{18}$$

For $|z| = 1$, since $|p(z)| \geq m$, we have for $|\lambda| < 1$,

$$|p(z)| - |\lambda| m > 0. \tag{19}$$

Applying Lemma 2.2 on the right-hand side of (18), and in view of (19), we have

$$|p(z_1) - \lambda m| = |p(z_1)| - |\lambda| m. \tag{20}$$

Using (20) to (18), and further using the fact that $|p(z_1)| \leq \max_{|z|=1} |p(z)|$, we get

$$\max_{|z|=1} |p(z) - \lambda m| \leq \max_{|z|=1} |p(z)| - |\lambda| m. \tag{21}$$

Combining (21) and (17), we get

$$\max_{|z|=1} |p(Rz)| - |\lambda| m \leq \frac{R^n + 1}{2} \left(\max_{|z|=1} |p(z)| - |\lambda| m \right),$$

i.e.

$$\max_{|z|=R} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)| - \frac{R^n - 1}{2} |\lambda| m. \tag{22}$$

For convenience, we denote the quantity on the right-hand side of the above inequality as

$$A(|\lambda|) = \frac{R^n + 1}{2} \max_{|z|=1} |p(z)| - \frac{R^n - 1}{2} |\lambda| m.$$

By derivative test, it follows that $A(|\lambda|)$ is a non-increasing function of $|\lambda| \in [0, 1)$, and hence

$$A(1) \leq A(|\lambda|) \leq A(0).$$

Also, we can see that $A(|\lambda|)$ is continuous with $|\lambda|$, and so it is evident that letting $|\lambda| \to 1$, it gives the most improved bound of (22) matching the bound of inequality (6) due to Aziz and Dawood [3]. While putting $|\lambda| = 0$, we get inequality (5) due to Ankeny and Rivlin [1].

For a better understanding of the nature of variation of bound (22), for simultaneous changes of $|\lambda|$ and R, we consider the example below.

Example 3.1. Consider the polynomial $p(z) = z^2 - 7z + 12$ with no zero in $|z| < 1$. For this polynomial, we have $\max\limits_{|z|=1} |p(z)| = 20$ and $\min\limits_{|z|=1} |p(z)| = 6$. Then it can be easily seen that for $R = 3$, by inequality (5), we have $\max\limits_{|z|=3} |p(z)| \leq 100$, while by inequality (6), $\max\limits_{|z|=3} |p(z)| \leq 76$, and an improvement of 24% over the bound obtained from (5).

Furthermore, it is of interest to depict graphically as in Figure 1, the variation of the bound of (22) which corresponds to the height of a point on the surface represented by the bound (22) for simultaneous changes of $|\lambda|$ and R, and it is clearly seen that for given R, the value of the bound keeps on decreasing as $|\lambda|$ increases, and the values of the improved bound for various R correspond to the heights of the boundary points of the surface intersected by the plane $|\lambda| = 1$.

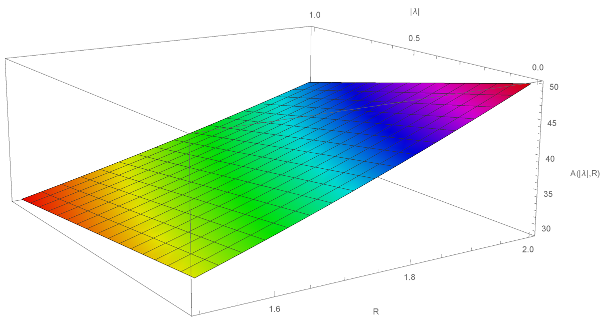

FIGURE 1. Surface graph of the function $(|\lambda|, R) \mapsto A(|\lambda|, R)$ for $0 \leq |\lambda| < 1$ and $1.5 \leq R \leq 2$.

Remark 3.2. Putting $\lambda = 0$ in (9) of Theorem 3.1, we obtain inequality (7) proved by Rahman and Schmeisser [14].

Remark 3.3. Our Theorem 3.1 could have some interesting implication concerning the integral bound of the reciprocal polynomial $z^n p(\frac{1}{z})$ or conjugate reciprocal polynomial $z^n \overline{p(\frac{1}{\bar{z}})}$ of a given class of polynomials $p(z)$ having all their zeros in $|z| \leq 1$. More precisely, suppose $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then both the reciprocal polynomials have all their zeros in $|z| \geq 1$, i.e. have no zero in $|z| < 1$, and hence we can well apply our theorem for finding the upper bound of the integral estimate of these reciprocal polynomials on the circle $|z| = R \geq 1$ in terms of the integral estimate on the unit circle $|z| = 1$.

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