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Semi-linear Differential Equations with Finite Delay via Densifiability Techniques

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Abstract. This article investigates the existence of solutions for certain types of semi-linear differential equations. Three specific problems are addressed: a problem involving semi-linear differential equations with finite delay, a neutral problem, and a semi-linear neutral type integro-differential problem with a nonlocal initial condition. The study utilizes a new fixed-point theorem based on the concept of nondensifiability degree, which is broader than the traditional measure of noncompactness and encompasses cases that were previously inaccessible. Additionally, an illustrative example is provided to support and clarify the findings.

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1. Introduction

Delay differential equations appear in models where the current state depends on its past. These equations are prevalent in various fields, including economics, physics, medicine, biology, and ecology. In some cases, knowing the solution at a single point is insufficient to describe the evolution over a time interval. For a thorough overview of the theory related to delay differential equations, refer to sources such as [3, 15, 5, 23, 1].

The role of delay in various models can vary: it might represent the incubation period of a contagious disease, the time required for accumulation, the time needed for cell maturation, or the transformation of one type of cell into another. Delay differential equations were introduced to model phenomena with a time lag between an action on the system and the system's response. For instance, in the birth processes of biological populations (such as cells or bacteria), a certain threshold must be reached before the system activates. V. Volterra initially introduced a general class of delay differential equations in [28] while studying the predator-prey model. The theory of delay differential equations saw significant development in the latter half of the last century, with notable contributions from Bellman and Cooke [22], as well as Lunel and Walther [5]. For further details, refer to [4].

In reference [27], the authors have investigated the existence of mild solutions to a second-order semilinear integrodifferential equation under compact conditions. Similarly, in [10], the authors have explored the existence, complete controllability, and approximate controllability of mild solutions for the same problem, incorporating

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measures of noncompactness. In both works, the authors utilized Grimmer's resolvent operators. For further insights into resolvent operators and integro-differential systems, interested readers are referred to [6, 7, 21, 22, 27, 9, 8] and the related references therein.

The authors introduced the concept of α-dense curves [12] in the 1980s . Cherruault [13] and Mora [24] were primarily responsible for its creation. Mora and Mira citeMoMi established the notion of the degree of nondensifiability (DND), which is based on α-dense curves. García [20, 18] demonstrated a novel fixed-point result using the DND that is more applicable than the Darbo fixed point theorem and its generalizations.

Evolution equations involving non-local initial conditions represent a generalization of evolution equations with classical initial conditions. Since they incorporate a broader scope of information, this concept offers a more comprehensive framework for describing natural phenomena compared to the classical approach. Nonlocal conditions, for instance, have the potential to be employed in the formulation of mathematical models for describing the progression of various phenomena. These encompass nonlocal neutral networks, nonlocal pharmacokinetics, nonlocal pollution, and nonlocal combustion. For additional insights into the importance of nonlocal conditions across various applied sciences, refer to [11, 29], and the related references therein.

In particular, Diop *et al.* [14] studied the existence of mild solutions for a class of nonlinear impulsive integro-differential equations with a nonlocal initial condition:

$$\begin{cases} \chi'(\rho) = \mathfrak{Z}\chi(\rho) + \aleph(\rho, \chi(\rho) + \int_0^\rho \Upsilon(\rho-\mu)\chi(\mu)d\mu, \ \text{if } 0 \le \rho \le T, \rho \ne \rho_\jmath, \\ \chi(0) + g(\chi) = \chi_0, \\ \chi(\rho_\jmath^+) - \chi(\rho_\jmath^-) = I_\jmath(\chi(\rho_\jmath)), \jmath = 1, 2, \dots, p, 0 < \rho_1 < \rho_2 < \cdots < \rho_p < T, \end{cases}$$

where $\mathfrak{Z}$ generates a C_0-semigroup on a Banach space $\mathfrak{W}$, $\Upsilon(\rho)$ is a closed linear operator on $\mathfrak{W}$ with time independent domain $D(\mathfrak{Z}) \subset D(\Upsilon)$. $\aleph : [0, T] \times \mathfrak{W} \to \mathfrak{W}$ and $g : PC([0, T], \mathfrak{W}) \to \mathfrak{W}$ are continuous functions where the set $PC([0, T], \mathfrak{W})$ is a Banach space.

Our findings fundamentally rely on the notable contributions made by Garcia [20, 19] in investigating the existence of mild solutions for semi-linear differential equations with finite delay in Banach spaces:

$$\begin{cases} \chi'(\rho) - \mathfrak{Z}(\rho)\chi(\rho) = \aleph(\rho, \chi_\rho), \ \text{if } \rho \in \Theta, \\ \chi(\rho) = \varphi(\rho), \ \text{if } \rho \in [-r, 0], \end{cases} \tag{1}$$

where $\Theta = [0, T], \aleph : \Theta \times C([-r, 0], \mathfrak{W}) \to \mathfrak{W}$ is a continuous function and the set $C([-r, 0], \mathfrak{W})$ is a Banach space of all continuous functions from $[-r, 0]$ into $\mathfrak{W}$. $(\mathfrak{W}, \|\cdot\|_{\mathfrak{W}})$ is a Banach space, and $\mathfrak{Z}(\rho) : \mathfrak{G}(\mathfrak{Z}(\rho)) \subset \mathfrak{W} \to \mathfrak{W}$, is closed linear operator on $\mathfrak{W}$, with dense domain $\mathfrak{G}(\mathfrak{Z}(\rho))$, which is independent of ρ, and $\varphi : [-r, 0] \to \mathfrak{W}$ is a continuous function. For any function χ defined on $[-r, T]$ and any $\rho \in [0, T]$ we denote by χ_ρ the element of $C([-r, 0], \mathfrak{W})$ defined by:

$$\chi_\rho(\theta) = \chi(\rho + \theta), \quad \theta \in [-r, 0].$$

The next step is to explore the existence of mild solutions for the following semi-linear neutral type differential equations with finite delay:

$$\begin{cases} \frac{d}{d\rho}[\chi(\rho) - g(\rho, \chi_\rho)] - \mathfrak{Z}(\rho)\chi(\rho) = \aleph(\rho, \chi_\rho), \ \text{if } \rho \in \Theta, \\ \chi(\rho) = \varphi(\rho), \ \text{if } \rho \in [-r, 0], \end{cases} \tag{2}$$

where $g : \Theta \times C([-r, 0], \mathfrak{W}) \to \mathfrak{W}$ is a continuous function.

Finally, we investigate the existence of mild solutions for the following semi-linear neutral type integro-differential equations with a nonlocal initial condition:

$$\begin{cases} \frac{d}{d\rho}[\chi(\rho) - g(\rho, \chi_\rho)] - \mathfrak{Z}(\rho)\chi(\rho) = \aleph(\rho, \chi_\rho) + \int_0^\rho \varpi(\rho, \mu)\chi(\mu)d\mu, \text{ if } \rho \in \Theta, \\ \chi(0) + f(\chi) = \chi_0 \in \mathfrak{W}, \\ \chi(\rho) = \varphi(\rho), \text{ if } \rho \in [-r, 0], \end{cases} \tag{3}$$

where $\varpi(\rho, \mu)$ is a closed linear operator on $\mathfrak{W}$ and $f : C([-r, T], \mathfrak{W}) \to \mathfrak{W}$ is continuous function.

The article is structured as follows: Section 2 introduces basic notations, preliminaries, and lemmas. In Section 3, we demonstrate the existence of mild solutions for the semi-linear differential problem (1). Sections 4 and 5 focus on the analysis of problems (2) and (3), respectively. Our research utilizes a novel fixed-point theorem based on the concept of DND. Finally, an example is provided in the last section to illustrate the results obtained.

2. Preliminaries

Let $\mathfrak{W}$ be a real Banach space with the norm $\|\cdot\|_{\mathfrak{W}}$ and $M_{\mathfrak{W}}$ is the class of non-empty and bounded subsets of $\mathfrak{W}$, let $B(\mathfrak{W})$ be the space of all bounded linear operators from $\mathfrak{W}$ into $\mathfrak{W}$, with the norm

$$\|T\|_{B(\mathfrak{W})} = \sup_{\|\chi\|_{\mathfrak{W}}=1} \|T(\chi)\|_{\mathfrak{W}}.$$

We denote by $(L^1(\Theta, \mathfrak{W}), \|\cdot\|_1)$ the Banach space of measurable functions that are Bochner integrable from $\Theta := [0, T]$ into $\mathfrak{W}$, with the norm

$$\|\chi\|_1 = \int_0^T \|\chi(\rho)\|_{\mathfrak{W}} d\rho.$$

$L^\infty(\Theta, \mathfrak{W})$, is the Banach space of measurable functions which are essentially bounded, with the norm

$$\|\cdot\|_\infty = \inf\{C > 0 : \|\chi(\rho)\|_{\mathfrak{W}} \leq C, \text{ a.e. } \rho \in \Theta\}.$$

By $C(\Theta, \mathfrak{W})$ we denote the Banach space of all continuous functions from Θ into $\mathfrak{W}$ with

$$\|\chi\| = \sup_{\rho\in\Theta} \|\chi(\rho)\|_{\mathfrak{W}}.$$

Let $C([-r, 0], \mathfrak{W})$ be the Banach space of all continuous functions from $[-r, 0]$ into $\mathfrak{W}$ with

$$\|\chi\|_{[-r,0]} = \sup_{\rho\in[-r,0]} \|\chi(\rho)\|_{\mathfrak{W}}.$$

By $C([-r, T], \mathfrak{W})$ we denote the Banach space of all continuous functions from $[-r, T]$ into $\mathfrak{W}$ with

$$\|\chi\|_{[-r,T]} = \sup_{\rho\in[-r,T]} \|\chi(\rho)\|_{\mathfrak{W}}.$$

Definition 2.1. ([24, 26]) Suppose that $\alpha \geq 0$ and $\Bbbk \in M_{\mathfrak{W}}$, a continuous mapping $\zeta : \wp := [0,1] \to \mathfrak{W}$ is an $\alpha-$dense curve in $\Bbbk$ if:

- $\zeta(\wp) \subset \Bbbk$.
- For any $\chi_1 \in \Bbbk$, there is $\chi_2 \in \zeta(\wp)$ such that $\|\chi_1 - \chi_2\|_{\mathfrak{W}} \leq \alpha$.

If for $\alpha > 0$, there is an $\alpha-$dense curve in $\Bbbk$, then $\Bbbk$ is densifiable.

Definition 2.2. ([25, 16]) Let $\alpha > 0$, and denote by $\Gamma_{\alpha,\Bbbk}$ the class of all $\alpha-$dense curves in $\Bbbk \in M_{\mathfrak{W}}$. The DND is a mapping $\varkappa : M_{\mathfrak{W}} \to \mathbb{R}_+$ defined as:

$$\varkappa(\Bbbk) = \inf\{\alpha \geq 0 : \Gamma_{\alpha,\Bbbk} \neq \varnothing\},$$

for each $\Bbbk \in M_{\mathfrak{W}}$.

Remark 2.1. It is important to highlight that a thorough examination of the degree of nondensifiability (DND) was conducted in [16]. Specifically, the study established that the DND does not function as a measure of noncompactness [16]. Nonetheless, it exhibits characteristics remarkably akin to those of MNC (see Proposition 2.6 in [16]).

Lemma 2.1 ([17, 16])**.** *Let $\Bbbk_1, \Bbbk_2 \in M_{\mathfrak{W}}$. Then, we have:*

(a) $\varkappa(\Bbbk_1) = 0 \iff \Bbbk_1$ *is a precompact set, for each nonempty, bounded and arc-connected subset* $\Bbbk_1$ *of* $\mathfrak{W}$.
(b) $\varkappa(\bar{\Bbbk}_1) = \varkappa(\Bbbk_1)$*, where* $\bar{\Bbbk}_1$ *denotes the closure of* $\Bbbk_1$.
(c) $\varkappa(\lambda\Bbbk_1) = |\lambda|\varkappa(\Bbbk_1)$*, for* $\lambda \in \mathbb{R}$.
(d) $\varkappa(x + \Bbbk_1) = \varkappa(\Bbbk_1)$*, for all* $x \in \mathfrak{W}$.
(e) $\varkappa(Conv\Bbbk_1) \leq \varkappa(\Bbbk_1)$ *and* $\varkappa(Conv\Bbbk_1 \cup \Bbbk_2) \leq \max\{\varkappa(Conv\Bbbk_1), \varkappa(Conv\Bbbk_2)\}$*, where*

$\varkappa(Conv\Bbbk_1)$ *represent the convex hull of* $\Bbbk_1$.
(f) $\varkappa(\Bbbk_1 + \Bbbk_2) \leq \varkappa(\Bbbk_1) + \varkappa(\Bbbk_2)$.

Let

$$\mathfrak{X} = \left\{ \begin{array}{c} h : \mathbb{R}_+ \to \mathbb{R}_+ : h \text{ is monotone increasing} \\ \text{and } \lim_{n\to\infty} h^n(\rho) = 0 \text{ for any } \rho \in \mathbb{R}_+ \end{array} \right\},$$

where $n \in \mathbb{N}$ and $h^n(\rho)$ denotes the $n-$th composition of h with itself.

The version of the Darbo fixed point theorem for the DND presented below plays a significant role in this paper.

Theorem 2.2. [20] *Let $\widehat{\Bbbk}$ be a nonempty, bounded, closed, and convex subset of a Banach space $\mathfrak{W}$, and let $\mho : \widehat{\Bbbk} \to \widehat{\Bbbk}$ be a continuous operator. Assume that there is $h \in \mathfrak{X}$ such that:*

$$\varkappa(\mho(\Bbbk)) \leq h(\varkappa(\Bbbk))$$

for any non-empty subset $\Bbbk$ of $\widehat{\Bbbk}$. Then, $\mho$ possesses at least one fixed point in $\widehat{\Bbbk}$.

Remark 2.2. It is important to observe that the fixed point theorem based on DND in [20] takes a form closely resembling the renowned Darbo fixed-point theorem [2]. Nevertheless, as demonstrated in [20, 17] through various examples, both outcomes are fundamentally distinct. The presented theorem in [20] operates under more inclusive conditions than the Darbo fixed-point theorem or its well-known generalizations.

Lemma 2.3. *([20]) Let $\Bbbk \subset C(\Theta, \mathfrak{W})$ be non-empty and bounded. Then:*

$$\sup_{\rho\in\Theta} \varkappa(\Bbbk(\rho)) \leq \varkappa(\Bbbk).$$

3. Existence of Mild Solutions for Semi-Linear Differential Equations

Definition 3.1. We say that a continuous function $\chi : [-r, T] \to \mathfrak{W}$ is a mild solution of problem (1), if χ satisfies the following integral equation

$$\begin{cases} \chi(\rho) = \Im(\rho, 0)\varphi(0) + \int_0^\rho \Im(\rho, \mu)\aleph(\mu, \chi_\mu)d\mu, \text{ if } \rho \in \Theta, \\ \chi(\rho) = \varphi(\rho), \text{ if } \rho \in [-r, 0], \end{cases}$$

where $\Im(\cdot, \cdot)$ is an evolution system of linear bounded operators on $\mathfrak{W}$ generated by the closed linear operator $\mathfrak{Z}$.

In order to obtain our existence result, we also need the following assumptions:

($\mathcal{A}_1$) The function $\aleph : \Theta \times C([-r, 0], \mathfrak{W}) \to \mathfrak{W}$ satisfies the Carathéodory conditions, and there exist $p_\aleph \in L^1(\Theta, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a nondecreasing continuous function such that:

$$\|\aleph(\rho, \chi)\|_{\mathfrak{W}} \le p_\aleph(\rho)\psi(\|\chi\|_{[-r,0]}), \text{ for } \chi \in C([-r, 0], \mathfrak{W}), \text{ and for a.e. } \rho \in \Theta.$$

($\mathcal{A}_2$) The bounded linear operator $\Im$ is uniformly continuous and there exists $M \ge 1$ such that

$$\|\Im(\rho, \mu)\|_{B(\mathfrak{W})} \le M, \text{ for every } \rho \in \Theta.$$

($\mathcal{A}_3$) There exist $K \in L^\infty(\Theta, \mathbb{R}_+)$ and $h \in \mathfrak{X}$ where for any non-empty, bounded and convex subset $\Bbbk \subset \mathfrak{W}$,

$$\varkappa(\aleph(\rho, \Bbbk)) \le K(\rho)h(\varkappa(\Bbbk)),$$

holds for a.e. $\rho \in \Theta$.

($\mathcal{A}_4$) There exists $\grave{r} > 0$ such that

$$\grave{r} \ge M \left[\|\varphi(0)\|_{\mathfrak{W}} + \psi(\grave{r})\|p_\aleph\|_{L^1}\right].$$

Theorem 3.1. *Assume that the conditions* $(\mathcal{A}_1) - (\mathcal{A}_4)$ *are satisfied and that*

$$TM\|K\|_\infty \le 1. \tag{4}$$

Thus, (1) has at least one mild solution defined on Θ.

Proof. Firstly, we consider the operator

$$\mho : C([-r, T], \mathfrak{W}) \to C([-r, T], \mathfrak{W})$$

defined by:

$$\mho\chi(\rho) = \begin{cases} \Im(\rho, 0)\varphi(0) + \int_0^\rho \Im(\rho, \mu)\aleph(\mu, \chi_\mu)d\mu, \text{ if } \rho \in \Theta \\ \varphi(\rho), \text{ if } \rho \in [-r, 0] \end{cases}$$

We consider the set

$$\widehat{\Bbbk} = \left\{\chi \in C([-r, T], \mathfrak{W}) : \|\chi\|_{C([-r,T]} \le \grave{r}\right\}.$$

We note that $\widehat{\Bbbk}$ is bounded, closed and convex subset.

Step 1 : We prove that $\mho\widehat{\Bbbk} \subset \widehat{\Bbbk}$.

Indeed for any $\chi \in \widehat{\Bbbk}$ and under $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_4)$ we obtain

$$\|\mho\chi(\rho)\|_{\mathfrak{W}} = \|\Im(\rho, 0)\varphi(0) + \int_0^\rho \Im(\rho, \mu)\aleph(\mu, \chi_\mu)d\mu\|_{\mathfrak{W}}$$

$$\begin{aligned}
&\leq \|\Im(\rho,0)\|_{B(\mathfrak{W})}\|\varphi(0)\|_{\mathfrak{W}} + \int_0^\rho \|\Im(\rho,\mu)\|_{B(\mathfrak{W})}\|\aleph(\mu,\chi(\mu))\|_{\mathfrak{W}} d\mu \\
&\leq M\|\varphi(0)\|_{\mathfrak{W}} + M\int_0^\rho p_\aleph(\mu)\psi(\|\chi(\rho)\|_{C([-r,0]})d\mu \\
&\leq M\,\|\varphi(0)\|_{\mathfrak{W}} + M\psi(\grave{r})\|p_\aleph\|_{L^1} \\
&\leq \grave{r}.
\end{aligned}$$

Thus $\mho(\widehat{\mathbb{k}}) \subset \widehat{\mathbb{k}}$. By $(\mathcal{A}_1)$ and the Lebesgue dominated convergence theorem, $\mho$ is continuous on $\widehat{\mathbb{k}}$.

Step 2 : We prove that $\mho$ satisfies the contractive condition.
Let F be any non-empty and convex subset of $\widehat{\mathbb{k}}$, and for each $\rho \in \Theta$, let $\alpha_\rho = \varkappa(F(\rho))$. Then $K \in L^\infty(\Theta, \mathbb{R}_+)$ and $h \in \mathfrak{X}$ where for a.e $\rho \in \Theta$,

$$\varkappa(\aleph(\rho, F(\rho))) \leq K(\rho)h(\alpha_\rho).$$

Therefore, for $\varepsilon \geq 0$, there is a continuous mapping $\zeta_\rho : \wp \to \mathfrak{W}$, with $\zeta_\rho(\wp) \subset \aleph(\rho, F(\rho))$, such that for all $\chi \in F$, there is $\eta \in \wp$ with

$$\|\aleph(\rho,\chi(\rho)) - \zeta_\rho(\eta)\|_{\mathfrak{W}} \leq K(\rho)h(\alpha_\rho) + \varepsilon, \text{ for a.e } \rho \in \Theta. \tag{5}$$

Construct now the mapping $\tilde{\zeta} : \wp \to ((C([-r,T],\mathfrak{W})), \|\cdot\|_\infty)$ as follows:

$$\eta \in \wp \to \tilde{\zeta}(\eta) = \Im(\rho,0)\varphi(0) + \int_0^\rho \Im(\rho,\mu)\zeta_\mu(\eta)d\mu, \text{ for a.e } \rho \in [-r,T].$$

So, $\tilde{\zeta}$ is continuous and $\tilde{\zeta}(\wp) \subset \mho(F)$. By (5), given $\chi \in F$ we have $\eta \in \wp$ where

$$\begin{aligned}
\|\mho\chi(\rho) - \tilde{\zeta}_\rho(\eta)\|_{\mathfrak{W}} &\leq \int_0^\rho \|\Im(\rho,\mu)\|_{B(\mathfrak{W})}\|\aleph(\mu,\chi(\mu)) - \zeta_\mu(\eta)\|d\mu \\
&\leq M\int_0^\rho (K(\mu)h(\alpha_\mu) + \varepsilon)d\mu.
\end{aligned}$$

Setting $\alpha := \varkappa(F)$, we can deduce that $h(\alpha_\rho) \leq h(\alpha)$ for a.e $\rho \in [-r,T]$, and

$$\begin{aligned}
\|\mho\chi(\rho) - \tilde{\zeta}_\rho(\eta)\|_{\mathfrak{W}} &\leq TM\|K\|_\infty h(\alpha) \\
&\leq h(\alpha).
\end{aligned}$$

Thus, from the arbitrariness of $\rho \in [-r,T]$, that $\varkappa(\mho F) \leq h(\alpha)$. □

Thus, by the Darbo fixed point theorem for the DND, the operator $\mho$ has a fixed point, which is a mild solution of (1).

4. Semi-Linear Neutral type Differential Equations with Finite Delay

Definition 4.1. We say that a function $\chi : [-r,T] \to \mathfrak{W}$ is a mild solution of problem (2), if χ satisfies the following integral equation

$$\begin{cases} \chi(\rho) = \Im(\rho,0)[\varphi(0) - g(0,\varphi(0))] + \int_0^\rho \Im(\rho,\mu)\aleph(\mu,\chi(\mu))d\mu, \text{ if } \rho \in \Theta \\ \chi(\rho) = \varphi(\rho), \text{ if } \rho \in [-r,0], \end{cases}$$

where $\Im(\cdot,\cdot)$ is an evolution system of linear bounded operators on $\mathfrak{W}$ generated by the closed linear operator $\mathfrak{Z}$.

Now, we assume the following hypotheses:

($\mathcal{B}_1$) The function $g : \Theta \times C([-r,0],\mathfrak{W}) \to \mathfrak{W}$ is Carathéodory, and there exist constants $L_1, L_2 > 0$ such that

$$\|g(\rho,\chi)\|_{\mathfrak{W}} \leq L_1\|\chi\|_{[-r,0]} + L_2, \text{ for } \rho \in \Theta, \ \chi \in C([-r,0],\mathfrak{W}).$$

($\mathcal{B}_2$) There exists $\grave{r} > 0$ such that

$$\grave{r} \geq M\left[\grave{r} + (L_1\grave{r} + L_2) + \psi(\grave{r})\|p_{\aleph}\|_{L^1}\right].$$

Theorem 4.1. *Assume that the conditions* $(\mathcal{A}_1) - (\mathcal{A}_3)$, $(\mathcal{B}_1) - (\mathcal{B}_2)$ *and (4) are satisfied. So, (2) has at least one solution defined on* Θ.

Proof. Let the operator $\mathcal{M} : C([-r,T],\mathfrak{W}) \to C([-r,T],\mathfrak{W})$ defined by:

$$\mathcal{M}\chi(\rho) = \begin{cases} \Im(\rho,0)[\varphi(0) - g(0,\varphi(0))] + \int_0^\rho \Im(\rho,\mu)\aleph(\mu,\chi(\mu))d\mu, & \text{if } \rho \in \Theta \\ \varphi(\rho), & \text{if } \rho \in [-r,0], \end{cases}$$

Step 1 : We prove $\mathcal{M}\widehat{\Bbbk} \subset \widehat{\Bbbk}$.
For any $\chi \in \widehat{\Bbbk}$ we obtain

$$\begin{aligned} \|\mathcal{M}\chi(\rho)\|_{\mathfrak{W}} &= \|\Im(\rho,0)[\varphi(0) - g(0,\varphi(0))] + \int_0^\rho \Im(\rho,\mu)\aleph(\mu,\chi(\mu))d\mu\|_{\mathfrak{W}} \\ &\leq \|\Im(\rho,0)\|_{B(\mathfrak{W})}\|\varphi(0) - g(0,\varphi(0)\|_{\mathfrak{W}} + \int_0^\rho \|\Im(\rho,\mu)\|_{B(\mathfrak{W})}\|\aleph(\mu,\chi_\mu)\|_{\mathfrak{W}}d\mu \\ &\leq M[\|\varphi(0)\|_{\mathfrak{W}} + \|g(0,\varphi(0))\|_{\mathfrak{W}}] + M\int_0^\rho p_{\aleph}(\mu)\psi(\|\chi\|_{[-r,0]})d\mu \\ &\leq M[\grave{r} + (L_1\grave{r} + L_2)] + M\psi(\grave{r})\|p_{\aleph}\|_{L^1} \\ &\leq \grave{r}. \end{aligned}$$

Thus $\mathcal{M}(\widehat{\Bbbk}) \subset \widehat{\Bbbk}$. Furthermore, combining assumption $(\mathcal{A}_1)$ and the Lebesgue dominated convergence theorem, $\mathcal{M}$ is continuous on $\widehat{\Bbbk}$.

Step 2 : Let $\digamma \subset \widehat{\Bbbk}$, and for each $\rho \in \Theta$, let $\alpha_\rho = \varkappa(\digamma(\rho))$. Thus, $K \in L^\infty(\Theta,\mathbb{R}_+)$ and $h \in \mathfrak{X}$ where for a.e $\rho \in \Theta$

$$\varkappa(\aleph(\rho,\digamma(\rho))) \leq K(\rho)h(\varkappa(\alpha_\rho)).$$

By the same technique of Step 2 in Theorem 3.1, we get:
$\tilde{\zeta}$ is continuous and $\tilde{\zeta}(\widetilde{\rho}) \subset \mathcal{M}(\digamma)$. By (5), given $\chi \in \digamma$ we can find $\eta \in \widetilde{\rho}$ where

$$\begin{aligned} \|\mathcal{M}\chi(\rho) - \tilde{\zeta}_\rho(\eta)\| &\leq \int_0^\rho \|\Im(\rho,\mu)\|_{B(\mathfrak{W})}\|\aleph(\mu,\chi(\mu)) - \zeta_\mu(\eta)\|d\mu \\ &\leq M\int_0^\rho K(\mu)h(\alpha_\mu) + \varepsilon d\mu. \end{aligned}$$

Setting $\alpha := \varkappa(\digamma)$, we can deduce that $h(\alpha_\rho) \leq h(\alpha)$ for a.e $\rho \in \Theta$, and

$$\begin{aligned} \|\mathcal{M}\chi(\rho) - \tilde{\zeta}_\rho(\eta)\| &\leq TM\|K\|_\infty h(\alpha) \\ &\leq h(\alpha). \end{aligned}$$

So, from the arbitrariness of $\rho \in \Theta$, that $\varkappa(\mathcal{M}\digamma) \leq h(\alpha)$.
Then χ is a fixed point of $\mathcal{M}$, which is a mild solution of (2). □

5. Integro-Differential Equations with Nonlocal Condition

Let us first consider the following linear Cauchy problem:

$$\begin{cases} \chi'(\rho) = \mathfrak{Z}(\rho)\chi(\rho) + \int_0^\rho \varpi(\rho,\mu)\chi(\mu)d\mu, & \text{for } \rho \geq 0, \\ \chi(0) = \chi_0 \in \mathfrak{W}. \end{cases} \tag{6}$$

Definition 5.1 ([22])**.** A resolvent operator for a Cauchy problem (6) is a bounded linear operator-valued function $\Im \in B(\mathfrak{W})$ for $\rho \geq 0$, verifying:
(1) $\Im(0) = I$ and $\|\Im(\rho)\|_{B(\mathfrak{W})} \leq Me^{\eta\rho}$ for $M > 0$ and $\eta \in \mathbb{R}$.
(2) For each $\chi \in \mathfrak{W}$, $\rho \to \Im(\rho)\chi$ is strongly continuous for $\rho \geq 0$.
(3) $\Im \in B(\mathfrak{W})$ for $\rho \geq 0$. For $\chi \in \mathfrak{W}$, $\Im(\cdot)\chi \in C^1(\mathbb{R}_+, \mathfrak{W}) \cap C(\mathbb{R}_+, \mathfrak{W})$ and

$$\begin{aligned} \Im'(\rho,\mu)\chi &= \mathfrak{Z}(\rho)\Im(\rho,0)\chi + \int_0^\rho \varpi(\rho,\mu)\Im(\mu)\chi d\mu \\ &= \Im(\rho,0)\mathfrak{Z}(\rho)\chi + \int_0^\rho \Im(\rho,\mu)\varpi(\mu)\chi d\mu, \end{aligned}$$

for $\rho \geq 0$.

From now on, we assume that:
$(Q1)$ The operator $\mathfrak{Z}$ is the infinitesimal generator of a uniformly continuous semigroup $\{T(\rho)\}_{\rho>0}$.
$(Q2)$ For $\rho \geq 0$, $\varpi(\rho,\mu)$ is closed linear operator from $\mathfrak{G}(\mathfrak{Z})$ to $\mathfrak{W}$ and $\varpi(\rho,\mu) \in B(\mathfrak{W})$. For any $\chi \in \mathfrak{W}$, the map $\rho \to \varpi(\rho,\mu)\chi$ is bounded, differentiable and the derivative $\rho \to \varpi'(\rho,\mu)\chi$ is bounded uniformly continuous on $\mathbb{R}_+$.

Theorem 5.1. *([21]) Assume that $(Q1) - (Q2)$ hold, then there exists a unique resolvent operator for the Cauchy problem (6).*

Definition 5.2. We say that a function $\chi(\cdot) \in C([-r,T],\mathfrak{W})$ is a mild solution of problem (3), if χ satisfies the following integral equation

$$\begin{cases} \chi(\rho) = \Im(\rho,0)[\chi_0 - g(0,\varphi(0)) - f(\chi)] + g(\rho,\chi_\rho) + \int_0^\rho \Im(\rho,\mu)\aleph(\mu,\chi(\mu))d\mu, & \rho \in \Theta, \\ \chi(\rho) = \varphi(\rho), \text{ if } \rho \in [-r,0], \end{cases}$$

Now, we assume the following hypotheses:
$(\mathcal{C}_1)$ The function $f : C([-r,T],\mathfrak{W}) \to \mathfrak{W}$ is continuous, and there exists a constant $\varsigma > 0$ such that

$$\|f(\chi)\|_{\mathfrak{W}} \leq \varsigma\|\chi\|_{C([-r,T],\mathfrak{W})}, \quad \chi \in C([-r,T],\mathfrak{W}).$$

$(\mathcal{C}_2)$ There exists $\sigma > 0$ such that

$$\sigma \geq M\left[\sigma(1 + L_1 + \varsigma) + L_2 + \psi(\sigma)\|p_\aleph\|_{L^1}\right] + L_1\sigma + L_2.$$

$(\mathcal{C}_3)$ There exist $\widehat{K} \in L^\infty(\Theta,\mathbb{R}_+)$ such that for any non-empty, bounded and convex subset $\Bbbk \subset \mathfrak{W}$,

$$\varkappa(g(\rho,\Bbbk)) \leq \widehat{K}(\rho)h(\varkappa(\Bbbk)),$$

holds for a.e. $\rho \in \Theta$.

Theorem 5.2. *Assume that the conditions $(\mathcal{A}_1)-(\mathcal{A}_3)$, $(\mathcal{B}_1)$, $(\mathcal{C}_1)-(\mathcal{C}_3)$ are satisfied and that*

$$TM\|K\|_\infty + \|\widehat{K}\|_\infty \leq 1. \tag{7}$$

Then, problem (3) has at least one solution defined on Θ.

Proof. Let the operator $\mathcal{Q} : C([-r,T],\mathfrak{W}) \to C([-r,T],\mathfrak{W})$ defined by:

$$\mathcal{Q}\chi(\rho) = \begin{cases} \Im(\rho,0)[\chi_0 - g(0,\varphi(0)) - f(\chi)] + g(\rho,\chi_\rho) + \int_0^\rho \Im(\rho,\mu)\aleph(\mu,\chi(\mu))d\mu, \ \rho \in \Theta, \\ \varphi(\rho), \ \text{if } \rho \in [-r,0], \end{cases}$$

We consider the bounded, closed and convex set

$$\overline{\Bbbk} = \left\{ \chi \in C([-r,T],\mathfrak{W}) : \|\chi\|_{C([-r,T]} \leq \sigma \right\}.$$

Step 1 : We prove that $\mathcal{Q}\overline{\Bbbk} \subset \overline{\Bbbk}$.
For any $\chi \in \overline{\Bbbk}$ we obtain

$$\begin{aligned}
\|\mathcal{Q}\chi(\rho)\|_{\mathfrak{W}} &= \|\Im(\rho,0)[\varphi(0) - g(0,\varphi(0)) - f(\chi)] + g(\rho,\chi_\rho) \\
&\quad + \int_0^\rho \Im(\rho,\mu)\aleph(\mu,\chi(\mu))d\mu\|_{\mathfrak{W}} \\
&\leq \|\Im(\rho,0)\|_{B(\mathfrak{W})}\|\varphi(0) - g(0,\varphi(0)) - f(\chi)\|_{\mathfrak{W}} \\
&\quad + \int_0^\rho \|\Im(\rho,\mu)\|_{B(\mathfrak{W})}\|\aleph(\mu,\chi_\mu)\|_{\mathfrak{W}}d\mu + \|g(\rho,\chi_\rho)\|_{\mathfrak{W}} \\
&\leq M[\|\varphi(0)\|_{\mathfrak{W}} + \|g(0,\varphi(0))\|_{\mathfrak{W}} + \varsigma\|\chi\|_{C([-r,T],\mathfrak{W})}] \\
&\quad + M\int_0^\rho p_\aleph(\mu)\psi(\|\chi\|_{[-r,0]})d\mu + L_1\|\chi\|_{[-r,0]} + L_2 \\
&\leq M[\sigma + (L_1\sigma + \varsigma\sigma + L_2)] + M\psi(\sigma)\|p_\aleph\|_{L^1} + L_1\sigma + L_2 \\
&\leq \sigma.
\end{aligned}$$

Thus $\mathcal{Q}(\overline{\Bbbk}) \subset \overline{\Bbbk}$. As in previous theorems, by $(\mathcal{A}_1)$ and the Lebesgue dominated convergence theorem, we deduce that $\mathcal{Q}$ is continuous on $\overline{\Bbbk}$.

Step 2 : Let $\digamma \subset \overline{\Bbbk}$, and for each $\rho \in \Theta$, let $\alpha_\rho = \varkappa(\digamma(\rho))$. Thus, $K \in L^\infty(\Theta,\mathbb{R}_+)$ and $h \in \mathfrak{X}$ where for a.e $\rho \in \Theta$ we have

$$\varkappa(\aleph(\rho,\digamma(\rho))) \leq K(\rho)h(\varkappa(\alpha_\rho)), \quad \text{and} \quad \varkappa(g(\rho,\digamma(\rho))) \leq \widehat{K}(\rho)h(\varkappa(\alpha_\rho)).$$

Therefore, for $\varepsilon \geq 0$, there is a continuous mapping $\zeta_\rho : \wp \to \mathfrak{W}$, with $\zeta_\rho(\wp) \subset \aleph(\rho,\digamma(\rho))$, such that for all $\chi \in \digamma$, there exist $\varepsilon_1, \varepsilon_2 > 0$ and $\eta \in \wp$ with

$$\|\aleph(\rho,\chi(\rho)) - \zeta_\rho(\eta)\|_{\mathfrak{W}} \leq K(\rho)h(\alpha_\rho) + \varepsilon_1, \tag{8}$$

and

$$\|g(\rho,\chi(\rho)) - \zeta_\rho(\eta)\|_{\mathfrak{W}} \leq \widehat{K}(\rho)h(\alpha_\rho) + \varepsilon_2, \ \text{for a.e } \rho \in \Theta. \tag{9}$$

Construct now the mapping $\tilde{\zeta} : \wp \to C([-r,T],\mathfrak{W})$ as follows:

$$\begin{aligned}
\eta \in \wp \to \tilde{\zeta}(\eta,\rho) &= \Im(\rho,0)[\varphi(0) - g(0,\varphi(0)) - f(\chi)] + \zeta_\rho(\eta) \\
&\quad + \int_0^\rho \Im(\rho,\mu)\zeta_\mu(\eta)d\mu, \ \text{for a.e } \rho \in [-r,T].
\end{aligned}$$

So, $\tilde{\zeta}$ is continuous and $\tilde{\zeta}(\wp) \subset \mho(F)$. By (8) and (9), given $\chi \in F$ we have $\eta \in \wp$ where

$$\|\mho\chi(\rho) - \tilde{\zeta}_\rho(\eta)\|_{\mathfrak{W}} \leq \|g(\rho,\chi(\rho)) - \zeta_\rho(\eta)\|_{\mathfrak{W}} + \int_0^\rho \|\Im(\rho,\mu)\|_{B(\mathfrak{W})}\|\aleph(\mu,\chi(\mu)) - \zeta_\mu(\eta)\|_{\mathfrak{W}} d\mu$$
$$\leq \widehat{K}(\rho)h(\alpha_\rho) + \varepsilon_2 + M\int_0^\rho (K(\mu)h(\alpha_\mu) + \varepsilon_1)d\mu.$$

Setting $\alpha := \varkappa(F)$, we can deduce that $h(\alpha_\rho) \leq h(\alpha)$ for a.e $\rho \in [-r,T]$, and

$$\|\mho\chi(\rho) - \tilde{\zeta}_\rho(\eta)\|_{\mathfrak{W}} \leq (TM\|K\|_\infty + \|\widehat{K}\|_\infty)h(\alpha)$$
$$\leq h(\alpha).$$

Thus, from the arbitrariness of $\rho \in [-r,T]$, that $\varkappa(\mho F) \leq h(\alpha)$. By the Darbo fixed point theorem for the DND, the operator $\mho$ has a fixed point, which is a mild solution of (3).

□

6. An Example

Consider the following class of partial differential system:

$$\begin{cases} \frac{\partial}{\partial\rho} z(\rho,\widetilde{\chi}) = \kappa(\rho,\widetilde{\chi})\frac{\partial}{\partial\widetilde{\chi}} z(\rho,\widetilde{\chi}) \\ \qquad -\frac{1}{1+e^\rho}\left(\frac{1}{1+\rho^2} + \ln(1+|z(\rho-r,\widetilde{\chi})|)\right) \text{ if } \rho\in\Theta=[0,1] \text{ and } \widetilde{\chi}\in(0,1), \\ z(\rho,0) = z(\rho,1) = 0, \quad \text{for } \rho \in \Theta, \\ z(0,\widetilde{\chi}) = e^{\widetilde{\chi}}, \quad \text{for } \widetilde{\chi}\in(0,1), \\ z(\rho,\widetilde{\chi}) = \varphi(\rho,\widetilde{\chi}), \quad \text{for } \rho\in[-r,0], \widetilde{\chi}\in(0,1), \end{cases} \tag{10}$$

where the function $\kappa(\cdot,\widetilde{\chi})$ is continuous for $\widetilde{\chi} \in [0,1]$ and $\kappa(\rho,\cdot)$ is uniformly Hölder continuous in $\delta \in \Theta$.

Let $\mathfrak{Z}$ be defined by

$$\mathfrak{Z}(\rho)\phi = \kappa(\rho,\widetilde{\chi})\phi',$$

and

$$\mathfrak{G}(\mathfrak{Z}) = \{z \in L^2(0,1) \ / \ z, \frac{\partial}{\partial\widetilde{\chi}} z \in L^2(0,1) \ ; \ z(0) = z(1) = 0\}.$$

The operator $\mathfrak{Z}$ is the infinitesimal generator of a C_0-semigroup on $L^2(0,1)$ with domain $\mathfrak{G}(\mathfrak{Z})$, the problem (10) has a resolvent operator $(\Im(\rho))_{\rho\geq 0}$ on $L^2(0,1)$ which is norm continuous.

Now, define

$$\chi(\rho)(\widetilde{\chi}) = z(\rho,\widetilde{\chi}),$$
$$\aleph(\rho,\chi(\rho))(\widetilde{\chi}) = \aleph(\rho,z(\rho))(\widetilde{\chi})$$

and $\aleph : \Theta \times L^2(0,1) \longrightarrow L^2(0,1)$ given by

$$\aleph(\rho,z(\rho))(\widetilde{\chi}) = \frac{1}{1+e^\rho}\left(\frac{1}{1+\rho^2} + \ln(1+|z(\rho-r,\widetilde{\chi})|)\right), \quad \text{for } \rho \in \Theta,$$

Now, for $\rho \in \Theta$, we have

$$\begin{aligned}\|\aleph(\rho, z(\rho))\|_{L^2} &= \|\frac{1}{1+e^{\rho}}\left(\frac{1}{1+\rho^2} + \ln(1+|z(\rho - r, \widetilde{\chi})|)\right)\|_{L^2} \\ &\leq \frac{1}{1+e^{\rho}}(1 + \|z(\rho, \widetilde{\chi})\|_{L^2}) \\ &\leq p_{\aleph}(\rho)\psi(\|z(\rho)\|_{L^2}).\end{aligned}$$

Therefore, assumption $(\mathcal{A}_1)$ is satisfied with

$$p_{\aleph}(\rho) = \frac{1}{1+e^{\rho}}, \ \rho \in \Theta \text{ and } \psi(\widetilde{\chi}) = 1 + \widetilde{\chi}, \ \widetilde{\chi} \in (0,1).$$

Now we shall check that condition of $(\mathcal{A}_4)$ is satisfied. Indeed, we have

$$\grave{r} \geq M + M(1+\grave{r}) + TML_1\grave{r}.$$

Thus

$$\grave{r} \geq \frac{M}{1 - 2M - TML_1}, \text{ where } 1 - 2M - TML_1 > 0.$$

For $\digamma \subset C(\Theta, L^2(0,1))$ and $\rho \in \Theta$ fixed, let ζ be an α_ρ-dense curve in $\digamma(\rho)$ for some $\alpha_\rho \geq 0$. Then, for $z \in \digamma$, there is $\eta \in \wp$ satisfying:

$$\|\chi(\rho) - \zeta(\eta, \rho)\|_{L^2} \leq \alpha_\rho.$$

Therefore, we have:

$$\begin{aligned}\|\aleph(\rho, z(\rho)) - \aleph(\rho, \zeta(\eta, \rho))\|_{L^2} &\leq \frac{1}{1+e^{\rho}}\|\ln(1+|z(\rho - r, \widetilde{\chi})|) - \ln(1+|\zeta(\eta, \rho - r)|)\|_{L^2} \\ &\leq \frac{1}{1+e^{\rho}}\left\|\ln\left(1 + \frac{|z(\rho - r, \widetilde{\chi}) - \zeta(\eta, \rho - r)|}{1+|\zeta(\eta, \rho - r)|}\right)\right\|_{L^2} \\ &\leq \frac{1}{1+e^{\rho}}\ln(1 + \|z(\rho - r, \widetilde{\chi}) - \zeta(\eta, \rho - r)\|_{L^2}) \\ &\leq \frac{1}{1+e^{\rho}}\ln(1+\alpha_\rho),\end{aligned}$$

and $h(\rho) = \ln(1+\rho)$. This function is continuous, and $h \in \mathfrak{X}$, so $(\mathcal{A}_4)$ is verified by $K(\rho) = \frac{1}{1+e^{\rho}}$. Consequently, all the hypotheses of Theorem 3.1 are satisfied and we conclude that the problem (10) has at least one solution $\chi \in C(\Theta, L^2(0,1))$.

References

[1] S. Abbas, M. Benchohra, *Advanced Functional Evolution Equations and Inclusions*, Developments in Mathematics **39**, Springer, Cham, 2015.

[2] A. Aghajani, J. Banaś, N. Sebzali, Some generalizations of Darbo fixed point theorem and applications. *Bull. Belg. Math. Soc. Simon Stevin* **20** (2013), 345-358.

[3] B. Balachandran, T. Kalmar-Nagy, D.E. Gilsinn, *Delay differential equations: Recent advances and new directions*, Springer, 2009.

[4] G. Ballinger, X. Liu, Boundedness for impulsive delay differential equations and applications in populations growth models, *Nonlinear Anal.* **53** (2003), 1041-1062.

[5] A. Bellen, M. Zennaro, *Numerical Methods for Delay Differential Equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003.

[6] N. Benkhettou, K. Aissani, A. Salim, M. Benchohra, C. Tunc, Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses, *Appl. Anal. Optim.* **6** (2022), 79-94.

[7] N. Benkhettou, A. Salim, K. Aissani, M. Benchohra, E. Karapınar, Non-instantaneous impulsive fractional integro-differential equations with state-dependent delay, *Sahand Commun. Math. Anal.* **19** (2022), 93-109. https://doi.org/10.22130/scma.2022.542200.1014

[8] A. Bensalem, A. Salim, B. Ahmad, M. Benchohra, Existence and controllability of integrodifferential equations with non-instantaneous impulses in Fréchet spaces, *CUBO. A Mathematical Journal* **25** (2023), no. 2, 231–250.

[9] A. Bensalem, A. Salim, M. Benchohra, Ulam-Hyers-Rassias stability of neutral functional integrodifferential evolution equations with non-instantaneous impulses on an unbounded interval, *Qual. Theory Dyn. Syst.* **22** (2023), Article no. 88.

[10] A. Bensalem, A. Salim, M. Benchohra, J. J. Nieto, Controllability results for second-order integro-differential equations with state-dependent delay. *Evol. Equ. Control Theory.* **12** (2023), no. 6, 1559-1576. http://dx.doi.org/10.3934/eect.2023026

[11] L. Byszewski, Existence and uniqueness of a classical solution to a functional-differential abstract nonlocal Cauchy problem, *Appl. Math. Stoch. Anal.* **12** (1999), 91-97.

[12] Y. Cherruault, A. Guillez, Une méthode pour la recherche du minimum global d'une fonctionnelle, *C. R. Acad. Sci. Paris Sér. I Math.* **296** (1983), 175-178.

[13] Y. Cherruault, G. Mora, *Optimisation Globale, Théorie des Courbes α-Denses*, Economica, Paris, France, 2005.

[14] M. A. Diop, M. Dieye, H. Hmoyed, K. Ezzinbi, On the existence of mild solutions for nonlocal impulsive partial integrodifferential equations in Banach spaces, *Le Matematiche* **74** (2019), no. 1, 13-34.

[15] T. Erneux, *Applied Delay Differential Equations, Surveys and Tutorials in the Applied Mathematical Sciences*, Springer, New York, 2009.

[16] G. García, G. Mora, The degree of convex nondensifiability in Banach spaces, *J. Convex Anal.* **22** (2015), 871-888.

[17] G. García, G. Mora, A fixed point result in Banach algebras based on the degree of nondensifiability and applications to quadratic integral equations, *J. Math. Anal. Appl.* **472** (2019), 1220-1235.

[18] G. García, A quantitative version of the Arzelà-Ascoli theorem based on the degree of nondensifiability and applications, *Appl. Gen. Topol.* **20** (2) (2019), 265-279.

[19] G. García, Existence of solutions for infinite systems of differential equations by densifiability techniques, *Filomat* **32** (2018), 3419-3428.

[20] G. García, Solvability of an initial value problem with fractional order differential equations in Banach space by α-dense curves, *Fract. Calc. Appl. Anal.* **20** (2017), 646–661.

[21] R. Grimmer, Resolvent opeators for integral equations in a Banach space, *Trans. Amer. Math. Soc.* **273** (1982), 333-349.

[22] R. Grimmer, A.J. Pritchard, Analytic resolvent operators for integral equations in a Banach space, *J. Differential Equations* **50** (1983), 234-259.

[23] V. Kolmanovskii, A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations,* Kluwer Academic Publishers, Dordrecht, 1999.

[24] G. Mora, Y. Cherruaul, Characterization and generation of α-dense curves, *Comput. Math. Appl.* **33** (1997), 83-91.

[25] G. Mora, J. A. Mira, α-dense curves in infinite dimensional spaces, *Int. J. Pure Appl. Math.* **5** (2003), 437-449.

[26] G. Mora, D. A. Redtwitz, Densifiable metric spaces, *Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Math.* **105** (2011), 71-83.

[27] S. Rezapour, K.S. Vijayakumar, H.R. Henriquez, V. Nisar, A. Shukla, A Note on existence of mild solutions for second-order neutral integro-differential evolution equations with state-dependent delay, *Fractal Fract.* **5** (2021), no. 3, 1-17.

[28] V. Volterra, Sur la Théorie Mathématiques des phénomènes héréditaires, *J.de Mathématiques pures et apliquées* **7** (1928), 249-298.

[29] X. Xue, Existence of solutions for semilinear nonlocal Cauchy problems in Banach spaces, *Electron. J. Differential Equations* **64** (2005), 1-7.

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