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A Parabolic Problem of Kirchhoff Type with Gradient Term and Nonlinear Boundary Condition

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Abstract. The object of this work is to study the existence of weak solutions for $(p_1(x), p_2(x))$-Laplacian parabolic Kirchhoff equation. We apply degree theory to operators of the type $T+S+C$, where T is maximal monotone, S is bounded pseudomonotone, and C is compact with $D(T) \subseteq D(C)$ and satisfies a sublinearity condition, to get our result within the context of Sobolev spaces with variable exponents.

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1. Introduction

In this research, we focus on the following nonlocal parabolic problem

$$
\begin{aligned}
& u_t - M_1(L_1(u))(\operatorname{div}(|\nabla u|^{p_1(x)-2}\nabla u) - |u|^{p_1(x)-2}u) \\
& \quad - M_2(L_2(u))(\operatorname{div}(|\nabla u|^{p_2(x)-2}\nabla u) - |u|^{p_2(x)-2}u) + f(x,t,u,\nabla u) = h(x,t), \\
& \qquad\qquad \text{in } Q = \Omega \times (0,T), \\
& \Big(M_1(L_1(u))|\nabla u|^{p_1(x)-2} + M_2(L_2(u))|\nabla u|^{p_2(x)-2}\Big)\frac{\partial u}{\partial \nu} + g(x,u) = 0, \\
& \qquad\qquad \text{on } \Sigma = \partial\Omega \times (0,T), \\
& u(x,0) = 0, \qquad x \in \Omega,
\end{aligned} \tag{1}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$, with a smooth boundary $\partial\Omega$, T is a fixed positive number, $p_i(x) \in C(\overline{\Omega})$ with $p_i(x) > 1$ for any $x \in \overline{\Omega}$ i=1,2 , $L_i(u) = \int_\Omega \frac{1}{p_i(x)}(|\nabla u|^{p_i(x)} + |u|^{p_i(x)})\,dx$, and M_i, f are functions that satisfy conditions which will be stated later.

It was in 1883 that Kirchhoff proposed the famous model

$$
\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}
$$

which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. When $M_1 = M_2 \equiv M, p_1 = p_2 = p$(constante) and $f(x,t,u,\nabla u) \equiv f(x,t,u)$, the problem (1) can be used to describe the motion of a nonstationary fluid or gas in a nonhomogeneous and

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anisotropic medium, and the nonlocal term M can describe a possible change in the global state of the fluid or gas caused by its motion in the considered medium, see [24]. The flux through the boundary, that depends nonlinearly on the function u, is described by the nonlinear boundary condition. In mathematical biology, one way to conceptualize problem (1) is as a model of bacterial spreading, where u represents the population density at the point (x,t) and the external source f represents the processes of birth and death, whose rates depend on the population. Owing to their numerous physical applications, PDEs with variable exponents have garnered increased attention in recent years due to their various physical applications. Indeed, they are capable of modeling a wide range of phenomena that come up in the study of thermo-rheological fluids [32, 33, 34], image restoration [14, 27], and electro-rheological fluids [32, 33, 34]. Numerous researchers have dedicated a significant amount of time to studying parabolic problems involving variable exponent growth condition, we refer the readers to [16, 21, 25, 35].

The nonlocal $(p_1(x), p_2(x))$-Laplace operator

$$\mathcal{L}(u) = -\sum_{i=1}^{2} M_i(L_i(u))(\mathrm{div}(|\nabla u|^{p_i(x)-2}\nabla u) - |u|^{p_i(x)-2}u).$$

is linked to so-called double-phase problems. In 1986, Zhikov [40] introduced for the first time in the literature a energy functional related to the double phase operator. For elliptic problems, with double phase operators along with a nonlinear boundary condition see [3, 28, 29]. As far as the $(p_1(x), p_2(x))$-Laplacian parabolic equations are concerned, few articles have appeared, we refer the reader to [4, 8, 13]. To solve quasilinear parabolic equations, most authors use techniques such as the theory of nonlinear semigroups, the discretization method, De Giorgi iteration technique, subdifferential calculus, theory of monotone operators, fixed point theorems and the classic Galerkin method. Differently from the above mentioned methods, in the present paper, to establish our main result we use the degree theory developed by Asfaw [6] for operators of type $T + S + C$, where T is a maximal monotone, S is bounded pseudomonotone and C is is compact con $D(T) \subseteq D(C)$ and satisfies a sublinearity condition. This is, as far as we are aware, the first attempt to solve a nonlocal $(p_1(x), p_2(x))$-Laplace parabolic problem with a nonlinear boundary condition and convection term, especially with degree theory.

We point out that the degree theory is one of the main tools for checking the solution existence of nonlinear elliptic PDEs, even in spaces of variable exponents, without resorting to usual variational methods. However, recently several scholars have implemented this methodology to solve evolution equations, see [22, 23, 31, 36, 37].

We provide the following organizational structure for this article. Section 2 includes some preliminary information on functional spaces of evolution and variable exponent Sobolev spaces, as well as the Asfaw topological degree needed for the proof of our main result. Section 3 contains the technical Lemmas and fundamental assumptions. The final section focuses on stating and demonstrating our main result about existence of weak solutions for problem (1).

2. Preliminaries

We need some theory on $W^{1,p(x)}(\Omega)$ also known as variable exponent Sobolev space, in order to study problem (1.1). (for more information, see [18]). $\mathbf{S}(\Omega)$ represents the set of all real functions that are measurable and defined on Ω. When two functions in $\mathbf{S}(\Omega)$ are equal almost everywhere, they are regarded as the same element of $\mathbf{S}(\Omega)$.

Let

$$C_+(\overline{\Omega}) = \{w : w \in C(\overline{\Omega}), w(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$w^- := \min_{\overline{\Omega}} w(x), \quad w^+ := \max_{\overline{\Omega}} w(x) \quad \text{for every } w \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{v \in \mathbf{S}(\Omega) : \int_\Omega |v(x)|^{p(x)}\, dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|v|_{L^{p(x)}(\Omega)} = |v|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{v(x)}{\lambda}|^{p(x)}\, dx \le 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{v \in L^{p(x)}(\Omega) : |\nabla v| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|v\|_{1,p(x),\Omega} := \|v\|_{W^{1,p(x)}(\Omega)} = |v|_{L^{p(x)}(\Omega)} + |\nabla v|_{L^{p(x)}(\Omega)}.$$

Proposition 2.1 ([18]). *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2 ([18]). *Set $\rho(v) = \int_\Omega |v(x)|^{p(x)}\, dx$. For any $v \in L^{p(x)}(\Omega)$, then*
(1) *for $v \neq 0$, $|v|_{p(x)} = \lambda$ if and only if $\rho(\frac{v}{\lambda}) = 1$;*
(2) *$|v|_{p(x)} < 1$ $(= 1; > 1)$ if and only if $\rho(v) < 1$ $(= 1; > 1)$;*
(3) *if $|v|_{p(x)} > 1$, then $|v|_{p(x)}^{p^-} \le \rho(v) \le |v|_{p(x)}^{p^+}$;*
(4) *if $|v|_{p(x)} < 1$, then $|v|_{p(x)}^{p^+} \le \rho(v) \le |v|_{p(x)}^{p^-}$;*
(5) *$\lim_{k\to+\infty} |v_k|_{p(x)} = 0$ if and only if $\lim_{k\to+\infty} \rho(v_k) = 0$;*
(6) *$\lim_{k\to+\infty} |v_k|_{p(x)} = +\infty$ if and only if $\lim_{k\to+\infty} \rho(v_k) = +\infty$.*

Proposition 2.3 ([19, 18]). *If $q \in C_+(\overline{\Omega})$ and $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.4 ([38]). *If $q \in C_+(\partial\Omega)$ and $q(x) \le p^\partial(x)$ $(q(x) < p^\partial(x))$ for $x \in \partial\Omega$, then there is a continuous (compact) embedding $W^{1,p(x)}(\partial\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.5 ([15]). *For any $u \in W^{1,p(x)}(\Omega)$, let*

$$\|u\|_\partial := |u|_{L^{p(x)}(\partial\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Then $\|u\|_\partial$ is a norm in $W^{1,p(x)}(\Omega)$, which is equivalent to $\|u\|_{1,p(x),\Omega}$.

Proposition 2.6 ([20, 18]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)}+\frac{1}{p(x)}=1$ holds a.e. in Ω. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\Big|\int_\Omega uv\,dx\Big|\le(\frac{1}{p^-}+\frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

As in [9], we extend a variable exponent $p:\overline{\Omega}\to[1,+\infty[$ to $\overline{Q}\to[1,+\infty[$ by setting $p(x,t)=p(x)$ for all $(x,t)\in\overline{Q}$. Now, we may consider the generalized Lebesgue space

$$L^{p(x)}(Q)=\{u:Q\to\mathbb{R}\quad\text{measurable such that }\int_Q|u(x,t)|^{p(x)}\,dxdt<\infty\}$$

which has similar properties to those of space $L^{p(x)}(\Omega)$.

Consider X a real reflexive Banach space, and X' its dual space; for each $x\in X$ and $x'\in X'$, the value $x'(x)$ is denoted by $\langle x',x\rangle$. We shall use the standard notation for Bochner spaces i.e. $L^r(0,T;X)$ is the space of strongly measurable function $u:\,]0,T[\to X$ for which $t\to\|u\|_X\ \in L^r(0,T)$, $r\ge1$.
We denote

$$w_M(x)=\max\{w_1(x),w_2(x)\}\quad,w_m(x)=\min\{w_1(x),w_2(x)\}$$

It is easy to prove that $w_M,w_m\in C_+(\overline{\Omega})$.

Next, we consider the space $X:=W^{1,p_1(x)}(\Omega)\bigcap W^{1,p_2(x)}(\Omega)$, equipped with the norm

$$\|u\|_X=\|u\|_{1,p_1(x),\Omega}+\|u\|_{1,p_2(x),\Omega}\text{ for all }\ u\in X$$

The space $W(Q)$ is defined as follows

$$W(Q)=\left\{u:[0,T]\to\cap_{i=1}^2W^{1,p_i(x)}(\Omega);u\in L^{p_i(x)}(Q):|\nabla u|\in L^{p_i(x)}(Q),i=1,2\right\},$$

where ∇u stands for the gradient of u with respect to the space variable x. It is is a Banach equipped with the norm

$$\|u\|:=\|u\|_{W(Q)}:=\sum_{i=1}^2\left(\|u\|_{L^{p_i(x)}(Q)}+\|\nabla u\|_{L^{p_i(x)}(Q)}\right),\quad\text{for all }u\in W(Q).$$

Suppose that

$$1+\frac{N}{N+1}<p_i(x)<p_M^*(x),\quad i=1,2.\tag{3}$$

Since $1+\frac{N}{N+1}-\frac{2N}{N+2}>0$, then $p_i(x)>\frac{2N}{N+2}$ which implies

$$W^{1,p_i(x)}(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow\left(W^{1,p_i(x)}(\Omega)\right)'\quad i=1,2,$$

where these embeddings are dense.

Remark 2.1. i)$(X,\|\cdot\|)$ is a reflexive and separable Banach space.
ii) For $q(x)\in C_+(\overline{\Omega})$ such that $q(x)<p_M^*(x)$ for any $x\in\overline{\Omega}$, we have $X:=W^{1,p_1(x)}(\Omega)\bigcap W^{1,p_2(x)}(\Omega)=W^{1,p_M(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$, the embedding is continuous and compact.
iii) By (3) and Theorem 1.5 in [5] $W(Q)\hookrightarrow L^{\hat{p}(x)}(Q)$ is compact, where $\hat{p}(x):=\max\{2,p_M(x)\}$.

We need the following concepts to state a fundamental theorem for our work.

For an operator $T : X \to 2^{X'}$ the graph of T, denoted $G(T)$, is defined by

$$G(T) = \{(x, y) : x \in D(T), y \in Tx\}$$

where $D(T)$ denotes the domain of T, given by $D(T) = \{x \in X : Tx \notin \emptyset\}$. We shall use "$\rightharpoonup$" ("$\to$") for the weak (strong) convergence.

An operator $T : X \supset D(T) \to 2^{X'}$ is said to be : a) monotone if for every $x, y \in D(T)$ and every $u \in Tx$ and $v \in Ty$ we have $\langle u - v, x - y\rangle \geq 0$, b) maximal monotone if T is monotone and $G(T)$ is maximal in $X \times X'$ when $X \times X'$ is partially ordered by the set inclusion, c) coercive if either $D(T)$ is bounded or there exists a function $\psi : [0, \infty[\to \mathbb{R}$ such that $\psi(t) \to \infty$ as $t \to \infty$ and

$$\langle u, x\rangle \geq \psi(\|x\|)\|x\|, \quad \forall x \in D(T), u \in Tx.$$

We observe that , in this setting, T is maximal monotone if and only if $R(T+\lambda I) = X'$ for all $\lambda \in [0, +\infty[$.

We recall that $T : X \supset D(T) \to Y$, where Y is a Banach space, is said to be: a) bounded if it maps bounded subsets of $D(T)$ onto bounded subsets of Y, b) compact if it maps bounded subsets of $D(T)$ onto relatively compact subsets in Y, and c) of type (S_+) if for any $(u_\nu) \subset D(T)$ with $u_\nu \rightharpoonup u$ and $\limsup\langle Tu_\nu, u_\nu - u\rangle \leq 0$, we have $u_\nu \to u$.

The following theorem is a generalization of the existence result due to Asfaw and Kartsatos [7] for the sum of two operators T and S, and yields the surjectivity of operators of type $T + S + C$.

Theorem 2.7 (Asfaw T. [6]). *Let X be real Banach space. Let $T : X \supseteq D(T) \to X'$ be maximal monotone with $0 \in T(0)$, $S : X \to X'$ be bounded and of type (S_+), and $C : D(C) \to X'$ be compact with $D(T) \subseteq D(C)$ and belonging to the class Γ^τ_σ (i.e there exist $\sigma \geq 0$ and $\tau \geq 0$ such that $\|Cx\| \leq \tau\|x\| + \sigma$ for all $x \in D(C)$). Assume, further that $T + S + C$ is coercive. Then $T + S + C$ is surjective.*

3. Hypotheses and technical Lemmas

First, we impose the conditions that allow us to achieve the solutions of problem (1). Concerning the functions M_1, M_2, we suppose that

(M_0) $M_i : [0, +\infty[\to]m_0, m_1[$ $(i = 1, 2)$ are continuous and nondecreasing functions with $m_0, m_1 > 0$.

Now, we can give the properties of the nonlocal $(p_1(x), p_2(x))$-Laplace operator

$$\mathcal{L}(u) = -\sum_{i=1}^{2} M_i(L_i(u))(\mathrm{div}(|\nabla u|^{p_i(x)-2}\nabla u) - |u|^{p_i(x)-2}u).$$

Consider the following functional $S_i : W(Q) \to W'(Q), i = 1, 2$ ($W'(Q)$ is the dual of $W(Q)$) given by

$$\langle S_i(u), v\rangle = \int_Q M_i(L_i(u)) \left(|\nabla u|^{p_i(x)-2}\nabla u \cdot \nabla v + |u|^{p_i(x)-2}uv\right) dxdt,$$

then $\mathcal{L} : W(Q) \to W'(Q)$ and

$$\langle \mathcal{L}(u), v\rangle := \sum_{i=1}^{2} \langle S_i(u), v\rangle, \ \forall u, v \in W(Q). \tag{4}$$

Consider the following functional

$$\Phi(u) = \int_0^T \widehat{M_1}\left(L_1(u)\right) dt + \int_0^T \widehat{M_2}\left(L_2(u)\right) dt := \Phi_1(u) + \Phi_2(u), \quad \forall u \in W(Q)$$

where $\widehat{M_i}(s) = \int_0^s M_i(t)\, dt$ for all $s \in [0, +\infty[$. Thus, it is simple to prove that $\Phi_i, i = 1, 2$ are well defined and continuously Gâteaux differentiable, and their Gâteaux derivatives at point $u \in W(Q)$ are given by

$$\langle \Phi_i'(u), v\rangle = \langle S_i(u), v\rangle$$

Thus

$$\langle \Phi'(u), v\rangle = \sum_{i=1}^{2} \langle S_i(u), v\rangle, \quad \text{for all } u, v \in W(Q).$$

Hence, the nonlocal $(p_1(x), p_2(x))$-Laplace operator is the derivative operator of Φ in the weak sense. Also it is obvious that Φ' is continuous.

Remark 3.1. By Theorem 2.1 in [17] we know that, for every $t \in]0, T[$ fixed, the operator $S_i : X \to X'$ is a homeomorphism strictly monotone and of type (S_+). Moreover, proceeding as in the proof of Theorem 2.2 in [28], we get that the operator $\sum_{i=1}^2 S_i : X \to X'$ is continuous, bounded, strictly monotone and of type (S_+).

Lemma 3.1. *If M satisfies (M_0), then*

(i) *$\mathcal{L} : W(Q) \to W'(Q)$ is a continuous, bounded and strictly monotone operator;*

(ii) *$\mathcal{L}$ is of type (S_+), i.e. if $u_\nu \rightharpoonup u$ in $W(Q)$ and*

$$\limsup_{\nu\to+\infty} \langle \mathcal{L}(u_n) - \mathcal{L}(u), u_\nu - u\rangle = 0, \ \text{then } u_\nu \to u \text{ in } W(Q);$$

Proof. i) We see that $\mathcal{L}$ is continuous because $\mathcal{L} = \Phi'$. Now, we prove that $\mathcal{L}$ is bounded. Let $\mathcal{B}$ be a bounded subset in $W(Q)$. It is obvious that $\{M_i(L_i(u)) : u \in \mathcal{B}\}$, $i = 1, 2$ is bounded, since M_i is continuous. Also if $u, v \in \mathcal{B}$, we have

$$\begin{aligned}
|\langle \mathcal{L}u, v\rangle| &= |\sum_{i=1}^{2} \int_Q M_i(L_i(u)) \left(|\nabla u|^{p_i(x)-2} \nabla u \cdot \nabla v + |u|^{p_i(x)-2} uv\right) dxdt| \\
&\le m_1 \sum_{i=1}^{2} \int_Q \left(|\nabla u|^{p_i(x)-1} |\nabla v| + |u|^{p_i(x)-1}|v|\right) dxdt \\
&\le 2m_1 \sum_{i=1}^{2} \left(\left\| |\nabla u|^{p_i(x)-1} \right\|_{p_i'(x),Q} \|\nabla v\|_{p_i(x),Q} + \left\| |u|^{p_i(x)-1} \right\|_{p_i'(x),Q} \|v\|_{p_i(x),Q}\right) \\
&\le C\left(\|u\|^{p_M^+} + 1\right) \|v\|
\end{aligned}$$

So, $\mathcal{L}$ is a bounded operator. Next, we will prove that $\mathcal{L}$ is a strictly monotone operator. Let $t \in]0,T[$, then from Remark 3.1 it follows that

$$\int_\Omega \big[(M_i(L_i(u))|\nabla u|^{p_i(x)-2}\nabla u - M_i(L_i(v))|\nabla v|^{p_i(x)-2}\nabla v)\cdot(\nabla u - \nabla v)$$
$$+ (|u|^{p_i(x)-2}u - |v|^{p_i(x)-2}v)(u-v)\big]\,dx \ge 0 \qquad \text{for } i=1,2. \qquad \forall u(.,t), v(.,t) \in X.$$

From this inequality, integrating over $[0,T]$ we obtain

$$\langle \mathcal{L}(u) - \mathcal{L}(v), u-v\rangle = \sum_{i=1}^{2}\langle S_i(u) - S_i(v), u-v\rangle$$
$$= \sum_{i=1}^{2}\int_0^T\int_\Omega \big[(M_i(L_i(u))|\nabla u|^{p_i(x)-2}\nabla u - M_i(L_i(v))|\nabla v|^{p_i(x)-2}\nabla v)\cdot(\nabla u - \nabla v)$$
$$+ (|u|^{p_i(x)-2}u - |v|^{p_i(x)-2}v)(u-v)\big]\,dxdt \ge 0, \qquad \forall u, v \in W(Q).$$

Next, we show that the operator $\mathcal{L}$ is of type (S_+). Let $(u_\nu)_\nu$ be a sequence in $W(Q)$ such that $u_\nu \rightharpoonup u$ and $\limsup_{\nu\to\infty}\langle \mathcal{L}(u_\nu) - \mathcal{L}(u), u_\nu - u\rangle \le 0$. Then

$$\lim_{\nu\to\infty}\langle \mathcal{L}(u_\nu) - \mathcal{L}(u), u_\nu - u\rangle = 0. \tag{5}$$

Now, proceeding similarly as in [17], we get

$$\begin{aligned}&\langle \mathcal{L}(u) - \mathcal{L}(v), u-v\rangle\\ &\ge m_0\sum_{i=1}^{2}\Big[\int_Q \frac{1}{2}(|\nabla u|^{p_i(x)-2} - |\nabla v|^{p_i(x)-2})(|\nabla u|^2 - |\nabla v|^2)\,dxdt \\ &+ \int_Q \frac{1}{2}(|u|^{p_i(x)-2}u - |v|^{p_i(x)-2}v)(u-v)\Big]\,dxdt \ge 0, \qquad \forall u, v \in W(Q).\end{aligned} \tag{6}$$

From (5) and (6), ∇u_ν converges in measure to ∇u and u_ν converges in measure to u in Q, thus we may find a subsequence (which we still denote by u_ν) satisfying $\nabla u_\nu \to \nabla u,\ u_\nu \to u,$ a.e on Q. Thanks to Fatou's Lemma we obtain

$$\liminf_{\nu\to\infty}\int_Q \frac{1}{p_i(x)}(|\nabla u_\nu|^{p_i(x)} + |u_\nu|^{p_i(x)})\,dxdt \ge \int_Q \frac{1}{p_i(x)}(|\nabla u|^{p_i(x)} + |u|^{p_i(x)})\,dxdt, \tag{7}$$

for i=1,2.

Since $u_\nu \rightharpoonup u$ we have

$$\lim_{\nu\to\infty}\langle \mathcal{L}(u_\nu), u_\nu - u\rangle = \lim_{\nu\to\infty}\langle \mathcal{L}(u_\nu) - \mathcal{L}(u), u_\nu - u\rangle = 0. \tag{8}$$

Moreover, after some calculations, according to (M_0), we obtain

$$\begin{aligned}\langle \mathcal{L}(u_\nu), u_\nu - u\rangle &\ge \sum_{i=1}^{2} M_i(L_i(u_\nu))\Big[\int_Q \frac{1}{p_i(x)}|\nabla u_\nu|^{p_i(x)}\,dxdt - \int_Q \frac{1}{p_i(x)}|\nabla u|^{p_i(x)}\,dxdt\Big]\\ &+ \sum_{i=1}^{2} M_i(L_i(u_\nu))\Big[\int_Q \frac{1}{p_i(x)}|u_\nu|^{p_i(x)}\,dxdt - \int_Q \frac{1}{p_i(x)}|u|^{p_i(x)}\,dxdt\Big]\\ &\ge m_0\sum_{i=1}^{2}\Big[\int_Q \frac{1}{p_i(x)}(|\nabla u_\nu|^{p_i(x)} + |u_\nu|^{p_i(x)})\,dxdt - \int_Q \frac{1}{p_i(x)}(|\nabla u|^{p_i(x)} + |\nabla u|^{p_i(x)})\,dxdt\Big]\end{aligned} \tag{9}$$

In view of (7)-(9) we get

$$\lim_{\nu\to\infty}\int_Q \frac{1}{p_i(x)}(|\nabla u_\nu|^{p_i(x)}+|u_\nu|^{p_i(x)})\,dxdt=\int_Q \frac{1}{p_i(x)}(|\nabla u|^{p_i(x)}+|u|^{p_i(x)})\,dxdt$$

for i=1,2,

which implies that the functions $\left\{\frac{1}{p_i(x)}(|\nabla u_\nu|^{p_i(x)}+|u_\nu|^{p_i(x)}\right\}_\nu, i=1,2$ have equi-absolutely continuous integrals. So, by Vitali's theorem we have

$$\lim_{\nu\to\infty}\int_Q (|\nabla u_\nu-\nabla u|^{p_i(x)}+|u_\nu-u|^{p_i(x)})\,dxdt=0 \qquad \text{for } i=1,2.$$

Therefore, by proposition 2.2, we conclude that $u_\nu\to u$ in $W(Q)$. □

Now we deal with the properties for the superposition operator induced by the functions f and g in (1). We assume that

(F_1) $f:\Omega\times(0,T)\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}$ is a Carathêodory function such that
i)$|f(x,t,\eta,\xi)|\le c_1(|\eta|+|\xi|)+k(x,t),\quad k\in L^\infty(0,T;L^{p(x)}(\Omega))$;
ii) $f(x,t,\eta,\xi)\eta\ge|\eta|^{r(x)}$, for all $(x,t)\in\Omega\times(0,T),\eta\in\mathbb{R}$ and $\xi\in\mathbb{R}^n$, $1\le r(x)\le 2$.

(G_1) $g:\Omega\times(0,T)\times\mathbb{R}\times\mathbb{R}^n\longrightarrow\mathbb{R}$ is a Carathêodory function such that
i)$|g(x,t,u)|\le c_2|u|+k_2(x,t)\quad$ for a.e. $(x,t)\in\Gamma\times(0,T)$, $k_2\in L^\infty(0,T;L^{p(x)}(\Sigma))$;
ii) $g(x,t,u)u\ge|u|^{s(x)},\ 1\le s(x)\le 2$.

Lemma 3.2. *Under assumptions $(F_1)-(G_1)$ the operator $C:W(Q)\supseteq D(C)\to W'(Q)$ given by*

$$\langle Cu,v\rangle=\int_Q f(x,t,u,\nabla u)v\,dxdt+\int_\Sigma g(x,t,u)v\,d\sigma dt \tag{10}$$

where $u\in D(C)=\{z\in W(Q):z_t\in W'(Q)\}$, is continuous and compact.

Proof. Let $C_1:W(Q)\to L^2(Q)$ and $C_2:W(Q)\to L^2(\Sigma)$ be two operators defined by

$$C_1(u)=f(x,t,u,\nabla u)\quad\text{and}\quad C_2(u)=g(x,t,u).$$

For any $u\in W(Q)$, let $u_\nu\to u$ in $W(Q)$. So, up to a subsequence, we have

$$\begin{cases} u_\nu\to u \quad\text{and}\quad \nabla u_\nu\to\nabla u \quad \text{a.e in } Q,\\ |u_\nu|\le l_1(x,t)\quad\text{and}\quad |\nabla u_\nu|\le l_2(x,t)\quad \text{a.e in} Q,\\ u_\nu\to u\quad\text{and}\quad |u_\nu|\le l_3(x,t)\quad\text{a.e in}\Sigma,\end{cases} \tag{11}$$

for some $l_1,l_2\in L^2(Q)$ and $l_3\in L^2(\sum)$. Thus, since f and g are Carathéodory functions, we get

$$\begin{cases} f(x,t,u_\nu,\nabla u_\nu)\to f(x,t,u,\nabla u)\quad\text{a.e in } Q\\ g(x,t,u_\nu)\to g(x,t,u)\quad\text{a.e in }\Sigma\end{cases} \tag{12}$$

From (F_1) *i)* and (G_1) *i)* it follows, respectively

$$\begin{cases} |f(x,t,u_\nu,\nabla u_\nu)|\le c_1\left(|l_1(x,t)|+|l_2(x,t)|\right)+|k(x,t)|\quad\text{a.e in } Q\\ |g(x,t,u_\nu)|\le c_2|l_3(x,t)|\quad\text{a.e in }\Sigma\end{cases} \tag{13}$$

Since

$$c_1\left(|l_1(x,t)|+|l_2(x,t)|\right)+|k(x,t)|\in L^2(Q) \qquad \text{and} \qquad c_2|l_3(x,t)|\in L^2(\Sigma),$$

we conclude with the dominated convergence theorem that

$$C_1(u_\nu)\to C_1(u) \quad \text{in } L^2(Q) \qquad \text{and} \qquad C_2(u_\nu)\to C_2(u) \quad \text{in } L^2(\Sigma).$$

Therefore, the entire sequences $(C_1(u_\nu), C_2(u_\nu))$ converges to $(C_1(u), C_2(u))$ in $L^2(Q)\times L^2(\Sigma)$. Thus, $G := (C_1, C_2)$ is contimuous on $W(Q)$.

Recall that the embedding $I : W(Q)\to L^2(Q)\times L^2(\Sigma)$ is continuous and compact (See Remark 2.1 iii)) and so the adjoint operator $I^* : L^2(Q)\times L^2(\Sigma)\to W'(Q)$ given by

$$\langle I^*(f,g),\varphi\rangle = \int_Q f\varphi\, dxdt + \int_\Sigma g\varphi\, d\sigma dt$$

is also compact. Therefore $C = I^*\circ G : W(Q)\supseteq D(C)\to W'(Q)$ is continuous and compact. ☐

Lemma 3.3. *Let $T : W(Q)\supseteq D(T)\to W'(Q)$ be defined by*

$$\langle Tu, v\rangle = -\int_Q uv_t\, dxdt, \quad \textit{for all } u\in D(T), v\in W(Q), \tag{14}$$

where $D(T) == \{z\in W(Q) : z_t\in W'(Q), z(0) = 0\}$. The linear operator T is generated by $(.)_t\equiv \partial/\partial t$ via the relation

$$\langle Tu, v\rangle = \int_0^T \langle u_t(t), v(t)\rangle_X \quad \textit{for all } u\in D(T), v\in W(Q).$$

Then T is a closed, densely defined, maximal monotone operator.

Proof. The proof can be established by adopting the arguments of [39] with little modifications. ☐

4. Main result

In this section, we give the notion of weak solution for the nonlinear parabolic problem (1) and we state the main result of this work.

Definition 4.1. A function $u\in W(Q)$ with $u_t\in W'(Q)$ is called a weak solution of problem (1) if:

$$\int_Q u_t\varphi, dx, dt + \int_Q \langle\mathcal{L}(u),\varphi\rangle dx, dt + \int_Q f(x,t,u,\nabla u)\varphi, dx, dt + \int_\Sigma g(x,u)\varphi, dS, dt = h,$$

for all $\varphi\in W(Q)$.

For the existence of a weak solution, we will use Theorem 2.7 to achieve our goal. Then, we only need to verify that all the conditions in this Theorem are fulfilled.

Theorem 4.1. *Let $h\in W'(Q)$. Suppose $(M_0), (F_1)$ and (G_1) hold. Then problem* (1) *admits at least one weak solution.*

Proof. In order to prove that problem (1) has a weak solution in $W(Q)$ it is sufficient to show that, for $h \in W'(Q)$. the equation

$$T + \mathcal{L} + C = h \quad \text{in } W'(Q) \tag{15}$$

is solvable, where $T, \mathcal{L}$ and C are given in (14), (4) and (10) respectively. To get this, we apply Asfaw's abstract theorem.

In view of Lemmas 3.1, 3.2 and 3.3, it remains to show that C lies in Γ_σ^τ and $T + \mathcal{L} + C$ is coercive. This is done in the following two claims.

Claim 1: C lies in Γ_σ^τ

By applying conditions (F_1), (G_1) and Holder's inequality, we have

$$\begin{aligned}
&|\langle Cu, v\rangle| = |\int_0^T \int_\Omega f(x,t,u,\nabla u)v\,dxdt + \int_0^T \int_{\partial\Omega} g(x,t,u)v\,dxdt| \\
&\leq c_1' \int_Q (|u| + |\nabla u|)|v|\,dxdt + c_2' \int_0^T (|k(t)|_{L^{p_M(x)}(\Omega)} + |k_2(t)|_{L^{p_M(x)}(\Sigma)})|v(t)|_X\,dt \\
&\leq c_3'\|u\|\|v\| + c_4' \left(\|k\|_{L^\infty(0,T;L^{p_M(x)}(\Omega))} + \|k_2\|_{L^\infty(0,T;L^{p_M(x)}(\Sigma))}\right)\|v\|,
\end{aligned}$$

for all $u, v \in W(Q)$.

Therefore, taking supremum over all $v \in W(Q)$ with $\|v\| \leq 1$ we get $\|Cu\|_{W'(Q)} \leq \tau\|u\| + \sigma$. So, $C \in \Gamma_\sigma^\tau$.

Claim 2: $T + \mathcal{L} + C$ is coercive

Now, by using (M_0), (F_1), (G_1) and monotonicity of T ($\langle Tu, u\rangle \geq 0, \quad$ for all $u, v \in D(T)$), we have for $\|u\| > 1$

$$\begin{aligned}
&\langle Tu + Su + Cu, u\rangle \\
&\geq \sum_{i=1}^{2} \int_0^T \int_\Omega M_i(L_i(u))\big(|\nabla u|^{p_i(x)} + |u|^{p_i(x)}\big)\,dxdt \\
&+ \int_0^T \int_\Omega f(x,t,u,\nabla u)u\,dxdt + \int_0^T \int_{\partial\Omega} g(x,t,u)u\,dxdt \\
&\geq m_0\|u\|^{p_m^-} + \|u\|^{r^-} + \|u\|^{s^-} \to +\infty, \text{ as } \|u\| \to +\infty.
\end{aligned}$$

Noting that $\left|\int_Q hu\,dxdt\right| \leq c_5'\|h\|_{L^{p_M'(x)}(Q)}\|u\|, \quad$ for all $u \in W(Q)$, we get $R = R(h) > 0$ such that

$$\langle Tu + Su + Cu - h, u\rangle > 0$$

for $u \in D(T) \cap \{v \in W(Q) : \|v\| = R\}$.

We conclude that the equation $Tu + Su + Cu = h$ has a weak solution in $D(T)$ since the criteria of Theorem 2.7 are verified. This implies that the problem (1) admits at least one weak solution. Consequently, the proof is complete. □

It is important to note that for the existence of weak solutions for (1), the monotonicity assumption on f with respect to u is not required. See the works [11, 26, 30, 12] and their references, for existence of weak solutions in elliptic as well as parabolic problems under monotone nonlinearities independent of ∇u.

It seems to be interesting to study problem (1) and the properties of its solutions for $\frac{2N}{N+1} < p_m(x) < 2$. We plan to address these questions in a future research.

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