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On the Properties of Strongly h_d Convex Functions

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Abstract. We study some optimization properties of h_d strongly convex functions. More precisely, we discuss the characterization properties/inequalities (existence and uniqueness) of minima of h_d strongly convex functions. Moreover, connections with polynomial norms are also presented.

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1. Introduction

In this paper we focus on the family of strongly convex functions, which is related with the positivity property of complete homogeneous symmetric polynomials with even degree. The study of these polynomial functions starts with the paper of Hunter [8] and was continued in [29], where was considered a new idea to establish this positiveness property. Moreover, two different techniques can be found in [26], one of them using a Schur-convexity argument and the other one based on a method with divided differences.

The family of complete homogeneous symmetric polynomials with n variables $x_1, \ldots, x_n$ and degree $d \in \mathbb{N}$ is given by

$$h_0(x_1, ..., x_n) = 1,$$
$$h_d(x_1, ..., x_n) := \sum_{1 \leq i_1 \leq \cdots \leq i_d \leq n} x_{i_1} \cdots x_{i_d} \qquad (d \geq 1).$$

On the other hand, it was proved that the properties of complete homogeneous symmetric polynomials with even degree can induce norm properties on complex matrices. See [4] and [7]. The simplest way to prove the positivity of h_d polynomial functions, for all even degrees $d \geq 2$, consists of using Schur-convexity and majorization techniques.

The concept of majorization is a powerful topic of research in last decades (see [13]): it was used to find a necessary and sufficient condition for a linear map to preserve group majorizations in [21]; new majorization results are studied in [9, 22];

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new results on superquadratic functions related to Jensen–Steffensen's inequality are given in [1]. See also [24, 25].

Our paper is related to the theory of uniformly convex functions, which allows the possibility to define the concept of majorization into the spaces of curved geometry in [17]. See also [14, 15, 16, 18, 19, 20].

In order to present the current settings let us introduce the main concepts we address in this paper: stronger and weaker h_d convexity for functions defined on $\mathbb{R}^n$. We define the class of convex functions by considering a perturbation of convex functions with complete homogeneous symmetric polynomials.

Definition 1.1. Let $C > 0$ and let $d \geq 2$ be an even natural number. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be h_d strongly convex with modulus C if the function $f(\cdot) - C\, h_d(\cdot)$ is convex. Similarly, a function $f : \mathbb{R}^n \to \mathbb{R}$ is called h_d weakly convex with modulus C if the function $f(\cdot) + C\, h_d(\cdot)$ is convex.

The relevance of this versions of uniformly convexity concepts is based on a positivity result given in [26], which asserts that: if $d \geq 2$ is an even natural index, then

$$h_d(x_1, x_2, ..., x_n) \geq 0 \qquad (x_1, ..., x_n \in \mathbb{R}). \tag{1}$$

Notice that, the above definition is inspired from the notion ω-m-star convex function (see, for instance, [12]). The motivation of the concept of h_2 strongly/weakly convex function is related with the one of uniformly convex function.

Definition 1.2. Let $C > 0$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be uniformly convex with modulus C if $f(\cdot) - C\, \|\cdot\|^2$ is convex. Equivalently, the function f is uniformly convex with modulus C if and only if the following inequality holds

$$f((1-\lambda)\mathbf{x} + \lambda\mathbf{y})) \leq (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y}) - C\lambda(1-\lambda)\|\mathbf{x} - \mathbf{y}\|^2, \tag{2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

We simply remark that, based on the following estimate

$$\frac{1}{2}\|x\|^2 \leq h_2(x) \leq \frac{n+1}{2}\|x\|^2 \qquad (x \in \mathbb{R}^n,\, n \in \mathbb{N}^*),$$

we have, in general, the equivalence between the concepts of uniformly convexity and h_2 strongly convexity. But, we cannot prove the existence of two positive constants C_1 and C_2 such that: a function is h_2 strongly convex with modulus C_1 if and only if it is uniformly convex with modulus C_2. We also recall other interesting consequences which confirm the relevance of h_d strongly convexity: for example, (2) holds similarly, even in the context of h_2 strongly convexity. See [11].

Proposition 1.1. *Let $C > 0$. Then, the function $f : \mathbb{R}^n \to \mathbb{R}$ is h_2 strongly convex with modulus C if and only if*

$$f((1-\lambda)\mathbf{x} + \lambda\mathbf{y})) \leq (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y}) - C\lambda(1-\lambda)h_2(\mathbf{x} - \mathbf{y}), \tag{3}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

The difference between the notions of uniformly convexity and strongly h_d convexity can be also presented in the case of a particular family of polynomial functions. See [11].

Proposition 1.2. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined as*

$$f(x, y, z) = \frac{a}{2}x^2 + ay^2 + \frac{a}{2}z^2 + (a - \alpha)xz + b \qquad (a, b, \alpha \in \mathbb{R}).$$

Then, for any $a \in (0, \infty)$ and $b \in \mathbb{R}$ there exists $C > 0$ and $\alpha > 0$ such that f is h_2 strongly convex with modulus C. Futhermore, for any $\varepsilon > 0$ there exist a, b, α such that f is not uniformly convex with modulus ε.

In the general case, for any even natural number $d \geq 2$, we get a natural but powerful extension of Proposition 1.1. See [11].

Theorem 1.3. *Let $C > 0$ and let $d \geq 2$ be an even natural number. Then, the function $f : \mathbb{R}^n \to \mathbb{R}$ is h_d strongly convex with modulus $C > 0$ if and only if*

$$f((1 - \lambda)\mathbf{x} + \lambda\mathbf{y})) \leq (1 - \lambda)f(\mathbf{x}) + \lambda f(\mathbf{y}) - C\lambda^{\frac{d}{2}}(1 - \lambda)^{\frac{d}{2}} h_d(\mathbf{x} - \mathbf{y}), \qquad (4)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

Moreover, for each $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ we have

$$h_d((1 - \lambda)\mathbf{x} + \lambda\mathbf{y}) - (1 - \lambda)h_d(\mathbf{x}) - \lambda h_d(\mathbf{y}) \leq -\lambda^{\frac{d}{2}}(1 - \lambda)^{\frac{d}{2}} h_d(\mathbf{x} - \mathbf{y}). \qquad (5)$$

We also recall an inequality of Jensen's type in the case of h_d strongly convex functions, for any even natural number $d \geq 2$. See [11].

Proposition 1.4. *(Jensen's type inequality for h_d strongly convexity) Let $C > 0$ and let $d \geq 2$ be an even natural number. If $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$ is a given function such that $F(x_1, ..., x_n) = f(x_1) + ... + f(x_n)$ is h_d strongly convex with modulus C on I^n then, for all $x_1, ..., x_n \in I$, the following inequality holds*

$$\begin{aligned} f\left(\frac{x_1 + ... + x_n}{n}\right) \leq & \frac{f(x_1) + ... + f(x_n)}{n} \\ & - C\frac{1}{n}\binom{n + d - 1}{d}\left(\left(\frac{x_1 + ... + x_n}{n}\right)^d - \frac{x_1^d + \cdots + x_n^d}{n}\right). \end{aligned} \qquad (6)$$

The concept of h_d strongly convexity can be seen in connection with other results existing in literature and we are confident that our paper gives the possibility to develop other interesting results on this topic, such as in other relevant papers (see, for instance [2, 3, 10, 12, 30]). More precisely, our research is also related to the class of ω-m-star convex functions, for which modulus function ω can be replaced with the polynomial function h_d. This is motivated by [12], where some similar properties are presented and for which we are able to express ω-m-star convex property, for some particular function ω, in terms of convexity of a suitable perturbed function. These ideas can also be extended on spaces related to other weaker notions (relative convexity, spaces with global nonpositive curvature, see [17, 18]).

On the other hand, new Ingham's type weighted inequalities are recently proved in [27, 28] by using the positivity of quadratic polynomials. Further research is to use these properties of h_d strongly/weakly convex functions theory developed in this paper to prove the positivity of a very general class of weighted symmetric polynomials.

Hence, the idea to consider positive symmetric polynomials instead of functions depending on the norm and the possibility to obtain similar results offer a new and fresh perspective within the topic of convexity. The aim of this paper is to study such properties in order to study more general class of optimization problems.

The rest of the paper is organised as follows: In Section 2 we present our main results, in the case of h_2 strongly convex functions, where we deduce similar results as the ones for uniformly convex functions. Section 3 is devoted to some connections of our h_d strongly convex functions with polynomial norms.

2. The main results

In this section we present some basic theoretical results on strongly convex functions with modulus $C > 0$. We recover some well-known classical results within uniform convex functions theory, which also holds in the case of strongly h_d convex functions.

Theorem 2.1. *Let $C > 0$, $d \geq 2$ an even natural number and let $f : \Omega \to \mathbb{R}$ be a h_d strongly convex with modulus C defined on a convex set $\Omega \subseteq \mathbb{R}^n$. Then the following statements hold true:*
- (i) *If Ω is an open set, then f is a continuous function on Ω.*
- (ii) *Any local minimizer of f is a global minimum for f.*
- (iii) *Moreover, the global minimizer of f is unique.*

Proof. (i) The continuity property follows from the fact that f is a convex function defined on a open and convex domain in $\mathbb{R}^n$. See, for instance, [19, 20].

(ii) Let us use the following identity

$$h_2(x_1, ..., x_n) = x_1^2 + ... + x_n^2 + \sum_{1 \leq i < j \leq n} x_i x_j. \tag{7}$$

Let $\mathbf{x} \in \Omega$ be a local minimizer for f and V neighborhood of $\mathbf{x}$ such that

$$f(\mathbf{x}) \leq f(\mathbf{v}) \qquad (\mathbf{v} \in V \cap \Omega).$$

Let us suppose that there exists $\mathbf{y} \in \Omega$ such that $f(\mathbf{y}) < f(\mathbf{x})$. Then for all $\lambda \in (0, 1)$, using Proposition 1.1 we have that

$$f\left((1-\lambda)\mathbf{x} + \lambda\mathbf{y}\right) \;=\; (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y}) - C\lambda(1-\lambda)h_2(\mathbf{x}-\mathbf{y}) < f(\mathbf{x}),$$

where we have used that $h_2(\mathbf{x} - \mathbf{y}) > 0$, for all $x, y \in \mathbb{R}^n$.

Now, taking λ sufficiently small we get that $(1-\lambda)\mathbf{x} + \lambda\mathbf{y} \in V$, which leads us to a contradiction.

(iii) Let $\mathbf{x}, \mathbf{y} \in \Omega$ two global minimizers of f. If we consider $\mathbf{x} \neq \mathbf{y}$, taking $\lambda \in (0, 1)$ we have that:

$$\begin{aligned} f((1-\lambda)\mathbf{x} + (\lambda\mathbf{y}) \;&<\; (1-\lambda)f(\mathbf{x}) + \lambda f(\mathbf{y}) - C\lambda(1-\lambda)h_2(\mathbf{x}-\mathbf{y}) \\ &<\; \min_{\mathbf{t}\in\Omega} f(\mathbf{t}), \end{aligned}$$

which leads us to a contradiction. □

Since for some optimization algorithms is is very useful to have nice estimates for such kind of functions we extend the notion of elliptic differentiable functions.

Definition 2.1. We say that a function $J : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is elliptic (α-elliptic) if it is differentiable on Ω and there exists an $\alpha > 0$ such that

$$\langle \nabla J(\mathbf{x}) - \nabla J(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq \alpha \|\mathbf{x} - \mathbf{y}\|^2 \qquad (\mathbf{x}, \mathbf{y} \in \Omega). \tag{8}$$

Definition 2.2. We say that a function $J : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is h_2-elliptic if it is differentiable on Ω with modulus C if

$$\langle \nabla J(\mathbf{x}) - \nabla J(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq Ch_2(x - y) \qquad (\mathbf{x}, \mathbf{y} \in \Omega). \tag{9}$$

By considering the convex function $g : \Omega \to \mathbb{R}$, where $g(\mathbf{x}) = J(\mathbf{x}) - Ch_2(\mathbf{x})$, and using in addition the well-known convex inequality

$$g(\mathbf{y}) \geq g(\mathbf{x}) + \langle \nabla g(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \qquad (\mathbf{x}, \mathbf{y} \in \Omega),$$

we can get easily the following result.

Theorem 2.2. *Let $J : \Omega \to \mathbb{R}$ be a differentiable function defined on the convex set $\Omega \subseteq \mathbb{R}^n$. Then the following affirmations are equivalent:*
(i) J is h_2 strongly convex with modulus C.
(ii) The following inequality holds true

$$J(\mathbf{x}) - J(\mathbf{y}) \geq \langle \nabla J(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + Ch_2(\mathbf{x} - \mathbf{y}) \qquad (\mathbf{x}, \mathbf{y} \in \Omega). \tag{10}$$

(iii) J is h_2-elliptic on Ω with modulus $2C$, i.e.

$$\langle \nabla J(\mathbf{x}) - \nabla J(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq 2Ch_2(\mathbf{x} - \mathbf{y}) \qquad (\mathbf{x}, \mathbf{y} \in \Omega).$$

Proof. $(i) \Rightarrow (ii)$. Let us consider a function $g : \Omega \to \mathbb{R}$ defined as it follows:

$$g(\mathbf{x}) = J(\mathbf{x}) - Ch_2(\mathbf{x}).$$

As g is convex we have that

$$g(\mathbf{y}) \geq g(\mathbf{x}) + \langle \nabla g(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle,$$

for all $\mathbf{x}, \mathbf{y} \in \Omega$.

Therefore, we have that

$$J(\mathbf{y}) - Ch_2(\mathbf{y}) \quad \geq \quad J(\mathbf{x}) - Ch_2(\mathbf{x}) + \langle \nabla J(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle - C\langle \nabla h_2(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle. \tag{11}$$

Notice that

$$\nabla h_2(\mathbf{x}) = (x_1 + S, ..., x_n + S),$$

where $S = \sum_{i=1}^{n} x_i$.

Hence by relation (11) we obtain that

$$\begin{aligned} J(\mathbf{y}) - J(\mathbf{x}) &\geq \langle \nabla J(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + C\left(h_2(\mathbf{y}) - h_2(\mathbf{x})\right) \\ &\quad -C(\alpha)\langle (x_1 + S, ..., x_n + S), \mathbf{y} - \mathbf{x} \rangle \\ &\geq \langle \nabla J(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle - CA(\mathbf{x}, \mathbf{y}), \end{aligned}$$

where $A(\mathbf{x}, \mathbf{y}) = \langle (x_1 + S, ..., x_n + S), \mathbf{y} - \mathbf{x} \rangle - (h_2(\mathbf{y}) - h_2(\mathbf{x}))$. Hence, we get

$$\begin{aligned} A(\mathbf{x}, \mathbf{y}) &= \sum_{i=1}^{n} x_i y_i + (x_1 + ... + x_n)(y_1 + ... + y_n) \\ &\quad -\sum_{i=1}^{n} x_i^2 - \sum_{1 \leq i < j \leq n} x_i x_j - \sum_{i=1}^{n} y_i^2 - \sum_{1 \leq i < j \leq n} y_i y_j. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} h_2(\mathbf{y}-\mathbf{x}) \;&=\; \sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n} y_i x_i + \sum_{i=1}^{n} x_i^2 + \\ &+ \sum_{1\le i<j\le n} y_i y_j - \sum_{1\le i<j\le n} y_i x_j - \sum_{1\le i<j\le n} y_j x_i + \sum_{1\le i<j\le n} x_i x_j. \end{aligned}$$

In order to prove inequality (10), we establish a certain connection between $A(\mathbf{x},\mathbf{y})$ and $h_2(\mathbf{y}-\mathbf{x})$. Indeed, we remark that

$$A(\mathbf{x},\mathbf{y}) + h_2(\mathbf{y}-\mathbf{x}) = 0.$$

Hence, we obtain that

$$J(\mathbf{y}) - J(\mathbf{x}) \;\ge\; \langle \nabla J(\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle + C h_2(\mathbf{x}-\mathbf{y}),$$

so (10) holds true.

$(ii) \Rightarrow (iii)$ Taking account of affirmation (ii), for any $x, y \in \mathbb{R}^n$ we have that

$$\begin{aligned} J(\mathbf{y}) &\ge J(\mathbf{x}) + \langle \nabla J(\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle + C h_2(\mathbf{x}-\mathbf{y}), \\ J(\mathbf{x}) &\ge J(\mathbf{y}) + \langle \nabla J(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle + C h_2(\mathbf{y}-\mathbf{x}). \end{aligned}$$

Adding the above inequalities we obtain that

$$\langle \nabla J(\mathbf{y}) - \nabla J(\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle \ge 2C h_2(\mathbf{y}-\mathbf{x}),$$

and (iii) is fulfilled.

$(iii) \Rightarrow (i)$ Similar estimates as the previous ones give the conclusion we need. □

We recall some classical well-known results in literature.

Theorem 2.3. *Let $U \subseteq \mathbb{R}^n$ a closed and nonempty set and $J : U \to \mathbb{R}$ a continuous and coercive functional if U is unbounded. Then there exists an $\mathbf{x} \in U$ such that*

$$J(\mathbf{x}) = \min_{\mathbf{y}\in U} J(\mathbf{y}). \tag{12}$$

Theorem 2.4. *Let $U \subset \mathbb{R}^n$ a nonempty and convex set. If f is differentiable at $\mathbf{x} \in U$ and*

$$\min_{\mathbf{y}\in U} f(\mathbf{y}) = f(\mathbf{x}), \tag{13}$$

then one have that

$$\langle \nabla f(\mathbf{y}), (\mathbf{x}-\mathbf{y})\rangle \ge 0, \qquad (\mathbf{y} \in U). \tag{14}$$

Conversely, if $\mathbf{x} \in U$ verifies relation (14) and f is convex then $\mathbf{x}$ is a global minimizer for f in U.

We are now in position to prove the following theoretical result.

Theorem 2.5. *If $U \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set, and J is h_2 strongly convex with modulus $C > 0$, then there exists an unique $\mathbf{x} \in U$ such that*

$$J(\mathbf{x}) = \min_{\mathbf{y}\in U} J(\mathbf{y}). \tag{15}$$

Proof. Taking $y = 0$ in (10) we get

$$\begin{aligned} J(\mathbf{x}) &\geq J(0) + \langle \nabla J(0), \mathbf{x} \rangle + C(\alpha) h_2(\mathbf{x}) \\ &\geq J(0) - \|\nabla J(0)\| \, \|\mathbf{x}\| + C(\alpha) h_2(\mathbf{x}) \\ &\geq J(0) - \|\nabla J(0)\| \|\mathbf{x}\| + \frac{C(\alpha)}{2} \|\mathbf{x}\|^2, \end{aligned} \tag{16}$$

using $\frac{1}{2}\|x\|^2 \leq h_2(x)$. By (16) it yields that

$$\lim_{\|\mathbf{x}\| \to \infty} J(\mathbf{x}) = +\infty,$$

so J is coercive.

We proceed now to prove that J is strictly convex. Let us consider $\mathbf{x}^1, \mathbf{x}^2 \in \Omega$ and $\lambda \in (0,1)$. Taking $\mathbf{x} = \mathbf{x}^1$ and $\mathbf{y} = \mathbf{x}^1 + \lambda\left(\mathbf{x}^2 - \mathbf{x}^1\right)$ in (10) we obtain that

$$J(\mathbf{x}^1) > J\left(\mathbf{x}^1 + \lambda(\mathbf{x}^2 - \mathbf{x}^1)\right) + \langle \nabla J\left(\mathbf{x}^1 + \lambda(\mathbf{x}^2 - \mathbf{x}^1)\right), \lambda(\mathbf{x}^2 - \mathbf{x}^1) \rangle. \tag{17}$$

Now, using (10) with $\mathbf{x} = \mathbf{x}^2$ and $\mathbf{y} = \mathbf{x}^1 + \lambda(\mathbf{x}^2 - \mathbf{x}^1)$ we get that

$$J(\mathbf{x}^2) > J\left(\mathbf{x}^1 + \lambda(\mathbf{x}^2 - \mathbf{x}^1)\right) + \langle \nabla J\left(\mathbf{x}^1 + \lambda(\mathbf{x}^2 - \mathbf{x}^1), (1-\lambda)(\mathbf{x}^2 - \mathbf{x}^1)\right) \rangle. \tag{18}$$

Adding (17) and (18) we obtain that

$$\lambda J(\mathbf{x}^2) + (1-\lambda) J(\mathbf{x}^1) > J\left((1-\lambda)\mathbf{x}^1 + \lambda \mathbf{x}^2\right),$$

therefore J is strictly convex.

In conclusion, by the coercivity of J we obtain the existence of a minimizer on U (using Theorem 2.3). The uniqueness of the minimizer is obtained by the fact that J is strictly convex, and the characterization is easily to observe using Theorem 2.4. □

3. Further connections with polynomial norms

In this section we discuss some connections between h_d polynomials, h_d strongly convex functions and polynomial norms.

We start this section by presenting some remarks concerning the possibility of defining a scalar product in terms of h_d symmetric polynomials. Thus, we define the map $\langle \cdot, \cdot \rangle_h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as follows

$$\langle \mathbf{x}, \mathbf{y} \rangle_h = \frac{h_2(\mathbf{x} + \mathbf{y}) - h_2(\mathbf{x} - \mathbf{y})}{4} \qquad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^n). \tag{19}$$

Let us compute

$$\begin{aligned} h_2(\mathbf{x} + \mathbf{y}) - h_2(\mathbf{x} - \mathbf{y}) &= \sum_{i=1}^{n} (x_i + y_i)^2 + \sum_{1 \leq i < j \leq n} (x_i + x_j)(x_j + y_j) \\ &\quad - \left(\sum_{i=1}^{n} (x_i - y_i)^2 + \sum_{1 \leq i < j \leq n} (x_i - y_i)(x_j - y_j) \right) = \end{aligned}$$

$$
\begin{aligned}
= \; & \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 + \sum_{1\le i<j\le n} x_i x_j + \sum_{1\le i<j\le n} x_i y_j \\
& + \sum_{1\le i<j\le n} y_i x_j + \sum_{1\le i<j\le n} y_i y_j - \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} y_i^2 \\
& - \sum_{1\le i<j\le n} x_i x_j + \sum_{1\le i<j\le n} x_i y_j + \sum_{1\le i<j\le n} y_i x_j - \sum_{1\le i<j\le n} y_i y_j \\
= \; & 2\sum_{i=1}^{n} x_i y_i + \sum_{1\le i<j\le n} y_i x_j + 2\sum_{i=1}^{n} x_i y_i + \sum_{1\le i<j\le n} x_i y_j + \sum_{1\le i<j\le n} y_i x_j \\
= \; & 4\sum_{i=1}^{n} x_i y_i + 2\sum_{1\le i<j\le n} x_i y_j + 2\sum_{1\le i<j\le n} y_i x_j .
\end{aligned}
$$

A straightforward computation gives

$$\langle \mathbf{x}, \mathbf{y}\rangle_h = \langle \mathbf{x}, \mathbf{y}\rangle + \frac{1}{2}\left(\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i y_i\right),$$

where $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n} x_i y_i$ denotes the usual scalar product in $\mathbb{R}^n$.

Notice that $\langle \mathbf{x}, \mathbf{y}\rangle_h$ satisfies the properties needed for a scalar product, i.e.

$$
\begin{aligned}
\langle \mathbf{x}, \mathbf{y}\rangle_h &= \langle \mathbf{y}, \mathbf{x}\rangle_h && (\mathbf{x}, \mathbf{y} \in \mathbb{R}^n),\\
\langle \alpha\mathbf{x}, \mathbf{y}\rangle_h &= \alpha\langle \mathbf{x}, \mathbf{y}\rangle_h && (\mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \alpha \in \mathbb{R}),\\
\langle \mathbf{x}+\mathbf{z}, \mathbf{y}\rangle_h &= \langle \mathbf{x}, \mathbf{y}\rangle_h + \langle \mathbf{z}, \mathbf{y}\rangle_h && (\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n),\\
\langle \mathbf{x}, \mathbf{x}\rangle_h &= h_2(\mathbf{x}) \ge 0 && (\mathbf{x} \in \mathbb{R}^n).
\end{aligned}
$$

Finally, if $h_2(\mathbf{x}) = 0$ we have

$$h_2(\mathbf{x}) = \frac{1}{2}(x_1 + \cdots + x_n)^2 + \frac{1}{2}(x_1^2 + \cdots + x_n^2) = 0,$$

which gives $\mathbf{x} = 0_n$.

Hence, $\langle \cdot, \cdot\rangle_h$ is a scalar product and a distance can be given by

$$d^2(\mathbf{x}, \mathbf{y}) = \sqrt{\langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle_h} \qquad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^n). \tag{20}$$

It is worth mentioning that even if h_d polynomials cannot induce a norm (for example, in majorization settings, we have that for any two vectors satisfying $x \prec y$, $h_2(y) \ge h_2(x) + h_2(y-x)$, see [12]) similar results (as in the uniform convexity settings) can be obtained. An interesting approach related to this idea was given in [23], for the case of strongly convexity and hence, further research can be now done in our settings. This is why we compare our results, all along the paper, with the ones obtained for classical strongly convexity.

Note that the following sentences are inspired from [5]. We present some polynomial norms, which means norms that are the d^{th} root of a homogeneous polynomial with degree d. An interesting connection between convexity and norm is given in the following theorem. See [5].

Theorem 3.1. *Let f be a form of degree d in n variables. The following statements are equivalent:*

(1) *The function $f^{\frac{1}{d}}$ is a norm on $\mathbb{R}^n$.*
(2) *The function f is convex and positive definite.*
(3) *The function f is strictly convex.*

Taking into account that not all norms are polynomial norms we are asking if we can generally approximate the norms by polynomial norms.

It was shown that, though not every norm is a polynomial norm, but any norm can be approximated to arbitrary precision by a polynomial norm. A related result is given by Barvinok in [6]. In [6] is was proved that any norm can be approximated by the d^{th} root of a nonnegative degree-d form, and quantifies the quality of the approximation as a function of n and d. The form he obtains however is not shown to be convex. In fact, in a later work it was pointed out that it would be an interesting question to know whether any norm can be approximated by the d^{th} root of a convex form with the same quality of approximation as for d^{th} roots of nonnegative forms.

The result below is a step in that direction, although the quality of approximation is weaker than that by Barvinok's [6]. We note that the form in Barvinok's construction is a sum of squares of other forms. Such forms are not necessarily convex. By contrast, the form that we construct is a sum of powers of linear forms and hence always convex.

Theorem 3.2. *Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Then, for any even integer $d \geq 2$:*
(i) There exists an n-variable convex positive definite form f_d of degree d such that

$$\frac{d}{n+d}\left(\frac{n}{n+d}\right)^{n/d}\|x\| \leq f_d^{1/d}(x) \leq \|x\|,\ (x \in \mathbb{R}^n). \tag{21}$$

In particular, for any sequence $\{f_d\}$ $(d = 2, 4, 6, ...)$ of such polynomials one has

$$\lim_{d\to\infty} \frac{f_d^{1/d}(x)}{\|x\|} = 1 \ \forall x \in \mathbb{R}^n.$$

(ii) One may assume without loss of generality that f_d in (i) is a nonnegative sum of d^{th} powers of linear forms.

Taking into account all the above results we consider that the theoretical facts presented in this paper will be of interest for further investigations in literature.

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