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Input-to-state Practical Partial Stability of Nonautonomous Nonlinear Systems

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Abstract. This paper deals with the asymptotic behaviors of solutions of nonautonomous nonlinear systems with respect a part of variables. We investigate the practical partial stability for a wide class of nonlinear time-varying systems. The study is based on the application of the Lyapunov indirect approach by using some new growth sufficient conditions which give rise to new classes of systems in presence of perturbations. Moreover, by utilizing scalar stable functions, the analysis achieves both asymptotic and input-to-state practical partial stability of a large class of nonlinear systems. The efficacy of the presented theorems is demonstrated through numerous numerical examples.

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1. Introduction

The concept of motion stability concerning specific variables, also referred to as partial stability, naturally emerges in various real-world applications. Rumyantsev, credited as the pioneer of partial stability theory for systems of ordinary differential equations with continuous right sides, laid the groundwork for this field. His seminal contributions, documented in various sources ([4]-[7], [17], [20]-[25], [32]-[36]), not only established fundamental results but also illustrated the practical relevance of his findings in addressing stability challenges in broader distributed-parameter systems. Following Rumyantsev's pioneering work, a considerable body of researchers has advanced the theory and methodologies surrounding the study of partial stability and stabilization. Their collective efforts have successfully addressed numerous significant practical challenges ([16], [18], [30]). The Lyapunov indirect approach, alternatively referred to as Lyapunov's second approach, stands out as a potent tool in the stability analysis and control system design, as outlined in [13]. By this method, if one can identify a positive definite function of the state such that its time derivative along the system trajectories is negative definite, it is argued that the system exhibits stability. Furthermore, by introducing various positive definiteness assumptions for the Lyapunov function and corresponding negative definiteness assumptions for its time derivative, distinct stability characteristics of the system under consideration can be inferred ([3], [9], [10]). Typically, for time-invariant systems, the strict negative definiteness requirement for the time derivative of the Lyapunov function can be relaxed

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to negative semi-definiteness, allowing the application of the Lasalle invariant principle [14]. In the context of time-varying systems, except for certain special cases, the applicability of the Lasalle invariant principle is either limited or poses challenges, as noted in [37]. The primary advantage of this approach lies in its flexibility regarding the requirement for the time derivative of the Lyapunov function: neither strict negative definiteness nor negative semi-definiteness is mandatory ([12], [19], [26, 31]). In the literature, the input-to-state stability property is frequently characterized by the Lyapunov function, see ([11],[27]-[29]) and the references therein. As usual, the time-derivative of the input-to-state stability-Lyapunov function is required to be negative definite under some additional condition on the input signal u. Thus, in addressing this issue, [15] proposed a valuable technique for converting a less stringent Lyapunov function with a negative semi-definite derivative into a Lyapunov function with a negative definite derivative. [13] used the same method to demonstrate the input-to-state stability of a time-varying system with a weak Lyapunov function. Besides, in many cases it is impossible or too costly to construct a feedback, ensuring input-to-state stability behavior of the closed-loop system. Therefore, [28] has introduced the notion of integral input-to-state stability, which is a nonlinear generalization of L2 stability. As is mentioned, the time derivative of the integral input-to-state stability-Lyapunov function is required to be negative definite [1]. The Gronwall inequality is frequently used to establish existence, uniqueness, and stability results for solutions of differential equations, particularly in the study of ordinary differential equations and partial differential equations. It allows researchers to derive crucial estimates that help in understanding the behavior of solutions over time. Recently, [38] the author provided a generalization of the Gronwall inequality in its differential form, which allows for the study of practical stability of a wide class of nonlinear systems.

In this paper, we address a new generalization of the Gronwall inequality in its differential form and with which we have ensured the stability of a high class of dynamical systems. The analysis of practical partial stability is facilitated through the utilization of the comparison principle and the introduction of scalar stable functions. These concepts encompass global practical uniform asymptotic partial stability, global uniform practical exponential partial stability, input-to-state practical partial stability, and integral input-to-state practical partial stability. The following sections of this note are structured as follows. Section 2 provides the system description and preliminary details. Our primary findings are outlined in Section 3, which is subdivided into three subsections focusing on practical asymptotic partial stability analysis, input-to-state practical partial stability analysis, and perturbed systems. Section 4 presents numerical examples to illustrate our results. Finally, we conclude our findings in Section 5.

2. System description and preliminaries

Throughout this paper the following notations: $\mathbb{R}_+ = [0, \infty[$ and $\mathbb{R}^n$ the $n-$dimensional Euclidean space with the Euclidean $2-$norm $|\cdot|$. Also, we use $\mathcal{BC}(\mathbb{R}_+, \mathbb{R})$ and $\mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ to represent, respectively, the space of $\mathbb{R}-$valued continuous bounded functions and piecewise continuous functions. We denote

$$\mathcal{L}_p^m(\mathbb{R}_+) = \left\{ f(\delta) : \mathbb{R}_+ \to \mathbb{R}^m \;\middle|\; \|f\|_p = \left(\int_{\mathbb{R}_+} |f(\delta)|^p \, \mathrm{d}\delta \right)^{\frac{1}{p}} < \infty \right\},$$

$$\mathcal{L}_{\infty}^{m}\left(\mathbb{R}_{+}\right)=\left\{f\left(\delta\right):\mathbb{R}_{+}\rightarrow\mathbb{R}^{m}\;\middle|\;\left\|f\right\|_{\infty}=\sup_{\delta\in\mathbb{R}_{+}}\left\{\left|f\left(\delta\right)\right|\right\}<\infty\right\},$$

where $p\in[1,\infty[$ is any integer. Moreover, if $m=1$, then the $\mathcal{L}_{p}^{m}\left(\mathbb{R}_{+}\right)$ and $\mathcal{L}_{\infty}^{m}\left(\mathbb{R}_{+}\right)$ will be respectively denoted by $\mathcal{L}_{p}\left(\mathbb{R}_{+}\right)$ and $\mathcal{L}_{\infty}\left(\mathbb{R}_{+}\right)$ for short. We also use $\|f\|_{[\delta_0,\delta]}=\sup\{\left|f\left(t\right)\right|, t\in[\delta_0,\delta]\subset\mathbb{R}_+\}$ to denote the truncation of the norm of f at δ. The following are definitions of comparison functions [13].

- The function $\sigma:\mathbb{R}_+\rightarrow\mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is continuous and strictly increasing and $\sigma(0)=0$. If σ is also such that $\sigma(r)\rightarrow+\infty$ as $r\rightarrow+\infty$, then it is said to belong to class $\mathcal{K}_\infty$.
- The continuous function $\lambda:\mathbb{R}_+\times\mathbb{R}_+\rightarrow\mathbb{R}_+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\lambda(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, $\lambda(r,s)\rightarrow 0$ as $s\rightarrow+\infty$ and the mapping $\lambda(r,s)$ is decreasing with respect to s.

Consider the following nonlinear time-varying system

$$\dot{x}\left(\delta\right)=f\left(\delta,x\left(\delta\right),u\left(\delta\right)\right) \tag{2.1}$$

where $f:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^m\rightarrow\mathbb{R}^n$ is piecewise continuous in δ, locally Lipschitz on x for bounded u. The input $u:\mathbb{R}_+\rightarrow\mathbb{R}^m$ is assumed to be locally essentially bounded. Let $\phi(\delta,\delta_0,x_0,u)$, be the unique solution of (2.1) passing through x_0 at time $\delta=\delta_0$ and $u\in\mathbb{R}^m$. For $x^T=(x_1,x_2,...x_n)\in\mathbb{R}^n$, let q be an integer such that $q\leq n$, $y^T=(x_1,x_2,...,x_q)\in\mathbb{R}^q$ and $z^T=(x_{q+1},x_{q+2},...,x_n)\in\mathbb{R}^{n-q}$. With this partition, the solution of (2.1) can be expressed for all $(\delta_0,x_0)\in\mathbb{R}_+\times\mathbb{R}^n$ *as* :

$$\phi(.,\delta_0,x_0,u)=(y^T(.,\delta_0,x_0,u),z^T(.,\delta_0,x_0,u))^T,$$

where $y(.,\delta_0,x_0,u)$ and $z(.,\delta_0,x_0,u)$ are respectively the unique solutions of the following time-varying systems

$$\dot{y}(\delta)=f_1(\delta,y(\delta),z(\delta),u(\delta))$$

and

$$\dot{z}(\delta)=f_2(\delta,y(\delta),z(\delta),u(\delta)),$$

with $f(\delta,x(\delta),u(\delta))=\big(f_1^T(\delta,y(\delta),z(\delta),u(\delta)),f_2^T(\delta,y(\delta),z(\delta),u(\delta))\big)^T$. Where, T denotes the transposition.

Throughout this paper, for any continuously differentiable function $L:\mathbb{R}_+\times\mathbb{R}^n\rightarrow\mathbb{R}_+$, we define the derivative of a function $L(\delta,x)$ along the solutions of system (2.1) as follows:

$$\dot{L}(\delta,x)=\frac{\partial L}{\partial\delta}(\delta,x)+\frac{\partial L}{\partial x}(\delta,x)f(\delta,x,u).$$

Now, let's consider the following definition.

Definition 2.1. Let $\xi,\psi\in\mathcal{PC}(\mathbb{R}_+,\mathbb{R})$ and $\alpha\in\mathbb{R}$. The function ξ is $(\psi,\alpha)-$globally uniformly practically exponentially stable $\big((\psi,\alpha)-GUPES\big)$ if there exist $\theta>0$, $\nu\geq 0$ and $\rho_\alpha>0$, such that, for all $\delta\geq\delta_0$,

$$\int_{\delta_0}^{\delta}\xi(s)ds\leq-\theta(\delta-\delta_0)+\nu,$$

and

$$\int_{\delta_0}^{\delta}|\psi(s)|\mu_\alpha(\delta,s)du\leq\rho_\alpha,$$

where $\mu_\alpha(\delta,s)=\exp\Big((1-\alpha)\displaystyle\int_s^\delta \xi(t)dt\Big)$.

Remark 2.1. In above Definition if $\alpha=0$, then the scalar ξ is $\psi-$globally uniformly practically exponentially stable, see Hadj Taieb [8] for details, therefore the Definition provided in [8] will be considered a particular case of Definition 2.1.

In the following sections, we will utilize the subsequent auxiliary result which taken from [13].

Lemma 2.1. [13]*(Comparison Lemma) Consider the differential equation*

$$\dot{x}(\delta)=f(\delta,x(\delta)),\qquad x(\delta_0)=x_0 \tag{2.2}$$

where $f(\delta,x(\delta))$ is piecewise continuous in s and locally Lipschitz in x, and for all $\delta\geq 0$, $x(\delta)\in J\subset\mathbb{R}$. Let $[\delta_0,T[$ (T could be infinity) be the maximal interval of existence of the solution $x(\delta)$, and suppose $x(\delta)\in J$ for all $\delta\in[\delta_0,T[$. Let $z(\delta)$ be a continuous function satisfies the differential inequality

$$\dot{z}(\delta)\leq f(\delta,z(\delta)),\qquad z(\delta_0)\leq x_0 \tag{2.3}$$

with $z(\delta)\in J$ for all $\delta\in[\delta_0,T[$. Then, $z(\delta)\leq x(\delta)$ for all $\delta\in[\delta_0,T[$.

Lemma 2.2. *(Young's inequality) For all non-negative real b and c, and all positive integers n and m satisfying $\frac{1}{n}+\frac{1}{m}=1$, it follows that*

$$bc\leq\frac{1}{n}b^n+\frac{1}{m}c^m.$$

Lemma 2.3. *Assume σ is a function belonging to the class $\mathcal{K}$. We have for all $b,c\geq 0$,*

$$\sigma(b+c)\leq\sigma(2b)+\sigma(2c).$$

Lemma 2.4. *For all $a\geq 1$ and for all two non-negative real b and c, we have:*

$$(b+c)^{\frac{1}{a}}\leq 2^{\frac{1}{a}}(b^{\frac{1}{p}}+c^{\frac{1}{a}}).$$

3. Main results

We began by introducing the following auxiliary result, which presents a new Generalized Gronwall-Bellman inequality in its differential form. The Generalized Gronwall-Bellman inequality outlined in [38] will serve as a specific case when $\alpha=0$.

Lemma 3.1. *Let $\xi,\psi\in\mathcal{PC}(\mathbb{R}_+,\mathbb{R})$, a constant $\alpha\in[0,1[$ and $y:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is a differentiable function, such that for all $\delta\geq\delta_0\geq 0$,*

$$\dot{y}(\delta)\leq\xi(\delta)y(\delta)+\psi(\delta)y^\alpha(\delta). \tag{3.1}$$

Then, for all $\delta\geq\delta_0\geq 0$, we have

$$y(\delta)\leq\Big[\Big(y(\delta_0)\exp\big(\int_{\delta_0}^\delta\xi(s)ds\big)\Big)^{1-\alpha}+(1-\alpha)\int_{\delta_0}^\delta\Big(\exp\big(\int_t^\delta\xi(s)ds\big)\Big)^{1-\alpha}\psi(t)dt\Big]^{\frac{1}{1-\alpha}}.$$

Proof. The proof is to apply Lemma 2.1 with the scalar differential equation

$$\dot{x}(\delta) = f(\delta, x(\delta)) = \xi(\delta)x(\delta) + \psi(\delta)x^{\alpha}(\delta), \qquad x(\delta_0) = x_0 = y(\delta_0). \tag{3.2}$$

It is clear that the function f satisfies all the conditions of the Lemma 2.1. Next, we begin by solve the equation 3.2.
For $\delta \geq \delta_0 \geq 0$, we consider the following tow cases:
Case 1: if $x_0 = 0$, then by using the uniqueness of the solution, we have $x(\delta_0) = 0$, for all $\delta \geq \delta_0$.
Case 2: if $x_0 \neq 0$, then by using the uniqueness of the solution, we have $x(\delta) \neq 0$, for all $\delta \geq \delta_0$. Now, we set $z(\delta) = x^{1-\alpha}(\delta) = (x(\delta))^{1-\alpha}$, hence $\dot{z}(\delta) = (1-\alpha)\dot{x}(\delta)x^{-\alpha}(\delta)$. Then, by dividing the equation (3.2) by $x^{\alpha}(\delta)$, we obtain:

$$\frac{\dot{x}(\delta)}{x^{\alpha}(\delta)} = \xi(\delta)x^{1-\alpha}(\delta) + \psi(\delta). \tag{3.3}$$

Thus, we have the following equation

$$\dot{z}(\delta) = \tilde{\xi}(\delta)z(\delta) + \tilde{\psi}(\delta), \tag{3.4}$$

where

$$\tilde{\xi}(\delta) = (1-\alpha)\xi(\delta) \ \text{ and } \ \tilde{\psi}(\delta) = (1-\alpha)\psi(\delta).$$

Then, using the equation (3.4) we obtain:

$$\frac{\partial}{\partial\delta}\Big(z(\delta)\exp\big(-\int_{\delta_0}^{\delta}\tilde{\xi}(s)ds\big)\Big) = \tilde{\psi}(\delta)\exp\big(-\int_{\delta_0}^{\delta}\tilde{\xi}(s)ds\big), \quad \text{for all } \ \delta \geq \delta_0 \geq 0. \tag{3.5}$$

It follows that, for all $\delta \geq \delta_0 \geq 0$.

$$z(\delta)\exp\big(-\int_{\delta_0}^{\delta}\tilde{\xi}(s)ds\big) - z(\delta_0) = \int_{\delta_0}^{\delta}\exp\big(-\int_{\delta_0}^{t}\tilde{\xi}(s)ds\big)\tilde{\psi}(t)dt \tag{3.6}$$

Then, by multiply the above equality by $\exp\big(\int_{\delta_0}^{\delta}\tilde{\xi}(s)ds\big)$, we obtain:

$$z(\delta) = z(\delta_0)\exp\big(\int_{\delta_0}^{\delta}\tilde{\xi}(s)ds\big) + \int_{\delta_0}^{\delta}\exp\big(\int_{t}^{\delta}\tilde{\xi}(s)ds\big)\tilde{\psi}(t)dt, \quad \text{for all } \ \delta \geq \delta_0 \geq 0. \tag{3.7}$$

Since $z(\delta) = x^{1-\alpha}(\delta)$, $\tilde{\xi}(\delta) = (1-\alpha)\xi(\delta)$ and $\tilde{\psi}(\delta) = (1-\alpha)\psi(\delta)$, for all $\delta \geq \delta_0 \geq 0$ we get:

$$\begin{aligned} x^{1-\alpha}(\delta) &= x^{1-\alpha}(\delta_0)\exp\big(\int_{\delta_0}^{\delta}(1-\alpha)\xi(s)ds\big) + \int_{\delta_0}^{\delta}\exp\big(\int_{t}^{\delta}(1-\alpha)\xi(s)ds\big)(1-\alpha)\psi(t)dt \\ &= \Big(x(\delta_0)\exp\big(\int_{\delta_0}^{\delta}\xi(s)ds\big)\Big)^{1-\alpha} + (1-\alpha)\int_{\delta_0}^{\delta}\Big(\exp\big(\int_{t}^{\delta}\xi(s)ds\big)\Big)^{1-\alpha}\psi(t)dt. \end{aligned}$$

Consequently, for all $\delta \geq \delta_0$ we have:

$$x(\delta) = \Big[\Big(y(\delta_0)\exp\big(\int_{\delta_0}^{\delta}\xi(s)ds\big)\Big)^{1-\alpha} + (1-\alpha)\int_{\delta_0}^{\delta}\Big(\exp\big(\int_{t}^{\delta}\xi(s)ds\big)\Big)^{1-\alpha}\psi(t)dt\Big]^{\frac{1}{1-\alpha}}.$$

As the function y satisfied the inequality 3.1 with $y(\delta_0) = x(\delta_0) = x_0$, then, all the conditions of Lemma 2.1 are satisfied, so one gets:

$$y(\delta) \leq \Big[\Big(y(\delta_0)\exp\big(\int_{\delta_0}^{\delta}\xi(s)ds\big)\Big)^{1-\alpha} + (1-\alpha)\int_{\delta_0}^{\delta}\Big(\exp\big(\int_{t}^{\delta}\xi(s)ds\big)\Big)^{1-\alpha}\psi(t)dt\Big]^{\frac{1}{1-\alpha}},$$

for all $\delta \geq \delta_0 \geq 0$. □

Remark 3.1. We can prove the Lemma 3.1 for all values of $\alpha \in \mathbb{R}$, using the same reasoning, but due to the differentiability issue, the most important cases are for $\alpha \in [0, 1[$.

3.1. Practical asymptotic partial stability analysis. In this subsection, we are interested to study the practical partial stability of systems (2.1), where $u \equiv 0$. Let's start by providing the following definition.

Definition 3.1. [2] System (2.1) is globally uniformly practically asymptotically $y-$stable $\big(GUPAy - S\big)$, if there exist $\gamma \in \mathcal{KL}$ and $r > 0$, such that for all $x_0 = (y_0, z_0) \in \mathbb{R}^n$, we have

$$|y(\delta, \delta_0, x_0, 0)| \leq \gamma(|x_0|, \delta - \delta_0) + r, \quad \text{for all } \ \delta \geq \delta_0 \geq 0. \tag{3.8}$$

Definition 3.2. [2] System (2.1) is globally uniformly practically exponentially $y-$stable $\big(GUPEy - S\big)$, if there exist $\theta > 0$, $k \geq 0$ and $r > 0$ such that for all $x_0 = (y_0, z_0) \in \mathbb{R}^n$, we have

$$|y(\delta, \delta_0, x_0, 0)| \leq k|x_0| \exp(-\theta(\delta - \delta_0)) + r, \ \text{ for all } \ \delta \geq \delta_0 \geq 0, \tag{3.9}$$

where θ is the rate of convergence.

Theorem 3.2. *Suppose there is a continuously differentiable function $L : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, two $\mathcal{K}_\infty$ functions $\sigma_i, i = 1, 2$, $a > 0$, $\alpha \in [0, 1[$, two functions $\xi, \psi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$, satisfying the following inequalities for all $\delta \geq 0$ and $x = (y, z) \in \mathbb{R}^n$:*

$$\sigma_1 \left(|y|\right) \leq L\left(\delta, x\right) \leq \sigma_2 \left(|x|\right) + a, \tag{3.10}$$

$$\dot{L}(\delta, x) \leq \xi\left(\delta\right) L\left(\delta, x\right) + \psi(\delta) L^\alpha\left(\delta, x\right). \tag{3.11}$$

Then, the system (2.1) with $u \equiv 0$ is $GUPAy - S$ if ξ is $(\psi, \alpha) - GUPES$.

Proof. Employing Lemma 3.1, we can establish that for all $\delta \geq \delta_0 \geq 0$,

$$\begin{aligned} L(\delta, \phi(\delta, \delta_0, x_0, 0)) \leq & \Big[\Big(L(\delta_0, x_0) \exp\Big(\int_{\delta_0}^{\delta} \xi(u)du\Big)\Big)^{1-\alpha} \\ & + (1 - \alpha) \int_{\delta_0}^{\delta} \Big(\exp\Big(\int_{u}^{\delta} \xi(s)ds\Big)\Big)^{1-\alpha} \psi(u)du\Big]^{\frac{1}{1-\alpha}} \\ \leq & \Big[\Big(L(\delta_0, x_0)\mu_0(\delta, \delta_0)\Big)^{1-\alpha} + (1 - \alpha) \int_{\delta_0}^{\delta} |\psi(u)|\mu_\alpha(\delta, u)du\Big]^{\frac{1}{1-\alpha}}, \end{aligned}$$

where $\mu_0(\delta, \delta_0) = \exp\Big(\int_{\delta_0}^{\delta} \xi(u)du\Big)$ and $\mu_\alpha(\delta, \delta_0) = \exp\Big((1 - \alpha) \int_{\delta_0}^{\delta} \xi(u)du\Big)$.
Therefore, by using the inequality (3.10) we have for all $\delta \geq \delta_0 \geq 0$,

$$\begin{aligned} \sigma_1(|y(\delta, \delta_0, x_0, 0)|) \leq & L(\delta, \phi(\delta, \delta_0, x_0, 0)) \\ \leq & \Big(\Big(L(\delta_0, x_0)\mu_0(\delta, \delta_0)\Big)^{1-\alpha} + (1 - \alpha) \int_{\delta_0}^{\delta} |\psi(u)|\mu_\alpha(\delta, u)du\Big)^{\frac{1}{1-\alpha}} \\ \leq & \sigma_\alpha\Big(\Big(L(\delta_0, x_0)\mu_0(\delta, \delta_0)\Big)^{1-\alpha} + (1 - \alpha) \int_{\delta_0}^{\delta} |\psi(u)|\mu_\alpha(\delta, u)du\Big), \end{aligned}$$

where $\sigma_\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, the function defined by $\sigma_\alpha(t) = t^{\frac{1}{1-\alpha}}$. It is evident that σ_α is a $\mathcal{K}_\infty$ function. Hence, by using Lemma 2.3, it follows that,

$$\begin{aligned}
\sigma_1(|y(\delta,\delta_0,x_0,0)|) &\leq \sigma_\alpha\Big(2\Big(L(\delta_0,x_0)\mu_0(\delta,\delta_0)\Big)^{1-\alpha}\Big)+\sigma_\alpha\Big(2(1-\alpha)\int_{\delta_0}^{\delta}|\psi(u)|\mu_\alpha(\delta,u)du\Big)\\
&\leq \Big(2\Big(L(\delta_0,x_0)\mu_0(\delta,\delta_0)\Big)^{1-\alpha}\Big)^{\frac{1}{1-\alpha}}+\Big(2(1-\alpha)\int_{\delta_0}^{\delta}|\psi(u)|\mu_\alpha(\delta,u)du\Big)^{\frac{1}{1-\alpha}}\\
&\leq 2^{\frac{1}{1-\alpha}}L(\delta_0,x_0)\mu_0(\delta,\delta_0)+\Big(2(1-\alpha)\int_{\delta_0}^{\delta}|\psi(u)|\mu_\alpha(\delta,u)du\Big)^{\frac{1}{1-\alpha}}\\
&\leq 2^{\frac{1}{1-\alpha}}\mu_0(\delta,\delta_0)\Big(\sigma_2\left(|x_0|\right)+a\Big)+\Big(2(1-\alpha)\int_{\delta_0}^{\delta}|\psi(u)|\mu_\alpha(\delta,u)du\Big)^{\frac{1}{1-\alpha}}
\end{aligned}$$

Since, ξ is $(\psi,\alpha)-GUPES$, then, for all $\delta \geq \delta_0$,

$$\begin{aligned}
\sigma_1(|y(\delta,\delta_0,x_0,0)|) &\leq^{\frac{1}{1-\alpha}} 2\exp\Big(-\theta(\delta-\delta_0)+\nu\Big)\Big(\sigma_2\left(|x_0|\right)+a\Big)+(2(1-\alpha)\rho_\alpha)^{\frac{1}{1-\alpha}}\\
&\leq 2^{\frac{1}{1-\alpha}}\exp(\nu)\sigma_2\left(|x_0|\right)\exp\big(-\theta(\delta-\delta_0)\big)+2^{\frac{1}{1-\alpha}}\exp(\nu)a+(2(1-\alpha)\rho_\alpha)^{\frac{1}{1-\alpha}}\\
&\leq d_1\sigma_2\left(|x_0|\right)\exp\big(-\theta(\delta-\delta_0)\big)+d_2,
\end{aligned}$$

where $d_1 = 2^{\frac{1}{1-\alpha}}\exp\big(\frac{\nu}{(1-\alpha)}\big)$ and $d_2 = 2^{\frac{1}{1-\alpha}}\exp(\nu)a+(2(1-\alpha)\rho_\alpha)^{\frac{1}{1-\alpha}}$. Consequently, for every $\delta \geq \delta_0$,

$$|y(\delta,\delta_0,x_0,0)| \leq \sigma_1^{-1}\Big(d_1\sigma_2\left(|x_0|\right)\exp\big(-\theta(\delta-\delta_0)\big)+d_2\Big).$$

Using Lemma 2.3, it follows that for all $\delta \geq \delta_0$,

$$|y(\delta,\delta_0,x_0,0)| \leq \sigma_1^{-1}\big(2d_1\sigma_2\left(|x_0|\right)\exp\big(-\theta(\delta-\delta_0)\big)\big)+\sigma_1^{-1}\big(2d_2\big).$$

Hence, the system (2.1) is $GUPAy-S$. ☐

Regarding the exponential case, we have the following result.

Theorem 3.3. *Suppose there is a continuously differentiable function* $L : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, *constants* $c_1 > 0$, $c_2 \geq 0$, $a > 0$, $\alpha \in [0,1[$, *a integer* $m \geq 1$, *two functions* $\xi, \psi \in \mathcal{PC}(\mathbb{R}_+,\mathbb{R})$, *satisfying the following inequalities for all* $\delta \geq 0$ *and* $x = (y,z) \in \mathbb{R}^n$:

$$c_1\left|y\right|^m \leq L\left(\delta,x\right) \leq c_2\left|x\right|^m + a, \tag{3.12}$$

$$\dot{L}(\delta,x) \leq \xi\left(\delta\right)L\left(\delta,x\right)+\psi(\delta)L^\alpha\left(\delta,x\right). \tag{3.13}$$

Then the system (2.1) with $u \equiv 0$ *is* $GUPEy-S$ *if* ξ *is* $(\psi,\alpha)-GUPES$.

Proof. Employing the same rationale as in Theorem 3.2, it follows that for all $\delta \geq \delta_0$,

$$|y(\delta,\delta_0,x_0,0)|^m \leq d_1\left|x_0\right|^m\exp\big(-\theta(\delta-\delta_0)\big)+d_2,$$

where $d_1 = \frac{(2c_2)^{\frac{1}{1-\alpha}}}{c_1}\exp(\nu)$ and $d_2 = \frac{2^{\frac{1}{1-\alpha}}c_2a}{c_1}\exp(\nu)+\frac{(2(1-\alpha)\rho_\alpha)^{\frac{1}{1-\alpha}}}{c_1}$. Consequently, with Lemma 2.4, it follows that for all $s \geq t$,

$$|y(s,t,x,0)| \leq (2d_1)^{\frac{1}{m}}|x_0|\exp\big(-\frac{\theta}{m}(\delta-\delta_0)\big)+(2d_2)^{\frac{1}{m}}.$$

Therefore, system (2.1) is $GUPEy-S$. ☐

3.2. Input-to-state practical partial stability analysis. In 1980, E. D. Sontag introduced the concept of input-to-state stability analysis [27]. Since its inception, this concept has become a cornerstone for analyzing and designing nonlinear systems. Its applications span across different domains, such as observer design, the small gain theorem, and stability testing of interconnected nonlinear systems [19, 27]. In the following subsection, we will illustrate the application of this concept in performing input-to-state practical partial stability analysis of nonlinear time-varying systems.

Definition 3.3. The nonlinear system (2.1) is said to be

(1) input-to-state practical $y-$stable $\big(ISPy-S\big)$ if there exist $\sigma\in\mathcal{KL}$, $\gamma_1\in\mathcal{K}$ and $r>0$ such that, for any $u\in\mathcal{L}_{\infty}^{m}\left(\mathbb{R}_{+}\right)$,

$$|y(\delta,\delta_0,x_0,u)|\leq\sigma\left(|x_0|,\delta-\delta_0\right)+\gamma_1\left(\|u\|_{[\delta_0,\delta]}\right)+r,\ \forall\delta\geq\delta_0\geq0.$$

(2) integral input-to-state practical $y-$stable $\big(IISPy-S\big)$ if there exist $\sigma\in\mathcal{KL}$, $\gamma_1,\gamma_2\in\mathcal{K}$ and $r>0$ such that, for any $u\in\mathcal{L}_{\infty}^{m}\left(\mathbb{R}_{+}\right)$,

$$|y(\delta,\delta_0,x_0,u)|\leq\sigma\left(|x_0|,\delta-\delta_0\right)+\gamma_1\left(\int_{\delta_0}^{\delta}\gamma_2\left(|u\left(s\right)|\right)\mathrm{d}s\right)+r,\ \forall\delta\geq\delta_0\geq0.$$

Initially, we present the subsequent result concerning the characterization of $ISPy-S$ for system (2.1).

Theorem 3.4. *Suppose there is a continuously differentiable function* $L:\mathbb{R}_{+}\times\mathbb{R}^{n}\to\mathbb{R}_{+}$, *three* $\mathcal{K}_{\infty}$ *functions* $\sigma_i, i=1,2,3$, $a>0$, $\alpha\in[0,1[$, *two functions* $\xi,\psi\in\mathcal{PC}(\mathbb{R}_{+},\mathbb{R})$, *such that, for all* $\delta\geq0$, $u\in\mathcal{L}_{\infty}^{m}\left(\mathbb{R}_{+}\right)$ *and* $x=(y,z)\in\mathbb{R}^{n}$:

$$\sigma_1\left(|y|\right)\leq L\left(\delta,x\right)\leq\sigma_2\left(|x|\right)+a, \tag{3.14}$$

$$\dot{L}(\delta,x)\leq\xi\left(\delta\right)L\left(\delta,x\right)+\psi(\delta)L^{\alpha}\left(\delta,x\right),\ \text{ if } L\left(\delta,x\right)\geq\sigma_3\left(|u\left(\delta\right)|\right). \tag{3.15}$$

Then the system (2.1) is $ISPy-S$ *if* ξ *is* $(\psi,\alpha)-GUPES$.

Proof. Let us consider the following inequality

$$L\left(\delta_1,\phi(\delta_1,\delta_0,x_0,u)\right)\geq\sigma_3\left(|u\left(\delta_1\right)|\right). \tag{3.16}$$

Now, consider tow cases.
Case 1: If (3.16) is true for almost all $\delta_1\in[\delta_0,\delta]\subset\mathbb{R}_{+}$. Then, it follows from condition (3.15) and Lemma 3.1, that for all $x=(y,z)\in\mathbb{R}^{n}$,

$$\begin{aligned}L\left(\delta,\phi(\delta,\delta_0,x_0,u)\right)\leq\ &2^{\frac{1}{1-\alpha}}\exp(\nu)\sigma_2\left(|x_0|\right)\exp\big(-\theta(\delta-\delta_0)\big)\\&+2^{\frac{1}{1-\alpha}}\exp(\nu)a+(2(1-\alpha)\rho_{\alpha})^{\frac{1}{1-\alpha}},\end{aligned} \tag{3.17}$$

Case 2: we assume that (3.16) does not hold true for almost all $t_1\in[t,s]\subset\mathbb{R}_{+}$.
Let us consider the set $\{\delta_1\in[\delta_0,\delta]:L\left(\delta_1,\phi(\delta_1,\delta_0,x_0,u)\right)\leq\sigma_3\left(|u\left(\delta_1\right)|\right)\}$ which is non-empty.
Denote $\delta^{*}=\sup\{\delta_1\in[\delta_0,\delta]:L\left(\delta_1,\phi(\delta_1,\delta_0,x_0,u)\right)\leq\sigma_3\left(|u\left(\delta_1\right)|\right)\}$.
Then, we have either $\delta^{*}=$ or $\delta^{*}<\delta$.
If $\delta^{*}=\delta$, then by using the definition of δ^{*} we get

$$L\left(\delta,\phi(\delta,\delta_0,x_0,u)\right)=L\left(\delta^{*},\phi(\delta^{*},\delta_0,x_0,u)\right)\leq\sup_{\delta_1\in[\delta_0,\delta]}\left\{\sigma_3\left(|u\left(\delta_1\right)|\right)\right\}=\sigma_3\left(\|u\|_{[\delta_0,\delta]}\right). \tag{3.18}$$

If $\delta^* < \delta$, then $L\left(\delta_1, \phi(\delta_1, \delta_0, x_0, u)\right) \geq \sigma_3\left(\left|u\left(\delta_1\right)\right|\right), \delta_1 \in [\delta^*, \delta]$, which, by (3.15), implies

$$\dot{L}\left(\delta, \phi(\delta, \delta_0, x_0, u)\right) \leq \xi\left(\delta\right) L\left(\delta, \phi(\delta, \delta_0, x_0, u)\right) + \psi(\delta) L^{\alpha}\left(\delta, \phi(\delta, \delta_0, x_0, u)\right),$$

Using Lemma 3.1 and by use the same argument as in Theorem 3.2 , we obtain the following result:

$$\begin{aligned}
L\left(\delta, \phi(\delta, \delta_0, x_0, u)\right) &\leq \Big(\Big(L(\delta^*, \phi(\delta^*, \delta_0, x_0, u)) \exp\Big(\int_{\delta^*}^{\delta} \xi(s)ds\Big)\Big)^{1-\alpha} \\
&\quad + (1-\alpha)\int_{\delta^*}^{\delta} \psi(t)\Big(\exp\Big(\int_{t}^{\delta} \xi(s)ds\Big)\Big)^{1-\alpha} dt\Big)^{\frac{1}{1-\alpha}} \\
&\leq \Big(\Big(L(\delta^*, \phi(\delta^*, \delta_0, x_0, u))\mu_0(\delta, \delta^*)\Big)^{1-\alpha} + (1-\alpha)\int_{\delta^*}^{\delta} |\psi(t)|\mu_\alpha(\delta, t)dt\Big)^{\frac{1}{1-\alpha}} \\
&\leq \Big(2\Big(L(\delta^*, \phi(\delta^*, \delta_0, x_0, u))\mu_0(\delta, \delta^*)\Big)^{1-\alpha}\Big)^{\frac{1}{1-\alpha}} + \Big(2(1-\alpha)\int_{\delta^*}^{\delta} |\psi(t)|\mu_\alpha(\delta, t)dt\Big)^{\frac{1}{1-\alpha}} \\
&\leq 2^{\frac{1}{1-\alpha}} L(\delta^*, \phi(\delta^*, \delta_0, x_0, u))\mu_0(\delta, \delta^*) + \Big(2(1-\alpha)\int_{\delta^*}^{\delta} |\psi(t)|\mu_\alpha(\delta, t)dt\Big)^{\frac{1}{1-\alpha}},
\end{aligned}$$

where $\mu_0(\delta, \delta^*) = \exp\Big(\int_{\delta^*}^{\delta} \xi(t)dt\Big)$ and $\mu_\alpha(\delta, t) = \exp\Big((1-\alpha)\int_{t}^{\delta} \xi(s)ds\Big)$. Then, since ξ is $(\psi, \alpha) - GUPES$ and by using the definition of δ^* the above inequality gives:

$$L\left(\delta, \phi(\delta, \delta_0, x_0, u)\right) \leq 2^{\frac{1}{1-\alpha}} \sigma_3\big(\|u\|_{[\delta_0, \delta]}\big) \exp\big(-\theta(\delta - \delta_0) + \nu\big) + \big(2(1-\alpha)\rho_\alpha\big)^{\frac{1}{1-\alpha}}.$$

Then, we obtain:

$$L\left(\delta, \phi(\delta, \delta_0, x_0, u)\right) \leq 2^{\frac{1}{1-\alpha}} \sigma_3\big(\|u\|_{[\delta_0, \delta]}\big) \exp(\nu) + \big(2(1-\alpha)\rho_\alpha\big)^{\frac{1}{1-\alpha}}. \tag{3.19}$$

Hence, we get from (3.17), (3.18) and (3.19) that

$$\begin{aligned}
L\left(\delta, \phi(\delta, \delta_0, x_0, u)\right) \leq{}& 2^{\frac{1}{1-\alpha}} \exp(\nu)\sigma_2\left(|x_0|\right) \exp\big(-\theta(\delta - \delta_0)\big) + 2^{\frac{1}{1-\alpha}} \exp(\nu) a \\
&+ \left(2(1-\alpha)\rho_\alpha\right)^{\frac{1}{1-\alpha}} + \big(2^{\frac{1}{1-\alpha}} \exp(\nu) + 1\big)\sigma_3\big(\|u)\|_{[\delta_0, \delta]}\big) \\
\leq{}& d_1 \sigma_2\left(|x_0|\right) \exp\big(-\theta_1(\delta - \delta_0)\big) + d_2 + (d_1 + 1)\sigma_3\big(\|u\|_{[\delta_0, \delta]}\big),
\end{aligned}$$

where $d_1 = 2^{\frac{1}{1-\alpha}} \exp(\nu)$ and $d_2 = 2^{\frac{1}{1-\alpha}} \exp(\nu) a + \left(2(1-\alpha)\rho_\alpha\right)^{\frac{1}{1-\alpha}}$.

Then, it follows from condition (3.14) and by using Lemma 2.3, we get

$$\begin{aligned}
|y(\delta, \delta_0, x_0, u)| \leq{}& \sigma_1^{-1}\left(2d_1\sigma_2(|x_0|) \exp\big(-\theta(\delta - \delta_0)\big) + 2d_2\right) \\
&+ \sigma_1^{-1}\Big(2(d_1 + 1)\sigma_3\big(\|u\|_{[\delta_0, \delta]}\big)\Big) \\
\leq{}& \sigma_1^{-1}\left(4d_1\sigma_2(|x_0|) \exp\big(-\theta(\delta - \delta_0)\big)\right) + \sigma_1^{-1}\left(4d_2\right) + \sigma_1^{-1}\Big(2(d_1 + 1)\sigma_3\big(\|u\|_{[\delta_0, \delta]}\big)\Big).
\end{aligned}$$

Hence, the system (2.1) is $ISPy - S$. The proof is finished. □

We subsequently introduce the ensuing result concerning the characterization of $IISPy - S$ for the system (2.1).

Theorem 3.5. *Suppose there is a continuously differentiable function* $L:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+$*, three* $\mathcal{K}_\infty$ $\sigma_i, i=1,2,3$*,* $a>0$*,* $\alpha\in[0,1[$*, two functions* $\xi,\psi\in\mathcal{PC}(\mathbb{R}_+,\mathbb{R})$ *such that, for all* $\delta\geq 0$*,* $u\in\mathcal{L}_\infty^m(\mathbb{R}_+)$ *and* $x=(y,z)\in\mathbb{R}^n$ *:*

$$\sigma_1(|y|)\leq L(\delta,x)\leq\sigma_2(|x|)+a, \tag{3.20}$$

$$\dot{L}(\delta,x)\leq\xi(\delta)\,L(\delta,x)+\big(\sigma_3(|u(\delta)|)+\psi(\delta)\big)L^\alpha(\delta,x). \tag{3.21}$$

Then, the system (2.1) is $IISPy-S$ *if* ξ *is* $(\psi,\alpha)-GUPES$*.*

Proof. By applying Lemma 3.1, it follows that for all $\delta\geq\delta_0$,

$$\begin{aligned}L(\delta,\phi(\delta,\delta_0,x_0,u))&\leq\Bigg(\Big(L(\delta_0,x_0)\exp\Big(\int_{\delta_0}^{\delta}\xi(t)dt\Big)\Big)^{1-\alpha}\\&\quad+(1-\alpha)\int_{\delta_0}^{\delta}\Big(\exp\Big(\int_s^{\delta}\xi(t)dt\Big)\Big)^{1-\alpha}\Big(\sigma_3(|u(s)|)+\psi(s)\Big)ds\Bigg)^{\frac{1}{1-\alpha}}\\&\leq\Bigg(\Big(L(\delta_0,x_0)\mu_0(\delta,\delta_0)\Big)^{1-\alpha}+(1-\alpha)\int_{\delta_0}^{\delta}(\sigma_3(|u(s)|)+|\psi(s)|)\mu_\alpha(\delta,s)ds\Bigg)^{\frac{1}{1-\alpha}},\end{aligned}$$

where $\mu_0(\delta,\delta_0)=\exp\Big(\int_{\delta_0}^{\delta}\xi(s)ds\Big)$ and $\mu_\alpha(\delta,\delta_0)=\exp\Big((1-\alpha)\int_{\delta_0}^{\delta}\xi(s)ds\Big)$.
Using the same arguments as the Theorem 3.2, we have for all $\delta\geq\delta_0$,

$$\begin{aligned}&\sigma_1(|y(\delta,\delta_0,x_0,u)|)\leq L(\delta,\phi(\delta,\delta_0,x_0,u))\\&\leq 2^{\frac{1}{1-\alpha}}\mu_0(\delta,\delta_0)\Big(\sigma_2(|x_0|)+a\Big)+\Big(2(1-\alpha)\int_{\delta_0}^{\delta}(\sigma_3(|u(s)|)+|\psi(s)|)\mu_\alpha(\delta,s)ds\Big)^{\frac{1}{1-\alpha}}\\&\leq 2^{\frac{1}{1-\alpha}}\sigma_2(|x_0|)\,\mu_0(\delta,\delta_0)+2^{\frac{1}{1-\alpha}}a\mu_0(\delta,\delta_0)\\&\quad+\Big(4(1-\alpha)\int_{\delta_0}^{\delta}|\psi(t)|\mu_\alpha(\delta,t)dt\Big)^{\frac{1}{1-\alpha}}+\Big(4(1-\alpha)\int_{\delta_0}^{\delta}\sigma_3(|u(t)|)\mu_\alpha(\delta,t)dt\Big)^{\frac{1}{1-\alpha}}.\end{aligned}$$

Since, ξ is $(\psi,\alpha)-GUPES$, then, for all $\delta\geq\delta_0$,

$$\begin{aligned}\sigma_1(|y(\delta,\delta_0,x_0,u)|)&\leq 2^{\frac{1}{1-\alpha}}\sigma_2(|x_0|)\exp(\nu)\exp(-\theta(\delta-\delta_0))+2^{\frac{1}{1-\alpha}}a\exp(\nu)\\&\quad+(4(1-\alpha)\rho_\alpha)^{\frac{1}{1-\alpha}}+\Big(4(1-\alpha)\int_{\delta_0}^{\delta}\sigma_3(|u(t)|)\exp((1-\alpha)\nu)dt\Big)^{\frac{1}{1-\alpha}}.\end{aligned}$$

Consequently, for every $\delta\geq\delta_0$, we have:

$$|y(\delta,\delta_0,x_0,u)|\leq\sigma_1^{-1}\Big(d_1\sigma_2(|x_0|)\exp\big(-\theta(\delta-\delta_0)\big)+d_2+d_3\Big(\int_{\delta_0}^{\delta}\sigma_3(|u(t)|)dt\Big)^{\frac{1}{1-\alpha}}\Big),$$

where

$$\begin{aligned}&d_1=2^{\frac{1}{1-\alpha}}\exp(\nu),\quad d_2=2^{\frac{1}{1-\alpha}}a\exp(\nu)+(4(1-\alpha)\rho_\alpha)^{\frac{1}{1-\alpha}}\\&d_3=\big(4(1-\alpha)\exp\big((1-\alpha)\nu\big)\big)^{\frac{1}{1-\alpha}}.\end{aligned}$$

By applying Lemma 2.3, it follows that for all $\delta\geq\delta_0$,

$$\begin{aligned}&|y(\delta,\delta_0,x_0,u)|\leq\sigma_1^{-1}\big(2d_1\sigma_2(|x_0|)\exp\big(-\theta(\delta-\delta_0)\big)+2d_2\big)+\sigma_1^{-1}\Big(2d_3\Big(\int_{\delta_0}^{\delta}\sigma_3(|u(t)|)dt\Big)^{\frac{1}{1-\alpha}}\Big)\\&\leq\sigma_1^{-1}\big(4d_1\sigma_2(|x_0|)\exp\big(-\theta(\delta-\delta_0)\big)\big)+\sigma_1^{-1}\Big(2d_3\Big(\int_{\delta_0}^{\delta}\sigma_3(|u(t)|)dt\Big)^{\frac{1}{1-\alpha}}\Big)+\sigma_1^{-1}(4d_2).\end{aligned}$$

Hence, system (2.1) is $IISPy-S$. □

3.3. Perturbed systems.

We consider the perturbed system of the form:

$$\dot{x} = f(\delta, x) + g(\delta, x, u), \tag{3.22}$$

where $x \in \mathbb{R}^n$ is the state, the input $u : \mathbb{R}_+ \to \mathbb{R}^m$ is assumed to be locally essentially bounded and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is piecewise continuous in δ, locally Lipschitz in x. The function $g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous in δ and locally Lipschitz in x and u. Let $\phi(., \delta_0, x_0, u)$, be the unique solution of (3.22) passing through (δ_0, x_0) such that $\phi(\delta_0, \delta_0, x_0, u) = x_0$ and $\phi(., \delta_0, x_0, u) = (y^T(., \delta_0, x_0, u), z^T(., \delta_0, x_0, u))^T$. In the actual literature, the synthesis of stability for the system (3.22) relies on the stability of the nominal system given by:

$$\dot{x} = f(\delta, x), \tag{3.23}$$

with $L(\delta, x)$ as a Lyapunov function candidate for the whole system provided that the size of perturbation is known.

Theorem 3.6. *Suppose there is a continuously differentiable function* $L : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, *constants* $c_1 > 0$, $c_2 \geq 0$, $a \geq 0$, $p > 1$, $\alpha \in [0, 1[$, $\sigma \in \mathcal{K}_\infty$, $\pi \in \mathcal{L}_1(\mathbb{R}_+)$, *a bounded function* λ, *two functions* $\xi, \psi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ *such that, for all* $\delta \geq 0$, $u \in \mathcal{L}^m_\infty(\mathbb{R}_+)$ *and* $x = (y, z) \in \mathbb{R}^n$:

$$c_1 |y|^p \leq L(\delta, x) \leq c_2 |x|^p + a, \tag{3.24}$$

$$\frac{\partial L}{\partial \delta}(\delta, x) + \frac{\partial L}{\partial x}(\delta, x) f(\delta, x) \leq \xi(\delta) L(\delta, x) + \psi(\delta) L^\alpha(\delta, x). \tag{3.25}$$

$$\frac{\partial L}{\partial x}(\delta, x) g(\delta, x, u) \leq \pi(\delta)|y|\sigma(|u(\delta)|) + \Big(\pi(\delta) + \lambda(\delta)\Big) L^\alpha(\delta, x). \tag{3.26}$$

Then the system (3.22) is $ISPy-S$ *if* ξ *is* $(\psi, \alpha)-GUPES$.

Proof. By using the inequalities (3.25) and (3.26), for all $\delta \geq 0$ and $x \in \mathbb{R}^n$ we have:

$$\begin{aligned}
&\frac{\partial L}{\partial \delta}(\delta, x) + \frac{\partial L}{\partial x}(\delta, x) f(\delta, x) + \frac{\partial L}{\partial x}(\delta, x) g(\delta, x, u) \\
&\leq \xi(\delta) L(\delta, x) + \psi(\delta) L^\alpha(\delta, x) + \pi(\delta)|y|\sigma(|u(\delta)|) + \Big(\pi(\delta) + \lambda(\delta)\Big) L^\alpha(\delta, x) \\
&\leq \xi(\delta) L(\delta, x) + \Big(\psi(\delta) + \pi(\delta) + \lambda(\delta)\Big) L^\alpha(\delta, x) + \pi(\delta)|y|\sigma(|u(\delta)|).
\end{aligned}$$

Utilizing Young's inequality, which asserts that for any $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we obtain:

$$|y|\sigma(|u(\delta)|) \leq \frac{|y|^p}{p} + \frac{(\sigma(|u(\delta)|))^q}{q}$$

Therefore, we have for all $\delta \geq 0$, and $x = (y, z) \in \mathbb{R}^n$,

$$\begin{aligned}
&\frac{\partial L}{\partial \delta}(\delta, x) + \frac{\partial L}{\partial x}(\delta, x) f(\delta, x) + \frac{\partial L}{\partial x}(\delta, x) g(\delta, x, u) \\
&\leq \xi(\delta) L(\delta, x) + \Big(\psi(\delta) + \pi(\delta) + \lambda(\delta)\Big) L^\alpha(\delta, x) + \frac{\pi(\delta)|y|^p}{p} + \frac{\pi(\delta)(\sigma(|u(\delta)|))^q}{q} \\
&\leq \left(\xi(\delta) + \frac{\pi(\delta)}{c_1 p}\right) L(\delta, x) + \frac{\pi(\delta)(\sigma(|u(\delta)|))^q}{q} + \Big(\psi(\delta) + \pi(\delta) + \lambda(\delta)\Big) L^\alpha(\delta, x) \\
&\leq \tilde{\xi}(\delta) L(\delta, x) + \tilde{\psi}(\delta) L^\alpha(\delta, x) \ \text{ if } L(\delta, x) \geq (\sigma(|u(\delta)|))^q,
\end{aligned}$$

where

$$\tilde{\xi}(\delta) = \xi(\delta) + \frac{\pi(\delta)}{c_1 p} + \frac{\pi(\delta)}{q} \quad \text{and} \quad \tilde{\psi}(\delta) = \psi(\delta) + \pi(\delta) + \lambda(\delta).$$

It is easy to see that $\tilde{\xi}$ is $(\tilde{\psi}, \alpha) - GUPES$ if ξ is $(\psi, \alpha) - GUPES$. Therefore, by applying Theorem 3.4, we have the system (3.22) is $ISPy - S$. $\square$

4. Illustrative Examples

In this section, we present several numerical examples to illustrate the effectiveness of the proposed stability theorems.

Example 1: Let's consider the following three-dimensional system

$$\begin{cases} \dot{x}_1 = \big(-1 + \delta^2 \exp(-\delta)\big)x_1 + \delta x_2 + \frac{\beta(\delta)\left(\frac{1}{2}\left(x_1^2+x_2^2\right)+\frac{1}{1+\delta}\right)^{\frac{1}{2}}}{(x_1^2+1)}, \\ \dot{x}_2 = \big(-1 + \delta^2 \exp(-\delta)\big)x_2 - \delta x_1 + \frac{\beta(\delta)\left(\frac{1}{2}x_1^2+\frac{1}{1+\delta}\right)^{\frac{1}{2}}}{(x_2^2+1)}, \\ \dot{x}_3 = x_3 + \delta^2 \exp(-\delta) x_1 + \delta x_2 \end{cases} \tag{4.1}$$

where $\delta \geq 0$, $x = (y^T, x_3) \in \mathbb{R}^3$, with $y^T = (x_1, x_2)$ and non-negative function $\beta \in \mathcal{L}_1(\mathbb{R}_+)$. We select a time-varying Lyapunov function

$$L(\delta, x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) + \frac{1}{1+\delta}.$$

Consequently, L fulfills the inequalities as indicated in (3.12), with $c_1 = \frac{1}{2}$, $c_2 = 1$, $a = 1$ and $m = 2$, for all $\delta \geq 0$.

Subsequently, the time derivative of $L(\delta, x)$ can be computed as

$$\begin{aligned} \dot{L}(\delta, x) &= -\frac{1}{(1+\delta)^2} + \left(x_1^2 + x_2^2\right)\Big[x_1\dot{x}_1 + x_2\dot{x}_2\Big] \\ &\leq x_1\Big[\big(-1 + \delta^2\exp(-\delta)\big)x_1 + \delta x_2 + \frac{\beta(\delta)\left(\frac{1}{2}\left(x_1^2+x_2^2\right)+\frac{1}{1+\delta}\right)^{\frac{1}{2}}}{(x_1^2+1)}\Big] \\ &\quad + x_2\Big[\big(-1 + \delta^2\exp(-\delta)\big)x_2 - \delta x_1 + \frac{\beta(\delta)\left(\frac{1}{2}x_1^2+\frac{1}{1+\delta}\right)^{\frac{1}{2}}}{(x_2^2+1)}\Big] \\ &\leq \big(-1 + \delta^2\exp(-\delta)\big)x_1^2 + \delta x_1 x_2 + \frac{\beta(\delta)x_1\left(\frac{1}{2}\left(x_1^2+x_2^2\right)+\frac{1}{1+\delta}\right)^{\frac{1}{2}}}{(x_1^2+1)} \\ &\quad + \big(-1 + \delta^2\exp(-\delta)\big)x_2^2 - \delta x_1 x_2 + \frac{\beta(\delta)x_2\left(\frac{1}{2}\left(x_1^2+x_2^2\right)+\frac{1}{1+\delta}\right)^{\frac{1}{2}}}{(x_2^2+1)} \\ &\leq \big(-1 + \delta^2\exp(-\delta)\big)\left(x_1^2+x_2^2\right) + \beta(\delta)\left(\frac{1}{2}\left(x_1^2+x_2^2\right)+\frac{1}{1+\delta}\right)^{\frac{1}{2}}\left[\frac{x_1}{(x_1^2+1)} + \frac{x_2}{(x_2^2+1)}\right] \\ &\leq \big(-1 + \delta^2\exp(-\delta)\big)\Big(\left(x_1^2+x_2^2\right) + \frac{1}{1+\delta} - \frac{1}{1+\delta}\Big) \\ &\quad + \beta(\delta)\left(\frac{1}{2}\left(x_1^2+x_2^2\right)+\frac{1}{1+\delta}\right)^{\frac{1}{2}}\left[\frac{x_1}{(x_1^2+1)} + \frac{x_2}{(x_2^2+1)}\right] \\ &\leq \big(-1 + \delta^2\exp(-\delta)\big)\Big(\left(x_1^2+x_2^2\right) + \frac{1}{1+\delta}\Big) + \frac{1}{1+\delta} \\ &\quad + \beta(\delta)\left(\frac{1}{2}\left(x_1^2+x_2^2\right)+\frac{1}{1+\delta}\right)^{\frac{1}{2}}\left[\frac{x_1}{(x_1^2+1)} + \frac{x_2}{(x_2^2+1)}\right] \end{aligned}$$

Since,

$$\frac{x}{1+x^2} \leq \frac{1}{2},$$

then,

$$\begin{aligned}
\dot{L}(\delta, x) &\leq \big(-1+\delta^2\exp(-\delta)\big)\Big((x_1^2+x_2^2)+\frac{1}{1+\delta}\Big)+\frac{1}{(1+\delta)^{\frac{1}{2}}}\big(\frac{1}{1+\delta}\big)^{\frac{1}{2}} \\
&\quad + \beta(\delta)\big(\frac{1}{2}\,(x_1^2+x_2^2)+\frac{1}{1+\delta}\big)^{\frac{1}{2}} \\
&\leq 2\big(-1+\delta^2\exp(-\delta)\big)\Big(\frac{1}{2}\,(x_1^2+x_2^2)+\frac{1}{2}\exp(-\delta)\Big) \\
&\quad + \big(\beta(\delta)+\frac{1}{(1+\delta)^{\frac{1}{2}}}\big)\big(\frac{1}{2}\,(x_1^2+x_2^2)+\exp(-\delta)\big)^{\frac{1}{2}} \\
&\leq \xi(\delta)L(\delta, x)+\psi(\delta)L^{\frac{1}{2}}(\delta, x),
\end{aligned}$$

where $\xi(\delta) = -2+2\delta^2\exp\big(-\delta\big)$ and $\psi(\delta) = \beta(\delta)+\frac{1}{(1+\delta)^{\frac{1}{2}}}$.
We can demonstrate that ξ is $(\psi, \frac{1}{2}) - GUPES$, namely it follows from Lemma 2.1 that there exist $\theta > 0$, $\nu \geq 0$ and $\rho_{\frac{1}{2}} > 0$, such that, for all $\delta \geq \delta_0$,

$$\int_{\delta_0}^{\delta} \xi(s)ds \leq -\theta(\delta-\delta_0)+\nu,$$

and

$$\int_{\delta_0}^{\delta} |\psi(s)|\mu_{\frac{1}{2}}(\delta, s)ds \leq \rho_{\frac{1}{2}},$$

where $\mu_{\frac{1}{2}}(\delta, \delta_0) = \exp\Big(\frac{1}{2}\displaystyle\int_{\delta_0}^{\delta} \xi(s)ds\Big)$.
On the one hand, we have for all $\delta \geq \delta_0 \geq 0$:

$$\begin{aligned}
\int_{\delta_0}^{\delta} \xi\,(s)\,\mathrm{d}s &= \int_{\delta_0}^{\delta} -2+2s^2\exp\big(-s\big)ds \\
&\leq -2(\delta-\delta_0)+\int_{0}^{+\infty} 2s^2\exp\big(-s\big)ds \leq -2(\delta-\delta_0)+4.
\end{aligned}$$

On the other hand for all $\delta \geq \delta_0 \geq 0$, we have

$$\begin{aligned}
\int_{\delta_0}^{\delta} |\psi(s)|\mu_{\frac{1}{2}}(\delta, s)ds &\leq \int_{\delta_0}^{\delta} \big(|\beta(s)|+\frac{1}{(1+s)^{\frac{1}{2}}}\big)\exp\big(-(\delta-s)+2\big)\mathrm{d}s \\
&\leq \exp(2)\Big(\int_{0}^{+\infty} |\beta(s)|\mathrm{d}s+\int_{\delta_0}^{\delta} \frac{1}{(1+s)^{\frac{1}{2}}}\exp\big(-(\delta-s)\big)\mathrm{d}s\Big) \\
&\leq \exp(2)\Big(\|\beta\|_1+\exp(-\delta)\int_{0}^{\delta} \exp(s)\mathrm{d}s\Big)\leq \exp(2)\Big(\|\beta\|_1+1\Big).
\end{aligned}$$

Therefore, all conditions outlined in Theorem 3.3 are met. Consequently, the system (4.1) is $GUPEy - S$.

Example 2: Consider the following nonlinear time-varying system

$$\begin{cases} \dot{x}_1 = \frac{1}{2}\left(\frac{1}{1+\delta^2+x_1^2+x_2^2} - \delta\left|\cos\delta\right|\right)x_1 + \frac{\delta|\cos\delta|}{3(1+x_1^2)}u(\delta) + \frac{\beta(\delta)\left(\frac{1}{2}\left(x_1^2+x_2^2\right)+\exp(-\delta)\right)^{\alpha}}{|x|^2+1}, \\ \\ \dot{x}_2 = \frac{1}{2}\left(\frac{1}{1+\delta^2+x_2^2+3x_3^2} - \delta\left|\cos\delta\right|\right)x_2 + \frac{\delta|\cos\delta|}{3(1+x_2^2)}u(\delta) + \frac{\beta(\delta)\left(\frac{1}{2}\left(x_1^2+x_3^2\right)\right)^{\alpha}}{2x_2^2+1}, \\ \\ \dot{x}_3 = \frac{1}{2}\left(\frac{1}{1+\delta^2+x_3^2+2x_1^2} - \delta\left|\cos\delta\right|\right)x_3 + \frac{\delta|\cos\delta|}{3(1+x_3^2)}u(\delta) + \frac{\beta(\delta)\left(\frac{1}{2}x_3^2+\exp\left(-\delta\right)\right)^{\alpha}}{x_3^2+1}, \\ \dot{x}_4 = \left(\frac{x_4}{1+\delta^2+x_3^2+2x_1^2}\right) + \frac{\delta|\cos\delta|}{3(1+x_3^2+x_4^2)}u(\delta) + \frac{\beta(\delta)\left(\frac{1}{2}x_3^2+\exp\left(-\delta\right)\right)^{\alpha}}{x_3^2+1}, \end{cases} \tag{4.2}$$

where $\delta \geq 0$, $x = (y^T, x_4)^T \in \mathbb{R}^3$, with $y = (x_1, x_2, x_3)^T$ and $u \in \mathcal{L}_\infty(\mathbb{R}_+)$, $\beta \in \mathcal{L}_1(\mathbb{R}_+)$ is non-negative function and $\alpha \in]0,1[$. We select a time-varying Lyapunov function

$$L(\delta, x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \exp(-\delta).$$

Consequently, L fulfills the inequalities as indicated in (3.14), with $\sigma_1(|y|) = \frac{1}{2}|y|^2$, $\sigma_2(|x|) = \frac{1}{2}|x|^2$ and $a = 1$ for all $\delta \geq 0$.
Subsequently, the time derivative of $L(\delta, x)$ can be computed as

$$\begin{aligned}
&\dot{L}(\delta, x) = -\exp(-\delta) + \dot{x_1}(\delta)x_1(\delta) + \dot{x_2}(\delta)x_2(\delta) + \dot{x_3}(\delta)x_3(\delta) \\
&\leq \Big[\frac{1}{2}\left(\frac{1}{1+\delta^2+x_1^2+x_2^2} - \delta\left|\cos\delta\right|\right)x_1 + \frac{\delta\left|\cos\delta\right|}{3(1+x_1^2)}u(\delta) + \frac{\beta(\delta)\big(\frac{1}{2}\left(x_1^2+x_2^2\right)+e^{-\delta}\big)^{\alpha}}{\left|x\right|^2+1}\Big]x_1(\delta) \\
&\quad + \Big[\frac{1}{2}\left(\frac{1}{1+\delta^2+x_1^2+3x_3^2} - \delta\left|\cos\delta\right|\right)x_2 + \frac{\delta\left|\cos\delta\right|}{3(1+x_2^2)}u(\delta) + \frac{\beta(\delta)\big(\frac{1}{2}\left(x_1^2+x_3^2\right)\big)^{\alpha}}{2x_2^2+1}\Big]x_2(\delta) \\
&\quad + \Big[\frac{1}{2}\left(\frac{1}{1+\delta^2+x_2^2+2x_1^2} - \delta\left|\cos\delta\right|\right)x_3 + \frac{\delta\left|\cos\delta\right|}{3(1+x_3^2)}u(\delta) + \frac{\beta(\delta)\big(\frac{1}{2}x_3^2+\exp(-\delta)\big)^{\alpha}}{x_3^2+1}\Big]x_3(\delta) \\
&\leq \left(\frac{1}{1+\delta^2} - \delta\left|\cos\delta\right|\right)\frac{1}{2}x_1^2 + \frac{\delta\left|\cos\delta\right|x_1}{3(1+x_1^2)}u(\delta) + \beta(\delta)\frac{1}{2}\big(\frac{1}{2}\left(x_1^2+x_2^2\right)+\exp(-\delta)\big)^{\alpha} \\
&\quad + \left(\frac{1}{1+\delta^2} - \delta\left|\cos\delta\right|\right)\frac{1}{2}x_2^2 + \frac{\delta\left|\cos\delta\right|x_2}{3(1+x_2^2)}u(\delta) + \beta(\delta)\frac{1}{2}\big(\frac{1}{2}\left(x_1^2+x_3^2\right)\big)^{\alpha} \\
&\quad + \left(\frac{1}{1+\delta^2} - \delta\left|\cos\delta\right|\right)\frac{1}{2}x_3^2 + \frac{\delta\left|\cos\delta\right|x_3}{3(1+x_3^2)}u(\delta) + \beta(\delta)\frac{1}{2}\big(\frac{1}{2}x_3^2+\exp(-\delta)\big)^{\alpha} \\
&\leq \left(\frac{1}{1+\delta^2} - \delta\left|\cos\delta\right|\right)\big(\frac{1}{2}(x_1^2+x_2^2+x_3^2)+\exp(-\delta)-\exp(-\delta)\big) + \frac{\delta\left|\cos\delta\right|\left|u(\delta)\right|}{2} \\
&+ \frac{3}{2}\beta(\delta)\big(\frac{1}{2}(x_1^2+x_2^2+x_3^2)+\exp(-\delta)\big)^{\alpha} \\
&\leq \left(\frac{1}{1+\delta^2} - \delta\left|\cos\delta\right|\right)L(\delta, x) + \frac{\delta\left|\cos\delta\right|\left|u(\delta)\right|}{2} + \delta\left|\cos\delta\right|\exp(-\delta) + \frac{3}{2}\beta(\delta)L^{\alpha}(\delta, x) \\
&\leq \left(\frac{1}{1+\delta^2} - \delta\left|\cos\delta\right|\right)L(\delta, x) + \frac{\delta\left|\cos\delta\right|\left|u(\delta)\right|}{2} + \delta\left|\cos\delta\right|\exp\big(-\delta(1-\alpha+\alpha)\big) + \frac{3}{2}\beta(\delta)L^{\alpha}(\delta, x) \\
&\leq \left(\frac{1}{1+\delta^2} - \delta\left|\cos\delta\right|\right)L(\delta, x) + \frac{\delta\left|\cos\delta\right|\left|u(\delta)\right|}{2} + \delta\left|\cos\delta\right|\exp\big(-\delta(1-\alpha)\big)L^{\alpha}(\delta, x) \\
&+ \frac{3}{2}\beta(\delta)L^{\alpha}(\delta, x) \leq \xi(\delta)L(\delta, x) + \psi(\delta)L^{\alpha}(\delta, x), \forall L(\delta, x) \geq \left|u(\delta)\right|,
\end{aligned}$$

where $\xi(\delta)=\frac{1}{1+\delta^2}-\frac{\delta|\cos(\delta)|}{2}$ and $\psi(\delta)=\frac{3}{2}\beta(\delta)+\delta\left|\cos\delta\right|\exp\big(-\delta(1-\alpha)\big)$. It is clear that $\psi\in\mathcal{L}_\infty\left(\mathbb{R}_+\right)$.
Then, we can show that ξ is $(\psi,\alpha)-GUPES$.
On the one hand, we have for all $\delta\geq\delta_0\geq 0$:

$$\begin{aligned}\int_{\delta_0}^{\delta}\xi\left(s\right)\mathrm{d}s&=\int_{\delta_0}^{\delta}\frac{1}{1+s^2}\mathrm{d}s-\int_{\delta_0}^{\delta}\frac{s\left|\cos(s)\right|}{2}\mathrm{d}s\\&\leq\int_{0}^{\infty}\frac{1}{1+s^2}\mathrm{d}s-\int_{\delta_0}^{\delta}\frac{s\left|\cos(s)\right|}{2}\mathrm{d}s\\&\leq\frac{\pi}{2}-\int_{\delta_0}^{\delta}\frac{s\left|\cos(s)\right|}{2}\mathrm{d}s.\end{aligned}$$

Similar to the approach outlined in Appendix A3 of [38], we can demonstrate that for all $\delta\geq\delta_0\geq 0$, we have:

$$-\int_{\delta_0}^{\delta}\frac{s\left|\cos(s)\right|}{2}\mathrm{d}s\leq-\frac{2}{3\pi}(\delta-\delta_0)+1.$$

Then, for all $\delta\geq\delta_0\geq 0$, we get:

$$\int_{\delta_0}^{\delta}\xi\left(s\right)\mathrm{d}s\leq-\frac{2}{3\pi}\left(\delta-\delta_0\right)+(\frac{\pi}{2}+1).$$

On the other hand for all $\delta\geq\delta_0\geq 0$, we have

$$\begin{aligned}&\int_{\delta_0}^{\delta}|\psi(\epsilon)|\exp\big((1-\alpha)\int_{\epsilon}^{\delta}\xi\left(t\right)\mathrm{d}t\big)d\epsilon\leq\int_{\delta_0}^{\delta}|\psi(\epsilon)|\exp\Big(-\frac{2}{3\pi}\left(\delta-\epsilon\right)+(\frac{\pi}{2}+1)\Big)\mathrm{d}\epsilon\\&\leq\|\psi(\delta)\|_\infty\exp\big((\frac{\pi}{2}+1)-\frac{2(1-\alpha)}{3\pi}\delta\big)\int_{0}^{\delta}\exp\Big(\frac{2(1-\alpha)}{3\pi}\epsilon\Big)\mathrm{d}\epsilon\\&\leq\frac{3\pi}{2(1-\alpha)}\|\psi(\delta)\|_\infty\exp(\frac{\pi}{2}+1).\end{aligned}$$

Hence, We have ξ is $(\psi,\alpha)-GUPES$ and all the conditions in Theorem 3.4 are satisfied. Then the system (4.1) is $ISPy-S$.
Example 3: Consider the following time-varying system:

$$\begin{cases}\dot{x}_1=\frac{1}{2}\left(\frac{1}{1+\delta+x_1^2}-\delta\left|\cos\delta\right|\right)x_1+\frac{\beta_1(\delta)\left(x_1^2+\exp(-\delta)\right)^{\frac{1}{2}}}{1+|x|}\\\qquad+\frac{\lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}}+(\lambda(\delta)+\beta_2(\delta))\left(x_1^2+\exp(-\delta)\right)^{\frac{1}{2}}}{1+x_1^2}+2\beta_1(\delta)x_2,\\\dot{x}_2=\frac{1}{2}\left(\frac{1}{1+\delta+x_2^2}-\delta\left|\cos\delta\right|\right)x_2+\frac{\beta_1(\delta)\left(x_1^2+x_2^2\right)^{\frac{1}{2}}}{1+|x|}\\\qquad+\frac{\lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}}+(\lambda(\delta)+\beta_2(\delta))\left(x_1^2+x_2^2+\exp(-\delta)\right)\big)^{\frac{1}{2}}}{1+x_2^2}-2\beta_1(\delta)x_1,\\\dot{x}_3=\frac{x_3\exp(t)}{1+\delta+x_2^2}+\frac{\beta_1(\delta)\left(x_1^2+x_2^2\right)^{\frac{1}{2}}}{1+|x|}+\frac{\lambda(\delta)\sigma(|u(\delta)|)|x|+(\lambda(\delta)+\beta_2(\delta))\left(x_1^2+x_2^2+\exp(-\delta)\right)\big)^{\frac{1}{2}}}{1+x_2^2},\\\dot{x}_4=\frac{1}{2}\left(\frac{1}{1+\delta+x_2^2}+3\delta\left|\cos\delta\right|\right)x_2+\frac{\beta_1(\delta)\left(x_1^2+x_2^2\right)^{\frac{1}{2}}}{1+|x|},\end{cases}\tag{4.3}$$

where $\delta\geq 0$, $x=(y^T,z^T)^T\in\mathbb{R}^4$, with $y=(x_1,x_2)^T$, $z=(x_3,x_4)^T$, $u\in\mathcal{L}_\infty\left(\mathbb{R}_+\right)$, β_1 and β_2 are two non-negative bounded functions, λ is a non-negative integrable function on $\mathbb{R}_+$ and $\sigma \in \mathcal{K}_\infty$. Then system has the form of (3.22) with

$$f(\delta,x)=\begin{pmatrix} \frac{1}{2}\left(\frac{1}{1+\delta+x_1^2}-\delta\left|\cos\delta\right|\right)x_1+\frac{\beta_1(\delta)\left(x_1^2+\exp(-\delta)\right)^{\frac{1}{2}}}{1+|x|} \\ \frac{1}{2}\left(\frac{1}{1+\delta+x_2^2}-\delta\left|\cos\delta\right|\right)x_2+\frac{\beta_1(\delta)\left(x_1^2+x_2^2\right)^{\frac{1}{2}}}{1+|x|} \\ \frac{x_3\exp(\delta)}{1+\delta+x_2^2}+\frac{\beta_1(\delta)\left(x_1^2+x_2^2\right)^{\frac{1}{2}}}{1+|x|} \\ \frac{1}{2}\left(\frac{1}{1+\delta+x_2^2}+3\delta\left|\cos\delta\right|\right)x_2+\frac{\beta_1(\delta)\left(x_1^2+x_2^2\right)^{\frac{1}{2}}}{1+|x|} \end{pmatrix}$$

and

$$g(\delta,x,u)=\begin{pmatrix} \frac{\lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}}+(\lambda(\delta)+\beta_2(\delta))\left(x_1^2+\exp(-\delta)\right)^{\frac{1}{2}}}{1+x_1^2}+2\beta_1(\delta)x_2 \\ \frac{\lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}}+(\lambda(\delta)+\beta_2(\delta))\left(x_1^2+x_2^2+\exp(-\delta)\right)^{\frac{1}{2}}}{1+x_2^2}-2\beta_1(\delta)x_1 \\ \frac{\lambda(\delta)\sigma(|u(\delta)|)|x|+(\lambda(\delta)+\beta_2(\delta))\left(x_1^2+x_2^2+\exp\left(-\delta\right)\right))^{\frac{1}{2}}}{1+x_2^2} \\ 0 \end{pmatrix}.$$

Then, let's consider the following Lyapunov function

$$L(\delta,x)=\frac{1}{2}(x_1^2+x_2^2)+e^{-\delta}.$$

It follows that L satisfies the inequalities given in (3.24), with $c_1=c_2=\frac{1}{2}$, $p=2$ and $a=1$ for all $\delta\geq 0$.
Then, the derivative of the function L along the trajectories of the nominal system $\dot{x}=f(\delta,x)$ is given by:

$$\begin{aligned}
\dot{L}(\delta,x) &= -e^{-\delta}+\dot{x_1}(\delta)x_1(\delta)+\dot{x_2}(\delta)x_2(\delta)\\
&\leq \left[\frac{1}{2}\left(\frac{1}{1+\delta+x_1^2}-\delta\left|\cos\delta\right|\right)x_1+\frac{\beta_1(\delta)\left(x_1^2+e^{-\delta}\right)^{\frac{1}{2}}}{1+|x|}\right]x_1(\delta)\\
&\quad+\left[\frac{1}{2}\left(\frac{1}{1+\delta+x_2^2}-\delta\left|\cos\delta\right|\right)x_2+\frac{\beta_1(\delta)\left(x_1^2+x_2^2\right)^{\frac{1}{2}}}{1+|x|}\right]x_2(\delta)\\
&\leq \left(\frac{1}{1+\delta}-\delta\left|\cos\delta\right|\right)\frac{1}{2}(x_1^2+x_2^2)+\frac{\beta_1(\delta)\left(x_1^2+e^{-\delta}\right)^{\frac{1}{2}}x_1(\delta)}{1+|x|}+\frac{\beta_1(\delta)\left(x_1^2+x_2^2\right)^{\frac{1}{2}}x_2(\delta)}{1+|x|}\\
&\leq \left(\frac{1}{1+\delta}-\delta\left|\cos\delta\right|\right)\left(\frac{1}{2}(x_1^2+x_2^2)+e^{-\delta}-e^{-\delta}\right)\\
&\quad+\beta_1(\delta)\left(x_1^2+x_2^2+e^{-\delta}\right)^{\frac{1}{2}}\left[\frac{x_1(\delta)}{1+|x|}+\frac{x_2(\delta)}{1+|x|}\right]\\
&\leq \left(\frac{1}{1+\delta}-\delta\left|\cos\delta\right|\right)L\left(\delta,x\right)+\delta\left|\cos\delta\right|e^{-\delta}+2\beta_1(\delta)\left(\frac{1}{2}(x_1^2+x_2^2)+e^{-\delta}\right)^{\frac{1}{2}}\left[1+1\right]\\
&\leq \left(\frac{2}{1+\delta}-\delta\left|\cos\delta\right|\right)L\left(\delta,x\right)+\delta\left|\cos\delta\right|e^{-\frac{1}{2}\delta}L^{\frac{1}{2}}(\delta,x)+4\beta_1(\delta)L^{\frac{1}{2}}\left(\delta,x\right)\\
&= \xi(\delta)L\left(\delta,x\right)+\psi(\delta)L^{\frac{1}{2}}\left(\delta,x\right),
\end{aligned}$$

where $\xi(\delta)=\frac{2}{1+\delta}-\delta\left|\cos\delta\right|$ and $\psi(\delta)=4\beta_1(\delta)+\delta\left|\cos\delta\right|e^{-\frac{1}{2}\delta}$.
We can show that ξ is $(\psi,\frac{1}{2})-GUPES$.

On the one hand, we have for all $\delta \geq \delta_0 \geq 0$:

$$\int_{\delta_0}^{\delta} \xi(\epsilon) d\epsilon \leq -\frac{4}{3\pi}(\delta-\delta_0) + 2\ln\big(1+\frac{3}{2}\pi\big)+2,$$

(see Appendix 3 in [38]). On the other hand for all $\delta \geq \delta_0 \geq 0$, we have

$$\begin{aligned}\int_{\delta_0}^{\delta} |\psi(\epsilon)| \exp\big(\frac{1}{2}\int_{\epsilon}^{\delta} \xi\,(t)\,\mathrm{d}t\big) d\epsilon &\leq \int_{\delta_0}^{\delta} |\psi(\epsilon)| \exp\Big(-\frac{2}{3\pi}(\delta-\epsilon) + \ln\left(1+\frac{3}{2}\pi\right)+1\Big) d\epsilon \\ &\leq \frac{3\pi}{2}\|\psi(\delta)\|_{\infty} \exp\Big(\ln\big(1+\frac{3}{2}\pi\big)+1\Big).\end{aligned}$$

Hence, all the conditions in Theorem 3.3 are satisfied. Then the nominal system is $GUPEy-S$. Moreover, for all $\delta \geq 0$ and $x \in \mathbb{R}^4$ we have:

$$\begin{aligned}\frac{\partial L}{\partial x}(\delta,x)g(\delta,x,u) &= \dot{x_1}(\delta)x_1(\delta) + \dot{x_2}(\delta)x_2(\delta) \\ &= \Big[\frac{\lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}} + (\lambda(\delta)+\beta_2(\delta))\big(x_1^2+e^{-\delta}\big)^{\frac{1}{2}}}{1+x_1^2} + 2\beta_1(\delta)x_2\Big]x_1(\delta) \\ &\quad + \Big[\frac{\lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}} + (\lambda(\delta)+\beta_2(\delta))\big(x_1^2+x_2^2+e^{-\delta}\big)^{\frac{1}{2}}}{1+x_2^2} - 2\beta_1(\delta)x_1\Big]x_2(\delta) \\ &= \Big[\lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}} + (\lambda(\delta)+\beta_2(\delta))\big(x_1^2+x_2^2+e^{-\delta}\big)^{\frac{1}{2}}\Big]\frac{x_1(\delta)}{1+x_1^2} \\ &\quad + \Big[\lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}} + (\lambda(\delta)+\beta_2(\delta))\big(x_1^2+x_2^2+e^{-\delta}\big)^{\frac{1}{2}}\Big]\frac{x_2(\delta)}{1+x_2^2}\end{aligned}$$

Since,

$$\frac{x}{1+x^2} \leq \frac{1}{2}.$$

Then, we obtain:

$$\begin{aligned}\frac{\partial L}{\partial x}(\delta,x)g(\delta,x,u) &\leq \lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}} + (\lambda(\delta)+\beta_2(\delta))\big(x_1^2+x_2^2+e^{-\delta}\big)^{\frac{1}{2}} \\ &\leq \lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}} + (\lambda(\delta)+\beta_2(\delta))\big(2\frac{1}{2}(x_1^2+x_2^2)+e^{-\delta}\big)^{\frac{1}{2}} \\ &\leq \lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}} + 2\big(\lambda(\delta)+\beta_2(\delta)\big)\big(\frac{1}{2}(x_1^2+x_2^2)+e^{-\delta}\big)^{\frac{1}{2}} \\ &\leq \lambda(\delta)\sigma(|u(\delta)|)(x_1^2+x_2^2)^{\frac{1}{2}} + 2\big(\psi(\delta)+\beta_2(\delta)\big)L^{\frac{1}{2}}\,(\delta,x)\,.\end{aligned}$$

Hence, all the conditions in Theorem 3.6 are satisfied. Consequently, the system (4.3) is $ISPy-S$.

5. Conclusion

This paper has studied practical partial stability analysis of a large class for nonlinear time-varying systems by using Lyapunov's second method. The proposed paractical partial stability theorem does not require that the time-derivative of the Lyapunov function is negative definite. The analysis of practical partial stability is facilitated through the comparison principle and the utilization of scalar stable functions. These concepts encompass various forms of practical partial stability, including asymptotic, exponential, input-to-state, and integral input-to-state practical partial stability. Some illustrative examples are provided to illustrate the effectiveness of the obtained conditions.

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