Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 33, 2006, Pages 54–66
ISSN: 1223-6934

# Solving some special functional equations by a general geometrical method

Alina Olteanu and Octav Olteanu

Abstract. In Section 1 we recall improved versions of some results of [19] concerning the equation

$$g(x) = g(f(x)) \quad \forall x \in A$$

where $f$ is the unknown strictly decreasing function (or operator), while $g$ is a given function (respectively operator), which satisfies some conditions. Such type results are proved in [19] and used in [16], [17], [8]. The existence of $f$ is proved by constructing it effectively. In Section 2 we apply the general results of Section 1 to some concrete functions and operators $g$. The corresponding special solutions $f$ have some additional nice properties, some of them being related to integers.

*2000 Mathematics Subject Classification.* 26A03, 26A06, 26A09, 26A48, 26C99, 26D07, 47A10, 47A60, 33B15, 11D41, 11D68.
*Key words and phrases.* the functional equation $g(x) = g(f(x))$, $\forall x \in A$, decreasing functions $f$ with nice properties vis-à-vis to integers, decreasing operators with nice properties.

## 1. General results

We recall the following improved versions of some results of [19]. These results are used in [16]-[17] to solve concrete operatorial equations (see also [7] and especially [8]).

**1.1. Theorem**. *Let $\alpha, \beta \in \bar{\mathbf{R}}$, $\alpha < \beta$, $a \in ]\alpha, \beta[$, $g : ]\alpha, \beta[ \to \mathbf{R}$ a continuous function. Assume that*

(a) $\lim_{x \searrow \alpha} g(x) = \lim_{x \nearrow \beta} g(x) = \lambda \in \bar{\mathbf{R}}$,

(b) *$g$ is strictly decreasing on $]\alpha, a]$ and strictly increasing on $[a, \beta[$.*

*Then there exists $f : ]\alpha, \beta[ \to ]\alpha, \beta[$ such that*

$$g(x) = g(f(x)), \quad \forall x \in ]\alpha, \beta[ \tag{1}$$

*and $f$ has the following properties*:

(i) *$f$ is strictly decreasing on $]\alpha, \beta[$,*

$$\lim_{x \searrow \alpha} f(x) = \beta, \quad \lim_{x \nearrow \beta} f(x) = \alpha;$$

(ii) *$a$ is the only fixed point of $f$;*

(iii) *we have $f^{-1} = f$ on $]\alpha, \beta[$;*

(iv) *$f$ is continuous*;

(v) *if we assume in addition that $g \in C^n(]\alpha, \beta[ \setminus \{a\})$, $n \in \mathbf{N} \cup \{\infty\}$, $n \geq 1$, then $g \in C^n(]\alpha, \beta[ \setminus \{a\})$;*

(vi) *if $g$ is derivable on $]\alpha, \beta[ \setminus \{a\}$, so is $f$;*

*Received*: December 12, 2004.

(vii) *if* $g \in C^2(]\alpha,\beta[)$, $g''(a) \neq 0$ *and there exists*

$$\rho_1 := \lim_{x\to a} f'(x) \in \bar{\mathbf{R}},$$

*then* $f \in C^1(]\alpha,\beta[) \cap C^2(]\alpha,\beta[\setminus\{a\})$ *and* $f'(a) = -1$;

(viii) *if* $g \in C^3(]\alpha,\beta[)$, $g''(a) \neq 0$ *and there exist*

$$\rho_1 := \lim_{x\to a} f'(x) \in \bar{\mathbf{R}}$$

*and*

$$\rho_2 := \lim_{x\to a} f''(x) \in \mathbf{R},$$

*then* $f \in C^2(]\alpha,\beta[) \cap C^3(]\alpha,\beta[\setminus\{a\})$ *and*

$$f''(a) = \rho_2 = -\frac{2}{3}\cdot\frac{g^{(3)}(a)}{g''(a)}; \tag{2}$$

(ix) *put* $g_l := g|_{]\alpha,a]}$, $g_r := g|_{[a,\beta[}$; *then for any* $x_0 \in ]\alpha,a]$, *we have*

$$f(x_0) = (g_r^{-1}\circ g_l)(x_0) = \sup\{x \in [a,\beta[\,; g_r(x) \leq g_l(x_0)\};$$

*for any* $x_0 \in [a,\beta[$, *we have*

$$f(x_0) = (g_l^{-1}\circ g_r)(x_0) = \inf\{x \in ]\alpha,a]\,; g_l(x) \leq g_r(x_0)\}.$$

For the proof of this theorem see [19].

Next we state the operatorial version of Theorem 1.1. Denote by $Izom_+(X)$ the set of all vector space izomorphisms $T : X \to X$ which apply $X_+$ onto itself.

**1.2. Theorem**. *Let* $X$ *be an order-complete vector lattice,* $a \in X$, $A_l$ *a convex subset such that*

$$a \in A_l \subset \{x \in X;\ x \leq a\}.$$

$A_r$ *a convex subset such that*

$$a \in A_r \subset \{x \in X;\ x \geq a\}.$$

*Let* $g_l : A_l \to X$ *be a convex operator such that* $\forall x \in A_l \setminus \{a\}$, *we have*

$$\partial g_l(x) \cap (-Izom_+(X)) \neq \Phi$$

*(for notations see* [19] *or* [9]*).*

*Let* $g_r : A_r \to X$ *be a convex operator such that* $\forall x \in A_r \setminus \{a\}$, *we have*

$$\partial g_r(x) \cap (Izom_+(X)) \neq \Phi.$$

*Assume also that*

$$g_l(a) = g_r(a) \quad \textit{and} \quad R(g_l) = R(g_r),$$

*where* $R(g)$ *is the range of* $g$.

*Let* $g : A := A_l \cup A_r \to X$ *be defined by*

$$g(x) := \begin{cases} g_l(x), & x \in A_l, \\ g_r(x), & x \in A_r. \end{cases}$$

*Then there exists* $F : A \to A$ *such that*

$$g(x) = g(F(x)), \quad \forall x \in A,$$

$F$ *is strictly decreasing on* $A$ *and has the properties*:

(a) *a is the only fixed point of* $F$;

(b) *there exists* $F^{-1}$ *and* $F^{-1} = F$ *on* $A$;

(c)

$$F(x_0) = g_r^{-1}(g_l(x_0)) = \sup\{x \in A_r; g_r(x) \leq g_l(x_0)\} \quad \forall x_0 \in A_l,$$

$$F(x_0) = g_l^{-1}(g_r(x_0)) = \inf\{x \in A_l; g_l(x) \leq g_r(x_0)\} \quad \forall x_0 \in A_r.$$

The proof of this theorem is similar to the proof of Theorem 1.10 [19], p.72-74. For the proof of (c) see also [9].

## 2. Applications

**2.1. Theorem**. *Let $p, q \in \mathbf{N} \setminus \{0\}$. Then there exists a function $f : \mathbf{R} \to \mathbf{R}$ such that*

$$x^{2p+1}(x-1)^{2q+1} = [f(x)]^{2p+1}[f(x)-1]^{2q+1}, \forall x \in \mathbf{R} \tag{1'}$$

*and $f$ has the following properties*:

(a) *$f$ is strictly decreasing on $\mathbf{R}$, $\lim_{x \searrow -\infty} f(x) = +\infty$, $\lim_{x \nearrow +\infty} f(x) = -\infty$;*

(b) $a := \dfrac{2p+1}{2(p+q+1)}$ *is the only fixed point of $f$;*

(c) $f^{-1} = f$ *on* $\mathbf{R}$;

(d) *$f$ is continuous on* $\mathbf{R}$;

(e) $f \in C^\infty(\mathbf{R} \setminus \{a\})$;

(f) *if there exists*

$$\rho_1 := \lim_{x \to a} f'(x) \in \bar{\mathbf{R}},$$

*then $f \in C^1(\mathbf{R})$ and $f'(a) = \rho_1 = -1$;*

(g) *if there exists $\rho_1$ and*

$$\rho_2 : \lim_{x \to a} f''(x) \in \mathbf{R},$$

*then $f \in C^2(\mathbf{R})$ and*

$$f''(a) = \frac{16}{3} \cdot \frac{(p+q+1)(q-p)}{(2p+1)(2q+1)}; \tag{2'}$$

(h) *if $q < p$, then there exists $\delta > 0$ sufficiently small such that*

$$f(x) + x \leq 2a \quad \forall x \in ]a-\delta, a+\delta[,$$

*and the inequality is strict for $x \neq a$; if $q > p$, then the opposite inequality holds*;

(i) *we have the following formulae for the construction of $f$*:

$$f(x_0) = \sup\{x; x \geq a, x^{2p+1}(x-1)^{2q+1} \leq x_0^{2p+1}(x_0-1)^{2q+1}\}, \quad \forall x_0 \leq a;$$

$$f(x_0) = \inf\{x; x \leq a, x^{2p+1}(x-1)^{2q+1} \leq x_0^{2p+1}(x_0-1)^{2q+1}\}, \quad \forall x_0 \geq a;$$

(j) *the straight line*

$$y = -x + \frac{2q+1}{p+q+1}$$

*is an asymptote for the graph of $f$ at $-\infty$ and at $+\infty$;*

(k) *if $\dfrac{2q+1}{p+q+1} \notin \mathbf{Z}$ (in particular if $q < p$), then there exists $M > 0$ sufficiently large such that*

$$m \in \mathbf{Z},\ |m| > M \Rightarrow f(m) \notin \mathbf{Z};$$

(l) *we have $f(0) = 1$, $f(1) = 0$, and*

$$f(2) \in \mathbf{Z}$$

*if and only if $p = q$; in this case,*

$$f(x) = 1 - x, \quad \forall x \in \mathbf{R}.$$

*(hence $f(\mathbf{Z}) = \mathbf{Z}$).*

*Proof.* One applies Theorem 1.1 to $\alpha = -\infty$, $\beta = +\infty$, $g(x) := x^{2p+1}(x-1)^{2q+1}$, $x \in \mathbf{R}$, $\lambda = +\infty$. We obviously have $g \in C^\infty(\mathbf{R})$ and

$$g'(x) = x^{2p}(x-1)^{2q}[2(p+q+1)x - (2p+1)], \quad x \in \mathbf{R}.$$

This implies $g'(x) < 0$ for $x < \dfrac{2p+1}{2(p+q+1)} =: a$, $x \neq 0$, $g'(x) = 0$ for $x = a$ and $g'(x) > 0$ for $x > a$, $x \neq 1$.

Thus $g$ satisfies the hypothesis of Theorem 1.1 and, by this Theorem, there exists $f : \mathbf{R} \to \mathbf{R}$ such that (1′) and (i) - (ix) of Theorem 1.1 hold. The conclusions (a) - (f) and (i) of Theorem 2.1 follow from the corresponding conclusions of Theorem 1.1. To prove (g), we have to compute $g''(a)$ and $g^{(3)}(a)$, where $a = \frac{2p+1}{2(p+q+1)}$ is the minimum point of $g$. A direct computation leads to:

$$\begin{aligned} g''(x) &= \{x^{2p}(x-1)^{2q}[2(p+q+1)x - (2p+1)]\}' = \\ &= [2px^{2p-1}(x-1)^{2q} + 2qx^{2p}(x-1)^{2q-1}][2(p+q+1)x - (2p+1)] + \\ &+ 2(p+q+1)x^{2p}(x-1)^{2q}. \end{aligned}$$

In particular, for $x = a = \dfrac{2p+1}{2(p+q+1)}$, we have

$$g''(a) = 2(p+q+1)\left(\frac{2p+1}{2(p+q+1)}\right)^{2p}\left(\frac{2p+1}{2(p+q+1)} - 1\right)^{2q} =$$

$$= \frac{(2p+1)^{2p}(2q+1)^{2q}}{2^{2p+2q-1}(p+q+1)^{2p+2q-1}}.$$

Derivating once again, one obtains

$$\begin{aligned} &g^{(3)}(a) = (g''(x))'|_{x=a} = \\ &= \{[2px^{2p-1}(x-1)^{2q} + 2qx^{2p}(x-1)^{2q-1}][2(p+q+1)x - (2p+1)] + \\ &+ 2(p+q+1)x^{2p}(x-1)^{2q}\}'|_{x=a} = \\ &= [2pa^{2p-1}(a-1)^{2q} + 2qa^{2p}(a-1)^{2q-1}] \cdot 2(p+q+1) + \\ &+ 2(p+q+1)[2pa^{2p-1}(a-1)^{2q} + 2qa^{2p}(a-1)^{2q-1}] = \\ &= 2(p+q+1)[4pa^{2p-1}(a-1)^{2q} + 4qa^{2p}(a-1)^{2q-1}] = \\ &= 8(p+q+1)a^{2p-1}(a-1)^{2q-1} \cdot [(p+q)a - p] = \\ &= 8(p+q+1)\left[(p+q)\frac{2p+1}{2(p+q+1)} - p\right]\frac{(2p+1)^{2p-1}}{2^{2p-1}(p+q+1)^{2p-1}} \cdot \\ &\cdot \left[\frac{2p+1-2p-2q-2}{2(p+q+1)}\right]^{2q-1} = \\ &= \frac{8(p+q+1)}{2(p+q+1)} \cdot (2p^2 + 2pq + p + q - 2p^2 - 2pq - 2p) \cdot \\ &\cdot \frac{(2p+1)^{2p-1}(-1)(2q+1)^{2q-1}}{2^{2p+2q-2}(p+q+1)^{2p+2q-2}} = \frac{4(p-q)(2p+1)^{2p-1}(2q+1)^{2q-1}}{2^{2(p+q-1)}(p+q+1)^{2(p+q-1)}}. \end{aligned}$$

Replacing these values of $g''(a)$, $g^{(3)}(a)$ into formula $f''(a) = -\frac{2}{3} \cdot \frac{g^{(3)}(a)}{g''(a)}$, we find

$$f''(a) = -\frac{2}{3} \cdot \frac{4(p-q)(2p+1)^{2p-1}(2q+1)^{2q-1}}{2^{2(p+q-1)}(p+q+1)^{2(p+q-1)}} \cdot$$

$$\cdot \frac{2^{2p+2q-1}(p+q+1)^{2p+2q-1}}{(2p+1)^{2p}(2q+1)^{2q}} = \frac{16}{3} \cdot \frac{(q-p)(p+q+1)}{(2p+1)(2q+1)}$$

Thus (g) is proved. In particular, if $q < p$, then $f''(a) < 0$ (and $f \in C^2(\mathbf{R})$). These lead to the existence of a $\delta > 0$ sufficiently small such that $f''(x) < 0 \; \forall x \in ]a-\delta, a+\delta[$, hence $f$ is strictly concave on $]a-\delta, a+\delta[$.

It follows that

$$f(x) - f(a) \leq f'(a)(x-a) = a - x, \quad \forall x \in ]a-\delta, a+\delta[.$$

Since $f(a) = a$, the last relation may be rewritten as

$$f(x) + x \leq 2a, \quad \forall x \in ]a-\delta, a+\delta[\;,$$

and the inequality is strict for $x \neq a$. If $q > p$, then $f$ is strictly convex on a neighbourhood of $a$, and the inequality

$$f(x) + x \geq 2a$$

holds on this neighbourhood. Thus (h) is proved. As (i) follows from Theorem 1.1, now we have to prove (j), i.e.

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = -1 \tag{3}$$

and

$$\lim_{x \to \pm\infty} [f(x) + x] = \frac{2q+1}{p+q+1}. \tag{4}$$

We start by rewriting $(1')$ as

$$1 = \left[\frac{f(x)}{x}\right]^{2p+1} \cdot \left[\frac{f(x)-1}{x-1}\right]^{2q+1} = \left[\frac{f(x)}{x}\right]^{2p+1} \cdot \left[\frac{\frac{f(x)}{x} - \frac{1}{x}}{1 - \frac{1}{x}}\right]^{2q+1}, \quad x \in \mathbf{R} \setminus \{0,1\}.$$

If $x_n \to -\infty$, then for any subsequence $\left(\frac{f(x_{k_n})}{x_{k_n}}\right)_n$ which converges in the compact $\bar{\mathbf{R}}$ to $\lambda$, we have

$$1 = \left[\frac{f(x_{k_n})}{x_{k_n}}\right]^{2p+1} \cdot \left[\frac{\frac{f(x_{k_n})}{x_{k_n}} - \frac{1}{x_{k_n}}}{1 - \frac{1}{x_{k_n}}}\right]^{2q+1} \to \lambda^{2p+1} \cdot \lambda^{2q+1} = \lambda^{2(p+q+1)}.$$

This leads to

$$\lambda \in \{-1, 1\}$$

and, since $\lim_{x_{k_n} \to -\infty} f(x_{k_n}) = +\infty$, we have

$$\lambda = -1.$$

This proves that

$$\lim_{x_n \to -\infty} \frac{f(x_n)}{x_n} = -1$$

for any sequence $(x_n)_n$ with $x_n \to -\infty$. Thus (3) is proved for $x \to -\infty$. Now we compute

$$\begin{aligned}
&\lim_{x\to-\infty}[f(x)+x] = \lim_{x\to-\infty}\frac{\frac{f(x)}{x}+1}{x^{-1}} = \frac{0}{0} = \\
&= \lim_{x\to-\infty}[f'(x)x - f(x)](-1) = \lim_{x\to-\infty}[f(x) - xf'(x)] = \\
&= \lim_{x\to-\infty}\left[f(x) - x\cdot\frac{f(x)[f(x)-1]}{x(x-1)}\cdot\frac{2(p+q+1)x-(2p+1)}{2(p+q+1)f(x)-(2p+1)}\right],
\end{aligned}$$

since from $(1')$, by derivation, we get

$$\begin{aligned}
f'(x) &= \frac{x^{2p}(x-1)^{2q}[(2p+1)(x-1)+(2q+1)x]}{[f(x)]^{2p}[f(x)-1]^{2q}[(2p+1)(f(x)-1)+(2q+1)f(x)]} = \\
&= \frac{f(x)[f(x)-1]x^{2p}(x-1)^{2q}}{[f(x)]^{2p+1}[f(x)-1]^{2q+1}}\cdot\frac{2(p+q+1)x-(2p+1)}{2(p+q+1)f(x)-(2p+1)} = \\
&\overset{(1')}{=} \frac{f(x)[f(x)-1]}{x(x-1)}\cdot\frac{2(p+q+1)x-(2p+1)}{2(p+q+1)f(x)-(2p+1)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{x\to-\infty}[f(x)+x] = \\
&= \lim_{x\to-\infty} f(x)\left\{1 - \frac{[f(x)-1][2(p+q+1)x-(2p+1)]}{(x-1)[2(p+q+1)f(x)-(2p+1)]}\right\} = \\
&= \lim_{x\to-\infty} f(x)\cdot\frac{2(p+q+1)[(x-1)f(x)-x(f(x)-1)]+(2p+1)[f(x)-x]}{(x-1)[2(p+q+1)f(x)-(2p+1)]} = \\
&= \lim_{x\to-\infty}\frac{f(x)}{x-1}\cdot\lim_{x\to-\infty}\frac{[x-f(x)](2q+1)}{2(p+q+1)f(x)-(2p+1)} = \\
&= [-(2q+1)]\cdot\lim_{x\to-\infty}\frac{1-\frac{f(x)}{x}}{2(p+q+1)\frac{f(x)}{x}-\frac{(2p+1)}{x}} = \\
&= -(2q+1)\cdot\frac{2}{-2(p+q+1)} = \frac{2q+1}{p+q+1},
\end{aligned}$$

so that (4) is proved for $x \to -\infty$.

Thus the straight line $y = -x + \frac{2q+1}{p+q+1}$ is an asymptote at $-\infty$ for the graph of $f$. On the other hand, since $f^{-1} = f$, the graph of $f$ is symmetrical with respect to the diagonal $\Delta = \{(x,x); x \in \mathbf{R}\}$, and so is the straight line $y = -x + \frac{2q+1}{p+q+1}$. These informations lead to the fact that the same straight line is an asymptote at $+\infty$ for the graph of $f$. Thus (j) is proved. To prove (k), assume that $\mu := \frac{2p+1}{p+q+1} \notin \mathbf{Z}$. Then we have

$$[\mu] < \mu < [\mu]+1.$$

¿From $\lim_{x\to\pm\infty}[f(x)+x] = \mu \in ][\mu],[\mu]+1[$, it follows that for $M > 0$ sufficiently large, we have $f(x)+x \in ][\mu],[\mu]+1[$, $\forall x$ such that $|x| > M$. If $m \in \mathbf{Z}$ and $|m| > M$, then

$$f(m)+m \in ][\mu],[\mu]+1[\ ,$$

which implies $f(m)+m\notin \mathbf{Z}$, i.e. $f(m)\notin \mathbf{Z}$. To finish the proof, we have to prove (l). From (1′) written for $x=0$, one obtains

$$[f(0)]^{2p+1}[f(0)-1]^{2q+1}=0,$$

which is equivalent to $f(0)\in\{0,1\}$. But $a:=\dfrac{2p+1}{2(p+q+1)}\in]0,1[$ is the only fixed point of $f$, so that $f(0)\neq 0$. It follows that $f(0)=1$. Similarly, $f(1)=0$. Assume now that

$$n:=f(2)\in\mathbf{Z}.$$

Then from (1′) written for $x=2$ one obtains

$$2^{2p+1}=n^{2p+1}(n-1)^{2q+1}. \tag{5}$$

On the other hand, $f$ being decreasing, we have

$$n:=f(2)<f(1)=0.$$

Thus $n$ is a negative integer. From this and from (5) we infer that

$$n=-1$$

and hence

$$2^{2p+1}\overset{(5)}{=}(-1)(-2)^{2q+1}=2^{2q+1}.$$

This leads to $p=q$. In this case, (1′) is equivalent to

$$x(x-1)=f(x)[f(x)-1],$$

which may be consider as an algebraic equation of second degree in the unknown $f(x)$, namely

$$[f(x)]^2-f(x)+x(1-x)=0.$$

The solution is given by

$$\begin{aligned} f(x) &\in \left\{\frac{1-(1-4x+4x^2)^{1/2}}{2},\frac{1+(1-4x+4x^2)^{1/2}}{2}\right\}= \\ &= \left\{\frac{1-(2x-1)}{2},\frac{1+(2x-1)}{2}\right\}=\{1-x,x\},\ x\in\mathbf{R}. \end{aligned}$$

Since $f$ is decreasing, we must have

$$f(x)=1-x,\quad x\in\mathbf{R}.$$

Of course, in this case we have

$$f(2)=-1\quad\text{and}\quad f(\mathbf{Z})=\mathbf{Z}.$$

Conversely, if $p=q$, then we have already observe that (1′) leads to $f(x)=1-x$, and hence $f(2)=-1\in\mathbf{Z}$. The proof is complete. □

Now we consider an application of Theorem 1.1 in which the interval $]\alpha,\beta[$ is bounded.

**2.2. Theorem**. *Let $\alpha\in]0,1[$, $\beta\in]1,2[$. Then there exists a function $f:[0,1]\to[0,1]$ such that*

$$x^\beta-x^\alpha=[f(x)]^\beta-[f(x)]^\alpha,\forall x\in[0,1]$$

*and*

(a) *$f$ is strictly decreasing on $[0,1]$, $f(0)=1$, $f(1)=0$;*

(b) *$a_1:=\left(\dfrac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}}$ is the only fixed point of $f$;*

(c) *$f^{-1}=f$ on $[0,1]$;*

(d) *$f$ is continuous on* $[0,1]$;
(e) $f \in C^\infty(\mathbf{R} \setminus \{a_1\})$;
(f) *if there exists*

$$\rho_1 : \lim_{x \to a_1} f'(x) \in \bar{\mathbf{R}},$$

*then* $f \in C^1(]0,1])$ *and* $f'(a_1) = -1$;
(g) *if there exists* $\rho_1 \in \bar{\mathbf{R}}$ *and*

$$\rho_2 := \lim_{x \to a_1} f''(x) \in \mathbf{R},$$

*then* $f \in C^2(]0,1[)$ *and*

$$f''(a_1) = -\frac{2}{3a_1} \cdot \frac{(\beta-1)(\beta-2) - (\alpha-1)(\alpha-2)}{\beta - \alpha} > 0;$$

(h) *if there exist* $\rho_1, \rho_2$ *as above, then there exists* $\delta > 0$ *such that*

$$f(x) + x \geq 2a_1 \quad \forall x \in ]a_1 - \delta, a_1 + \delta[\ ;$$

(i) *we have*

$$f(x_0) = \sup\{x \in [a_1, 1[\ ; x^\beta - x^\alpha \leq x_0^\beta - x_0^\alpha\} \quad \forall x_0 \in ]0, a_1]\ ;$$

$$f(x_0) = \inf\{x \in ]0, a_1]\ ; x^\beta - x^\alpha \leq x_0^\beta - x_0^\alpha\} \quad \forall x_0 \in [a_1, 1[\ .$$

(j) *Put* $(\mathbf{N}^*)^{-1} := \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$. *If* $\alpha \in \mathbf{Q}$ *and* $\beta = \alpha + 1$, *then: there exists* $n \in \mathbf{N} \setminus \{0,1\}$ *such that*

$$f\left(\frac{1}{n}\right) \in (\mathbf{N}^*)^{-1}$$

*if and only if*

$$\alpha = \frac{1}{n-1}.$$

*Proof.* We apply Theorem 1.1 to $\alpha = 0$, $\beta = 1$, $g(x) := x^\beta - x^\alpha$, $\lambda = 0$. Obviously, $g \in C^\infty([0,1])$. Then

$$g'(x) = x^{\alpha-1}(\beta x^{\beta-\alpha} - \alpha), \quad x \in [0,1],$$

which leads to

$$g'(x) < 0 \quad \text{for} \quad 0 \leq x < \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}} =: a_1,$$

$$g'(x) = 0 \quad \text{for} \quad x = a_1,$$

$$g'(x) > 0 \quad \text{for} \quad x \in ]a_1, 1].$$

Thus $g$ satisfies the hypothesis of Theorem 1.1 for $a := a_1 = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}} \in ]0,1[$ and, from Theorem 1.1 we infer that there exists $f : ]0,1[ \to ]0,1[$ such that

$$g(x) = x^\beta - x^\alpha = g(f(x)) = [f(x)]^\beta - [f(x)]^\alpha, \quad x \in ]0,1[\ ,$$

$\lim_{x \searrow 0} f(x) = 1$, $\lim_{x \nearrow 1} f(x) = 0$ and the properties (i)-(ix) of Theorem 1.1 hold. These results lead to the fact that $f$ satisfies (a)-(f) of Theorem 2.2.

To prove $(g)$, we compute

$$g''(x) = \beta(\beta-1)x^{\beta-2} - \alpha(\alpha-1)x^{\alpha-2} > 0, \quad \forall x > 0,$$

$$g^{(3)}(x) = \beta(\beta-1)(\beta-2)x^{\beta-3} - \alpha(\alpha-1)(\alpha-2)x^{\alpha-3}$$

and

$$\begin{aligned} f''(a_1) &\overset{(2)}{=} -\frac{2}{3}\cdot\frac{g^{(3)}(a_1)}{g''(a_1)} = \\ &= -\frac{2}{3}\cdot\frac{\beta(\beta-1)(\beta-2)a_1^{\beta-3}-\alpha(\alpha-1)(\alpha-2)a_1^{\alpha-3}}{\beta(\beta-1)a_1^{\beta-2}-\alpha(\alpha-1)a_1^{\alpha-2}} = \\ &= -\frac{2}{3}\cdot\frac{a_1^{\alpha-3}[\beta(\beta-1)(\beta-2)a_1^{\beta-\alpha}-\alpha(\alpha-1)(\alpha-2)]}{a_1^{\alpha-2}[\beta(\beta-1)a_1^{\beta-\alpha}-\alpha(\alpha-1)]} = \\ &= -\frac{2}{3\cdot a_1}\cdot\frac{\beta(\beta-1)(\beta-2)\cdot\dfrac{\alpha}{\beta}-\alpha(\alpha-1)(\alpha-2)}{\beta(\beta-1)\cdot\dfrac{\alpha}{\beta}-\alpha(\alpha-1)} = \\ &= -\frac{2}{3\cdot a_1}\cdot\frac{(\beta-1)(\beta-2)-(\alpha-1)(\alpha-2)}{\beta-\alpha}. \end{aligned}$$

Thus

$$\begin{aligned} &\operatorname{sign} f''(a_1) = \operatorname{sign}\left(-\frac{2}{3a_1(\beta-\alpha)}\right)\cdot \\ &\cdot\operatorname{sign}\,[(\beta-1)(\beta-2)-(\alpha-1)(\alpha-2)] = (-1)(-1) = +1, \end{aligned}$$

which finishes the proof of (g).

Now (h) follows easily from $(g)$, since $f''(a_1) > 0$ and $f \in C^2(]0,1[)$ imply the strictly convexity of $f$ on an interval $]a_1-\delta, a_1+\delta[$. This leads to

$$f(x) \geq f(a_1) + f'(a_1)(x-a_1) = a_1 - (x-a_1) = 2a_1 - x,$$

i.e.

$$f(x) + x \geq 2a_1, \quad \forall x \in ]a_1-\delta, a_1+\delta[$$

and we have equality if and only if $x = a_1$.

The assertion (i) of Theorem 2.2 follows from (ix) Theorem 1.1. The proof will be finished if we prove (j). Assume that there exist $n, p \in \{2, 3, \ldots,\}$ such that

$$f\left(\frac{1}{n}\right) = \frac{1}{p},$$

i.e.

$$\frac{1}{n^\beta} - \frac{1}{n^\alpha} = \frac{1}{p^\beta} - \frac{1}{p^\alpha}, \tag{6}$$

where $\alpha = \dfrac{l}{k} \in \mathbf{Q}$ $(l, k \in \mathbf{N}\setminus\{0\})$, $\beta = \alpha + 1 = \dfrac{l}{k} + 1 = \dfrac{l+k}{k}$. In these conditions, (6) may be rewritten as

$$\frac{1}{n^\alpha}\left(\frac{1}{n} - 1\right) = \frac{1}{p^\alpha}\left(\frac{1}{p} - 1\right)$$

or, equivalently

$$\left(\frac{p}{n}\right)^\alpha = \frac{\frac{1}{p}-1}{\frac{1}{n}-1} = \frac{p-1}{n-1}\cdot\frac{n}{p},$$

or further

$$\begin{aligned} \frac{p^{\alpha+1}}{n^{\alpha+1}} = \frac{p-1}{n-1} &\Leftrightarrow (n-1)p^{\frac{l+k}{k}} = (p-1)n^{\frac{l+k}{k}} \Leftrightarrow \\ &\Leftrightarrow (n-1)^k p^{l+k} = (p-1)^k n^{l+k}, \end{aligned} \tag{6'}$$

where $n, p \in \{2, 3, \ldots\}$. Since $(n, n-1) = (p, p-1) = 1$, any prime divisor of $p$ is a divisor of $n$ and any prime divisor of $n$ is a divisor of $p$. It follows easily from (6′) that

$$p = n.$$

It follows that

$$f\left(\frac{1}{n}\right) = \frac{1}{n},$$

which implies (via (b)), that

$$a_1 := \frac{\alpha}{\alpha+1} = \frac{1}{n},$$

which is equivalent to

$$\alpha = \frac{1}{n-1}.$$

Thus an implication of (j) is proved. Conversely, assume that $\beta = \alpha+1$ and $\alpha = \dfrac{1}{n-1}$, where $n \in \{2, 3, \ldots\}$. Then

$$f\left(\frac{1}{n}\right) = f\left(\frac{\alpha}{\alpha+1}\right) = f(a_1) \overset{(b)}{=} a_1 = \frac{\alpha}{\alpha+1} = \frac{1}{n} \in (\mathbf{N}^*)^{-1}.$$

The proof is complete. □

Next we prove an operatorial version of Theorem 2.2, as an application of Theorem 1.2.

Let $H$ be a Hilbert space. Denote by $\mathcal{A}(H)$ the real vector space of all self-adjoint operators acting on $H$. Let $T$ be a fixed element of $\mathcal{A}(H)$. Put

$$\mathcal{A}_1 = \mathcal{A}_1(T) := \{U \in \mathcal{A}(H); UT = TU\},$$
$$X := \{U \in \mathcal{A}_1; UV = VU \quad \forall V \in \mathcal{A}_1\}$$
$$(\text{see [5], p.}303-305)$$
$$X_+ := \{U \in X; \langle U(h), h\rangle \geq 0, \quad \forall h \in H\}.$$

It is known that $X$ is an order-complete vector lattice and a commutative algebra of operators.

**2.3. Theorem**. *Let $\alpha \in ]0,1[$, $\beta \in ]1,2[$, $a_1 := \left(\dfrac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}}$. Let*

$$A_l := \{U \in X;\ \sigma(U) \subset ]0, a_1[\} \cup \{a_1 I\},$$
$$A_r := \{U \in X;\ \sigma(U) \subset ]a_1, 1[\} \cup \{a_1 I\},$$

*where $\sigma(U)$ is the spectrum of $U$ and $I$ is the identity operator on $H$.*

*Put $a := a_1 I \in ]0, I[$. Let*

$$A := A_l \cup A_r.$$

*Then there exists a strictly decreasing map*

$$F : A \to A$$

*such that*

$$U^\beta - U^\alpha = [(F(U)]^\beta - [F(U)]^\alpha \quad \forall U \in A$$

*and $F$ has the following properties*:

(i) *$a =: a_1 I$ is the only fixed point of $F$;*

(ii) *$F$ is invertible and $F^{-1} = F$ on $A$;*

(iii) *$F$ can be constructed by formulae*

$$F(U_0) = \sup\{U \in A_r;\ U^\beta - U^\alpha \leq U_0^\beta - U_0^\alpha\} \quad \forall U_0 \in A_l,$$

$$F(U_0) = \inf\{U \in A_l;\ U^\beta - U^\alpha \leq U_0^\beta - U_0^\alpha\} \quad \forall U_0 \in A_r.$$

*Proof.* We apply Theorem 1.2 to $X, a, A$ defined above and to $g : A \to X$,

$$g(U) := U^\beta - U^\alpha, \quad U \in A.$$

In [19] p. 79-80 we proved that

$$U \mapsto U^n$$

is convex on $X_+$ (where $n \in \mathbf{N} \setminus \{0\}$). That proof has not used that $n \in \mathbf{N}$, but only the convexity of the map

$$x \mapsto x^\beta, \quad x \in \mathbf{R}_+,$$

which is valid for any real $\beta \geq 1$. Thus

$$U \mapsto U^\beta, \quad U \in X_+ \quad (\beta \geq 1),$$

is a convex operator by the same proof.

Similarly,

$$U \to U^\alpha,\ U \in X_+, \quad \alpha \in ]0,1[,$$

is concave on $X_+$ by the concavity of the elementary function

$$x \mapsto x^\alpha, \quad x \in [0,\infty[, \quad \alpha \in ]0,1[\ .$$

Thus

$$g(U) = U^\beta - U^\alpha, \quad U \in X_+,$$

is convex as a sum of two convex operators. We have

$$g'(U)(V) = (\beta U^{\beta-1} - \alpha U^{\alpha-1})V, \quad U \in X_+,\ V \in X.$$

We have to prove that

$$U \in A_l \setminus \{a\} \Rightarrow g'(U) \in -\mathrm{Izom}_+(X).$$

Let $U \in A_l \setminus \{a\}$. Then $\sigma(U) \subset ]0, a_1[$. Thus for any $t \in \sigma(U)$ we have $0 < t < a_1 = \left(\dfrac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}}$, hence $0 < t^{\beta-\alpha} < \dfrac{\alpha}{\beta}$, which implies $-\alpha < \beta t^{\beta-\alpha} - \alpha < 0$. These relations yield:

$$-\alpha t^\alpha < \beta t^\beta - \alpha t^\alpha < 0,$$

which imply

$$\beta t^{\beta-1} - \alpha t^{\alpha-1} < 0 \quad (t \in \sigma(U)).$$

Thus

$$\sigma(\beta U^{\beta-1} - \alpha U^{\alpha-1}) \subset ]-\infty, 0[\ .$$

This leads to the fact that $\beta U^{\beta-1} - \alpha U^{\alpha-1}$ is invertible and

$$\sigma((\beta U^{\beta-1} - \alpha U^{\alpha-1})^{-1}) \subset ]-\infty, 0[\ ,$$

i.e.

$$(\beta U^{\beta-1} - \alpha U^{\alpha-1})^{-1} < 0.$$

Using the commutativeness of $X$, from this, we obtain

$$g'(U)(V) = (\beta U^{\beta-1} - \alpha U^{\alpha-1})V < 0,$$
$$(g'(U))^{-1}(V) = (\beta U^{\beta-1} - \alpha U^{\alpha-1})^{-1}V < 0, \quad U, V \in X_+$$

(the product of two permutable operators, one of which being positive and the other one being negative, is a negative operator).

Thus

$$g'(U), \quad (g'(U))^{-1} \in -\mathrm{Izom}_+(X) \quad \forall U \in A_l.$$

Similarly,

$$g'(U), \quad (g'(U))^{-1} \in \mathrm{Izom}_+(X) \quad \forall U \in A_r.$$

Now we prove that

$$R(g_l) = R(g_r).$$

Let $g_l(U_1) \in R(g_l)$ be such that $U_1 \in A_l \setminus \{a\}$. Let $f : [0,1] \to [0,1]$ be the function constructed in Theorem 2.2. Let

$$U_2 := F(U_1),$$

where $F(U_1)$ is as in Lemma 3.3.1 [4], p.227 (functional calculus applied to $f$). Then

$$\sigma(U_2) = \sigma(F(U_1)) = f(\sigma(U_1)) \subset ]a_1, 1[$$

$$(\sigma(U_1) \subset ]0, a_1[ \Rightarrow f(\sigma(U_1)) \subset ]a_1, 1[$$

since $f$ applies $]0, a_1[$ onto $]a_1, 1[$). Thus $U_2 \in A_r$. On the other hand, the construction of $f$ implies

$$g(t_1) = g(f(t_1)) \quad \forall t_1 \in ]0, 1[ \; .$$

We integrate this equality on the spectrum $\sigma(U_1) \subset ]0, a_1[ \subset ]0, 1[$, with respect to the spectral measure attached to $U_1$, one obtains:

$$g_l(U_1) = g_r(F(U_1)) = g_r(U_2) \in R(g_r)$$

(since $U_2 := F(U_1) \in A_r$). Thus $g_l(U_1) \in R(g_r)$, $\forall U_1 \in A_l \setminus \{a\}$, which means that $R(g_l) \subset R(g_r)$. Similarly, we have $R(g_r) \subset R(g_l)$, so that we have

$$R(g_l) = R(g_r).$$

Now all conditions from the hypothesis of Theorem 1.2 are accomplished, so that the conclusion follows and the proof is complete. □

## References


[1] Boboc, N., *Analiză Matematică* I. Ed. Universităţii din Bucureşti, 1999.

[2] Colojoară, I., *Analiză Matematică*. Ed. Didactică şi Pedagogică, Bucureşti, 1983.

[3] Colojoară, I., *Elemente de Teorie Spectrală*. Ed. Academiei Române, Bucureşti, 1968.

[4] Cristescu, R., *Analiză Funcţională*, Ediţia a II-a. Ed. Didactică şi Pedagogică, Bucureşti, 1970.

[5] Cristescu, R., *Ordered Vector Spaces and Linear Operators*. Abacus Press, Tunbridge Wells, Kent, 1976.

[6] Cristescu, R., Grigore, Gh., Niculescu, C., Păltineanu, G., Popa, N. and Vuza, D.T., *Structuri de Ordine în Analiza Funcţională*, Vol. **3**. Ed. Academiei Române, Bucureşti, 1992.

[7] Drăguşin, C., Olteanu, O., Gavrilă, M., *Analiză Matematică, Probleme*. Volumul I, Ed. MATRIX ROM, Bucureşti, (2006).

[8] Drăguşin, C., Olteanu, O., Gavrilă, M., *Analiză Matematică, Probleme*. Volumul II, Ed. MATRIX ROM, Bucureşti, (in print).

[9] Dumitrescu, C., Olteanu, O., *The inverse of a convex operator. Newton's method for convex operators*. U.P.B. Sci. Bull., Mechanical Engineering, **53**, 3-4, (1991), 23-29.

[10] Gussi, Gh., *Itinerar în Analiza Matematică*, Lyceum 95. Ed. Albatros, Bucureşti, 1970.

[11] Hardy, G.H. and Wright, E.M., *An Introduction to the Theory of Numbers*, Fifth Edition. Oxford University Press Inc. New York, 1979.

[12] Kurosh, A., *Higher Algebra*. Mir Publishing, Moscow, 1972.

[13] Nicolescu, M., Dinculeanu, N., and Marcus, S., *Analiză Matematică* (I, II). Ed. Didactică şi Pedagogică, Bucureşti, 1971.

[14] Niculescu, C., *Fundamentele Analizei Matematice*, I, *Analiză pe Dreapta Reală*. Ed. Academiei Române, Bucureşti, 1996.

[15] Olariu, V., and Olteanu, O., *Analiză Matematică*. Ed. SEMNE, Bucureşti, 1999.

[16] Olteanu, A., and Olteanu, O., *Some exponential-type functional equations solved by a geometrical general method*. U.P.B. Sci. Bull. Series A, **67**, *1* (2005), 35-48.

[17] Olteanu, A., and Olteanu, O., *Algebraic functional equations solved by a geometrical general method*. Proceedings of the 3-rd International Qolloquium "Mathematics and Engineeering and Numerical Physics", October 7-9, 2004, Bucharest, pp. 208-217, Balkan Society of Geometers, Geometry Balkan Press, 2005.

[18] Olteanu, A., and Olteanu, O., *Solving some special functional equations by a general "geometrical" method, and an approach of the complex case*. Proceedings of the International Conference on Complex Analysis and Related Topics. The X-th Romanian-Finnish Seminar (Cluj-Napoca, Romania, August, 14-19, 2005); Rev. Roumaine Math. Pures Appl. **51**, *5-6* (2006) (in print).

[19] Olteanu, O., and Simion, Gh., *A new geometric aspect of the implicit function principle and Newton's method for operators*. Math. Reports, **5(55)**, 1(2003), 61-84.

[20] Rudin, W., *Principles of Mathematical Analysis*. Mc. Graw-Hill, New York, 1964.

[21] Sireţchi, Gh., *Calcul Diferenţial şi Integral, I. Noţiuni Fundamentale*. Ed. Ştiinţifică şi Enciclopedică, Bucureşti, 1985.

[22] Stănăşilă, O., *Analiză Matematică*. Ed. Didactică şi Pedagogică, Bucureşti, 1981.

[23] Şilov, G.E., *Analiză Matematică. Funcţii de o Variabilă*. Ed. Ştiinţifică şi Enciclopedică, Bucureşti, 1985.

[24] Vasilescu, Fl.-H., *Iniţiere în Teoria Operatorilor Liniari*. Ed. Tehnică, Bucureşti, 1987.

(Alina Olteanu) University Politehnica of Bucharest
Department of Mathematics I
Splaiul Independentei 313
060032 Bucharest, Romania
*E-mail address*: alinaolteanu001@yahoo.ie

(Octav Olteanu) University Politehnica of Bucharest
Department of Mathematics I
Splaiul Independentei 313
060032 Bucharest, Romania
*E-mail address*: oolteanu@mathem.pub.ro