Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 52(2), 2025, Pages 555–565, DOI: 10.52846/ami.v52i2.2030
ISSN: 1223-6934

Conformal Change of Douglas Space of Second Kind with Generalized (α, β)-metric

Brijesh Kumar Tripathi and Sadika Khan

Abstract. In 2008, I. Y. Lee [7] defined the Douglas space of the second kind of Finsler space with (α, β)-metric and examined the condition that a Finsler space with a (α, β)-metric is a Douglas space of the second kind. In this paper, we established the condition that a Douglas space of the second kind with generalized (α, β)-metric is conformally changed to a Douglas space of the second kind. Furthermore, we derived various results that indicate the second kind of Douglas space with different (α, β)-metrics, such as exponential type metric, and some special (α, β)-metrics are invariant under conformal change.

2010 Mathematics Subject Classification. 53B40, 53C60.
Key words and phrases. Finsler space, Conformal change, Douglas space, Douglas space of second kind, Generalized (α, β)-metric, Conformally invariant.

1. Introduction

S. Báscó and M. Matsumoto [3] introduced the concept of Douglas space as a generalization of Berwald space based on geodesic equations. They also consider Landsberg space to be a generalization of Berwald space. In 2002 [4], S. Báscó and B. Szilági introduced the idea of weakly-Berwald space, which is another generalization of Berwald space. In a recent study, I. Y. Lee [7] identified conditions for a Finsler space with Matsumoto metric to be a Douglas space of the second kind.

In 1929, M. S. Knbelman [9] introduced the conformal theory of Finsler space which has been thoroughly studied by M. Hashiguchi [8]. Later, Y. D. Lee [6] and B. N. Prasad [14] discovered conformally invariant tensors in Finsler space with (α, β) metric. In [16], the authors investigated that a Douglas space of second kind with generalized Kropina metric and Matsumoto metric is conformal Douglas space of second kind. Recently, the authors of [2, 15] shown that the Douglas space of second kind with special (α, β)-metric is conformally transformed to a Douglas space of second kind.

In 1972, Matsumoto [12] introduced the concept of a (α, β)-metric, $L(\alpha, \beta)$ and had been studied by several authors [11, 13, 17, 18, 19]. If L is a positively homogeneous function of α and β of degree one, where α is a Riemannian metric and β is a one-form on a n-dimensional manifold M, then a Finsler metric $L(x, y)$ is known as (α, β)-metric $L(\alpha, \beta)$. The Randers, Kropina, and Matsumoto metrics are the three most interesting instances of (α, β)-metrics. Theories of Finsler spaces with (α, β)-metrics have contributed significantly to the development of Finsler geometry [1, 11].

Received October 3, 2024. August 9, 2025.

In the present paper, we considered the generalized (α,β)-metric in the form of $L = c_1\alpha e^{\frac{\beta}{(\alpha+\beta)}} + c_2\beta e^{\frac{\beta}{(\alpha+\beta)}}$ on a n-dimensional manifold M and studied the criteria of second kind Douglas space with (α,β)-metric and conformal transformation of second kind Douglas space with (α,β)-metric. In Section 5, we have proved that a Douglas space of the second kind is conformally changed to a Douglas space of the second kind with a specified metric. Also, we have obtained certain results that show that exponential type metric and some special (α,β)-metrics are conformally changed to a second kind of Douglas space.

2. Preliminaries

Let $F^n = (M, L(\alpha,\beta))$ be a Finsler space with an (α,β)-metric, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is Riemannian metric and $\beta = b_i(x)y^i$ is 1-form. The space $R^n = (M,\alpha)$ is called the Riemannian space associated with F^n. We use the following symbols [11]:

$$r_{ij} = \tfrac{1}{2}\{b_{i;j} + b_{j;i}\}, \quad r^i_j = a^{ih} r_{hj}, \quad r_j = b_i r^i_j$$

$$s_{ij} = \tfrac{1}{2}\{s_{i;j} - s_{j;i}\}, \quad s^i_j = a^{ih} s_{hj}, \quad s_j = b_i s^i_j, \quad b^i = a^{ih} b_h, \quad b^2 = b^i b_i.$$

Let $B\Gamma = \{G^i_{jk}(x,y), G^i_j\}$ be the Berwald connection of Finsler space F^n, which plays an important role in the present study. B^i_{jk} represents the difference tensor of G^i_{jk} and χ^i_{jk} as well as

$$G^i_{jk}(x,y) = B^i_{jk}(x,y) + \chi^i_{jk}(x).$$

With the subscript 0 and contracting by y^i, we get

$$G^i_j = B^i_j + \chi^i_{0j} \quad \text{and} \quad 2G^i = 2B^i + \chi^i_{00}.$$

and then $B^i_{jk} = \dot{\partial}_k B^i_j$ and $B^i_j = \dot{\partial}_j B^i$. Let $F^n = (M,F)$ be a Finsler space, the geodesics of F^n are given by the system of differential equations [8]

$$\frac{d^2x^i}{dt^2}y^j - \frac{d^2x^j}{dt^2}y^i + 2(G^i y^j - G^j y^i) = 0, \quad y^i = \frac{dx^i}{dt},$$

in a parameter t. The function $G^i(x,y)$ is given by

$$2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F) = \chi^i_{jk}(x,y) y^j y^k,$$

where $\partial_i = \frac{\partial}{\partial x^i}, F = \frac{L^2}{2}, \chi^i_{jk}$ are Christoffel symbols constructed from $g_{ij}(x,y)$ with respect to x^i and $g^{ij}(x,y)$ is the inverse of fundamental metric tensor $g_{ij}(x,y)$.

In [3], it has been shown that F^n is a Douglas space if and only if the Douglas tensor

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n+1}(G_{ijk}y^h + G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j)$$

vanishes identically, where $G^h_{ijk} = \dot{\partial}_k G^h_{ij}$ is the hv-curvature tensor of the Berwald connection $B\Gamma$ [11].

The space F^n is said to be a Douglas space [3] if

$$D^{ij} = G^i(x,y)y^j - G^j(x,y)y^i \tag{2.1}$$

are homogeneous polynomials in y^i of degree three. Differentiating (2.1) with respect to y^h, y^k, y^p *and* y^q, we have $D^{ij}_{hkpq} = 0$, which are equivalent of $D^{im}_{hkpm} = (n+1)D^i_{hkp} = 0$. Thus if a Finsler space F^n satisfies the condition $D^{ij}_{hkpq} = 0$, which are equivalent to $D^{im}_{hkpm} = (n+1)D^i_{hkp} = 0$, we call it a Douglas space. Further differentiating (2.1) by y^m and contracting m and j in the obtained equation, we have $D^{im}_m = (n+1)G^i - G^m_m y^i$. Thus, we have

Definition 2.1. [10] A Finsler space F^n is said to be a Douglas space of second kind if

$$D^{im}_m = (n+1)G^i - G^m_m y^i \tag{2.2}$$

are homogeneous polynomials in (y^i) of degree two.

Further, a Finsler space with an (α, β)-metric is said to be a Douglas space of the second kind if and only if

$$B^{im}_m = (n+1)B^i - B^m_m y^i$$

are homogeneous polynomials in (y^i) of degree two, where B^m_m is given by [10]. Furthermore, differentiating the above with respect to y^h, y^j *and* y^k, we have

$$B^{im}_{hjkm} = B^i_{hjk} = 0.$$

Thus, we have

Definition 2.2. A Finsler space F^n with (α, β)-metric is said to be a Douglas space of second kind if it satisfies the condition that $B^{im}_m = (n+1)B^i - B^m_m y^i$ are homogeneous polynomials in (y^i) of degree two.

3. Douglas space of second kind with (α, β)-metric.

In the present section, we discuss the conditions for a Finsler space with a (α, β)-metric to be a Douglas space of the second kind.

Let us consider the function $G^i(x, y)$ of F^n with an (α, β)-metric. According to [11], $G^i(x, y)$ can be written as

$$2G^i = \chi^i_{00} + 2B^i,$$

$$B^i = \left(\alpha \frac{L_\beta}{L_\alpha}\right) s^i_0 + C^* \left[\frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta}\right)\right], \tag{3.1}$$

where

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha) - 2\alpha s_0 L_\beta}{2(\beta^2 L_\alpha + \alpha\chi^2 L_{\alpha\alpha})}, \tag{3.2}$$

$$\chi^2 = b^2\alpha^2 - \beta^2, \quad b^i = a^{ij}b_j, \quad b^2 = a^{ij}b_i b_j.$$

Since $\chi^i_{00} = \chi^i_{jk}(x)y^j y^k$ are homogeneous polynomials in (y^i) of degree two, equation (3.1) yields

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i). \tag{3.3}$$

by means of (2.1) and (3.3), we have the following lemma [10]:

Lemma 3.1. *A Finsler space F^n with an (α, β)-metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are $hp(3)$.*

Further, differentiating (3.3) by y^m and contracting m and j in the obtained equation, we obtain

$$B^{im}_m = \dot{\partial}_m\left(\frac{\alpha L_\beta}{L_\alpha}\right)(s^i_0 y^m - s^m_0 y^i) + \frac{\alpha L_\beta}{L_\alpha}\dot{\partial}_m(s^i_0 y^m - s^m_0 y^i) + \dot{\partial}_m\left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)$$
$$C^*(b^i y^m - b^m y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}(\dot{\partial}_m C^*)(b^i y^m - b^m y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*\dot{\partial}_m(b^i y^m - b^m y^i). \tag{3.4}$$

Using (2.2) and the homogeneity of (y^i), we obtain

$$\dot{\partial}_m\left(\frac{\alpha L_\beta}{L_\alpha}\right)(s^i_0 y^m - s^m_0 y^i) = \left(\frac{\alpha L_\beta}{L_\alpha}\right)s^i_0 - \frac{\alpha^2 L L_{\alpha\alpha} s_0}{(\beta L_\alpha)^2}y^i, \tag{3.5}$$

$$\frac{\alpha L_\beta}{L_\alpha}\dot{\partial}_m(s^i_0 y^m - s^m_0 y^i) = \frac{n\alpha L_\beta}{L_\alpha}s^i_0, \tag{3.6}$$

$$\dot{\partial}_m\left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)C^*(b^i y^m - b^m y^i) = \frac{\chi^2\{\alpha L_\alpha L_{\alpha\alpha\alpha} + (2L_\alpha - \alpha L_{\alpha\alpha})L_\alpha\}C^*}{(\beta L_\alpha)^2}y^i, \tag{3.7}$$

$$(\dot{\partial}_m C^*)y^m = 2C^*, \tag{3.8}$$

$$\begin{aligned}(\dot{\partial}_m C^*)b^m =& \frac{1}{2\alpha\beta\Omega^2}[\Omega\{\beta(\chi^2 + 2\beta^2)M + 2\alpha^2\beta^2 L_\alpha r_0 - \alpha\beta\chi^2 L_{\alpha\alpha} r_{00}\\ &- 2\alpha(\beta^3 L_\beta + \alpha^2\chi^2 L_{\alpha\alpha})s_0\} - \alpha^2\beta M(2b^2\beta^2 L_\alpha - \chi^4 L_{\alpha\alpha\alpha}\\ &- b^2\alpha\chi^2 L_{\alpha\alpha})],\end{aligned} \tag{3.9}$$

$$\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*\dot{\partial}_m(b^i y^m - b^m y^i) = \frac{(n-1)\alpha^2 L_{\alpha\alpha}C^*}{\beta L_\alpha}b^i, \tag{3.10}$$

where

$$M = (r_{00}L_\alpha - 2\alpha s_0 L_\beta),\ \ Y_i = a_{ir}y^r,\ \ s_{00} = 0,\ \ b^r s_r = 0,\ \ a^{ij}s_{ij} = 0.$$

Substituting (3.5), (3.6), (3.7), (3.8), (3.9) and (3.10) into (3.4), we have

$$\begin{aligned}B^{im}_m =& \frac{(n+1)\alpha L_\beta}{L_\alpha}s^i_0 + \frac{\alpha\{(n+1)\alpha^2\Omega L_{\alpha\alpha}b^i + \beta\chi^2 A y^i\}}{2\Omega^2}r_{00}\\ &- \frac{\alpha^2\{(n+1)\alpha^2\Omega L_\beta L_{\alpha\alpha}b^i + B y^i\}}{L_\alpha\Omega^2}s_0 - \frac{\alpha^3 L_{\alpha\alpha}y^i}{\Omega}r_0,\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}&\Omega = (\beta^2 L_\alpha + \alpha\chi^2 L_{\alpha\alpha}),\ \ \Omega \neq 0,\\ &A = \alpha L_\alpha L_{\alpha\alpha\alpha} + 3L_\alpha L_{\alpha\alpha} - 3\alpha(L_{\alpha\alpha})^2,\\ &B = \alpha\beta\chi^2 L_\alpha L_\beta L_{\alpha\alpha\alpha} + \beta\{(3\chi^2 - \beta^2)L_\alpha - 4\alpha\chi^2 L_{\alpha\alpha}\}L_\beta L_{\alpha\alpha} + \Omega L L_{\alpha\alpha}.\end{aligned} \tag{3.12}$$

We use the following result [7]:

Theorem 3.2. *The necessary and sufficient condition for a Finsler space F^n with an (α, β)-metric to be a Douglas space of the second kind is that, B^{im}_m is given by (3.11) and (3.12), provided that $\Omega \neq 0$.*

4. Conformal change of Douglas space of second kind with (α,β)-metric

In the present section, we studied the condition for a Douglas space of second kind with (α,β)-metric being invariant under conformal change.

Let $F^n = (M, L)$ and $\bar{F^n} = (M, \bar{L})$ be two Finsler space on the same underlying manifold M. If we have a function $\sigma(x)$ in each coordinate neighbourhood of M such that $\bar{L}(x,y) = e^{\sigma}L(x,y)$, then F^n is called conformal to $\bar{F^n}$ and the change $L \to \bar{L}$ of metric is called conformal change.

As to (α,β)-metric, $\bar{L} = e^{\sigma}L(\alpha,\beta)$, is equivalent to $\bar{L} = L(e^{\sigma}\alpha, e^{\sigma}\beta)$ by homogeneity. Therefore, a conformal change of (α,β)-metric is expressed as $(\alpha,\beta) \to (\bar{\alpha},\bar{\beta})$, where $\bar{\alpha} = e^{\sigma}\alpha$, $\bar{\beta} = e^{\sigma}\beta$. Therefore, we have

$$\begin{aligned} \bar{a}_{ij} &= e^{2\sigma}a_{ij}, \quad \bar{b_i} = e^{\sigma}b_i, \\ \bar{a}^{ij} &= e^{-2\sigma}a^{ij}, \quad \bar{b^i} = e^{-\sigma}b^i. \end{aligned} \tag{4.1}$$

and $b^2 = a^{ij}b_ib_j = \bar{a}^{ij}\bar{b}_i\bar{b}_j$. Thus, we state the following:

Proposition 4.1. [11] *A Finsler space with (α,β)-metric and the length b of b_i with respect to the Riemannian metric α is invariant under any conformal change of (α,β)-metric.*

From (4.1), it follows that the conformal change of Christoffel symbols is given by [8]:

$$\bar{\chi}^i_{jk} = \chi^i_{jk} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - \sigma^i a_{jk}, \tag{4.2}$$

where $\sigma_j = \partial_j\sigma$ and $\sigma^i = a^{ij}\sigma_j$.

From (4.1) and (4.2), we have the following identities:

$$\begin{aligned} \bar{\nabla}_j\bar{b_i} &= e^{\sigma}(\rho a_{ij} - \sigma_i b_j + \nabla_j b_i), \\ \bar{r}_{ij} &= e^{\sigma}\left[\rho a_{ij} + r_{ij} - \frac{1}{2}(b_j\sigma_i + b_i\sigma_j)\right], \\ \bar{s}_{ij} &= e^{\sigma}\left[\frac{1}{2}(b_i\sigma_j - b_j\sigma_i) + s_{ij}\right], \\ \bar{s}^i_j &= e^{-\sigma}\left[\frac{1}{2}(b^i\sigma_j - b_j\sigma^i) + s^i_j\right], \\ \bar{s}_j &= s_j + \frac{1}{2}(b^2\sigma_j - \rho b_j), \end{aligned} \tag{4.3}$$

where $\rho = \sigma_r b^r$.

From (4.2) and (4.3), we can easily obtain the following:

$$\begin{aligned} \bar{\chi}^i_{00} &= \chi^i_{00} + 2\sigma_0 y^i - \alpha^2\sigma_j, \\ \bar{r}_{00} &= e^{\sigma}(\rho\alpha^2 + r_{00} - \sigma_0\beta), \\ \bar{s}^i_0 &= e^{-\sigma}\left[\frac{1}{2}(\sigma s_0 b^i - \beta\sigma^i) + s^i_0\right], \\ \bar{s}_0 &= \frac{1}{2}(\sigma_0 b^i - \rho\beta) + s_0. \end{aligned} \tag{4.4}$$

Next, we find the conformal change of B^{ij} given in (3.3), since $\bar{L}(\alpha,\beta) = e^{\sigma}L(\alpha,\beta)$ and

$$\bar{L}_{\bar{\alpha}} = L_{\alpha}, \ \bar{L}_{\bar{\alpha}\bar{\alpha}} = e^{-\sigma}L_{\alpha\alpha}, \ \bar{L}_{\bar{\beta}} = L_{\beta}, \ \bar{\chi^2} = e^{2\sigma}\chi^2. \tag{4.5}$$

By using (3.2), (4.4), (4.5) and lemma (3.1), we obtain

$$\bar{C}^* = e^{\sigma}(C^* + D^*),$$

where

$$D^* = \frac{\alpha\beta\{(\beta\alpha^2 - \sigma_0\beta)L_\alpha - \alpha(b^2\sigma_0 - \rho\beta)L_\beta\}}{2(\beta^2 L - \alpha + \alpha\chi^2 L_{\alpha\alpha})}.$$

Here under the conformal change, B^{ij} can be written as

$$\begin{aligned}\bar{B}^{ij} &= \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^i_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i) \\ &+ \left(\frac{\alpha\sigma_0 L_\beta}{L_\alpha} + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} D^*\right)(b^i y^j - b^j y^i) - \frac{\alpha\beta L_\beta}{2L_\alpha}(\sigma^i y^j - \sigma^j y^i) \\ &= B^{ij} + C^{ij},\end{aligned}$$

where

$$C^{ij} = \left(\frac{\alpha\sigma_0 L_\beta}{L_\alpha} + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} D^*\right)(b^i y^j - b^j y^i) - \frac{\alpha\beta L_\beta}{2L_\alpha}(\sigma^i y^j - \sigma^j y^i). \tag{4.6}$$

From (3.12), we have

$$\bar{\Omega} = e^{2\sigma}\Omega, \ \bar{A} = e^{-\sigma}A, \ \bar{B} = e^{2\sigma}B.$$

Now, we apply conformal change to B^{im}_m, and obtain

$$\bar{B}^{im}_m = B^{im}_m + K^{im}_m, \tag{4.7}$$

where

$$\begin{aligned}2K^{im}_m &= \frac{(n+1)\alpha L_\beta}{L_\alpha}(\sigma_0 b^i - \beta\sigma^i) + \frac{(n+1)\alpha^3\Omega L_{\alpha\alpha} b^i + \alpha\beta\chi^2 A y^i}{\Omega^2}(\rho\alpha^2 - \sigma_0\beta) \\ &- \left[\frac{\alpha^2\{(n+1)\alpha^2\Omega L_\beta L_{\alpha\alpha} b^i + B y^i\}}{L_\alpha \Omega^2} - \frac{\alpha^3 L_{\alpha\alpha} y^i}{\Omega}\right](b^2\sigma_0 - \rho\beta). \end{aligned} \tag{4.8}$$

Thus, we have

Theorem 4.2. [16] *The necessary and sufficient condition for a Douglas space of second kind with (α, β)-metric to be a Douglas of second kind under conformal change is that K^{im}_m is homogeneous polynomial in (y^i) of degree two.*

5. Conformal change of Douglas space of second kind with generalized (α, β)-metric

In this section, we shall study the conformal change of Douglas space of second kind with generalized (α, β)-metric. Let us consider a Finsler space with generalized (α, β)-metric as follows:

$$L = c_1 \alpha e^{\frac{\beta}{\alpha+\beta}} + c_2 \beta e^{\frac{\beta}{\alpha+\beta}} \tag{5.1}$$

where c_1 and c_2 are constants. The partial derivative of (5.1) with respect to α and β are given by

$$\begin{aligned}
L_\alpha &= e^{\frac{\beta}{\alpha+\beta}}\left(\frac{c_1(\alpha^2+\alpha\beta+\beta^2)-c_2\beta^2}{(\alpha+\beta)^2}\right),\\
L_{\alpha\alpha} &= -\beta^2 e^{\frac{\beta}{\alpha+\beta}}\left(\frac{c_1(\alpha+2\beta)-c_2(2\alpha+3\beta)}{(\alpha+\beta)^4}\right),\\
L_{\alpha\alpha\alpha} &= \beta^2 e^{\frac{\beta}{\alpha+\beta}}\left(\frac{c_1(3\alpha^2+11\alpha\beta+\beta^2)-c_2(6\alpha^2+18\alpha\beta+13\beta^2)}{(\alpha+\beta)^6}\right),\\
L_\beta &= e^{\frac{\beta}{\alpha+\beta}}\left(\frac{c_2(\alpha^2+3\alpha\beta+\beta^2)+c_1\alpha^2}{(\alpha+\beta)^2}\right).
\end{aligned}\tag{5.2}$$

Using (3.12), we obtain

$$\begin{aligned}
\Omega =&\frac{\beta^2 e^{\frac{\beta}{\alpha+\beta}}}{(\alpha+\beta)^4}\left\{c_1\alpha^4+3c_1\alpha^3\beta+4c_1\alpha^2\beta^2-c_2\alpha^2\beta^2-c_1\alpha^2\chi^2+2c_2\alpha^2\chi^2\right.\\
&\left.+3c_1\alpha\beta^3-2c_2\alpha\beta^3-2c_1\alpha\beta\chi^2+3c_2\alpha\beta\chi^2+c_1\beta^4-c_2\beta^4\right\},\\
A =&\frac{e^{\frac{2\beta}{\alpha+\beta}}}{(\alpha+\beta)^8}\left\{-c_1^2\alpha^4\beta^3+3c_1c_2\alpha^4\beta^3-10c_1^2\alpha^3\beta^4-12c_2^2\alpha^3\beta^4+26c_1c_2\alpha^3\beta^4\right.\\
&-25c_1^2\alpha^2\beta^5-39c_2^2\alpha^2\beta^5+66c_1c_2\alpha^2\beta^5-24c_1^2\alpha\beta^6-38c_2^2\alpha\beta^6+62c_1c_2\alpha\beta^6\\
&\left.-6c_1^2\beta^7-9c_2^2\beta^7+15c_1c_2\beta^7\right\},\\
B =&\frac{\beta^2 e^{\frac{3\beta}{\alpha+\beta}}}{(\alpha+\beta)^{10}}\left\{-c_1^3\alpha^8\beta^2+2c_1^2c_2\alpha^8\beta^2-6c_1^3\alpha^7\beta^3+11c_1^2c_2\alpha^7\beta^3-16c_1^3\alpha^6\beta^4\right.\\
&-4c_1c_2^2\alpha^6\beta^4+30c_1^2c_2\alpha^6\beta^4+7c_1c_2^2\alpha^6\beta^2\chi^2-2c_1^2c_2\alpha^6\beta^2\chi^2-26c_1^3\alpha^5\beta^5\\
&-18c_1c_2^2\alpha^5\beta^5+54c_1^2c_2\alpha^5\beta^5-5c_1^3\alpha^5\beta^3\chi^2-12c_2^3\alpha^5\beta^3\chi^2+39c_1c_2^2\alpha^5\beta^3\chi^2\\
&-5c_1^2c_2\alpha^5\beta^3\chi^2-28c_1^3\alpha^4\beta^6-2c_2^3\alpha^4\beta^6-35c_1c_2^2\alpha^4\beta^6+66c_1^2c_2\alpha^4\beta^6\\
&-16c_1^3\alpha^4\beta^4\chi^2-79c_2^3\alpha^4\beta^4\chi^2+137c_1c_2^2\alpha^4\beta^4\chi^2-21c_1^2c_2\alpha^4\beta^4\chi^2\\
&-20c_1^3\alpha^3\beta^7+7c_2^3\alpha^3\beta^7-40c_1c_2^2\alpha^3\beta^7+54c_1^2c_2\alpha^3\beta^7-16c_1^3\alpha^3\beta^5\chi^2\\
&-179c_2^3\alpha^3\beta^5\chi^2+283c_1c_2^2\alpha^3\beta^5\chi^2-77c_1^2c_2\alpha^3\beta^5\chi^2-9c_1^3\alpha^2\beta^8\\
&+8c_2^3\alpha^2\beta^8-27c_1c_2^2\alpha^2\beta^8+28c_1^2c_2\alpha^2\beta^8-2c_1^3\alpha^2\beta^6\chi^2-171c_2^3\alpha^2\beta^6\chi^2\\
&+279c_1c_2^2\alpha^2\beta^6\chi^2-104c_1^2c_2\alpha^2\beta^6\chi^2-2c_1^3\alpha\beta^9+3c_2^3\alpha\beta^9-8c_1c_2^2\alpha\beta^9\\
&+7c_1^2c_2\alpha\beta^9-65c_2^3\alpha\beta^7\chi^2+104c_1c_2^2\alpha\beta^7\chi^2-42c_1^2c_2\alpha\beta^7-9c_2^3\beta^8\chi^2\\
&\left.+15c_1c_2^2\beta^8\chi^2-6c_1^2c_2\beta^8\chi^2\right\}.
\end{aligned}\tag{5.3}$$

Thus, using (5.2) and (5.3), K^{im}_m can be written as

$$\begin{aligned}
2K^{im}_m =&\frac{\alpha(n+1)\{c_2(\alpha^2+3\alpha\beta+\beta^2)+c_1\alpha^2\}(\sigma_0 b^i-\beta\sigma^i)}{(c_1(\alpha^2+\alpha\beta+\beta^2)-c_2\beta^2}+Z_1+Z_2+Z_3\\
&+Z_4+Z_5,
\end{aligned}\tag{5.4}$$

where

$$
\begin{aligned}
Z_1 =& \frac{\alpha^3(n+1)\{-(c_1(\alpha+2\beta)-c_2(2\alpha+3\beta))b^i\}(\rho\alpha^2-\sigma_0\beta)}{Q_1},\\
Z_2 =& \frac{\alpha\chi^2 Q_2 y^i(\rho\alpha^2-\sigma_0\beta)}{Q_1^2},\\
Z_3 =& \frac{\alpha^4(n+1)Q_3 b^i(b^2\sigma_0-\rho\beta)}{Q_1(c_1(\alpha^2+\alpha\beta+\beta^2)-c_2\beta^2)},\\
Z_4 =& \frac{-\alpha^2 Q_4 y^i(b^2\sigma_0-\rho\beta)}{Q_1^2(c_1(\alpha^2+\alpha\beta+\beta^2-c_2\beta^2))},\\
Z_5 =& \frac{\alpha^3\{-(c_1(\alpha+2\beta)-c_2(2\alpha+3\beta))\}y^i(b^2\sigma_0-\rho\beta)}{Q_1},
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1 =& c_1\alpha^4+3c_1\alpha^3\beta+4c_1\alpha^2\beta^2-c_2\alpha^2\beta^2-c_1\alpha^2\chi^2+2c_2\alpha^2\chi^2\\
&+3c_1\alpha\beta^3-2c_2\alpha\beta^3-2c_1\alpha\beta\chi^2+3c_2\alpha\beta\chi^2+c_1\beta^4-c_2\beta^4,\\
Q_2 =& -c_1^2\alpha^4+3c_1c_2\alpha^4-10c_1^2\alpha^3\beta-12c_2^2\alpha^3\beta+26c_1c_2\alpha^3\beta\\
&-25c_1^2\alpha^2\beta^2-39c_2^2\alpha^2\beta^2+66c_1c_2\alpha^2\beta^2-24c_1^2\alpha\beta^3\\
&-38c_2^2\alpha\beta^3+62c_1c_2\alpha\beta^3-6c_1^2\beta^4-9c_2^2\beta^4+15c_1c_2\beta^4,\\
Q_3 =& c_1^2\alpha^3-2c_2^2\alpha^3-c_1c_2\alpha^3+2c_1^2\alpha^2\beta-9c_2^2\alpha^2\beta+2c_1c_2\alpha^2\beta-11c_2^2\alpha\beta^2\\
&+7c_1c_2\alpha\beta^2-3c_2^2\beta^3+2c_1c_2\beta^3,\\
Q_4 =& -c_1^3\alpha^8+2c_1^2c_2\alpha^8-6c_1^3\alpha^7\beta+11c_1^2c_2\alpha^7\beta-16c_1^3\alpha^6\beta^2\\
&-4c_1c_2^2\alpha^6\beta^2+30c_1^2c_2\alpha^6\beta^2+7c_1c_2^2\alpha^6\chi^2-2c_1^2c_2\alpha^6\chi^2-26c_1^3\alpha^5\beta^3\\
&-18c_1c_2^2\alpha^5\beta^3+54c_1^2c_2\alpha^5\beta^3-5c_1^3\alpha^5\beta\chi^2-12c_2^3\alpha^5\beta\chi^2+39c_1c_2^2\alpha^5\beta\chi^2\\
&-5c_1^2c_2\alpha^5\beta\chi^2-28c_1^3\alpha^4\beta^4-2c_2^3\alpha^4\beta^4-35c_1c_2^2\alpha^4\beta^4+66c_1^2c_2\alpha^4\beta^4\\
&-16c_1^3\alpha^4\beta^2\chi^2-79c_2^3\alpha^4\beta^2\chi^2+137c_1c_2^2\alpha^4\beta^2\chi^2-21c_1^2c_2\alpha^4\beta^2\chi^2\\
&-20c_1^3\alpha^3\beta^5+7c_2^3\alpha^3\beta^5-40c_1c_2^2\alpha^3\beta^5+54c_1^2c_2\alpha^3\beta^5-16c_1^3\alpha^3\beta^3\chi^2\\
&-179c_2^3\alpha^3\beta^3\chi^2+283c_1c_2^2\alpha^3\beta^3\chi^2-77c_1^2c_2\alpha^3\beta^3\chi^2-9c_1^3\alpha^2\beta^6\\
&+8c_2^3\alpha^2\beta^6-27c_1c_2^2\alpha^2\beta^6+28c_1^2c_2\alpha^2\beta^6-2c_1^3\alpha^2\beta^4\chi^2-171c_2^3\alpha^2\beta^4\chi^2\\
&+279c_1c_2^2\alpha^2\beta^4\chi^2-104c_1^2c_2\alpha^2\beta^4\chi^2-2c_1^3\alpha\beta^7+3c_2^3\alpha\beta^7-8c_1c_2^2\alpha\beta^7\\
&+7c_1^2c_2\alpha\beta^7-65c_2^3\alpha\beta^5\chi^2+104c_1c_2^2\alpha\beta^5\chi^2-42c_1^2c_2\alpha\beta^5-9c_2^3\beta^6\chi^2\\
&+15c_1c_2^2\beta^6\chi^2-6c_1^2c_2\beta^6\chi^2.
\end{aligned}
$$

This shows that K^{im}_m is a second degree homogeneous polynomial in y^i. Thus, we have the following:

Theorem 5.1. *A second kind Douglas space with generalized (α,β)-metric $L = c_1\alpha e^{\frac{\beta}{\alpha+\beta}} + c_2\beta e^{\frac{\beta}{\alpha+\beta}}$, where c_1 and c_2 are constants is conformally changed to a second kind Douglas space.*

According to Theorem 5.1, one can show that a second kind Douglas space with a Finsler space of some (α,β)-metric is conformally transformed into a second kind Douglas space. We discussed the following cases:

Case: (i) If $c_1 = 1$ and $c_2 = 0$, the generalized (α,β)-metric becomes $L = \alpha e^{\frac{\beta}{\alpha+\beta}}$. In this instance, $2K^{im}_m$ can be written as

$$
\begin{aligned}
2K^{im}_m =& \frac{\alpha^3(n+1)(\sigma_0 b^i - \beta\sigma^i)}{\alpha^2+\alpha\beta+\beta^2} + \frac{\alpha^3(n+1)(2\beta-\alpha)b^i(\rho\alpha^2-\sigma_0\beta)}{\alpha^4+3\alpha^3\beta+4\alpha^2\beta^2-\alpha^2\chi^2+3\alpha\beta^3-2\alpha\beta\chi^2+\beta^4}\\
&- \frac{\alpha\chi^2(\alpha^4+10\alpha^3\beta+25\alpha^2\beta^2+24\alpha\beta^3+6\beta^4)y^i(\rho\alpha^2-\sigma_0\beta)}{(\alpha^4+3\alpha^3\beta+4\alpha^2\beta^2-\alpha^2\chi^2+3\alpha\beta^3-2\alpha\beta\chi^2+\beta^4)^2}\\
&- \frac{\alpha^6(n+1)(\alpha+2\beta)b^i(b^2\sigma_0-\rho\beta)}{(\alpha^4+3\alpha^3\beta+4\alpha^2\beta^2-\alpha^2\chi^2+3\alpha\beta^3-2\alpha\beta\chi^2+\beta^4)(\alpha^2+\alpha\beta+\beta^2)}\\
&- \left\{\frac{\alpha^2(\alpha^8+6\alpha^7\beta+16\alpha^6\beta^2-26\alpha^5\beta^3+5\alpha^5\beta\chi^2+28\alpha^4\beta^4+16\alpha^4\beta^2\chi^2)y^i(b^2\sigma_0-\rho\beta)}{(\alpha^4+3\alpha^3\beta+4\alpha^2\beta^2-\alpha^2\chi^2+3\alpha\beta^3-2\alpha\beta\chi^2+\beta^4)^2(\alpha^2+\alpha\beta+\beta^2)}\right.\\
&\left.+\frac{\alpha^2(20\alpha^3\beta^5+15\alpha^3\beta^3\chi^2+9\alpha^2\beta^6+2\alpha^2\beta^4\chi^2+2\alpha\beta^7)y^i(b^2\sigma_0-\rho\beta)}{(\alpha^4+3\alpha^3\beta+4\alpha^2\beta^2-\alpha^2\chi^2+3\alpha\beta^3-2\alpha\beta\chi^2+\beta^4)^2(\alpha^2+\alpha\beta+\beta^2)}\right\}\\
&- \frac{\alpha^3(\alpha+2\beta)y^i(b^2\sigma_0-\rho\beta)}{\alpha^4+3\alpha^3\beta+4\alpha^2\beta^2-\alpha^2\chi^2+3\alpha\beta^3-2\alpha\beta\chi^2+\beta^4}.
\end{aligned}
\tag{5.5}
$$

which shows that K^{im}_m is $hp(2)$. Thus, we have the following:

Corollary 5.2. *A second kind Douglas space with exponential type metric $L = \alpha e^{\frac{\beta}{\alpha+\beta}}$ is invariant under conformal change.*

Case: (ii) If $c_1 = 0$ and $c_2 = 1$, then generalized (α,β)-metric becomes $L = \beta e^{\frac{\beta}{\alpha+\beta}}$. In this instance, $2K^{im}_m$ can be witten as

$$
\begin{aligned}
2K^{im}_m =& -\frac{\alpha(n+1)(\alpha^2+3\alpha\beta+\beta^2)(\sigma_0 b^i-\beta\sigma^i)}{\beta^2}\\
&+ \frac{\alpha^3(n+1)(2\alpha+3\beta)b^i(\rho\alpha^2-\sigma_0\beta)}{\alpha^2\beta^2+\alpha^2\chi^2-\alpha\beta\chi^2-\beta^4}\\
&+ \frac{\alpha\chi^2(12\alpha^3\beta+39\alpha^2\beta^2+38\alpha\beta^3+9\beta^4)y^i(\rho\alpha^2-\sigma_0\beta)}{(\alpha^2\beta^2+\alpha^2\chi^2-\alpha\beta\chi^2-\beta^4)^2}\\
&+ \frac{\alpha^4(n+1)(2\alpha+3\beta)(\alpha^2+3\alpha\beta+\beta^2)b^i(b^2\sigma_0-\rho\beta)}{\beta^2(\alpha^2\beta^2+\alpha^2\chi^2-\alpha\beta\chi^2-\beta^4)}\\
&- \left\{\frac{\alpha^2(12\alpha^5\chi^2+2\alpha^4\beta^3+79\alpha^4\beta\chi^2-7\alpha^3\beta^4+179\alpha^3\beta^2\chi^2-8\alpha^2\beta^5)y^i(b^2\sigma_0-\rho\beta)}{\beta(\alpha^2\beta^2+\alpha^2\chi^2-\alpha\beta\chi^2-\beta^4)^2}\right.\\
&\left.+\frac{\alpha^2(-8\alpha^2\beta^5+171\alpha^2\beta^3\chi^2-3\alpha\beta^6+9\beta^5\chi^2)y^i(b^2\sigma_0-\rho\beta)}{\beta(\alpha^2\beta^2+\alpha^2\chi^2-\alpha\beta\chi^2-\beta^4)^2}\right\}\\
&+ \frac{\alpha^3(2\alpha+3\beta)y^i(b^2\sigma_0-\rho\beta)}{\alpha^2\beta^2+\alpha^2\chi^2-\alpha\beta\chi^2-\beta^4}.
\end{aligned}
\tag{5.6}
$$

which shows that K^{im}_m is $hp(2)$. Thus, we have the following:

Corollary 5.3. *A second kind Douglas space with special (α,β)-metric $L=\beta e^{\frac{\beta}{\alpha+\beta}}$ is invariant under conformal change.*

Case: (iii) If $c_1 = 1$ and $c_2 = 1$, the generalized metric becomes $L=(\alpha+\beta)e^{\frac{\beta}{\alpha+\beta}}$ [5]. In this instance, $2K^{im}_m$ can be written as

$$\begin{aligned}
2K^{im}_m &= \frac{(n+1)(2\alpha^2+3\alpha\beta+\beta^2)(\sigma_0 b^i-\beta\sigma^i)}{\alpha+\beta}\\
&- \frac{\alpha^2(n+1)(3\alpha+5\beta)b^i(\rho\alpha^2-\sigma_0\beta)}{\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\alpha\chi^2+\beta^3+\beta\chi^2}\\
&+ \frac{\alpha\chi^2(2\alpha^2+4\alpha\beta+2\beta^2)y^i(\rho\alpha^2-\sigma_0\beta)}{(\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\alpha\chi^2+\beta^3+\beta\chi^2)^2}\\
&- \frac{\alpha^3(n+1)(2\alpha^2+3\alpha\beta+\beta^2)b^i(b^2\sigma_0-\rho\beta)}{\alpha^4+3\alpha^3\beta+3\alpha^2\beta^2+\alpha^2\chi^2+\alpha\beta^3+\alpha\beta\chi^2}\\
&- \left\{\frac{\alpha^2(\alpha^5+5\alpha^4\beta+10\alpha^3\beta^2+5\alpha^3\chi^2+10\alpha^2\beta^3+17\alpha^2\beta\chi^2)y^i(b^2\sigma_0-\rho\beta)}{(\alpha+\beta)(\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\alpha\chi^2+\beta^3+\beta\chi^2)^2}\right.\\
&+\left.\frac{\alpha^2(\alpha\beta^4+21\alpha\beta^2\chi^2+\beta^5+11\beta^3\chi^2)y^i(b^2\sigma_0-\rho\beta)}{(\alpha+\beta)(\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\alpha\chi^2+\beta^3+\beta\chi^2)^2}\right\}\\
&- \frac{\alpha^2(3\alpha+5\beta)y^i(b^2\sigma_0-\rho\beta)}{\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\alpha\chi^2+\beta^3+\beta\chi^2}. \qquad (5.7)
\end{aligned}$$

which shows that K^{im}_m is $hp(2)$. Thus, we have the following:

Corollary 5.4. *A second kind Douglas space with special (α,β)-metric $L=(\alpha+\beta)e^{\frac{\beta}{\alpha+\beta}}$ [5] is invariant under conformal change.*

6. Conclusion

In Finsler geometry, a conformal transformation of a second kind of Douglas space involves mapping one Douglas space onto another while preserving certain geometric properties up to a conformal factor. Specifically, a conformal transformation preserves angles but not necessarily distances. The use of conformal transformations in Finsler geometry allows for the exploration of relationships between different Finsler metrics and the study of geometric properties that are invariant under conformal changes. This approach is particularly useful for analyzing curvature properties, geodesics, and other geometric structures in Finsler spaces. Additionally, conformal transformations find applications in physics, optimization problems, and computational geometry.

In the present paper, we have proved that a Douglas space of the second kind is conformally changed to a Douglas space of the second kind with generalized (α,β)-metric (5.1). Also, we have obtained certain results that show that exponential type metric and some special (α,β)-metrics are conformally changed to a second kind of Douglas space.

References

[1] P.L. Antonelli, R.S. Ingarden, M. Matsumoto, *The Theory of sprays and Finsler spaces with Applications in Physics and Biology*, Kluwer Acad. Publ., Dordrecht, 1993.

[2] A.R. Ajay Kumar, P. Kumar, On the Conformal transformation of Douglas space of second kind with generalized (α, β)-metric, *Advances in Dynamical Systems and Applications* **17** (2022), no. 1, 299-310.

[3] S. Báscó, M. Matsumoto, On Finsler spaces of Douglas type. A generalization of the notion of Berwald space, *Publ. Math. Debrecen* **51** (1997), 385-406.

[4] S. Báscó, B. Szilagi, On a Weakly Berwald space of Kropina type, *Mathematica Pannonica* **13** (2002), no. 1, 91-95.

[5] V.K. Chaubey et al., Characterization of Finsler Space with Rander's-Type Exponential-Form Metric, *Mathematics* **13** (2025), 1063-1077.

[6] Y.D. Lee, Conformal transformations of difference tensors of Finsler space with an (α, β)-metric, *Comm. Korean Math. Soc.* **21** (1997), no. 4, 975-984.

[7] I.Y. Lee, Douglas space of the second kind of Finsler space with Matsumoto metric, *Journal of the Chungcheong Mathematical Society* **21** (2008), no. 4, 209-221.

[8] M. Hashiguchi, On conformal transformations of Finsler metrics, *J. Math. Kyoto Univ.* **16** (1976), 25-50.

[9] M.S. Knebelman, Conformal geometry of generalized metric spaces, *Proc. Nat. Acad. Sci. USA* **15** (1929), 376-379.

[10] M. Matsumoto, Finsler spaces with (α, β)-metric of Douglas type, *Tensor, N. S.* **60** (1998), 123-134.

[11] M. Matsumoto, Theory of Finsler spaces with (α, β)-metric, *Rep. Math. Phys.* **31** (1992), 43-83.

[12] M. Matsumoto, On a C-reducible Finsler space, *Tensor, N. S.* **24** (1972), 29-37.

[13] S.K. Narsimhamurthy, Ajith, C.S. Bagewadi, Conformal change of Douglas space of second kind with (α, β)-metric, *Journal of Mathematics Analysis* **3** (2012), no. 2, 25-30.

[14] B.N. Prasad, B.N. Gupta, D.D. Singh, On Conformal transformation in Finsler spaces with an (α, β)-metric, *Indian J. Pure Appl. Math.* **18** (1987), no. 4, 290-301.

[15] R. Ranjan, P.N. Pandey, P. Ajith, Conformal transformation of Douglas space of second kind with special (α, β)-metric, *Arab Journal of Mathematical Sciences* **87** (2009), 359-370.

[16] G. Shanker, D. Choudhary, On the conformal change of Douglas space of second kind with certain (α, β)-metrics, *International Journal of Pure and Applied Mathematics* **103** (2015), no. 4, 613-624.

[17] B.K. Tripathi, S. Khan, V.K. Chaubey, On Projectively Flat Finsler Space with a special cubic (α, β)- metric, *Filomat* **37** (2023), no. 26, 8975-8982.

[18] B.K. Tripathi, S. Khan, On weakly Berwald space with a special cubic (α, β)-metric, *Surveys in Mathematics and its Applications* **18** (2023), 1-11.

[19] B.K. Tripathi, Hypersurface of a Finsler space with exponential form of (α, β)-metric, *Annals of the University of Craiova-Mathematics and Computer Science Series* **47** (2020), no. 1, 132-140.

[20] B.K. Tripathi, D. Patel, T.N. Pandey,On weakly Berwald space with special exponential (α, β)-metric, *Journal of Rajasthan Academy of Physical Sciences* **22** (2023), no. 1-2, 97-113.

(Brijesh Kumar Tripathi) Department of Mathematics, L. D. College of Engineering, Ahmedabad, Gujarat-380015, India
E-mail address: `brijeshkumartripathi4@gmail.com`

(Sadika Khan) Department of Mathematics, L. D. College of Engineering, Ahmedabad, Gujarat-380015, India
E-mail address: `sadikakhan013@gmail.com`