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A Mathematical Analysis and Numerical Simulation of Fish and Zooplankton Model with Age-structured in the Prey population Taking Account the Fishing Effect

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Abstract. The work presented in this paper is part of the general framework of mathematical ecology regarding the complex dynamics in predatory prey model. The main objective is the formulation and study of a prey-predator model to describe the dynamics of fish and zooplankton population taking into account two phenomenons: the catching effect in the fish population and the age-structure in the zooplankton dynamics. We present some mathematical results concerning the positive solution existence, the stability and persistence of the model equilibria using a positive initial condition. The total population of zooplankton and the total population of fish are uniformly weakly persistent if $\mathbf{R}_0 > 1$ and $\mathbf{R}_\star < 1$. Some numerical simulations according to the mathematical conditions are performed using the finite volume method, are done to illustrate those results in the different fish exploited and inexploited areas.

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1. Introduction

The economic importance of fishing for fish, crustaceans, molluscs and other aquatic organisms, the interest of fishermen in maximizing yields from natural stands, the need for responsible authorities to ensure the safeguarding of stocks through measures are powerful motives for prioritizing research on dynamics of populations exploited in fisheries,[2, 4]. The problem of the management of bio-diversitie resources in generally and particularly, the resources halieutic management, interest many researchers. Moreover, the existence of statistics on large tonnages followed year after year facilitates the work of biologists by providing them with a mass of information which makes it possible to follow the evolution of the population exploited [4, 5]. In marine ecosystems, most aquatic life relies on plankton. The zooplankton is the animal component of the plankton and it is consumed by the fish population and other aquatic animals [7, 12, 13, 28].
Thus, plankton forms the basis of all aquatic food chains and it has an essential role in the study of marine ecology [28, 30]. Fishing plays an important socio-economic role in most countries in maritime vocation. It is an essential lever in the development of these countries and contributes significantly to the growth objectives of the national economy including the deficit of the balance of payments, it also contributes to job

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creation and food needs satisfaction [1, 13]. Along with the high demand for fish products in the absence of adequate and coherent policies for sustainable management of exploitable resources, a situation of overfishing has settled in certain operating areas [4, 5, 8, 15].
Some fish species are becoming increasingly rare. Due to high demand on the international market, the amount of fish caught in commercial value in certain operating areas declined by over 80 % for some fish species. At the present stage of knowledge, research on fish, raises a set of problems in terms of eco-biology and population dynamics. These issues related to biology and species ecology are all ambiguous areas for a comprehensive study and reliable use of existing knowledge to the applied level (management and planning). Current trends of the evolution of stocks of the species show worrying signs regarding the average size of fish caught, the reduction of catch per unit of fishing effort. It appears therefore that essential and relevant control mechanisms should be put in place in order to observe a biological rest which will enable the juveniles to reach the age of sexual maturity before being captured. The central problem of dynamics of this species is the declining of the inventories. The main causes of the problem are different among which there are the increase in fishing effort, the lack of a management plan and layout, changing climatic and hydrological factors, the strong external demand etc.. And the fact of encountering these crises affects this sector and reflections should be undertaken to advocate the sustainable management of this resource[7, 12, 21, 24, 28, 15].
It is in this context that we are interested in this work to study the dynamics of fish, taking into account the notion of fishing effect. age structure in her food population. To deepen our study, we will introduce a age structure in their resource which is the zooplankton population. Note that our study will focus on a mathematical age-structured PDE model that we will build starting from Lotka Volterra equations on the predator-prey models[1, 9, 11, 17, 20]. It is about actually watching carefully the effect of predation on the one hand and on the other hand looking at the effect of the fishery regarding the dynamics of the population in order to take adequate measures concerning the preservation of the species. In particular, the realistic case of extinction of the population may occur. Indeed, we proved that, depending on the age distribution of the fertility rate and of the mortality rate of the preys, the total population tends to disappear[22, 23, 24, 25, 26]. This phenomenon happens when a zooplankton will produce, in average, less than one direct offspring during its lifespan, translated by $\mathbf{R}_0 < 1$ and $\mathbf{R}_0 > 1$. In fact, the mathematical and numerical results prove that if $\mathbf{R_0} < 1$, the equilibria without fishery is globally asymptomatically stable. In the opposite case, when $\mathbf{R_0} > 1$, we proved that under the assumption that the initial zooplankton population is young enough then the total population is uniformly weakly persistent with fishery[18, 29, 33, 34].
This work is structured as follows. In section 2, we present the mathematical model which will be the subject of our study. Section 3 provides some mathematical results of the model in its general version. Also, we express some standard notions from mathematical ecology by formulating the conditions of the solution persistence. Computational simulations are performed in section 4 and finally, in the last section, we end with some conclusion remarks and future works.

2. Mathematical age-structured model formulation

In this section, a model is proposed for to describe the dynamics of the fish-zooplankton system. We will take into account two fundamental ecological aspects, namely age structured and fishing effect, in our modelling[1, 9, 11].

2.1. The food chain typology of the system. The aim is the formulation of some prey-predator models to describe fish and zooplankton dynamics, with age-structure of the zooplankton[9, 22].
Let us consider $u(t,a)$ the density of zooplankton at age $a \geq 0$ at time $t \geq 0$. Note by $B(t)$, the total population of zooplankton at time t. This quantity can be written as follows

$$B(t) = \int_0^{+\infty} u(t,a)da$$

We consider the following functions:

- $\mu(a) \geq 0$ the basic mortality rate of the zooplankton at age a
- $\beta(a, B(t)) \geq 0$ the natural fertility rate of the zooplankton at age a.

The reproduction processes of the zooplankton population is given as follows:

$$u(t,0) = \int_0^{+\infty} \beta(a, B(t))u(t,a)da, \quad (t,a) \in \mathbb{R}^+ \times \mathbb{R}^+$$

The dynamics of the zooplankton at $t = 0$ is given by

$$u(0,a) \quad = \quad u_0(a), \quad a \in \mathbb{R}^+.$$

So, the density of the zooplankton without the fish dynamic is modelled by the following PDE equation:

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) = -\mu(a)u(t,a), \quad (t,a) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ \qquad u(t,0) = \int_0^{\infty} \beta(a, B(t))u(t,a)da, \quad (t,a) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ \qquad u(0,a) = u_0(a), \quad a \in \mathbb{R}^+. \end{cases} \tag{2.1}$$

Now we are introduce in our modelling the fish dynamic take into account the fishing effect in the fish population,[25, 28, 30]. Then, let us consider $v(t)$ the density of the fish population at time t. State model parameters are:

- $\gamma(a)$, the a age-dependent function that represent the predation rate of the fish population on the zooplankton,
- $\alpha, \in (0,1)$ the constant parameters that denote the assimilation coefficient of ingested zooplankton,
- α_1, the constant parameter that denote the basic mortality rate of the fish population,
- $q(a) \geq 0$ the a age-dependent function that represent the fishing catchability at age a,
- $E(t) \geq 0$ the time dependent function at time t that represent the fishing effort on the fish population.

The dynamics of zooplankton-fish system can be represented by the following figure:

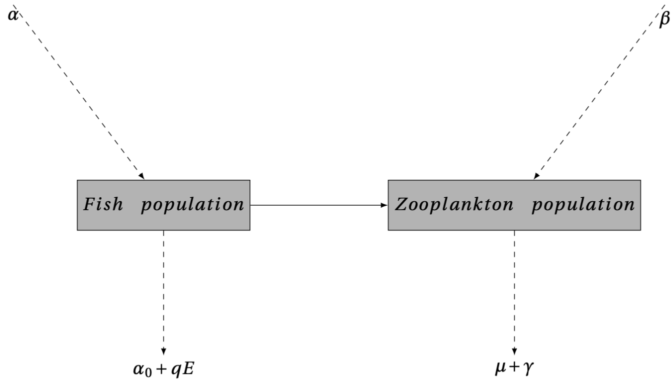

FIGURE 1. Compartmental representation for fish zooplankton model with predator take into account fishing effort on the fish dynamics.

2.2. A age structured PDE model of the system. We develop the final PDE model describing the zooplankton and fish dynamics,[9, 11, 15]. Then according to the Figure 1, we obtain the following PDE system with $q(0) = q_0$ and $E(0) = E_0$:

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) = -\mu(a)u(t,a) - v(t)\gamma(a)u(t,a), \\ v'(t) = \alpha v(t)\int_0^\infty \gamma(a)u(t,a)da - \alpha_1 v(t) - q(a)E(t)v(t), \\ u(t,0) = \int_0^\infty \beta(a,B(t))u(t,a)da, \quad (t,a) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(0,a) = u_0(a), \quad a \in \mathbb{R}^+ \quad and \quad v(0) = v_0. \end{cases} \tag{2.2}$$

where

- $\mu \in \mathbf{L}^\infty(\mathbb{R}_+)$, $\beta \in \mathbf{L}^\infty(\mathbb{R}_+)$ and $\gamma \in \mathbf{L}^\infty(\mathbb{R}_+)$,
- $\beta \geq 0$, $\mu \geq 0$, $\alpha_1 > 0$,
- $\alpha \in (0,1)$

3. Main theoretical results

In all that follows, we assume the following on parameter μ, q and E:

$\exists \quad \mu > 0,$ sach that $\quad \mu(a) \geq \mu_0, \forall a \geq 0, q(a) = q \geq 0, \forall a \geq 0, E(t) = E \geq 0, \forall t \geq 0, (H_0)$

A consequence of (H_0) is that

$$\int_.^\infty \mu(a)da$$

implying that $l(a) = exp(-\int_0^a \mu(i)di)$ is a probability function, this latter describing the survival until age a.

3.1. Positivity and well posedness of the system. The solution of the problem (2.2) is a function w defined by

$$w(t) = \begin{pmatrix} u(t,.) \\ v(t) \end{pmatrix}$$

In all that follows, consider the Banach space

$$\mathbf{X} = \mathbf{L}^1(\mathbb{R}_+) \times \mathbb{R}$$

with the product norm and his nonnegative cone defined by

$$\mathbf{X}_+ = \mathbf{L}^1_+(\mathbb{R}_+) \times \mathbb{R}$$

Let us consider the following differential operator,[9, 10, 29]

$$\mathbf{T} : D(\mathbf{T}) \subset \mathbf{X} \to \mathbf{X}$$

where

$$D(\mathbf{T}) = \left\{ (\phi, \psi) \in \mathbf{X}, \phi \in W^{1,1}(\mathbb{R}_+) \text{ and } \phi(0) = \int_0^\infty \beta(a)\phi(a)da \right\}, \mathfrak{D}\phi = -\frac{d\phi}{da} - \mu\phi$$

$$\mathbf{T} = \begin{pmatrix} \mathfrak{D} & 0 \\ 0 & -\alpha_1 - qE \end{pmatrix}$$

and the function $Q : \mathbf{X} \to \mathbf{X}$ given by

$$Q(\phi, \psi) = \begin{pmatrix} -\psi\gamma(.)\phi(.) \\ \alpha \int_0^\infty \gamma(a)\phi(a)da \end{pmatrix}$$

so that the (2.2) rewrites as the following abstract Cauchy Problem :

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \mathbf{T}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + Q(u(t), v(t)), \\ (u(0), v(0)) = (u_0(.), v_0) \in \mathbf{X} \end{cases} \tag{3.3}$$

The nonlinear part Q of system (3.3) is quadratic,[9, 29]. So it is clearly a locally Lipschitz continuous function on $\mathbf{X}$. For every $(u_0(.), v_0) \in \mathbf{X}$, there exists $t_{max} \leqslant +\infty$ such that system (3.3) has a unique mild solution $w \in \mathbf{C}([0, t_{max}(u_0(.), v_0)), \mathbf{X})$ where $t_{max}(u_0(.), v_0) \leqslant \infty$. Furthermore, this solution is defined in a classical sense whenever $(u_0(.), v_0) \in \mathbf{D}(\mathbf{T})$.
We are proved that for any initial condition $(u_0(.), v_0) \in \mathbf{X}_+$ the corresponding solution remains nonnegative on $[0, t_{max})$. Let us define

$$\mathbf{A_m} = \{(\phi, \psi) \in \mathbf{X}, \quad \|(\phi, \psi)\|_{\mathbf{X}} \leqslant m\}, \quad m > 0.$$

Proposition 3.1. *For every $(u_0(.), v_0) \in \mathbf{X}_+$, there exists $t_{max}(u_0, v_0) \leqslant \infty$ such that system (3.3) has a unique mild solution $w \in \mathbf{C}([0, t_{max}(u_0(.), v_0)), \mathbf{X}_+)$.*

Proof. Let us consider $m > 0$ and $\lambda_m \geqslant m\|\gamma\|_{\mathbf{L}^\infty(\mathbb{R}_+)}$. We define the differential operator

$$\mathbf{T}_m = \mathbf{T} - \lambda_m \mathbf{I} \quad \text{and the function} \quad \mathbf{Q}_m = \mathbf{Q} + \lambda_m \mathbf{I}.$$

Then $\mathbf{T}_m$ is the infinitesimal generator of a positive [29, 33] $\mathcal{C}_0$-semigroup $\{\mathbf{R}_{\mathbf{T}_m}(t)\}_{t\geq 0}$ on $\mathbf{X}_+$ that satisfies

$$\|\mathbf{R}_{\mathbf{T}_m}(t)\|_{\mathbf{X}} \leqslant e^{-(\lambda_m+\omega)t}, \quad \forall t \geq 0.$$

Let us consider

$$r_m \quad = \quad 2\|(u_0, v_0)\|_{\mathbf{X}} \sup_{s\in[0,1]} \|\mathbf{R}_{\mathbf{T}_m}(s)\| > 0,$$

then suppose m large enough to haver $r_m \leq m$ and we denote by

$$\mathbf{X}_+^{r_m} \quad = \quad \mathbf{X}_+ \cap \mathbf{A}_{r_m} \subset \mathbf{A}_m.$$

We consider $\Lambda > 0$ such that

$$\Lambda \leq min\left\{1, \frac{1}{2(Kr_m + \lambda_m)\sup_{s\in[0,1]}\|\mathbf{R}_{\mathbf{T}_m}(s)\|_{\mathbf{X}}}\right\}, K = 2\|\gamma\|_{\mathbf{L}^\infty(\mathbb{R}^+)}$$

The linear operator,[10, 14]

$$\mathfrak{R} : \mathbf{C}([0,\Lambda],\mathbf{X}) \to \mathbf{C}([0,\Lambda],\mathbf{X})$$

$$\mathfrak{R}(u(t,.),v(t)) = \mathbf{R}_{\mathbf{T}_m}(t).w_0 + \int_0^t \mathbf{R}_{\mathbf{T}_m}(t-s)\mathbf{Q}_{\mathbf{m}}w(s)ds$$

is a 1/2-shrinking operator on $\mathbf{C}([0,\Lambda],\mathbf{X}_+^{r_m})$ that preserves this latter space. The Banach-Picard theorem,[26, 32, 33] and some classical time extending properties of the solution then yield the proposition. □

The following result ensure the global existence of the system (3.3) solution.

Proposition 3.2. *For every $(u_0,v_0) \in \mathbf{X}_+$, system (3.3) has a unique mild solution $w \in \mathbf{C}(\mathbb{R}_+,\mathbf{X}_+)$.*

Proof. Consider $w \in \mathbf{C}([0,t_{max}),\mathbf{X}_+)$, the solution of system (3.3) and suppose by contradiction that $t_{max} < \infty$. Let us first prove that for every $t \geq 0$, $\|u(t,.)\|_{\mathbf{L}^1(\mathbb{R}_+)}$. A direct consequence of the positivity is that

$$\partial_t u(t,a) + \partial_a u(t,a) \quad \leq \quad -\mu(a)u(t,a).$$

By using the characteristics of the PDE equation, that an implicit formulation of the solution of

$$\partial_t u(t,a) + \partial_a u(t,a) \quad = \quad -\mu(a)u(t,a)$$

that satisfies the loopback boundary condition in (2.2) is given by

$$u(t,a) = \begin{cases} u_0(a-t)e^{-\int_{a-t}^a \mu(s)ds}, & a \geq t \\ \Phi(t-a)e^{-\int_0^a \mu(s)ds}, & a < t \end{cases} \tag{3.4}$$

where $\Phi(t) = u(t,0)$ verified

$$\begin{aligned} \Phi(t) &= \int_0^t \beta(i)\Phi(t-i)e^{-\int_0^i \mu(s)ds}di + \int_t^\infty \beta(i)u_0(i-t)e^{-\int_{i-t}^i \mu(s)ds}di \\ &= \int_0^t \Phi(i)\beta(t-i)e^{-\int_0^{t-i} \mu(s)ds}di + \int_0^\infty \beta(i+t)u_0(i)e^{-\int_i^{i+t} \mu(s)ds}di \end{aligned} \tag{3.5}$$

Using the equation (3.5) we define two operators,[26, 27, 15]

$$\mathbf{T}_1 : \mathbf{L}^1(0,t) \to \mathbf{L}^1(0,t), \quad \mathbf{T}_2 : \mathbf{L}^1(\mathbb{R}_+) \to \mathbf{L}^1(0,t)$$

$$\text{for any} \quad \Phi \in \mathbf{L}^1(0,t), \phi \in \mathbf{L}^1(\mathbb{R}_+) \quad \text{and} \quad \xi \in [0,t] \quad \text{by}$$

$$\mathbf{T}_1\Phi(\xi) = \int_0^\xi \Phi(v)\beta(\xi-v)e^{-\int_0^{\xi-v} \mu(s)ds}dv$$

$$\mathbf{T}_2\phi(\xi) = \int_0^\infty \phi(v)\beta(\xi+v)e^{-\int_v^{\xi+v} \mu(s)ds}dv$$

Consequently we formally get the following representation:

$$u(t,a) = \begin{cases} u_0(a-t)e^{-\int_{a-t}^a \mu(s)ds}, & a \geq t \\ (I-\mathbf{T}_1)^{-1}\mathbf{T}_2u_0(t-a)e^{-\int_0^a \mu(s)ds}, & a < t \end{cases} \tag{3.6}$$

The function $u(t,a)$ define in (3.6) is well defined. According to the works in [10, 14], $\mathbf{T}_1$ is a Volterra operator, then for all $\lambda \in \mathbb{C} \setminus \{0\}$ and $\Phi \in \mathbf{L}^1(0,t)$ fixed, we have a unique function

$$\varphi \in \mathbf{L}^1(0,t) \quad \text{such that} \quad (\lambda I - \mathbf{T}_1)\varphi = \Phi$$

Thus $(I-\mathbf{T}_1)^{-1}$ is well defined from $\mathbf{L}^1(0,t)$ to $\mathbf{L}^1(0,t)$. Since

$$u_0 \in \mathbf{L}^1(\mathbb{R}_+) \quad \text{then} \quad (I-\mathbf{T}_1)^{-1}\mathbf{T}_2 u_0 \in \mathbf{L}^1(0,t).$$

Consequently for all $t \geq 0$ we have

$$\|u(t,.)\|_{\mathbf{L}^1(\mathbb{R}_+)} \leq \int_t^\infty u_0(a-t)da + \int_0^t (I-\mathbf{T}_1)^{-1}\mathbf{T}_2 u_0(a-t)da < \infty$$

Moreover, straightforward upper bounds imply that

$$v'(t) \leq \alpha C v(t)\|\gamma\|_{\mathbf{L}^\infty(\mathbb{R}_+)} \forall t \geq 0, \quad \text{where} \quad C = max_{s\in[0,t_{max}]}\|u(s,.)\|_{\mathbf{L}^1(\mathbb{R}_+)} < \infty.$$

Thus $t_{max} < \infty$, an integration of the differential inequality would lead to

$$v(t) \leq v_0 e^{\alpha C t_{max}\|\gamma\|_{\mathbf{L}^\infty}} < \infty,$$

implying a contradiction with the fact that we have either

$$\lim_{t\to t_{max}} \|u(s,.)\|_{\mathbf{L}^1(\mathbb{R}_+)} = \infty \quad \text{or} \quad \lim_{t\to t_{max}} \|v(t)\| = \infty.$$

Thus $t_{max} = \infty$ and the solution is global in time. □

Now our goal is to analyse the asymptotic behaviour of the solutions.

3.2. Asymptotic behaviour analysis of the solutions. Let us define

$$\rho_1 = \sup\{a \geq 0 : \quad |\operatorname{supp}(\gamma)\bigcap(0,a) = 0\} < \infty$$

Consequently

$$\exists \quad \gamma_\star > 0 \quad \text{and} \quad \rho_2 > \rho_1 \quad \text{such} \quad \text{that} \quad \int_{\rho_1}^{\rho_2} \gamma(a)da \geq \gamma_\star.$$

The case $\rho_1 > 0$ translates the fact that the youngest zooplankton are not considered as a resource availability for the fish population. We define the following thresholds,[22, 25]

$$\mathbf{R_0} = \int_0^\infty \beta(a)e^{-\int_0^a \mu(s)ds}da \quad \text{and} \quad \mathbf{R}_\star = \int_0^{\rho_1} \beta(a)e^{-\int_0^a \mu(s)ds}da$$

Let us consider the positive initial condition $(u_0, v_0) \in \mathbf{X}_+^*$ where

$$\mathbf{X}_+^* = \{(u_0,v_0) \in \mathbf{X}_+ : \int_0^\infty u_0(a)da > 0 \quad \text{and} \quad v_0 > 0\}$$

Remark 3.1. Similarly to the basic reproductive number in the epidemiological case, we have:

(a) The $\mathbf{R_0}$ value represents the average number of offspring that is produced over the lifetime by one zooplankton and in a context with no predation.

(b) The $\mathbf{R}_\star$ value represents the offspring produced by one zooplankton from his birth, until it begins to be hunted by the fish population.

Proposition 3.3. *Concerning to the system (2.2) equilibria, we have:*

(1) $E_0 = (0,0)$ *is a unique trivial equilibrium if* $\mathbf{R}_0 < 1$ *or if* $\mathbf{R_0} > 1$ *and* $\mathbf{R}_\star \geq 1$.

(2) *For all* $\varepsilon \in [0,\infty)$, $E_{1,\varepsilon} = (u^*_{1,\varepsilon}, 0)$ *is an equilibrium if* $\mathbf{R_0} = 1$ *and* $\mathbf{R}_\star < 1$ *where* $u^*_{1,\varepsilon}(a) = \varepsilon e^{-\int_0^a \mu(s)ds}$.

(3) *If* $\mathbf{R}_0 = 1$ *and* $\mathbf{R}_\star = 1$, *then for all* $\varepsilon \geq 0, E_{1,\varepsilon}$ *and* $E_{2,\varepsilon} = (u^*_{2,\varepsilon}, \varepsilon)$ *are equilibria of* (2.2) *with*

$$u^*_{2,\varepsilon}(a) = u^*_{2,\varepsilon}(0)e^{-\int_0^a \mu(s)ds - \varepsilon\int_0^a \gamma(s)ds},$$
$$u^*_{2,\varepsilon}(0) = \frac{\alpha_1 - qE}{\alpha}\left[\int_0^\infty \gamma(a)e^{-\int_0^a \mu(s)ds-\varepsilon\int_0^a\gamma(s)ds}da\right]^{-1}.$$

(4) *If* $\mathbf{R_0} > 1$ *and* $\mathbf{R}_\star < 1$, *then besides the trivial equilibrium* E_0 *there is a positive non trivial equilibrium* $E_2 = (u_2^*, v^*) = (u_{2,v^*}, v^*)$ *with*

$$\int_0^\infty \beta(a)e^{-\int_0^a \mu(s)ds - v^*\int_0^a\gamma(s)ds}da = 1 \tag{3.7}$$

Proof. The point $(u^*, v^*) \in \mathbf{X}$ is an equilibrium point if it is a solution of the system

$$\begin{cases} (u^*)'(a) &= -\mu(a)u^*(a) - v^*\gamma(a)u^*(a), \\ 0 &= \alpha v^*\int_0^\infty \gamma(a)u^*(a)da - \alpha_1 v^* - qEv^*, \\ u^*(0) &= \int_0^\infty \beta(a)u^*(a)da. \end{cases}$$

After an integration, we have the following system

$$\begin{cases} u^*(a) = u^*(0)e^{-\int_0^a \mu(s)ds - v^*\int_0^a\gamma(s)ds}, \\ u^*(0)\left[1 - \int_0^\infty \beta(a)e^{-\int_0^a\mu(s)ds - v^*\int_0^a\gamma(s)ds}da\right] = 0, \\ v^*\left[\alpha\int_0^\infty \gamma(a)da - \alpha_1 - qE\right] = 0. \end{cases}$$

□

Let us define the following expression, [10, 29, 15]:

(1) The growth rate $\omega_0(\mathbf{T})$ of $\{\mathbf{R_T}(t)\}_{t\geq 0}$ define by

$$\omega_0(\mathbf{T}) = \omega_0(\mathbf{R_T}(t)\}_{t\geq 0}) := \lim_{t\to+\infty}\frac{\ln(\|\mathbf{R_T}(t)\|_{\mathbf{X}})}{t}$$

(2) We consider $\mathbf{K}(\mathbf{X}) \subset \mathbf{L}(\mathbf{X})$ and define the following expression for all $\mathbf{F} \in \mathbf{L}(\mathbf{X})$,

$$\|F\|_{ess} := \inf_{H\in\mathbf{K}(\mathbf{X})}\|F - H\|_{\mathbf{X}} \text{ and } \omega_{ess}(\mathbf{T}) = \omega_{ess}(\mathbf{R_T}(t)\}_{t\geq 0}) := \lim_{t\to+\infty}\frac{\ln(\|\mathbf{R_T}(t)\|_{ess})}{t}$$

the essential growth rate of $\{\mathbf{R_T}(t)\}_{t\geq 0}$,

3.3. Stability of system equilibria. To perform the stability analysis of the system (2.2), we exhibit some spectral properties,[22, 27] of the differential operator $\mathbf{T}$ and of the semigroup $\{\mathbf{R_T}(t)\}_{t\geq 0}$. For more details about spectral theory and stability results we can look at [10, 11, 14]. For every equilibrium point $E = (u^*, v^*) \in \mathbf{X}$ the differential of Q at an equilibrium E can be written as

$$D_E Q = (D_E Q)_1 + (D_E Q)_2 = \begin{pmatrix} -v^*\gamma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -u^*\gamma \\ \alpha v^* J_\gamma(.) & \alpha\int_0^\infty \gamma(a)u^*(a)da. \end{pmatrix}$$

where J_γ is the operator defined for some integrable function g on $\mathbb{R}_+$ by $J_\gamma : g \to \int_0^\infty \gamma(a)g(a)da.$

3.3.1. *Stability of E_0 study.* The linearised system to study is $X'(t) = \mathbf{T}X(t)$. The differential of Q at the point E_0 is the null matrix. Using [10, 14, 15] and since $\omega_{ess}(\mathbf{T}) < 0$, we just need to study eigenvalues of $\mathbf{T}$. We thus try to solve the system

$$\begin{cases} \partial_t u(t,a) &= -\partial_a u(t,a) - \mu(a)u(t,a), \\ v'(t) &= -\alpha_1 v(t) - qEv(t), \\ u(t,0) &= \int_0^\infty \beta(a)u(t,a)da \end{cases} \tag{3.8}$$

Then the solutions of (3.8) take the following form:

$$u(t,a) = \overline{u}(a)e^{\rho t}, \quad v(t) = \overline{v}e^{\rho t} \quad \text{and} \quad \rho \in \mathbb{C}.$$

So, according to the first equation of the system (3.8), we obtain the following expressions:

$$\begin{cases} \overline{u}(a) &= \overline{u}(0)e^{-\int_0^a[\rho+\mu(s)]ds}, \\ \lambda\overline{v} &= -\alpha_1\overline{v} - qE\overline{v}, \\ \overline{u}(0) &= \int_0^\infty \overline{u}(a)\beta(a)da. \end{cases}$$

The second equation only admits $-(\alpha_1 + qE)$ as eigenvalue, which is real and negative. Then, using the third equation, we obtain the following characteristic equation

$$\int_0^\infty \beta(a)e^{-\int_0^a[\rho+\mu(s)]ds} = 1$$

Consequently, we have the following proposition 3.4 concerning to the stability of E_0 [9, 10].

Proposition 3.4. *The state E_0 is globally asymptotically stable in $\mathbf{X}_+$ if* $\mathbf{R}_0 < 1$ *and unstable if* $\mathbf{R}_0 > 1$.

Proof. (1) Proof of E_0 is globally asymptotically stable. Suppose that $\mathbf{R}_0 < 1$. We have the real part of the characteristic equation:

$$\int_0^\infty \beta(a)e^{-\mathrm{Re}(\rho)a}\cos(-\mathrm{Im}(\rho)a)e^{-\int_0^a \mu(s)ds}da = 1.$$

Then, if $\mathrm{Re}(\rho) \geq 0$, we get $\mathbf{R}_0 \geq 1$ that is absurd, so $\omega_0(\mathbf{T}) < 0$, and E_0 is locally exponentially asymptotically stable,[10, 11, 14]. Now we prove the global stability. Let $(u_0, v_0) \in \mathbf{X}_+$ be the initial condition, then the solution of (2.2) at time t is given by the Duhamel formula,[10, 11, 14].

$$\begin{pmatrix} u(t,.) \\ v(t) \end{pmatrix} = \mathbf{R_T}(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \int_0^t \mathbf{R_T}(t-s)f\begin{pmatrix} u(t,.) \\ v(t) \end{pmatrix}$$

We have

$$\lim_{t\to\infty}\left\|\mathbf{R_T}(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right\|_{\mathbf{X}} = 0 \quad \text{and} \quad \lim_{t\to\infty}\|u(t,.)\|_{\mathbf{L}^1} = 0$$

From the second equation of (2.2), we get $\lim_{t\to\infty} v(t) = 0$ and the global stability of E_0 follows.

(2) Now we show that suppose that E_0 is unstable if $\mathbf{R}_0 > 1$. In fact, under the hypothesis $\mathbf{R}_0 > 1$, we define the following function

$$f : \rho \to \int_0^\infty \beta(a)e^{-\rho a}e^{-\int_0^\infty \mu(s)ds}da$$

It is clear to see that f is strictly decreasing and $f(0) = \mathbf{R}_0 > 1$. There consequently exists $\overline{\rho} > 0$ such that $f(\overline{\rho}) = 1$, so we have $\omega_0(\mathbf{T}) > 0$. Since $\omega_{ess}(\mathbf{T}) \leq 0$. So E_0 is unstable, [1, 2, 13, 10, 11, 14, 22].

□

3.3.2. *Stability of $E_{1,\varepsilon}$ study.* The differential of Q at $E_{1,\varepsilon}$ is given by

$$D_{E_{1,\varepsilon}}Q = \begin{pmatrix} 0 & -u^*_{1,\varepsilon}\gamma \\ 0 & \alpha \int_0^\infty \gamma(a)u^*_{1,\varepsilon}(a)da. \end{pmatrix}$$

and the linearized at $E_{1,\varepsilon}$ is thus $\dfrac{dw}{dt}(t) = (\mathbf{T} + D_{E_{1,\varepsilon}}Q)w(t)$

Once again we just need to study eigenvalues of the operator $\mathbf{T} + D_{E_{1,\varepsilon}}$, so we study the system

$$\begin{cases} \overline{u}(a) &= \overline{u}(0)e^{-\int_0^a[\rho+\mu(s)]ds} - \gamma(a)u^*_{1,\varepsilon}(a)\overline{v}, \\ \rho\overline{v} &= -\alpha_1\overline{v} - qE\overline{v} + \alpha\overline{v}\int_0^\infty u^*_{1,\varepsilon}(a)\gamma(a)da, \\ \overline{u}(0) &= \int_0^\infty \overline{u}(a)\beta(a)da. \end{cases} \tag{3.9}$$

Proposition 3.5. *The stability of $E_{1,\varepsilon}$ is given by the following results.*

(1) *The equilibrium $E_{1,\varepsilon}$ is unstable if $\alpha\varepsilon > (\alpha_1 + qE)\left[\int_0^\infty \gamma(a)e^{-\int_0^a \mu(s)ds}da\right]^{-1}$.*
(2) *If $\varepsilon > 0$ then for any $\xi > 0$ there exist $\overline{\varepsilon}(\xi)$ such that $E_{1,\overline{\xi}} \in B(E_{1,\varepsilon}, \xi)$*

Proof. (1) We suppose that $\alpha\varepsilon > (\alpha_1 + qE)\left[\int_0^\infty \gamma(a)e^{-\int_0^a \mu(s)ds}da\right]^{-1}$ and we use the second equation of (3.9) for to get the following expression:

$$\begin{aligned} \rho\overline{v} &= \left(-\alpha_1 - qE + \alpha\int_0^\infty \varepsilon\gamma(a)e^{-\int_0^a \mu(s)ds}da\right)\overline{v}. \quad \text{Let us consider} \\ \overline{\rho} &= -\alpha_1 - qE + \alpha\int_0^\infty \varepsilon\gamma(a)e^{-\int_0^a \mu(s)ds}da > 0. \end{aligned}$$

According to the first and third equation of (3.9), we have

$$\left(1 - \int_0^\infty \beta(a)e^{-\int_0^a(\mu(s)+\overline{\rho})ds}da\right)\overline{u}(0) + \varepsilon\overline{v}\int_0^\infty \beta(a)\gamma(a)e^{-\int_0^a \mu(s)ds}da = 0.$$

We obtain directly that

$$\int_0^\infty \beta(a)e^{-\int_0^a(\mu(s)+\overline{\rho})ds}da < \mathbf{R_0} \quad \text{since} \quad \overline{\rho} > 0.$$

We know by definition of $E_{1,\varepsilon}$ that $\mathbf{R_0} = 1$. Consequently

$$\left(1 - \int_0^\infty \beta(a)e^{-\int_0^a(\mu(s)+\overline{\rho})ds}da\right) > 0.$$

If $\overline{v} = 1$ and

$$\overline{u}(0) = \frac{-\varepsilon\int_0^\infty \gamma(a)\beta(a)e^{-\int_0^a \mu(s)ds}da}{1 - \int_0^\infty \beta(a)e^{-\int_0^a(\mu(s)+\overline{\rho})ds}da}$$

So we define

$$\overline{u}(a) = \overline{u}(0)e^{-\int_0^a[\mu(s)+\overline{\rho}]ds} - \gamma(a)u^*_{1,\varepsilon}(a).$$

So we find a positive eigenvalue and the equilibrium $E_{1,\varepsilon}$ is unstable,[9, 10].

(2) Let $\varepsilon > 0, \xi > 0$ and define $\overline{\varepsilon} = \varepsilon + \xi\mu_0$. We obtain $E_{1,\varepsilon} \in B(E_{1,\varepsilon}, \xi)$ since we have

$$\|E_{1,\overline{\varepsilon}} - E_{1,\varepsilon}\|_{\mathbf{X}} = \|u^*_{1,\overline{\varepsilon}} - u^*_{1,\varepsilon}\|_{\mathbf{L}^1(\mathbb{R}_+)} = \mid \overline{\varepsilon} - \varepsilon \mid \int_0^\infty e^{-\int_0^a \mu(s)ds} da \leq \xi\mu_0 \int_0^\infty e^{-\mu_0 a} da \leq \xi.$$

This latter inequality prevents all equilibria $E_{1,\varepsilon}$ for any $\varepsilon \geq 0$ to be locally asymptotically stable.

□

3.3.3. *Stability of E_2 study.* The differential of Q at he equilibrium $E_2 = (u_2^*, v^*)$ is given by

$$D_{E_2}Q = \begin{pmatrix} -v^*\gamma & -u_2^*\gamma \\ \alpha v^* J_\gamma(.) & \alpha_1 + qE \end{pmatrix}$$

The equation $\dfrac{dX(t)}{dt} = (\mathbf{T} + D_{E_2}Q)X(t)$ is the linearised system,[11, 29]. We consider the following system:

$$\begin{cases} \overline{u}'(a) & = & [\rho + \mu(a) + v^*\gamma(a)]\overline{u}(a) - \gamma(a)u_2^*(a)\overline{v}, \\ \rho\overline{v} & = & \alpha v^* \int_0^\infty \overline{u}(a)\gamma(a)da, \\ \overline{u}(0) & = & \int_0^\infty \overline{u}(a)\beta(a)da. \end{cases} \tag{3.10}$$

If we consider

$$m_{11} = 1 - \int_0^\infty \beta(a) e^{-\int_0^a (\mu(s) + \rho + v^*\gamma(s))ds} da,$$

$$m_{12} = \frac{\alpha_1 + qE}{\alpha\theta} \int_0^\infty \beta(a) e^{-\int_0^a [\mu(s) + v^*\gamma(s)]ds} \int_0^a \gamma(w) e^{-\rho(a-w)} dw da,$$

$$m_{21} = \alpha v^* \int_0^\infty \gamma(a) e^{-\int_0^a (\mu(s) + \rho + v^*\gamma(s))ds} da,$$

$$m_{22} = -\rho - \frac{(\alpha_1 + qE)v^*}{\theta} \int_0^\infty \gamma(a) e^{-\int_0^a [\mu(s) + v^*\gamma(s)]ds} \int_0^a \gamma(w) e^{-\rho(a-w)} dw da$$

with

$$\theta = \int_0^\infty \gamma(a) e^{-\int_0^a [\mu(s) + v^*\gamma(s)]ds} da.$$

We finally have to solve the system $MU = V$ where

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad V = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} \overline{u}(0) \\ \overline{v} \end{pmatrix}.$$

We can compute the eigenvalues in the specific case where γ is constant.

Proposition 3.6. *If there exist γ_0 such that $\gamma(a) = \gamma_0$ for all $a \geq 0$, then $\rho = \pm\sqrt{v^*\gamma_0(\alpha_1 + qE)}$.*

Proof. We need to have $\det(M) = 0$ to get a non zero solution of the system $MU = V$.

$$\det(M) = 0 \implies m_{11}m_{22} = m_{21}m_{12} \implies m_{12} = \frac{\gamma_0 m_{11}(\alpha_1 + qE)}{\alpha\theta\rho}$$

$$\text{and} \quad m_{22} = \frac{\gamma_0 m_{21}(\alpha_1 + qE)}{\alpha\theta\rho}$$

So we obtain

$$-\rho - \frac{(\alpha_1 + qE)v^*\gamma_0^2}{\rho\theta}\left(\int_0^\infty e^{-\int_0^a[\mu(s)+v^*\gamma_0]ds}da - \int_0^\infty e^{-\int_0^a[\mu(s)+v^*\gamma_0+\rho]ds}da\right) = \frac{(\alpha_1 + qE)v^*\gamma_0^2}{\rho\theta}\int_0^\infty e^{-\int_0^a[\mu(s)+v^*\gamma_0+\rho]ds}da.$$

We finally get $-\rho = \dfrac{(\alpha_1 + qE)v^*\gamma_0}{\rho}$ according to the expression of θ. □

3.4. Persistence of the solution analysis. Her, express some standard notions from mathematical ecology by formulating, in the context of (2.2), the definition of persistence,[9, 11, 14, 15].

We assume the the following hypothesis:

$$(H_1): \quad \exists \quad \kappa_1 > 0, \quad \exists \quad 0 < \underline{a} < \overline{a} < \infty$$

$$\text{such that} \quad \beta(a) \geq \kappa_1 \text{ for almost every (f.a.e)} \quad a \in (\underline{a}, \overline{a})$$

$$(H_2): \quad \exists \quad \kappa_2 > 0, \quad \exists \quad 0 \leq t_1 < t_2 < \underline{a}: \quad \int_{t_1}^{t_2} u_0(a)da \geq \kappa_2.$$

Without lost of generality, we assume that $\mid t_1 - t_2 \mid < \mid \underline{a} - \overline{a} \mid$ even if we reduce κ_2.

Remark 3.2. On the zooplankton population, in the biological point of view we have the following observations:

(a) The hypothesis (H_1) means that the zooplankton population of a certain range of age have a high ability to reproduce.
(b) The hypothesis (H_2) together with property (H_1) means that there is initially a high enough quantity of young zooplankton that will be able to reproduce later.

Proposition 3.7. [10, 11, 14] *For any initial condition* $(u_0, v_0) \in \mathbf{X_{H_2}}$ *and if* $\mathbf{R}_\star > 1$, *then zooplankton population and fish population explode with* $\mathbf{X}_{H_2} = \{(u_0, v_0) \in \mathbf{X}_+^* :$ *that satisfies* $H_2\}$.

Proof. We have the following equation

$$\partial_t u(t,a) + \partial_a u(t,a) = -(\mu(a) + v(t)\gamma(a))u(t,a), \quad \forall t \geq 0, \quad \forall a \geq 0.$$

This equation leads to:

$$u(t,a) \geq \begin{cases} u_0(a-t)e^{-\int_{a-t}^a(\mu(s)+v(t-a+s)\gamma(s))ds}, & a \geq t \\ \Phi(t-a)e^{-\int_0^a(\mu(s)+v(t-a+s)\gamma(s))ds}, & a < t. \end{cases} \tag{3.11}$$

- We prove firstly that there exists $t^\star$ such that $\Phi(t) > 0, \forall t \in [t^\star, t^\star + \rho_1]$. We know that

$$\Phi(t) \geq \int_{t_1}^{t_2} \beta(w+t)u_0(w)e^{-(\|\mu\|_{\mathbf{L}^\infty} + K\|\gamma\|_{\mathbf{L}^\infty})t}dw, \quad \forall t \in [\underline{a} - t_1, \overline{a} - t_2].$$

where $K = \max_{w \in [0, \overline{a} - t_2]} v(w) < \infty$. So under the hypothesis (H_1) and (H_2), we have

$$\Phi(t) \geq \sigma_1 := \kappa_1\kappa_2 e^{-(\|\mu\|_{\mathbf{L}^\infty} + K\|\gamma\|_{\mathbf{L}^\infty})(\overline{a} - t_2)}, \quad \forall t \in [\underline{a} - t_1, \overline{a} - t_2].$$

Either $\underline{a} - t_2 - (\overline{a} - t_1) \geq \rho_1$ and this step is done, or we continue by defining $\epsilon \in (0, \underline{a} - t_2 - (\overline{a} - t_1))$. Then we prove that

$$\Phi(t) \geq \sigma_2 := \kappa_1 \sigma_1 (t_2 - t_1) e^{-(\|\mu\|_{\mathbf{L}^\infty} + \widehat{K}\|\gamma\|_{\mathbf{L}^\infty})(2\overline{a} - t_2 - \epsilon)} > 0,$$
$$\forall t \in [2\underline{a} - t_1 + \epsilon, \overline{a} - t_2 - \epsilon], \quad \widehat{K} = \max_{w \in [0, 2\overline{a} - t_2 - \epsilon]}.$$

So, if $\overline{a} - t_2 - (\underline{a} - t_1) + (\overline{a} - \underline{a} - \epsilon) \geq \rho_1$ we stop, else we continue. Since we get each time a bigger interval on which Φ is positive then we get what we wanted.

- Now, we can prove that $\Phi(t) > 0, \forall t \geq t^\star$. Indeed, since $\mathbf{R}_\star > 0$, there exist

$$\epsilon \in (0, \rho_1) \quad \text{such that} \quad \int_\epsilon^{\rho_1} \beta(w) dw > 0.$$

Then $\forall \overline{\epsilon} \in [0, \epsilon]$, we have

$$\Phi(t^\star + \rho_1 + \overline{\epsilon}) \geq \int_{t^\star + \overline{\epsilon}}^{t^\star + \rho_1} \Phi(w) \beta(t^\star + \rho_1 + \overline{\epsilon} - w) e^{-(t^\star + \rho_1 + \overline{\epsilon} - w)\|\mu\|_{\mathbf{L}^\infty}} dw > 0.$$

So, we have $\Phi(t) > 0, \forall t \geq t^\star$.

- We prove that $\lim_{t \to \infty} \Phi(t) = \infty$. Note that

$$\Phi(t^\star + \rho_1) \geq \Phi(\underline{t}) \mathbf{R}_\star \quad \text{with} \quad \Phi(\underline{t}) = \min_{w \in [t^\star, t^\star + \rho_1]} \Phi(w).$$

Moreover for every $l > 0$ we have

$$\Phi(t^\star + \rho_1 + l) \geq \mathbf{R}_\star \min_{w \in [t^\star + l, t^\star + a_1 + l]} \Phi(w) \geq \Phi(\underline{t}) \mathbf{R}_\star$$

Then, we have

$$\Phi(t^\star + 2\rho_1 + l) \geq \mathbf{R}_\star \min_{w \in [t^\star + \rho_1 + l, t^\star + 2\rho_1 + l]} \Phi(w) \geq \Phi(\underline{t}) \mathbf{R}_\star^{\mathbf{2}}, \quad \forall l > 0.$$

- We have

$$u(t, a) = \Phi(t - a) e^{-\int_0^a \mu(s) ds} \quad \text{for any} \quad a \in [0, \rho_1] \quad \text{and} \quad \forall t > a.$$

Then $\lim_{t \to \infty} u(t, a) = \infty, \quad \forall a \in [0, \rho_1]$. Consequently we have

$$\lim_{t \to \infty} \|u(t, .)\|_{\mathbf{L}^1(\mathbb{R}_+)} = \infty.$$

Now, let us suppose that there exists $K > 0$ such that for every $t \geq 0$ we have $v(t) \leq K$. Then a lower bound of (2.2) gives for every $t \geq 0$

$$u(t, a) \geq \begin{cases} u_0(a - t) e^{-(\|\mu\|_{\mathbf{L}^\infty} + K\|\gamma\|_{\mathbf{L}^\infty})t}, & a \geq t \\ \Phi(t - a) e^{-(\|\mu\|_{\mathbf{L}^\infty} + K\|\gamma\|_{\mathbf{L}^\infty})a}, & a < t \end{cases}$$

We have $\lim_{t \to \infty} \Phi(t) = \infty$ and so for every $\overline{K} > 0$, there exist $t^\star > 0$ such that $\Phi(t) \geq \overline{K}$ for any $t \geq t^\star$. Consequently, we can write

$$u(t, a) \geq \overline{K} e^{-(\|\mu\|_{\mathbf{L}^\infty} + K\|\gamma\|_{\mathbf{L}^\infty})\rho_2} =: \overline{K} S, \forall t \geq t^\star + \rho_2 \quad \text{and every} \quad a \in (\rho_1, \rho_2).$$

We can deduce from (2.2) that

$$v'(t) \geq \left[\alpha \int_{\rho_1}^{\rho_2} \gamma(a) u(t, a) da - \alpha_1 - qE \right] v(t) \geq [\alpha \gamma - \overline{K} S - \alpha_1 - qE] v(t) =: \overline{S} v(t),$$
$$\forall t \geq t^\star + \rho_2.$$

Taking $\overline{K}$ big enough, we get $S > 0$. Finally, an integration of the latter equation gives

$$v(t) \geq v(t^\star + \rho_2)e^{-\overline{S}(t^\star+\rho_2)}e^{\overline{S}t} \to \infty, \quad \text{where} \quad t \to \infty, \quad \forall t \geq t^\star + \rho_2$$

which is a contradiction with the fact that v is bounded.

- Consequently there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} t_n = \infty = \lim_{n\to\infty} v(t_n).$$

Thus for every $M > 0$, there exists $n^\star \in \mathbb{N}$ such that $v(t_n) \geq M, \quad \forall n \geq n^\star$. Suppose that $v(t)$ does not go to ∞ when t goes to ∞. Then there would exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$, there exists $\bar{t} > t_n$ that verifies $v(\bar{t}) < \epsilon$. Obviously there exists $n^\star \in \mathbb{N}$ such that meaning that

$$\bar{t} \in [t_{n^\star}, t_{n^\star+1}], \text{meaning that} \quad v(t_{n^\star}) \geq K, \quad v(\bar{t}) < \epsilon, \quad v(t_{n^\star+1}) \geq K.$$

Let

$$K \geq M := \epsilon e^{(\alpha_1+qE)(\rho_2-\rho_1)}, \quad \text{since} \quad v'(t) \geq -(\alpha_1+qE)v(t),$$

and by using the continuity of v, we can find an interval

$$[t^\star, t^\star + (\rho_2-\rho_1)] \subset [t_{n^\star}, t_{n^\star+1}] \quad \text{such that} \quad v'(t^\star + (\rho_2-\rho_1)) < 0$$
$$\text{and} \quad v(t) \leq M, \forall t \in [t^\star, t^\star + (\rho_2-\rho_1)].$$

Moreover, by definition of ρ_1, we can find ρ_2 such that $\rho_2 - \rho_1 < \rho_1$ even if we reduce $\gamma_\star$. Since we have

$$\lim_{n\to\infty} u(t,a) = \infty, \quad \forall a \in [0, \rho_1],$$

then there exists $\underline{t} > 0$ such that

$$\alpha\gamma_\star u(t,a) \geq (\alpha_1+qE)e^{(\rho_2-\rho_1)(\|\mu\|_{\mathbf{L}^\infty}+K\|\gamma\|_{\mathbf{L}^\infty})}, \forall t > \underline{t} \forall a \in [\rho_1 - (\rho_2-\rho_1), \rho_1].$$

We consider n big enough such that $t^\star \geq \underline{t}$. Consequently we get the

$$\alpha\gamma_\star u(t\star + (\rho_2-\rho_1), a)] \geq (\alpha_1+qE) \quad \forall a \in [\rho_1 - (\rho_2-\rho_1), \rho_1]$$

and then $\quad v'(t^\star + (\rho_2-\rho_1)) \geq 0$. So we have $\lim_{t\to\infty} v(t) = \infty$.

□

Proposition 3.8. *If* $\mathbf{R_0} > 1$ *and* $\mathbf{R}_\star < 1$, *then the total population of zooplankton and the total population of fish are uniformly weakly persistent.*

Proof. • Since we have

$$\mathbf{R}_0 > 1, \quad \text{then there exists} \quad a^\star > 0 \quad \text{such that} \quad P := \int_0^{a^\star} \beta(x)e^{-\int_0^x \mu(s)ds}dx > 1.$$

Let us consider

$$\overline{P} \in (1, P) \quad \text{and define} \quad \overline{K} = \frac{1}{a^\star\|\gamma\|_{\mathbf{L}^\infty}}(\ln(P) - \ln(\overline{P})) > 0$$

Consider ϵ small enough, then leads to $v(t) \leq \overline{K}$, for every $t \geq \bar{t}$ big enough. Moreover v is bounded by a positive constant K and, using the last proof, we obtain $\Phi(t) > 0, \forall t \geq t^\star$. Consequently, defining

$$\widehat{t} = max\{t^\star, \bar{t}\}, \text{we obtain} \quad \Phi(\widehat{t} + a^\star) \geq \Phi(\underline{t})\overline{P} \quad \text{where} \quad \Phi(\underline{t}) = \overline{P} \min_{w\in[\widehat{t}, \widehat{t}+a^\star]} \Phi(w).$$

Using the last proof with respectively $\widehat{t}, a^\star, \overline{P}$ instead of $t, \rho_1, \mathbf{R}_\star$: we get $\lim_{t\to\infty} \Phi(t) = \infty$. So we have

$$\lim_{t\to\infty} \|u(t,.)\|_{\mathbf{L}^1(\mathbb{R}_+)} = \infty, \quad \lim_{t\to\infty} v(t) = \infty$$

since v is bounded, which is a contradiction,[10, 14]. Thus we get the persistence result for the zooplankton.

- Taking $\epsilon > 0$ small enough we get

$$\|u(t,.)\|_{\mathbf{L}^1(\mathbb{R}_+)} < \frac{\alpha_1 + qE}{\alpha\|\gamma\|_{\mathbf{L}^\infty}} \quad \text{for all} \quad t \geq \bar{t} \quad \text{big enough, so that} \quad \lim_{t\to\infty} v(t) = 0$$

using the second equation of (2.2) and we then get a contradiction by using the first point.

□

4. Numerical simulations

In this section, we present a series of numerical simulations of the (2.2) model. The main objective is to understand the fishing effect which are the main factors influencing the stability of the system. To do this, we will vary some values of the fishing effort coefficients E and the capturability coefficient q, in accordance with the mathematical study, to look at the impact on population. The following numerical simulations, that are performed using the finite volume method, aim at investigating other behavior of the dynamical system (2.2) under some biologically reasonable parameters,[1, 2, 8, 12, 28, 30].

The parameters used are given in the following tables:

4.1. The numerical scheme. We define the intervals $a \in [0, a_{max}]$ and $t \in [0, T]$ then we note $\triangle a$ and $\triangle t$ respectively the age and the time steps. We define

$$a_{j+1/2} = j \triangle a, \ t^n = n \triangle t \ \text{ and } K_j = (a_{j-\frac{1}{2}}, a_{j+\frac{1}{2}})$$

for $j, n \in \mathbb{N}$.
We denote by u_j^n the approximation of the average of $u(t^n, a)$ over K_j and we compute the initial states

$$u_j^n \approx \frac{1}{\Delta a}\int_{K_j} u(t^n, a)da, \ u_j^0 = \frac{1}{\Delta a}\int_{K_j} u_0(a)da \text{ and } v^0 = v_0, \ \forall j \geq 1.$$

We then set $\alpha, \alpha_1, q, E > 0$ and once β, μ, γ are chosen, we compute the data:

$$\beta_j = \frac{1}{\Delta a}\int_{K_j} \beta(a)da, \ \gamma_j = \frac{1}{\Delta a}\int_{K_j} \gamma(a)da \text{ and } \mu_j = \frac{1}{\Delta a}\int_{K_j} \mu(a)da$$

We define

$$v_{n+1} = \frac{v_n(1 + \alpha \triangle t\tau(\gamma u^n))}{1 + (\alpha_1 + qE)\Delta t}, \quad \forall n \geq 0 \quad \text{where} \quad \tau(\gamma u^n) = \Delta a \sum_{j\geq 1} \gamma_j u_j^n.$$

Integrating the first equation of (2.2) regarding a over K_j and supposing x regular enough, we get

$$\partial_t \int_{K_j} u(t,a)da + u(t, a_{j+\frac{1}{2}}) - u(t, a_{j-\frac{1}{2}}) = -\int_{K_j} (\mu(a) + v(t)\gamma(a))u(t,a)da.$$

Then, an Euler's scheme,[8, 12, 28, 30] and the integrals estimates give us

$$\begin{cases} u_{j+1}^{n+1} = \dfrac{u_{j+1}^{n} + \frac{\triangle t}{\triangle a} u_j^{n+1}}{1 + \frac{\triangle t}{\triangle a} + \triangle t \mu_{j+1} + \triangle t v^{n+1} \gamma_{j+1}}, \quad \forall j \geq 0, \quad \forall n \geq 1, \\ u_0^{n+1} = \triangle a \sum_{j \geq 1} \beta_j u_j^n, \quad \forall n \geq 1. \end{cases}$$

4.2. Numerical results. Let us use the following functions:

$$\mu(a) = \mu_0 + \frac{\mu_0 a}{1 + as} \quad \text{with} \quad \mu_0 > 0 \quad \text{and} \quad s \in \mathbb{R}_+^*,$$

$$\mu(a) = \mu_0 + \frac{\mu_0 a}{1 + as} \quad \text{with} \quad \mu_0 > 0 \quad \text{and} \quad s \in \mathbb{R}_+^*,$$

$$\gamma(a) = \gamma_0 (1 - ape^{1-pa}) \quad \text{with} \quad \gamma_0 > 0, p > 0.$$

For our simulations we take the parameters values in the following Table 1.

Parameters	Value	Reference
a_{max}	20.00	[2, 8]
$\triangle a$	0.1	[8, 11]
α	0.70	[1, 8]
α_1	0.10	[28, 9]
μ_0	0.05	[8, 12]
γ_0	0.50	[2, 8]
s	1.00	[2, 12]
c	1.00	[2, 8]
p	0.25	[12, 28]
q	1.00	[6, 11]
P_0	150	[28, 30]
F_0	100	[28, 30]

Table 1. Parameters values for the numerical simulation of the system.

We represent the trajectory of the solution with on the x-axis the quantity of the fish population and on the y-axis the total quantity $\|u\|_{\mathbf{L}^1}$ of zooplankton population. Firstly we confirm the existence of positive component equilibria. we examine the behavior of the system by varying the mortality values due to fishing. We consider q and E. But as the mortality due to fishing depends on q and E, we will consider the catchability coefficient $q = 1$ and making vary the value of the fishing effort. Our aim of this experiment is to understand how the zooplankton-fish dynamics system behaves. Figures 2 shows the stability of the positive equilibrium. If $\mathbf{R_0} > 1$ and $\mathbf{R}_\star < 1$, then the total population of zooplankton and the total population of fish are uniformly weakly persistent. We take $\beta_0 = 7$ in the Figure 2. Then the simulations make us suppose that the solution is bounded for any positive initial conditions $(u_0, v_0) \in \mathbf{X_{H_2}}$. Moreover the hypothesis $(H_1) - (H_2)$ is verified. Consequently of Proposition 3.8, the total populations are uniformly weakly persistent and the solution will either converge to the equilibrium E_2. The existence of centers confirms the existence of the fish population despite the predation and the fishing in the area. We can still have an equilibrium. Thus, we talk about the phenomenon of subsistence.

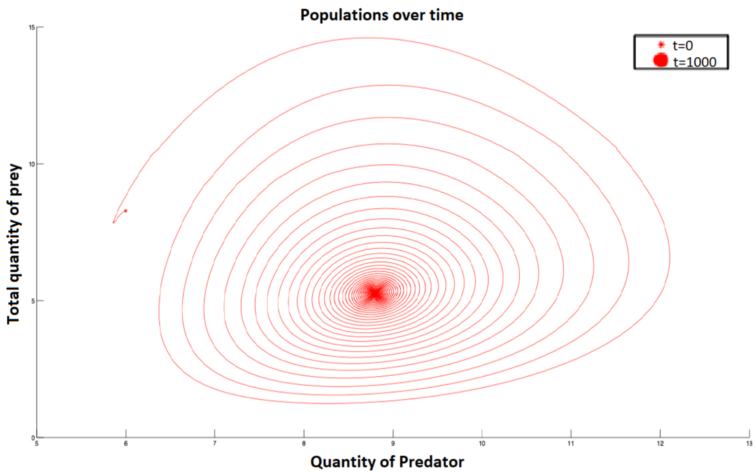

FIGURE 2. The phase portrait of the system (2.2) when $\beta_0 = 7$ and $E = 0.2$. We take her $\mathbf{R_0} = 2.82 > 1$ and $\mathbf{R}_\star = 0.56 < 1$ and bounded positive initial conditions $(u_0, v_0) \in \mathbf{X_{H_2}}$. The total populations are uniformly weakly persistent and the solution will either converge to E_2.

We continuous our numerical analysis by considering now, the population evolve in the area with $E = 0.2$ and we consider $\beta_0 = 4$ in the Figure 3. Then the simulations make us suppose that the solution is bounded for any positive initial conditions $(u_0, v_0) \in \mathbf{X_{H_2}}$. Moreover the hypothesis $(H_1) - (H_2)$ is verified. Consequently of Proposition 3.8, the total populations are uniformly weakly persistent and the solution will either converge to the periodic function.

In the Figure 4, we show the convergence to $E_0 = (0, 0)$ for any initial condition $(u_0, v_0) \in \mathbf{x}_+$. We take $\beta_0 = 1$ and $E = 0.2$, we have $\mathbf{R_0} = 0.32 < 1$, so by the Proposition 3.4 the solution will converge to $E_0 = (0, 0)$. Now the question that arises is: what can unbalanced the system? To answer this question, we continue our simulation by considering areas in which we allocate values related to the fishing effort.

All the last cases, we have studied numerically the behaviour using all parameters in the Table 1. In fact, we have use the fishing effort $E = 0.2$ in the all numerical results. Now, we are going to use different exploited area for to look at the behaviour of the system zooplankton-fish population according to the mathematical analysis. We consider $\beta_0 = 7$ in the Figure 5. Then the simulations make us suppose that the solution is bounded for any positive initial conditions $(u_0, v_0) \in \mathbf{X_{H_2}}$. Moreover the hypothesis $(H_1) - (H_2)$ is verified. Consequently of Proposition 3.8, the total populations are uniformly weakly persistent and the solution will either converge to the periodic function. The figure presented to illustrate the behaviour of the system zooplankton-fish population are: $E = 0.34(Fig5, a)$; $E = 0.43(Fig5, b)$; $E = 0.63(Fig5, c)$; $E = 0.84(Fig5, d)$; $E = 1.42(Fig5, e)$; $E = 1.72(Fig5, f)$.

Those results allow us to say that even if the area is exploited with such a value for the fishing effort, despite predation, we can see that the observed dynamics is

FIGURE 3. The phase portrait of the system (2.2) when $\beta_0 = 4$ and $E = 0.2$. We take her $\mathbf{R_0} = 2.82 > 1$ and $\mathbf{R}_\star = 0.56 < 1$ and bounded positive initial conditions $(u_0, v_0) \in \mathbf{X_{H_2}}$. The total populations are uniformly weakly persistent and the solution will either converge to the periodic function.

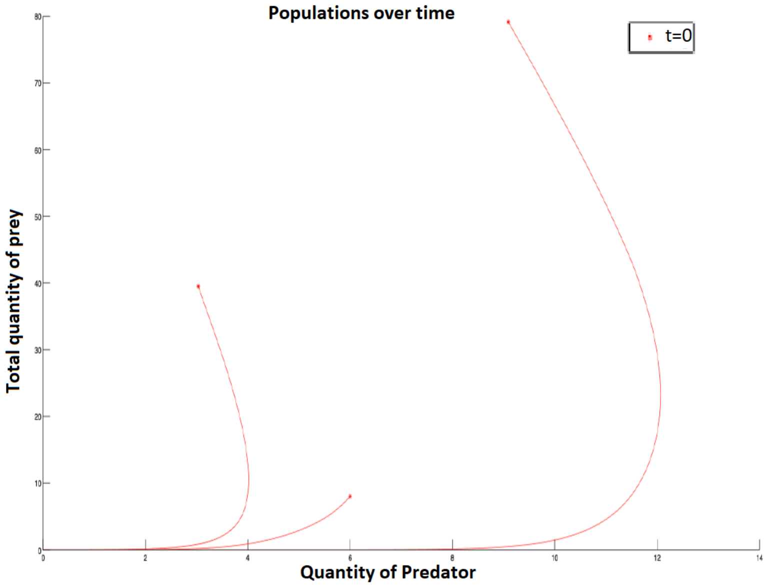

FIGURE 4. The phase portrait of the system (2.2) when $\beta_0 = 1$ and $E = 0.2$. We take $\mathbf{R_0} = 0.32 < 1$ and $\mathbf{R}_\star = 0.56 < 1$ and bounded positive initial conditions $(u_0, v_0) \in \mathbf{X}_+$. The solution will converge to $E_0 = (0, 0)$.

very close to the one encountered in the case of a fish population living in an pristine area[28, 30]. It allows us to say that if the zone is exploited with such a value of

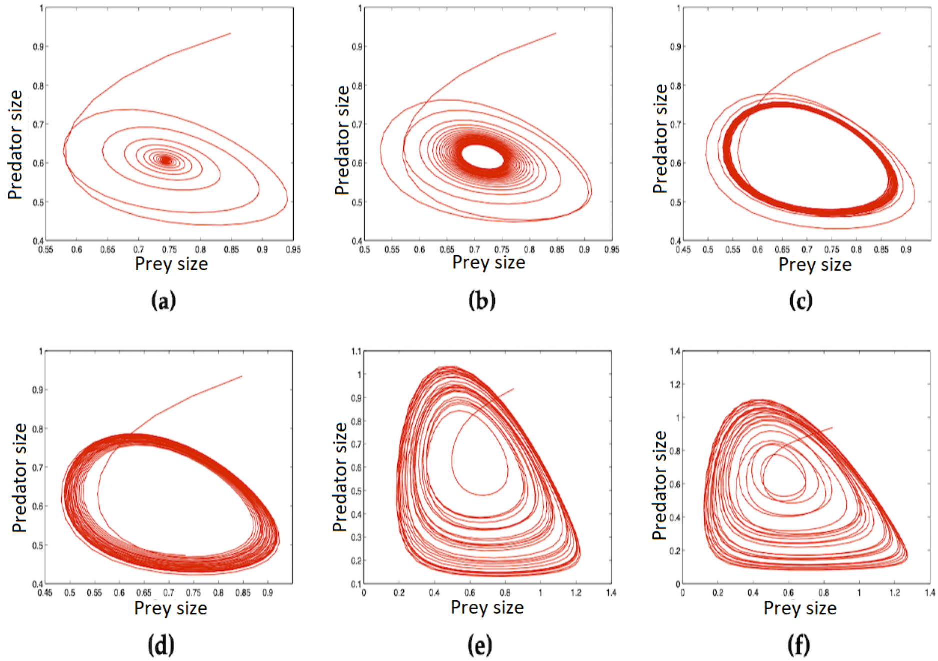

FIGURE 5. The phase portrait of the system (2.2) when $\beta_0 = 7$. We take $\mathbf{R_0} = 2.82 > 1$ and $\mathbf{R_\star} = 0.56 < 1$ and bounded positive initial conditions $(u_0, v_0) \in \mathbf{X_{H_2}}$. The total populations are uniformly weakly persistent and the solution will either converge to the periodic function. Different cases are presented: $E = 0.34(Fig5, a) - E = 0.43(Fig5, b) - E = 0.63(Fig5, c) - E = 0.84(Fig5, d) - E = 1.42(Fig5, e) - E = 1.72(Fig5, f)$.

fishing effort, the fish populations are not at risk. The persistence of the convergence towards a center of those dynamics, despite fishing shows that the area is normally exploited.

Remark 4.1. [28, 30] We have the following observation:

(a) Even if there is the predation in the fish population and if the area is exploited with a value of fishing effort less than or equal to $E \leq 0.5$, there is no risk for the fish population. We are talking about a normally exploited area.

(b) In such cases of the area is exploited with $E > 0.5$ (figure 4), an efficient management policy of fishing must be urgently adopted , otherwise there is a real risk for the fish population.

5. Conclusion

In this paper we dealt with the fish population dynamics under a diet on a zooplankton base. The mathematical model associated with this dynamics is based on PDE systems. The mathematical study allowed us to show that despite predation on fish population, positive component equilibria exist and its persist weakly. We can say that predation does not negatively influence the aquatic ecosystem. Additionally, the simulations allowed us to have an idea about the behaviour of the dynamics based on different values of the fishing effort. We could observe through those numerical results that when the area is exploited with some fishing effort values, an efficient management policy must be adopted otherwise it is likely to be catastrophic for the fish population.

In our future works, we will continue our study, focusing on the impact of the diffusion on the fish-zooplankton dynamics.

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