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Local Fractional Hilbert-type Inequalities in a Half-discrete Case

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Abstract. We provide a comprehensive analysis of fractal Hilbert-type inequalities in this study. More specifically, we prove a generalised half-discrete Hilbert-type inequality with weight functions and a general local fractional continuous kernel. We examine certain selections of power weight functions and homogeneous kernels as an application.

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1. Introduction

Let p and q be a pair of non-negative conjugate parameters i.e. $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. In essence, the famous Hilbert integral inequality (see [4]) asserts that

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y}dxdy \leq \frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p\|g\|_q, \tag{1}$$

where $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$ are non-negative functions. The constant $\frac{\pi}{\sin\frac{\pi}{p}}$, appearing on the right-hand side of (1), is the best possible in the sense that it can not be replaced by a smaller positive constant so that the inequality still holds.

Following its discovery, the Hilbert inequality was the subject of in-depth research by numerous authors. There were several generalisations available, including extensions to multidimensional instances, weight functions and integration domains, inequalities with wider kernels, and refinements of the original Hilbert inequality. Even though it is classical, many writers find the Hilbert inequality to be an interesting subject. A complete analysis of the early evolution of the Hilbert inequality is given in [4], and a set of more recent results is given in [6].

Here we present a brief overview of the key concepts in local fractional calculus to aid the reader's understanding. The concepts of the local fractional derivative and local fractional integral discussed in [10] (see also [11]) will be mostly covered in this section.

Let $\mathbb{R}^\alpha$, $0<\alpha\leq 1$, be an α-type fractal set of real line numbers. We define addition and multiplication operations on $\mathbb{R}^\alpha$ by $a^\alpha+b^\alpha := (a+b)^\alpha$ and $a^\alpha \cdot b^\alpha = a^\alpha b^\alpha := (ab)^\alpha$, $a^\alpha, b^\alpha \in \mathbb{R}^\alpha$. Obviously, with these two operations, $\mathbb{R}^\alpha$ is a field with an additive identity 0^α and a multiplicative identity 1^α.

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The basic step in developing the local fractional calculus on $\mathbb{R}^\alpha$ is the notion of the local fractional continuity. A non-differentiable function $f : \mathbb{R} \to \mathbb{R}^\alpha$ is said to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon^\alpha$. Throughout this paper $C_\alpha(I)$ stands for the set of local fractional continuous functions on interval I.

The local fractional derivative of f of order α at the point $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Gamma(1+\alpha)(f(x) - f(x_0))}{(x - x_0)^\alpha},$$

where Γ is the usual Gamma function. Alternatively, we write $f^{(\alpha)}(x) = D_x^\alpha f(x)$. Moreover, if $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \dots D_x^\alpha}^{k+1} f(x)$ is well-defined for every $x \in I$, then we say that f belongs to $D_{(k+1)\alpha}(I)$, $k = 0, 1, 2, \dots$.

The local fractional integral can be defined for a class of local fractional continuous functions. Let $f \in C_\alpha[a, b]$ and let $P = \{t_0, t_1, \dots, t_N\}$, $N \in \mathbb{N}$, be a partition of interval $[a, b]$ such that $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$. Furthermore, for this partition P, let $\Delta t_j = t_{j+1} - t_j$, $j = 0, \dots, N-1$, and $\Delta t = \max\{\Delta t_1, \Delta t_2, \dots, \Delta t_{N-1}\}$. In this setting, the local fractional integral of f on the interval $[a, b]$ of order α (denoted by ${}_aI_b^{(\alpha)} f(x)$) is defined by

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\alpha!}\int_a^b f(x)(dx)^\alpha = \frac{1}{\alpha!}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,$$

where $\alpha! := \Gamma(1+\alpha)$. If for any $x \in [a, b]$, there exists ${}_aI_x^{(\alpha)} f(x)$, then we denote by $f(x) \in I_x^{(\alpha)}[a, b]$.

Similarly to the Riemann integral, we have an analogue of the Newton-Leibnitz formula on the fractal space. Namely, if $f = g^{(\alpha)} \in C_\alpha[a, b]$, then ${}_aI_b^{(\alpha)} f(x) = g(b) - g(a)$. For example, if $f(x) = x^\gamma$, $\gamma > 0$, then

$${}_aI_b^{(\alpha)} x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}(b^{\gamma+\alpha} - a^{\gamma+\alpha}).$$

For more details about the above presented concepts of fractional differentiability and integrability, the reader is referred to [1], [5], [7] and [8] and references therein.

The starting point in establishing Hilbert-type inequalities is the well-known Hölder inequality. A fractal version of the Hölder inequality (see [11]) asserts that if $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then the inequality

$${}_aI_b^{(\alpha)} f(x)g(x) \le \left[{}_aI_b^{(\alpha)} f^p(x)\right]^{\frac{1}{p}} \left[{}_aI_b^{(\alpha)} g^q(x)\right]^{\frac{1}{q}} \tag{2}$$

holds for all $f, g \in C_\alpha(a, b)$.

Besides, we introduce the following notation and definition (see [3]).

Definition 1.1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^\alpha$. If the following inequality

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda^\alpha f(x_1) + (1-\lambda)^\alpha f(x_2) \tag{3}$$

holds for any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, then f is said to be a generalized convex function on I.

Mo et al. [9] proved the following generalized Hermite-Hadamard inequality for local fractional integral: let $f \in I_x^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a<b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\, {}_aI_b^{(\alpha)} f \le \frac{f(a)+f(b)}{2^\alpha}. \tag{4}$$

Applying above inequality we proved next two lemmas (see also [12]).

Lemma 1.1. *If $f \in I_x^{(\alpha)}(\mathbb{R}_+)$, $f^{(\alpha)}(t)<0$, $f^{(2\alpha)}(t)>0$ $(t \in (1/2,\infty))$, then we have*

$$\frac{1}{\Gamma(1+\alpha)}\int_1^\infty f(t)(dt)^\alpha \le \frac{1}{\Gamma(1+\alpha)}\sum_{n=1}^\infty f(n) \le \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^\infty f(t)(dt)^\alpha. \tag{5}$$

Lemma 1.2. *Let $r>0$, $m,n \in \mathbb{N}$, and $K(x,y)$ be strictly decreasing and generalized convex function in both variables on $\mathbb{R}_+$. Then*

$$K(m,y)y^{-\alpha r} \quad \textit{and} \quad K(x,n)x^{-\alpha r}$$

are strictly decreasing and generalized convex function on $\mathbb{R}_+$.

In this research, a new Hilbert-type inequality with a general homogeneous kernel and best constant is obtained by the use of weight functions and local fractional calculus technique.

2. Main results

The basic step in researching Hilbert-type inequalities is the well-known Hölder's inequality. A half-discrete fractal version of Hölder's inequality is proved in the following lemma.

Lemma 2.1. *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let h, F, $G \in C_\alpha(\mathbb{R}_+^2)$ be non-negative functions. If*

$$0 < \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F^p(x,n)(dx)^\alpha < \infty,$$

and

$$0 < \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)G^q(x,n)(dx)^\alpha < \infty,$$

then the following inequality holds

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F(x,n)G(x,n)(dx)^\alpha\\ &\le \left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F^p(x,n)(dx)^\alpha\right)^{\frac{1}{p}}\\ &\times \left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)G^q(x,n)(dx)^\alpha\right)^{\frac{1}{p}}.\end{aligned} \tag{6}$$

Proof. The inequality (6) is trivially true in the case when h or F or G is identically equal to zero. Suppose that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F^p(x,n)(dx)^\alpha \times \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)G^q(x,n)(dx)^\alpha \neq 0.$$

Applying the known $\alpha-$Young's inequality

$$x^{\frac{\alpha}{p}}y^{\frac{\alpha}{q}} \leq \frac{x^\alpha}{p^\alpha}+\frac{y^\alpha}{q^\alpha},\ x,y\geq 0,\ \text{and}\ \frac{1}{p}+\frac{1}{q}=1,\ p>1,$$

to

$$x^\alpha := \frac{h(x,n)F^p(x,n)}{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F^p(x,n)(dx)^\alpha}$$

and

$$y^\alpha := \frac{h(x,n)G^q(x,n)}{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)G^q(x,n)(dx)^\alpha}$$

obtaining

$$\frac{[h(x,n)]^{\frac{1}{p}}F(x,n)\,[h(x,n)]^{\frac{1}{q}}G(x,n)}{\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F^p(x,n)(dx)^\alpha\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)G^q(x,n)(dx)^\alpha\right)^{\frac{1}{p}}}$$
$$\leq \frac{1}{p^\alpha}\frac{h(x,n)F^p(x,n)}{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F^p(x,n)(dx)^\alpha}$$
$$+\frac{1}{q^\alpha}\frac{h(x,n)G^q(x,n)}{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)G^q(x,n)(dx)^\alpha}.$$

Intergrating and summarizing both side of the above inequality, we have

$$\frac{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F(x,n)G(x,n)(dx)^\alpha}{\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F^p(x,n)(dx)^\alpha\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)G^q(x,n)(dx)^\alpha\right)^{\frac{1}{p}}}$$
$$\leq \frac{1}{p^\alpha}\frac{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F^p(x,n)(dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)F^p(x,n)(dx)^\alpha}$$
$$+\frac{1}{q^\alpha}\frac{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)G^q(x,n)}{\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty h(x,n)G^q(x,n)(dx)^\alpha} = \frac{1}{p^\alpha}+\frac{1}{q^\alpha}=1^\alpha.$$

□

The previous lemma will serve us to prove the main result.

Theorem 2.2. *Let* $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, *and let* $(a_n)_{n\in\mathbb{N}}$ *be non-negative real sequence. If* φ, f, $\psi \in C_\alpha(\mathbb{R}_+)$ *and* $K\in C_\alpha(\mathbb{R}_+)^2$ *is non-negative decreasing function in both variables on* $\mathbb{R}_+$, *then the following inequalities hold and are equivalent*

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty K(x,n)\,f(x)\,a_n^\alpha(dx)^\alpha$$
$$\leq \left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty (\varphi\omega_1 f)^p(x)(dx)^\alpha\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty (\psi\omega_2)^q(n)\,a_n^{\alpha q}\right)^{\frac{1}{q}}, \tag{7}$$

and

$$\left(\sum_{n=1}^{\infty}(\psi\,\omega_2)^{-p}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(x,n)f(x)(dx)^{\alpha}\right)^{p}\right)^{\frac{1}{p}} \leq \left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}(\varphi\omega_1 f)^{p}(x)(dx)^{\alpha}\right)^{\frac{1}{p}}, \tag{8}$$

where

$$\omega_1^{p}(x) := \sum_{n=1}^{\infty}K(x,n)\,\psi^{-p}(n) \tag{9}$$

and

$$\omega_2^{q}(n) := \frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(x,n)\,\varphi^{-q}(x)(dx)^{\alpha}. \tag{10}$$

Proof. The left-hand side of inequality (7) can be presented differently in this manner:

$$\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=1}^{\infty}K(x,n)f(x)\,a_n^{\alpha}(dx)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=1}^{\infty}K(x,n)f(x)\frac{\varphi(x)}{\psi(n)}\,a_n^{\alpha}\,\frac{\psi(n)}{\varphi(x)}(dx)^{\alpha}.$$

Now, applying the half-discrete Hölder's inequality (6) to the above relation gives

$$\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=1}^{\infty}K(x,n)f(x)\,a_n^{\alpha}(dx)^{\alpha} \leq \left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=1}^{\infty}K(x,n)f^{p}(x)\frac{\varphi^{p}(x)}{\psi^{p}(n)}(dx)^{\alpha}\right)^{\frac{1}{p}} \times \left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=1}^{\infty}K(x,n)\,a_n^{\alpha q}\,\frac{\psi^{q}(n)}{\varphi^{q}(x)}(dx)^{\alpha}\right)^{\frac{1}{q}}.$$

At last, using the Fubini theorem and definitions of functions ω_1 and ω_2 we obtain (7).

Now, we are going to prove the equivalence of inequalities (7) and (8). For that reason, suppose that inequality (7) holds. Defining the sequence $(a_n)_{n\in\mathbb{N}}$ by

$$a_n^{\alpha} := (\psi\,\omega_2)^{-p}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(x,n)f(x)(dx)^{\alpha}\right)^{p-1}$$

and using (7), we have

$$\begin{aligned}&\sum_{n=1}^{\infty}(\psi\omega_2)^{-p}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(x,n)f(x)(dx)^{\alpha}\right)^p\\&=\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=1}^{\infty}K(x,n)f(x)\,a_n^{\alpha}(dx)^{\alpha}\\&\le\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}(\varphi\omega_1 f)^p(x)(dx)^{\alpha}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}(\psi\omega_2)^q(n)\,a_n^{\alpha q}\right)^{\frac{1}{q}}\\&=\left(\sum_{n=1}^{\infty}(\psi\omega_2)^{-p}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(x,n)f(x)(dx)^{\alpha}\right)^p\right)^{\frac{1}{q}},\end{aligned}$$

that is, we get (8).

Now, suppose that inequality (8)) holds. In that case, another use of the fractal discrete Hölder's inequality (see also [12]) yields

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=1}^{\infty}K(x,n)f(x)\,a_n^{\alpha}(dx)^{\alpha}\\&=\sum_{n=1}^{\infty}(\psi\omega_2)^{-1}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(x,n)f(x)(dx)^{\alpha}\right)(\psi\omega_2)(n)\,a_n^{\alpha}\\&\le\left(\sum_{n=1}^{\infty}(\psi\,\omega_2)^{-p}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(x,n)f(x)(dx)^{\alpha}\right)^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}(\psi\omega_2)^q(n)\,a_n^{\alpha q}\right)^{\frac{1}{q}}\\&\le\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}(\varphi\omega_1 f)^p(x)(dx)^{\alpha}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}(\psi\omega_2)^q(n)\,a_n^{\alpha q}\right)^{\frac{1}{q}},\end{aligned}$$

which implies (7). Hence, inequalities (7) and (8) are equivalent. □

In order to obtain an application of Theorem 2.2, we need the following lemma.

Lemma 2.3. *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, and let $K\in C_{\alpha}(\mathbb{R}_+^2)$ be a non-negative homogeneous function of degree $-\alpha s$, $s>0$. If K is decreasing and generalized convex function in both variables on $\mathbb{R}_+$, then*

$$\begin{aligned}\varpi_1^p(x):=&\sum_{n=1}^{\infty}K(x,n)n^{-\alpha pA_2}\\&\le\Gamma(1+\alpha)x^{\alpha-\alpha pA_2-\alpha s}k(pA_2),\quad x>0,\end{aligned}\tag{11}$$

and

$$\begin{aligned}\varpi_2^q(n):=&\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(x,n)x^{-\alpha qA_1}(dx)^{\alpha}\\&\le\Gamma(1+\alpha)n^{\alpha-\alpha s-\alpha qA_1}k(2-s-qA_1),\end{aligned}\tag{12}$$

where $A_1\in\left(\max\{\frac{1-s}{q},0\},\frac{1}{q}\right)$ and $A_2\in\left(\max\{\frac{1-s}{p},0\},\frac{1}{p}\right)$.

Proof. Using Lemmas 1.1 and 1.2, we obtain

$$\varpi_1^p(x) \le \Gamma(1+\alpha)\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(x,t)t^{-\alpha p A_2}(dt)^\alpha.$$

Furthermore, using homogeneity of function K and obvious substitution $u = \frac{t}{x}$ we have

$$\begin{aligned}\varpi_1^p(x) &\le \Gamma(1+\alpha)x^{\alpha-\alpha pA_2-\alpha s}\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(1,u)u^{-\alpha pA_2}(du)^\alpha \\ &= \Gamma(1+\alpha)x^{\alpha-\alpha pA_2-\alpha s}k(pA_2),\end{aligned}$$

which implies (11). Similarly, we obtain (12). □

The main results are stated below.

Theorem 2.4. *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. Let $f \in C_\alpha(\mathbb{R}_+)$ and let $(a_n)_{n\in\mathbb{N}}$ be non-negative real sequence. If $K(x,y)$, A_1, A_2 are defined as in Lemma 2.3,then the following inequalities hold and are equivalent*

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty K(x,n)f(x)\,a_n^\alpha(dx)^\alpha \qquad (13)\\ &\le L\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(1-s)+\alpha p(A_1-A_2)}f^p(x)(dx)^\alpha\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{\alpha(1-s)+\alpha q(A_2-A_1)}a_n^{\alpha q}\right)^{\frac{1}{q}},\end{aligned}$$

and

$$\begin{aligned}&\left(\sum_{n=1}^\infty n^{\alpha(s-1)(p-1)+\alpha p(A_1-A_2)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(x,n)f(x)(dx)^\alpha\right)^p\right)^{\frac{1}{p}} \qquad (14)\\ &\le L\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha(1-s)+\alpha p(A_1-A_2)}f^p(x)(dx)^\alpha\right)^{\frac{1}{p}},\end{aligned}$$

where $L = [\Gamma(1+\alpha)k(pA_2)]^{\frac{1}{p}}\,k(2-s-qA_1)^{\frac{1}{q}}$.

We now study some of the interesting choices of the parameters A_1 and A_2. More precisely, let the parameters A_1 and A_2 fulfil the requirement

$$pA_2 + qA_1 = 2 - s. \qquad (15)$$

Then, the constant L from Theorem 2.4 turns into

$$L^* = [\Gamma(1+\alpha)]^{\frac{1}{p}}\,k(pA_2). \qquad (16)$$

Further, the inequalities (13) and (14) take form

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty K(x,n)f(x)\,a_n^\alpha(dx)^\alpha \qquad (17)\\ &\le L^*\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{-\alpha+\alpha pqA_1}f^p(x)(dx)^\alpha\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-\alpha+\alpha pqA_2}a_n^{\alpha q}\right)^{\frac{1}{q}},\end{aligned}$$

and

$$\left(\sum_{n=1}^{\infty} n^{\alpha(p-1)(1-pqA_2)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty} K(x,n)f(x)(dx)^{\alpha}\right)^p\right)^{\frac{1}{p}} \le L^*\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty} x^{-\alpha+\alpha pqA_1} f^p(x)(dx)^{\alpha}\right)^{\frac{1}{p}}. \tag{18}$$

In the next theorem we prove that, if the parameters A_1 and A_2 satisfy condition (15), then one obtains the best possible constant.

Theorem 2.5. *Let s, A_1, A_2 and $K(x,y)$ be defined as in Theorem 2.4. If the parameters A_1 and A_2 satisfy condition (15), then the constant $L^* = [\Gamma(1+\alpha)]^{1/q} k(pA_2)$ in inequalities (17) and (18) is the best possible.*

Proof. For this reason, put $\widetilde{f}(x) = x^{-\alpha qA_1-\frac{\alpha\varepsilon}{p}}\chi_{[1,+\infty)}$ and $\widetilde{a}_n^{\alpha} = n^{-\alpha pA_2-\frac{\alpha\varepsilon}{q}}$ where $0<\varepsilon<\frac{1-pA_2}{q}$. Let us suppose that the inequality (17) is valid. First, we obtain

$$\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty} x^{-\alpha+\alpha pqA_1}\widetilde{f}^p(x)(dx)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_1^{\infty} x^{-\alpha-\alpha\varepsilon}(dx)^{\alpha} = \frac{1}{\varepsilon^{\alpha}\Gamma(1+\alpha)}. \tag{19}$$

By using Lemma 1.1 we have

$$\begin{aligned}\frac{1}{\Gamma(1+\alpha)\varepsilon^{\alpha}} &= \frac{1}{\Gamma(1+\alpha)}\int_1^{\infty} u^{-\alpha-\alpha\varepsilon}(du)^{\alpha} \le \frac{1}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} n^{-\alpha-\alpha\varepsilon}\\ &= \frac{1}{\Gamma(1+\alpha)}\sum_{n=1}^{\infty} n^{-\alpha+\alpha pqA_2}\widetilde{a}_n^{\alpha q}\\ &\le \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^{\infty} u^{-\alpha-\alpha\varepsilon}(du)^{\alpha} + \frac{1}{\Gamma(1+\alpha)}\int_1^{\infty} u^{-\alpha-\alpha\varepsilon}(du)^{\alpha}.\end{aligned}$$

Hence, we obtain

$$\sum_{n=1}^{\infty} n^{-\alpha+\alpha pqA_2}\widetilde{a}_n^{\alpha q} \le \frac{1}{\varepsilon^{\alpha}} + O(1). \tag{20}$$

Now, let suppose that there exits a positive constant M, $M < L^*$, such that the inequality (17)) is still valid if we replace L^* with M. Hence, if we insert relations (19) and (20) in inequality (17), with the constant M instead of L^*, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_1^{\infty}\sum_{n=1}^{\infty} K(x,n)\widetilde{f}(x)\,\widetilde{a}_n^{\alpha}\,(dx)^{\alpha} \le \frac{1}{\varepsilon^{\alpha}\Gamma^{\frac{1}{p}}(1+\alpha)}(M+o(1)). \tag{21}$$

On the other hand, we estimate the left-hand side of inequality (17). Namely, if we insert the above defined function $(\widetilde{f}(x)$ and sequence $(\widetilde{a}_n^{\alpha})_{n\in\mathbb{N}}$ in the left-hand side of

(17), we get the inequality

$$I_\varepsilon := \frac{1}{\Gamma^2(1+\alpha)} \int_0^\infty \sum_{n=1}^\infty K(x,n)\widetilde{f}(x)\,\widetilde{a}_n^\alpha\,(dx)^\alpha \tag{22}$$

$$\geq \frac{1}{\Gamma(1+\alpha)} \int_1^\infty x^{-\alpha q A_1 - \frac{\alpha\varepsilon}{p}} \left(\frac{1}{\Gamma(1+\alpha)} \int_1^\infty K(x,y) y^{-\alpha p A_2 - \frac{\alpha\varepsilon}{q}} (dy)^\alpha \right) (dx)^\alpha,$$

where we used Lemma 1.1. By using the substitution $u = \frac{y}{x}$ we obtain

$$I_\varepsilon \geq \frac{1}{\Gamma(1+\alpha)} \int_1^\infty x^{-\alpha-\alpha\varepsilon} \left(\frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{x}}^\infty K(1,u) u^{-\alpha p A_2 - \frac{\alpha\varepsilon}{q}} (du)^\alpha \right) (dx)^\alpha. \tag{23}$$

Since the kernel K is strictly decreasing in both variables, it follows that $K(1,0) \geq K(1,t)$, for $t > 0$, so we have

$$\frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{x}}^\infty K(1,u) u^{-\alpha p A_2 - \frac{\alpha\varepsilon}{q}} (du)^\alpha$$

$$\geq \frac{1}{\Gamma(1+\alpha)} \int_0^\infty K(1,u) u^{-\alpha p A_2 - \frac{\alpha\varepsilon}{q}} (du)^\alpha - \frac{K(1,0)}{\Gamma(1+\alpha)} \int_0^{\frac{1}{x}} K(1,u) u^{-\alpha p A_2 - \frac{\alpha\varepsilon}{q}} (du)^\alpha$$

$$= k\left(pA_2 + \frac{\varepsilon}{q}\right) - \frac{K(1,0)}{\Gamma(1+\alpha)(1-pA_2-\frac{\varepsilon}{q})^\alpha} x^{\alpha p A_2 + \frac{\alpha\varepsilon}{q} - \alpha}$$

and consequently

$$J_\varepsilon \geq \frac{1}{\varepsilon^\alpha} \frac{k\left(pA_2 + \frac{\varepsilon}{q}\right)}{\Gamma(1+\alpha)} + \frac{K(1,0)}{\Gamma^2(1+\alpha)} \frac{1}{(1-pA_2-\frac{\varepsilon}{q})^\alpha (pA_2 - \frac{\varepsilon}{p} - 1)^\alpha}. \tag{24}$$

Stated differently, the relations (22), (23) and (24) yield the estimate for the left-hand side of inequality (17):

$$\frac{1}{\Gamma(1+\alpha)} \int_0^\infty \sum_{n=1}^\infty K(x,n)\widetilde{f}(x)\,\widetilde{a}_n^\alpha\,(dx)^\alpha \leq \frac{1}{\varepsilon^\alpha \Gamma^{\frac{1}{p}}(1+\alpha)} (L^* + o(1)). \tag{25}$$

In the end, by comparing (21) and (25), and by letting $\varepsilon \longrightarrow 0^+$, we get that $L^* \leq M$, which contradicts with the assumption that the constant M is smaller than L^*.

The equivalence of inequalities (17) and (18) means that the constant L^* is the best possible in the inequality (18). The proof is now completed. □

The following results are corollaries of Theorem 2.4. We used the kernel $K_1(x,y) = (x+y)^{-\alpha s}$, $s > 0$, to process. By using local fractional calculus, we obtain

$$\frac{\partial^\alpha}{\partial x^\alpha} \frac{1}{(x+y)^{\alpha s}} = -\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-1)\alpha)} \frac{1}{(x+y)^{\alpha(s+1)}} \leq 0, \quad x,y > 0,$$

and similarly

$$\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \frac{1}{(x+y)^{\alpha s}} = \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s-1)\alpha)} \frac{1}{(x+y)^{\alpha(s+2)}} \geq 0, \quad x,y > 0.$$

Applying Lemma 1.2 we obtain

$$\frac{\partial^\alpha}{\partial x^\alpha} K_1(x,y) x^{-\alpha r} \leq 0 \quad \text{and} \quad \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} K_1(x,y) x^{-\alpha r} \geq 0 \tag{26}$$

for $r > 0$.

As we proceed, we assume that

$$A_1 = \frac{2-s}{2q}, \quad A_2 = \frac{2-s}{2p}. \tag{27}$$

Then, the constant L^* from Theorem 2.5 becomes

$$\begin{aligned} L^* &= \Gamma^{\frac{1}{p}}(1+\alpha)k(pA_2) \\ &= \Gamma^{\frac{1}{p}}(1+\alpha)k\left(1-\frac{s}{2}\right) = \Gamma^{\frac{1}{p}}(1+\alpha)\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{u^{-\alpha-\frac{\alpha s}{2}}}{(1+u)^{\alpha s}}(du)^\alpha \\ &= \Gamma^{\frac{1}{p}}(1+\alpha)B_\alpha\left(\frac{s}{2},\frac{s}{2}\right). \end{aligned}$$

Now, from Theorem 2.5 we get the following result.

Corollary 2.6. *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $0<s<2$, and $f \in C_\alpha(\mathbb{R}_+)$ be non-negative function and $(a_n)_{n\in\mathbb{N}}$ be non-negative real sequence. Then the following inequalities hold and are equivalent*

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=1}^\infty \frac{f(x)\,a_n^\alpha}{(x+y)^{\alpha s}}(dx)^\alpha \\ &\le L_1\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha p(1-\frac{s}{2})-\alpha} f^p(x)(dx)^\alpha\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{\alpha q(1-\frac{s}{2})-\alpha}a_n^{\alpha q}\right)^{\frac{1}{q}}, \end{aligned} \tag{28}$$

and

$$\begin{aligned} &\left(\sum_{n=1}^\infty n^{\frac{\alpha p s}{2}-\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \frac{f(x)}{(x+y)^{\alpha s}}(dx)^\alpha\right)^p\right)^{\frac{1}{p}} \\ &\le L_1\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty x^{\alpha p(1-\frac{s}{2})-\alpha} f^p(x)(dx)^\alpha\right)^{\frac{1}{p}}, \end{aligned} \tag{29}$$

where the constant $L_1 = \Gamma^{1/p}(1+\alpha)B_\alpha\left(\frac{s}{2},\frac{s}{2}\right)$ is the best possible.

Remark 2.1. The constant appearring in our next example is expressed in terms of a local fractional hypergeometric function defined by

$${}_2F_1^\alpha(a,b;c;z) = \frac{1}{B_\alpha(b,c-b)}\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{(b-1)\alpha}(1-t)^{(c-b-1)\alpha}(1-zt)^{-a\alpha}(dt)^\alpha,$$

where $c > b > 0$, $|z| \le 1$.

To find the corresponding constant, let $K_2(x,y)$ be defined by $K_2(x,y) = (x+y+\max\{x,y\})^{-\alpha s}$, $0<s<2$, and let A_1, A_2 be defined by (27). Lemma 1.2 is utilised to establish that the functions $K_2(x,y)x^{-\alpha r}$ and $K_2(x,y)y^{-\alpha r}$, $r>0$, are decreasing and generalized convex functions for any fixed $x \in \mathbb{R}_+$ or $y \in \mathbb{R}_+$, respectively.

According to Theorem 2.5, the related inequalities are given with the best possible constant

$$L_2 = 2^{1-\alpha s}B_\alpha\left(\frac{s}{2},1\right)\ {}_2F_1^\alpha\left(s,\frac{s}{2};\frac{s}{2}+1;-\frac{1}{2}\right)$$

(see also [12]).

Remark 2.2. With regards to the best constants, another intriguing aspect appears when considering certain operator expressions closely connected to Hardy-Hilbert-type inequalities (18). For the sake of making things simple, we deal here with inequality (18) for $A_1 = A_2 = \frac{1}{pq}$ and $s = 1$. Given this setting, inequality (18) reduce to

$$\|\mathcal{L}_1 f\|_{l^{\alpha p}} \leq k\Big(\frac{1}{q}\Big)\|f\|_{L^{\alpha p}(\mathbb{R}_+)} \tag{30}$$

where $\mathcal{L}_1 : L^p(\mathbb{R}_+) \to l^{\alpha p}$ is linear operator

$$(\mathcal{L}_1 f)_n = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty K(x,n) f(x) (dx)^\alpha, \; n \in \mathbb{N}.$$

As a consequence of inequality (30), the operator $\mathcal{L}_1$ is well-defined and bounded, as well. In addition, since $k\big(\frac{1}{q}\big)$ is the best constant in (30), we are able to determine norms of $\mathcal{L}_1$. Namely, exploiting this fact, it follows that

$$\|\mathcal{L}_1\| = \sup_{f \neq 0} \frac{\|\mathcal{L}_1 f\|_{l^{\alpha p}}}{\|f\|_{L^{\alpha p}(\mathbb{R}_+)}} = k\Big(\frac{1}{q}\Big).$$

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