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Growth of Meromorphic Solutions to Homogeneous Complex Linear Delay Differential Equations

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Abstract. This paper is devoted to study the growth and oscillation of meromorphic solutions of homogeneous complex linear delay differential equations of the form

$$L(z,f)=\sum_{s=0}^{n}\sum_{j=0}^{m}A_{sj}(z)f^{(j)}(z+c_s)=0,$$

where $c_s, s=0,...,n$ are distinct complex numbers and $A_{sj}(z), s=0,...,n, j=0,...,m$, $n,m\in\mathbb{N}$ are entire or meromorphic functions with the same order. We extend some results based on those of Lan-Chen and Wu-Zheng.

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1. Introduction

Throughout this paper, we assume that the readers are familiar with the standard notations and the fundamental results of Nevanlinna value distribution theory of meromorphic functions see ([9], [11], [12], [23]). We say that a meromorphic function $a(z)$ is a small function of $f(z)$ if $T(r,a)=o(T(r,f))$ as $r\to+\infty$ outside of a possible exceptional set of finite logarithmic measure. Let $\mathbb{N}=\{1,2,\cdots\}$ denote the set of positive natural numbers. We recall the definitions of standard notations which are use in this paper.

Definition 1.1. ([11], [23]) The order $\rho(f)$ of a meromorphic function f is defined by

$$\rho(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r},$$

and the hyper-order of f is defined by

$$\rho_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.

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Definition 1.2. ([11], [23]) The exponent of convergence of the sequence of zeros of a meromorphic function f is defined by

$$\lambda(f)=\limsup_{r\to+\infty}\frac{\log N(r,\frac{1}{f})}{\log r},$$

where $N(r,\frac{1}{f})$ is the integrated counting function of zeros of f in $\{z:|z|\leq r\}$.

Research into the properties of meromorphic solutions of complex difference equations have become a subject of great interest from the point of view of Nevanlinna's theory and its difference analogues. Several authors have examined the growth properties of meromorphic solutions of complex linear difference equations

$$A_n(z)f(z+c_n)+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=0, \tag{1}$$

where $n\in\mathbb{N}, c_j, j=1,...,n$, are distinct non-zero complex numbers and $A_j(z), j=0,...,n$ are entire or meromorphic functions. Chiang-Feng [7] studied the growth of meromorphic solutions of homogeneous linear difference equations in the case where there exists only one coefficient with the maximal order. Then, Laine-Yang [13] showed that if the leading coefficient depends on the type but not on the order, then they obtained the following result.

Theorem 1.1. ([13]) *Let $A_0(z),...,A_n(z)$ be entire functions of finite order such that among those having the maximal order*

$$\rho=\max_{0\leq j\leq n}\{\rho(A_j)\},$$

exactly one has its type strictly greater than the others. Then for any meromorphic solution f of (1), we have

$$\rho(f)\geq\rho+1.$$

Recently in [14], Lan-Chen have investigated the growth properties of meromorphic solutions of equation (1), imposing a few restrictions on the coefficients of the difference equations when there is no dominant coefficient and achieved the following result.

Theorem 1.2. ([14]) *Let c_j, $j=1,...,n$, be different complex constants and let*

$$A_j(z)=P_j(z)\exp\{h_j(z)\}+Q_j(z),\ j=1,...,n,$$

where $h_j(z)$ are polynomials of degree $k\geq1$, $P_j(z)\,(\not\equiv0)$ and $Q_j(z)$ are entire functions whose order is lower than k. Among the leading coefficients of $h_j(z)$, $j=1,...,n$, with the maximal modulus, there exists a term unequal to the other terms. If $f(z)(\not\equiv0)$ is a meromorphic solution of equation

$$A_n(z)f(z+c_n)+\cdots+A_1(z)f(z+c_1)=0,$$

then

$$\rho(f)\geq k+1.$$

In this case, Wu-Zheng [22] have investigated a more particular problem on the coefficients $A_j(z), j=0,...,n$ than the coefficients of Theorem 1.2, and they got the following result, which extend and enhance the previous result for equation (1).

Theorem 1.3. ([22]) *Let* $n, k \in \mathbb{N}$ *and*

$$A_j(z) = B_j(z)\exp\{P_j(z)\} + D_j(z)\exp\{Q_j(z)\} + R_j(z), j = 0, 1, ..., n,$$

where $P_j(z) = a_{jk}z^k + \cdots + a_{j0}, Q_j(z) = b_{jk}z^k + \cdots + b_{j0}, j = 0, 1, ..., n$ *are polynomials with degree* k *and satisfy* $|a_{jk}| \geq |b_{jk}| > 0, B_j(z), D_j(z), R_j(z), j = 0, ..., n,$ *are meromorphic functions and satisfy* $\max_{0\leq j\leq n}\{\rho(B_j), \rho(D_j), \rho(R_j)\} = \sigma < k, A_j(z) - R_j(z) \not\equiv 0, j = 0, 1, ..., n$. *Let* $c_j, j = 1, ..., n$ *be distinct non-zero complex constants. If there exists an* $i \in \{0, 1, ..., n\}$ *such that for all* $j \neq i$, $|a_{ik}| \geq |a_{jk}|$*and*

$$\arg(a_{ik}) \neq \arg(a_{jk}), \ \ \textit{or} \ \arg(a_{ik}) = \arg(a_{jk}), |a_{ik}| > |a_{jk}|$$

and

$$\arg(a_{ik}) \neq \arg(b_{jk}), \ \ \textit{or} \ \arg(a_{ik}) = \arg(b_{jk}), |a_{ik}| > |b_{jk}|$$

hold simultaneously, then every meromorphic solution $f(z)(\not\equiv 0)$ *of equation* (1) *satisfies* $\rho(f) \geq k+1$. *Further, if* $\varphi(z)(\not\equiv 0)$ *is a meromorphic function with* $\rho(\varphi) < k+1$, *then for every meromorphic solution* $f(z)(\not\equiv 0)$ *of equation* (1) *with* $\rho_2(f) < 1$, *we have* $\lambda(f - \varphi) = \rho(f) \geq k + 1$.

Historically, Naftalevič in [18], was used operator theory and the iteration method to investigate meromorphic solutions to complex delay-differential equations. Further, there have been few studies on complex delay-differential equation fields using Nevanlinna. Lately, in their book [16], Liu, Laine and Yang presented developments and new results on complex delay-differential equations, an area with important and interesting applications that is also gaining attention (see [15], [17], [19], [20], [21]).

Recently, Wu-Zheng [21], Zhou-Zheng [24] and Chen-Zheng [6] have investigated the growth of solutions of homogeneous and non-homogeneous linear delay differential equations

$$L(z, f) = \sum_{s=0}^{n}\sum_{j=0}^{m} A_{sj}(z)f^{(j)}(z + c_s) = 0 \tag{2}$$

and

$$L(z, f) = \sum_{s=0}^{n}\sum_{j=0}^{m} A_{sj}(z)f^{(j)}(z + c_s) = F(z), \tag{3}$$

where $A_{sj}(z)(s = 0, ..., n, j = 0, ..., m), F(z)$ are entire or meromorphic functions in the case when only one coefficient is a dominant coefficient, and they have estimated the lower bound of the order of the meromorphic solutions. Later, Belaïdi [1], Bellaama-Belaïdi [2], [3] and Dahmani-Belaïdi [8] focused then the study on the order of growth of equations (2) and (3) if there exists a coefficient of maximal logarithmic order (lower order) or logarithmic type (lower type) than other coefficients and obtained some valuable results.

2. Statement of Main Results

In this paper, by combining the complex difference and the complex differential equations, we expand the results according to the assumptions on the coefficients in Theorem 1.2 and Theorem 1.3, for the equation (2), and we get two results for complex linear delay differential equations.

Theorem 2.1. *Let $n, m \in \mathbb{N}$ and $c_s, s = 0, ..., n$ be distinct complex constants and let*

$$A_{sj}(z) = P_{sj}(z)\exp\{h_{sj}(z)\} + Q_{sj}(z), s = 0, ..., n, j = 0, ..., m, \tag{4}$$

where $h_{sj}(z) = a_{sjk}z^k + a_{sjk-1}z^{k-1} + \cdots + a_{sj0}, (a_{sjk} \neq 0)$ are polynomials of degree $k \geq 1$, $P_{sj}(z)(\not\equiv 0)$ and $Q_{sj}(z)$ are entire functions whose order is lower than k. Suppose that $|a_{l0k}| > \max\{|a_{sjk}|, 0 \leq s \leq n, 0 \leq j \leq m, (s, j) \neq (l, 0)\}$. If $f(z)(\not\equiv 0)$ is a meromorphic solution of equation (2), then

$$\rho(f) \geq k + 1.$$

Example 2.1. The entire function $f(z) = \exp\{z^2\}$ is a solution of the delay differential equation

$$A_{12}(z)f''(z+i) + A_{02}(z)f''(z) + A_{11}(z)f'(z+i) + A_{01}(z)f'(z)$$
$$+A_{10}(z)f(z+i) + A_{00}(z)f(z) = 0,$$

where

$$A_{00}(z) = -2(2z^2 + 4iz - 1)\exp\{(3+4i)z - 1\},$$
$$A_{10}(z) = 2(z+i)\exp\{4iz\} - (2z^3 + 2iz^2 - 2z - 2i),$$
$$A_{01}(z) = (2z^2+1)\exp\{-iz+1\} - (2\pi z^2 + \pi), \quad A_{11}(z) = -(\exp\{4iz\} - (z^2 - 1)),$$
$$A_{02}(z) = -(z\exp\{-iz+1\} - \pi z), \quad A_{12}(z) = \exp\{(3+2i)z\}.$$

In this equation, we have $\rho(P_{sj}(z)) = \rho(Q_{sj}(z)) = 0, (s = 0, 1, j = 0, 1, 2)$ and

$$|a_{001}| = |3+4i| = 5 > \max\{|a_{sjk}|, (s,j) \neq (0,0)\}$$
$$= \max\{|4i|, |-i|, |3+2i|\} = 4.$$

Thus, the conditions of Theorem 2.1 are verified. We see that for $s = 0, 1, j = 0, 1, 2,$ we have

$$\rho(f) = 2 = \rho(A_{sj}) + 1 = \deg(h_{sj}) + 1 = 1 + 1 = 2.$$

Theorem 2.2. *Let $k, n, m \in \mathbb{N}$ and*

$$A_{sj}(z) = P_{sj}(z)\exp\{h_{sj}(z)\} + Q_{sj}(z), \ s = 0, ..., n, j = 0, ..., m,$$

where $h_{sj}(z) = a_{sjk}z^k + a_{sjk-1}z^{k-1} + \cdots + a_{sj0}, (a_{sjk} \neq 0)$, are polynomials with degree k, and $P_{sj}(z)(\not\equiv 0), Q_{sj}(z)$ are meromorphic functions satisfying

$$\max\{\rho(P_{sj}), \rho(Q_{sj}), 0 \leq s \leq n, 0 \leq j \leq m\} = \sigma < k,$$

let $c_s, s = 0, ..., n$ be distinct non-zero complex constants. Suppose that for $(s, j) \neq (l, 0)$, we have

$$\arg(a_{l0k}) \neq \arg(a_{sjk}), \ |a_{l0k}| \geq |a_{sjk}|,$$

or

$$\arg(a_{l0k}) = \arg(a_{sjk}), \ |a_{l0k}| > |a_{sjk}|,$$

then every meromorphic solution $f(z)(\not\equiv 0)$ of equation (2) satisfies

$$\rho(f) \geq k + 1.$$

Furthermore, if $\varphi(z)(\not\equiv 0)$ is a meromorphic function with $\rho(\varphi) < k+1$, then for every meromorphic solution $f(z)$ of equation (2), we have

$$\lambda(f - \varphi) = \rho(f) \geq k + 1.$$

Example 2.2. The entire function $f(z) = \exp\{z^2\}$ is a solution of the delay differential equation

$$A_{12}(z)f''(z-2i) + A_{02}(z)f''(z+1+2i) + A_{11}(z)f'(z-2i) + A_{01}(z)f'(z+1+2i) + A_{10}(z)f(z-2i) + A_{00}(z)f(z+1+2i) = 0,$$

where

$$A_{00}(z) = 2\frac{z^2-1}{z^2+2i\pi z-1}\exp\{-3iz+3-4i\} + \frac{2(z^2-1)}{iz^2-4\pi},$$

$$A_{10}(z) = -\left(\frac{2z^3}{z^2-i\pi}\exp\{(4+4i)z+1\} + \frac{2iz^3}{z-2i}\right),$$

$$A_{01}(z) = -\left(\frac{z+i\pi}{z^2+(1+2i)z}\exp\{(-2-4i)z-4i\} + \frac{2(z^2-1)}{(iz^2-4\pi)(z+1+2i)}\right),$$

$$A_{11}(z) = -\left(\frac{z^2-1}{(z-2i)(z^2+2i\pi z-1)}\exp\{(2+5i)z+4\} + \frac{2i}{z^2-4iz-4}\right),$$

$$A_{02}(z) = \frac{z^3}{(z^2-i\pi)(2z^2+(4+8i)z+8i-5)}\exp\{(2-4i)z-4i\} + \frac{z^2-1}{(iz^2-4\pi)(2z^2+(4+8i)z+8i-5)},$$

$$A_{12}(z) = \frac{z+i\pi}{2z^3-8iz^2-7z}\exp\{4iz+1\} + \frac{iz^3+2i}{(z-2i)(2z^2-8iz-7)}.$$

We have

$$|a_{101}| = |4+4i| = 4\sqrt{2} > \max\{|a_{sjk}|, (s,j) \neq (1,0)\} = \max\{|-3i|, |-2-4i|, |2+5i|, |2-4i|, |4i|\} = |2+5i| = \sqrt{29},$$

and

$$\arg(a_{101}) \neq \arg(a_{sjk}), (s,j) \neq (1,0).$$

Moreover $\rho(P_{sj}(z)) = \rho(Q_{sj}(z)) = 0, s = 0,1, j = 0,1,2$. Hence, the conditions of Theorem 2.2 are satisfied, we see that

$$\rho(f) = 2 = \rho(A_{sj}) + 1 = 1+1, \quad 0 \le s \le 1, 0 \le j \le 2.$$

Set $\varphi(z) = \exp\{z\} + 1$, with $\rho(\varphi) = 1 < 2$. Then

$$\lambda(f-\varphi) = \rho(f) = 2.$$

Example 2.3. The function $f(z) = \exp\{\exp z\}$ is a solution of the delay differential equation

$$A_{11}(z)f'(z+2i) + A_{01}(z)f'(z-i) + A_{10}(z)f(z+2i) + A_{00}(z)f(z-i) = 0,$$

where

$$A_{00}(z) = \frac{z}{iz^2-12z+6i}\exp\{(4+4i)z\}, \quad A_{10}(z) = -\frac{2z^3+2iz}{4z^4+i\pi}\exp\{3z+2i\},$$

$$A_{01}(z) = -\frac{z}{iz^2-12z+6i}\exp\{(3+4i)z+i\}, \quad A_{11}(z) = \frac{2z^3+2iz}{4z^4+i\pi}\exp\{2z\}.$$

We have $\rho(P_{sj}(z)) = 0, s = 0,1, j = 0,1$ and

$$|a_{001}| = |4+4i| = 4\sqrt{2} > \max\{|a_{sjk}|, (s,j) \neq (0,0)\} = \max\{|3+2i|, |3+4i|, |2|\} = 5,$$

and

$$\arg(a_{001}) \neq \arg(a_{sjk}), (s,j) \neq (0,0).$$

Hence, the conditions of Theorem 2.2 are satisfied, we see that

$$\rho(f) = +\infty > \rho(A_{sj}) + 1 = 1 + 1 = 2,\ 0 \leq s \leq 1, 0 \leq j \leq 1.$$

Set $\varphi(z) = \cos(2z)$, with $\rho(\varphi) = 1 < 2$. Then

$$\lambda(f - \varphi) = \rho(f) = +\infty.$$

3. Some Auxiliary Lemmas

Lemma 3.1. ([5]) *Let f be a meromorphic function with $\rho(f) = \rho < \infty$. Then for any given $\varepsilon > 0$, there is a set $E \subset (1, +\infty)$ of finite logarithmic measure, such that*

$$|f(z)| \leq \exp\{r^{\rho+\varepsilon}\}$$

holds for all z satisfying $|z| = r \notin [0,1] \cup E, r \to +\infty$.

Lemma 3.2. ([7]) *Let η_1 and η_2 be two arbitrary complex numbers and let f be a meromorphic function of finite order ρ. For given $\varepsilon > 0$, there exists a subset $E \subset (0, +\infty)$ of finite logarithmic measure such that, for all z satisfying the relation $|z| = r \notin [0,1] \cup E$, the following double inequality is holds*

$$\exp\{-r^{\rho-1+\varepsilon}\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\{r^{\rho-1+\varepsilon}\}.$$

Lemma 3.3. ([10]) *Let f be a meromorphic function of finite order ρ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E$, and for all k, j, $0 \leq j < k$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq r^{(k-j)(\rho-1+\varepsilon)}.$$

Lemma 3.4. ([4]) *Suppose that $P(z) = (\alpha + i\beta)z^k + \cdots (\alpha, \beta$ are real numbers such that $|\alpha| + |\beta| \neq 0)$ is a polynomial with degree $k(\geq 1)$, and $\omega(z)(\not\equiv 0)$ is a meromorphic function with $\rho(\omega) < k$. Set $g(z) = \omega(z)\exp\{P(z)\}, z = r\exp\{i\theta\}, \delta(P,\theta) = \alpha\cos(n\theta) - \beta\sin(n\theta)$. Then for any given $\varepsilon > 0$, there exists a set $H_0 \subset [0, 2\pi)$ with linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (H_0 \cup H_1)$,there exists $r_0 = r_0(\theta, \varepsilon)(> 0)$ such that for $|z| = r > r_0$, we have*
(i) If $\delta(P,\theta) > 0$, then

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^k\} < |g(r\exp\{i\theta\})| < \exp\{(1+\varepsilon)\delta(P,\theta)r^k\}.$$

(ii) If $\delta(P,\theta) < 0$, then

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^k\} < |g(r\exp\{i\theta\})| < \exp\{(1-\varepsilon)\delta(P,\theta)r^k\},$$

where $H_1 = \{\theta \in [0, 2\pi) : \delta(P,\theta) = 0\}$ is a finite set.

Remark 3.1. ([4]) Let $P(z) = a_k z^k + \cdots + a_0$ ($a_k \neq 0$) be a polynomial with degree $k(\geq 1)$ and $z = r\exp\{i\theta\}$, we denote

$$\delta(P,\theta) = |a_k|\cos(\arg a_k + k\theta).$$

Lemma 3.5. *Let $n, m \in \mathbb{N}$ and $F(z), A_{sj}(z), s = 0, ..., n, j = 0, ..., m$, be meromorphic functions of finite order such that $A_{nm}(z)F(z) \not\equiv 0$ and let $c_s, s = 0, ..., n$ be distinct non-zero complex constants. Suppose that $f(z)(\not\equiv 0)$ is a meromorphic solution of equation (3). If*

$$\rho(f) > \max\{\rho(F), \rho(A_{sj}), s = 0, ..., n, j = 0, ..., m\}, \tag{5}$$

then we have $\lambda(f) = \rho(f)$.

Proof. Suppose that $\lambda(f) < \rho(f)$. From Hadamard's theorem, we can write f in the form

$$f(z) = \frac{H_1(z)}{H_2(z)}\exp\{h(z)\}, \tag{6}$$

where $H_1, H_2(\not\equiv 0)$ are the canonical product formed by zeros (poles) of f such that

$$\lambda(H_1) = \rho(H_1) = \lambda(f) < \rho(f), \tag{7}$$

$$\lambda(H_2) = \rho(H_2) = \lambda(\frac{1}{f}) < \rho(f). \tag{8}$$

In addition, we have

$$h(z) = a_p z^p + \cdots + a_1 z + a_0,$$

is a polynomial with $\deg h(z) = p \geq 1$, where $a_p, a_{p-1}, ..., a_0$ are complex constants such that $a_p \neq 0$. Furthermore, from (6) we can rewrite $f^{(j)}(z), j = 1, ..., m$ in the following form

$$f^{(j)}(z) = \left(\frac{H_1(z)}{H_2(z)}\exp\{h(z)\}\right)^{(j)} = \psi_j(z)\exp\{h(z)\}, \tag{9}$$

where $\psi_j(z)(j = 1, ..., m)$ are meromorphic functions formed by $H_1, H_2, h(z)$ and their derivatives. Substituting (9) into (3), we obtain

$$\sum_{s=0}^{n}\sum_{j=0}^{m} A_{sj}(z)\psi_j(z + c_s)\exp\{h(z + c_s)\} = F(z). \tag{10}$$

By dividing both sides of (10) by $\exp\{a_p z^p\}$, we get

$$\sum_{s=0}^{n}\sum_{j=0}^{m} A_{sj}(z)\psi_j(z + c_s)\exp\{(pa_p c_s + a_{p-1})z^{p-1} + \cdots\} = F(z)\exp\{-a_p z^p\}. \tag{11}$$

From (5), (7) and (8), from the left side of (11), we obtain

$$\begin{aligned}
&\rho\left(\sum_{s=0}^{n}\sum_{j=0}^{m} A_{sj}(z)\psi_j(z + c_s)\exp\{(pa_p c_s + a_{p-1})z^{p-1} + \cdots\}\right)\\
&\leq \max_{0\leq s\leq n, 0\leq j\leq m}\{\rho(A_{sj}), \rho(H_1), \rho(H_2), \rho(\exp{(pa_p c_s + a_{p-1})z^{p-1}} + \cdots)\}\\
&= \deg((pa_p c_s + a_{p-1})z^{p-1} + \cdots) = p - 1,\ \ 0 \leq s \leq n,
\end{aligned}$$

and on the other hand, from the right side of (11), we have

$$\rho\left(F(z)\exp\{-a_p z^p\}\right) = p.$$

This is a contradiction, hence

$$\lambda(f) = \rho(f).$$

□

Example 3.1. We consider the entire function $f(z) = \exp\{z^2\} - 1$ which is a solution to the following delay-differential equation

$$A_{11}(z)f'(z+i) + A_{01}(z)f'(z+1) + A_{10}(z)f(z+i) + A_{00}(z)f(z+1) = F(z),$$

where

$$A_{00}(z) = \frac{z^3-\pi}{z+1}, A_{01}(z) = -\frac{z^3-\pi}{2(z+1)^2}, A_{10}(z) = -\frac{2iz^2}{z+i}\exp\{-2iz\},$$

$$A_{11}(z) = \frac{iz^2}{(z+i)^2}\exp\{-2iz\}, F(z) = \frac{2iz^2}{z+i}\exp\{-2iz\} - \frac{z^3-\pi}{z+1},$$

and

$$\max_{0\le s\le 1, 0\le j\le 1}\{\rho(A_{sj}), \rho(F)\} = 1 < \rho(f) = 2.$$

Hence, the conditions of Lemma 3.5 are satisfied. We see that

$$\lambda(f) = \rho(f) = 2.$$

4. Proofs of Theorems

Proof of Theorem 2.1.

Proof. Contrary to our assertion, we assume that $\rho = \rho(f) < k+1$. Let

$$h_{sj}(z) = a_{sjk}z^k + h^*_{sj}(z), \tag{12}$$

where $a_{sjk} \neq 0$ are complex constants and $h^*_{sj}(z)$ are polynomials with $\deg(h^*_{sj}) \le k-1$, $s = 0, ..., n, j = 0, ..., m$. We set

$$|a_{l0k}| > |a_{sjk}|,\ \theta_{l0} \neq \theta_{sj},\ \theta_{sj} = \arg(a_{sjk}) \in [0, 2\pi), 0 \le s \le n, 0 \le j \le m, (s,j) \neq (l,0).$$

We now choose θ such that

$$\cos(k\theta + \theta_{l0}) = 1. \tag{13}$$

Thus, by $\theta_{sj} \neq \theta_{l0}$, $(s,j) \neq (l,0)$, we find

$$\cos(k\theta + \theta_{sj}) < 1, 0 \le s \le n, 0 \le j \le m, (s,j) \neq (l,0). \tag{14}$$

Denote

$$a = |a_{l0k}|,\ b = \max_{(s,j)\neq(l,0)}\{|a_{sjk}|\},\ c = \max_{(s,j)\neq(l,0)}\{b\cos(k\theta + \theta_{sj})\} < a, \tag{15}$$

and

$$\beta = \max_{0\le s\le n, 0\le j\le m}\{\rho(P_{sj}), \rho(Q_{sj})\} < k.$$

Clearly $\rho\left(\frac{P_{sj}}{P_{l0}}\right) \le \max\{\rho(P_{sj}), \rho(P_{l0})\} = \beta, 0 \le s \le n, 0 \le j \le m, (s,j) \neq (l,0)$ and

$$\rho\left(\frac{Q_{sj}}{P_{l0}}\right) \le \max\{\rho(Q_{sj}), \rho(P_{l0})\} = \beta,\ 0 \le s \le n, 0 \le j \le m.$$

By Lemma 3.1, for any given ε, with

$$0 < 2\varepsilon < \min\{1, k+1-\rho, k-\beta, a-c\},$$

there is a set $E_1 \subset (1,+\infty)$ with a finite logarithmic measure such that, for all z satisfying $|z|=r\notin E_1\cup[0,1]$, we obtain

$$\left|\frac{P_{sj}(z)}{P_{l0}(z)}\right| \le \exp\{r^{\beta+\varepsilon}\},\ (s,j)\ne(l,0), \left|\frac{Q_{sj}(z)}{P_{l0}(z)}\right| \le \exp\{r^{\beta+\varepsilon}\},\ 0\le s\le n, 0\le j\le m. \tag{16}$$

It is clear that the functions $\exp\{-h^*_{l0}(z)\}$ and $\exp\{h^*_{sj}(z)\}$, $0\le s\le n, 0\le j\le m, (s,j)\ne(l,0)$, are of regular order whose degree is lower than $k-1$ respectively. Thus, for all large z, $|z|=r$, we get

$$|\exp\{-h^*_{l0}(z)\}|\le \exp\{r^{k-1+\varepsilon}\},\quad \left|\exp\{h^*_{sj}(z)\}\right|\le \exp\{r^{k-1+\varepsilon}\},\ (s,j)\ne(l,0). \tag{17}$$

Applying Lemma 3.2 and Lemma 3.3 to f, we conclude that there is a set $E_2\subset(1,+\infty)$ with finite logarithmic measure such that, for all z satisfying $|z|=r\notin E_2\cup[0,1]$, we can write for $(s,j)\in\{0,...,n\}\times\{0,...,m\}$

$$\left|\frac{f^{(j)}(z+c_s)}{f(z+c_l)}\right| = \left|\frac{f^{(j)}(z+c_s)}{f(z+c_s)}\right|\left|\frac{f(z+c_s)}{f(z+c_l)}\right| \le \begin{cases} r^{j(\rho-1+\varepsilon)}\exp\{r^{\rho-1+\varepsilon}\},\ s\ne l,\\ r^{j(\rho-1+\varepsilon)},\ s=l.\end{cases} \tag{18}$$

Equation (2) gives

$$-A_{l0}(z)=\sum_{s=0,s\ne l}^{n}\sum_{j=0}^{m}A_{sj}(z)\frac{f^{(j)}(z+c_s)}{f(z+c_l)}+\sum_{j=1}^{m}A_{lj}(z)\frac{f^{(j)}(z+c_l)}{f(z+c_l)}. \tag{19}$$

By substituting (4) into equation (19), we obtain

$$\begin{aligned}-\exp\{a_{l0k}z^k\} \ =\ & \sum_{s=0,s\ne l}^{n}\sum_{j=0}^{m}\frac{f^{(j)}(z+c_s)}{f(z+c_l)}\exp\{-h^*_{l0}(z)\}\\ &\times\left(\frac{P_{sj}(z)}{P_{l0}(z)}\exp\{a_{sjk}z^k+h^*_{sj}(z)\}+\frac{Q_{sj}(z)}{P_{l0}(z)}\right)+\sum_{j=1}^{m}\frac{f^{(j)}(z+c_l)}{f(z+c_l)}\\ &\times\exp\{-h^*_{l0}(z)\}\left(\frac{P_{lj}(z)}{P_{l0}(z)}\exp\{a_{ljk}z^k+h^*_{lj}(z)\}+\frac{Q_{lj}(z)}{P_{l0}(z)}\right)\\ &+\exp\{-h^*_{l0}(z)\}\frac{Q_{l0}(z)}{P_{l0}(z)}.\end{aligned} \tag{20}$$

Let $z=r\exp\{i\theta\}$, where $r\notin[0,1]\cup E_1\cup E_2$. Substituting (13)-(18) into (20), we find

$$\begin{aligned}\exp\{|a_{l0k}|r^k\}\le &\sum_{s=0,s\ne l}^{n}\sum_{j=0}^{m}r^{j(\rho-1+\varepsilon)}\exp\{r^{k-1+\varepsilon}+r^{\rho-1+\varepsilon}+r^{\beta+\varepsilon}\}\\ &\times\left(\exp\{b\cos(k\theta+\theta_{sj})r^k+r^{k-1+\varepsilon}\}+1\right)\\ &+\sum_{j=1}^{m}r^{j(\rho-1+\varepsilon)}\exp\{r^{k-1+\varepsilon}+r^{\beta+\varepsilon}\}\left(\exp\{b\cos(k\theta+\theta_{lj})r^k+r^{k-1+\varepsilon}\}+1\right)\\ &+\exp\{r^{k-1+\varepsilon}+r^{\beta+\varepsilon}\}.\end{aligned}$$

Thus

$$\begin{aligned}\exp\{ar^k\}&\le(n+1)(m+1)r^{m(\rho-1+\varepsilon)}\exp\{cr^k+2r^{k-1+\varepsilon}+r^{\rho-1+\varepsilon}+r^{\beta+\varepsilon}\}\\ &\le(n+1)(m+1)r^{m(\rho-1+\varepsilon)}\exp\{(c+2\varepsilon)r^k\}.\end{aligned} \tag{21}$$

Dividing both sides of (21) by $(n+1)(m+1)r^{m(\rho-1+\varepsilon)}\exp\{(c+2\varepsilon)r^k\}$ and letting $r\to+\infty$, we get $+\infty\le 1$. Thus, this is a contradiction, hence $\rho(f)\ge k+1$. □

Proof of Theorem 2.2.

Proof. If $\rho(f)=+\infty$, then the theorem holds. We suppose $\rho(f)=\rho<+\infty$. Set

$$I_1=\{(s,j)\neq(l,0),\arg a_{l0k}\neq\arg a_{sjk},\ |a_{l0k}|\ge|a_{sjk}|\},$$

$$I_2=\{(s,j)\neq(l,0),\arg a_{l0k}=\arg a_{sjk},\ |a_{l0k}|>|a_{sjk}|\}.$$

It is clear that I_1,I_2 do not intersect with each other and $I=I_1\cup I_2=\{0,...,n\}\times\{0,...,m\}\setminus\{(l,0)\}$. Now, we may choose $\theta_0\in(0,2\pi)$ such that $\cos(\arg a_{l0k}+k\theta_0)=1$ (if $k=1$ and $\arg a_{l0k}=0$, then we replace $[0,2\pi)$ in Lemma 3.4 by $[-\frac{\pi}{2},\frac{3\pi}{2})$, if $k\ge 2$ this kind of θ_0 can always be chosen). Set $z=re^{i\theta},\theta\in[0,2\pi)$.
For $(s,j)\in I_1$, there exists sufficiently small $\varepsilon_1>0$ such that for all $\theta\in(\theta_0-\varepsilon_1,\theta_0+\varepsilon_1)\subset(0,2\pi)$, we have

$$\cos(\arg a_{l0k}+k\theta)>\max_{(s,j)\in I_1}\{\cos(\arg a_{sjk}+k\theta),0\}.$$

Since $|a_{l0k}|\ge|a_{sjk}|>0,(s,j)\neq(l,0)$, then we see that

$$\delta(h_{l0},\theta)>\max_{(s,j)\in I_1}\{\delta(h_{sj},\theta),0\}.$$

For $(s,j)\in I_2$, there exists sufficiently small $\varepsilon_2>0$ such that for all $\theta\in(\theta_0-\varepsilon_2,\theta_0+\varepsilon_2)\subset(0,2\pi)$, we have

$$\cos(\arg(a_{l0k})+k\theta)=\cos(\arg(a_{sjk})+k\theta)>0,\ \ 0\le s\le n,0\le j\le m,(s,j)\neq(l,0).$$

Since $|a_{l0k}|>|a_{sjk}|,(s,j)\neq(l,0)$, we see that

$$\delta(h_{l0},\theta)>\max_{(s,j)\in I_2}\{\delta(h_{sj},\theta)\}>0.$$

Let $\varepsilon_0=\min\{\varepsilon_1,\varepsilon_2\}$. Then for any $\theta\in(\theta_0-\varepsilon_0,\theta_0+\varepsilon_0)\subset(0,2\pi)$, we have

$$\delta(h_{l0},\theta)=\delta_1>\max_{(s,j)\in I}\{\delta(h_{sj},\theta),0\}=\delta_2. \tag{22}$$

By Lemma 3.1, for any given $\varepsilon(0<\varepsilon<\min\{1,\frac{\rho-1-k}{2},\frac{k-\sigma}{2},\frac{\delta_1-\delta_2}{2\delta_1+\delta_2}\})$, there exists a set $E_3\subset(1,+\infty)$ with a finite linear measure, such that for all z satisfying $|z|=r\notin[0,1]\cup E_3$ and $r\to+\infty$, we have

$$|P_{sj}(z)|\le\exp\{r^{\sigma+\varepsilon}\},\ |Q_{sj}(z)|\le\exp\{r^{\sigma+\varepsilon}\},s=0,...,n,j=0,...,m. \tag{23}$$

By Lemma 3.4, (22) and (23), for the above ε, there exists $E_4\subset[0,2\pi)$ with linear measure zero and a finite set $E_5=\{\theta\in[0,2\pi):\delta(h_{l0},\theta)=0\}$, such that for all $\theta\in(\theta_0-\varepsilon_0,\theta_0+\varepsilon_0)\setminus(E_4\cup E_5)$, there exists $r_0=r_0(\theta,\varepsilon)>0$, such that for $r>r_0=r_0(\theta,\varepsilon)$ and $0\le s\le n,0\le j\le m$, $(s,j)\neq(l,0)$ we have

$$\begin{aligned}|A_{sj}(z)|&\le|P_{sj}(z)\exp\{h_{sj}(z)\}|+|Q_{sj}(z)|\\&\le\exp\{(1+\varepsilon)\delta_2r^k\}+\exp\{r^{\sigma+\varepsilon}\}\\&\le 2\exp\{(1+\varepsilon)\delta_2r^k+r^{\sigma+\varepsilon}\},\end{aligned} \tag{24}$$

and

$$\begin{aligned}|A_{l0}(z)|&\ge|P_{l0}(z)\exp\{h_{l0}(z)\}|-|Q_{l0}(z)|\\&\ge\exp\{(1-\varepsilon)\delta_1r^k\}-\exp\{r^{\sigma+\varepsilon}\}.\end{aligned} \tag{25}$$

By applying Lemma 3.2 and Lemma 3.3 to f, for the above ε, we conclude that there is a set $E_6 \subset (1,+\infty)$ with finite logarithmic measure such that, for all z satisfying $|z| = r \notin [0,1] \cup E_6$, we can write for $0 \le s \le n, 0 \le j \le m$,

$$\left|\frac{f^{(j)}(z+c_s)}{f(z+c_l)}\right| = \left|\frac{f^{(j)}(z+c_s)}{f(z+c_s)}\right|\left|\frac{f(z+c_s)}{f(z+c_l)}\right| \le \begin{cases} r^{j(\rho-1+\varepsilon)}\exp\{r^{\rho-1+\varepsilon}\}, & s \ne l, \\ r^{j(\rho-1+\varepsilon)}, & s = l. \end{cases} \tag{26}$$

Equation (2) gives

$$|A_{l0}(z)| \le \sum_{s=0, s\ne l}^{n}\sum_{j=0}^{m}|A_{sj}(z)|\left|\frac{f^{(j)}(z+c_s)}{f(z+c_l)}\right| + \sum_{j=1}^{m}|A_{lj}(z)|\left|\frac{f^{(j)}(z+c_l)}{f(z+c_l)}\right|. \tag{27}$$

Let $z = r\exp\{i\theta\}$, where $\arg z = \theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \setminus (E_4 \cup E_5), |z| = r \notin [0,1] \cup E_3 \cup E_6$. Substituting (24)-(26) into (27), for sufficiently large r, we have

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta_1 r^k\} - \exp\{r^{\sigma+\varepsilon}\} \\ &\le 2n(m+1)r^{m(\rho-1+\varepsilon)}\exp\{(1+\varepsilon)\delta_2 r^k + r^{\sigma+\varepsilon} + r^{\rho-1+\varepsilon}\} \\ &\quad + 2mr^{m(\rho-1+\varepsilon)}\exp\{(1+\varepsilon)\delta_2 r^k + r^{\sigma+\varepsilon}\} \\ &\le (2n(m+1)+2m)r^{m(\rho-1+\varepsilon)}\exp\{(1+\varepsilon)\delta_2 r^k + r^{\sigma+\varepsilon} + r^{\rho-1+\varepsilon}\}. \end{aligned} \tag{28}$$

Since $(0 < \varepsilon < \min\{1, \frac{\rho-1-k}{2}, \frac{k-\sigma}{2}, \frac{\delta_1-\delta_2}{2\delta_1+\delta_2}\})$, then we get

$$(1+\varepsilon)\delta_2 < (1-2\varepsilon)\delta_1, \sigma+\varepsilon < k-\varepsilon,$$

and

$$\frac{\exp\{r^{\sigma+\varepsilon}\}}{\exp\{(1-\varepsilon)\delta_1 r^k\}} \to 0, r \notin [0,1] \cup E_3 \cup E_6, r \to +\infty.$$

Then, from (28) for sufficiently large r, we obtain

$$\frac{1}{2}\exp\left\{(1-\varepsilon)\delta_1 r^k\right\} \le (2n(m+1)+2m)r^{m(\rho-1+\varepsilon)}\exp\{(1+\varepsilon)\delta_2 r^k + r^{\sigma+\varepsilon} + r^{\rho-1+\varepsilon}\},$$

i.e.,

$$\exp\left\{(1-\varepsilon)\delta_1 r^k - (1+\varepsilon)\delta_2 r^k - r^{\sigma+\varepsilon}\right\} \le 2(2n(m+1)+2m)r^{m(\rho-1+\varepsilon)}\exp\{r^{\rho-1+\varepsilon}\},$$

which gives

$$\exp\{\frac{\varepsilon}{2}\delta_1 r^k\} \le 2(2n(m+1)+2m)r^{m(\rho-1+\varepsilon)}\exp\{r^{\rho-1+\varepsilon}\} \le \exp\{r^{\rho-1+2\varepsilon}\}. \tag{29}$$

Thus, (29) implies $k \le \rho - 1 + 2\varepsilon$, and by $\varepsilon > 0$ is arbitrary, we obtain

$$\rho(f) \ge k+1.$$

Set $g(z) = f(z) - \varphi(z)$, by substituting g into equation (2), then g solve the equation

$$\sum_{s=0}^{n}\sum_{j=0}^{m}A_{sj}(z)g^{(j)}(z+c_s) = -\sum_{s=0}^{n}\sum_{j=0}^{m}A_{sj}(z)\varphi^{(j)}(z+c_s).$$

Since $\rho(\varphi) < k+1$, then $\varphi(z)(\not\equiv 0)$ does not solve (2), that is,

$$\sum_{s=0}^{n}\sum_{j=0}^{m}A_{sj}(z)\varphi^{(j)}(z+c_s) \not\equiv 0,$$

and

$$\rho\left(-\sum_{s=0}^{n}\sum_{j=0}^{m} A_{sj}(z)\varphi^{(j)}(z+c_s)\right) \leq \max\{\rho(A_{sj}), 0 \leq s \leq n, 0 \leq j \leq m, \rho(\varphi)\},$$
$$< k+1 \leq \rho(f).$$

Therefore, by Lemma 3.5, we obtain $\lambda(g) = \rho(g)$. Hence

$$\lambda(g) = \lambda(f-\varphi) = \rho(f).$$

□

5. Conclusion

Throughout this article, by combining the complex difference and complex differential equations, we have investigated the properties of growth and oscillation of solutions of some homogeneous complex linear delay differential equations. We obtained two results which give estimates of the lower bound of the order of growth and the exponent of convergence of the difference between the solution and a meromorphic function of small growth of such homogeneous complex linear delay differential equations.

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