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# Correspondence between Different Types Of Quadratic Differentials

Monica Roşiu and Peter Kessler

Abstract. In this paper we develope the correspondence between quadratic differentials defined on a Klein surface , its double covering

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## 1. Introduction

For all notions and properties, not explained in this paper we will refer to [R]. In the beginning we will review the main notions and notations which will be used in this paper.

*A Klein surface* is a pair $(X,\ \mathcal{A})$ , where $X$ is a surface and $\mathcal{A}$ is a maximal dianalytic atlas on $X$, such that $\mathcal{A}$ does not contain any analytic subatlas. In this paper, we will consider nonorientable Klein surfaces and without border, as such is specified in N. Alling and N.Greenleaf$'$ papers, which are nonorientable Riemann surfaces in Teichmuller$'$s meaning, without specifying these characteristics all the time .

Let $\mathcal{O}_2$ be a Riemann surface. A mapping $\mathbf{k} : \mathcal{O}_2 \to \mathcal{O}_2$ with property $\mathbf{k} \circ \mathbf{k} = Id$, where $Id$ is the identity of $\mathcal{O}_2$, is an involution of $\mathcal{O}_2$.

*A symmetric Riemann surface* is a pair $(\mathcal{O}_2, \mathbf{k})$, consisting of a Riemann ( orientable) surface $\mathcal{O}_2$ and an antianalytic involution, $\mathbf{k} : \mathcal{O}_2 \to \mathcal{O}_2$ having no fixed points.

## 2. Results

Let $(X, \mathcal{A})$ be a Klein surface and $(\mathcal{O}_2, k)$ the symmetric Riemann surface associated to $X$ which is the double covering of $X$. Then $(X, \mathcal{A})$ is dianalytic equivalent with $\mathcal{O}_2/\mathcal{H}$, where $\mathcal{H}$ is the group generated of $k$, with respect to the composition of functions. Let $\mathcal{B}_1$ the maximal analytic atlas on $\mathcal{O}_2$ and $\mathcal{B}_2$ the maximal analytic atlas on $\mathbf{k}(\mathcal{O}_2)$. $\mathbf{k}(\mathcal{O}_2)$ is $\mathcal{O}_2$ endowed with the second orientation. Then $k : \mathcal{O}_2 \to k(\mathcal{O}_2)$ is an antianalytic isomorphism. So $\mathcal{O}_2 = (\mathcal{O}_2, \mathcal{B}_1)$ and $\mathbf{k}(\mathcal{O}_2) = (\mathcal{O}_2, \mathcal{B}_2)$.

Thus the canonical projection $q : \mathcal{O}_2 \to X$ is a dianalytic mapping which mix the two structures on $\mathcal{O}_2$. Indeed, if $P \in \mathcal{O}_2$, then its $\mathcal{H}$- orbit consists of two elements $P$ and $k(P)$. Then, $\widetilde{P} = P \cup k(P)$ and the mapping $\{\ P\ ,\ k(P)\ \} \to \widetilde{P}$ is a dianalytic isomorphism between $\mathcal{O}_2/\mathcal{H}$ and $X$. Thus $X$ can be identified with $\mathcal{O}_2/\mathcal{H}$, up to a dianalytic isomorphism.

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Let $X$ be a Klein surface. Then there is a simply connected Riemann surface $\widehat{X}$ and a covering transformation $\pi : \widehat{X} \to X$ (see [F], pg.32; [BG], pg. 7). Also $\widehat{X}$ satisfy the universal property (see [F], pg.31], so $\pi$ is the universal covering of $X$.

Let $\mathcal{G}$ be the covering transformations group of $\pi$. Because $X$ is nonorientable, $\mathcal{G}$ will contain analytic automorphisms of $\widehat{X}$ and antianalytic automorphisms of $\widehat{X}$. Let $\mathcal{G}_1$ be the subgroup of the analytic automorphisms of $\mathcal{G}$. Then $\mathcal{G}_1$ is a subgroup of $\mathcal{G}$ and for every $S \in \mathcal{G} \setminus \mathcal{G}_1$, $\mathcal{G} = \mathcal{G}_1 \cup S\mathcal{G}_1$, $\mathcal{G}_1 \cap S\mathcal{G}_1 = \emptyset$, where $S\mathcal{G}_1 = \{S \circ T \mid T \in \mathcal{G}_1\}$. If $P \in \widehat{X}$, then we denote with $\widetilde{P}$ (respectively $P$ ) its $\mathcal{G}$- orbit ( respectively its $\mathcal{G}_1$-orbit ). So

$$\widetilde{P} = \{G(\widehat{P}) \mid G \in \mathcal{G}\} \quad \text{and } P = \{T(\widehat{P}) \mid T \in \mathcal{G}_1\}.$$

The quotient space $\widehat{X}/\mathcal{G} = \{\widetilde{P} \mid \widehat{P} \in \widehat{X}\}$ has a Klein surface structure and the covering projection $\pi : \widehat{X} \to \widehat{X}/\mathcal{G}$, $\widehat{P} \xrightarrow{\pi} \widetilde{P}$ is a dianalytic covering mapping, by the definition of $\widetilde{P}$.

We can identify in a topological way $X$ with $\widehat{X}/\mathcal{G}$, by the homeomorphism $\alpha : X \to \widehat{X}/\mathcal{G}$, such that for every $\widetilde{P} \in X$, if $\widehat{P} \in \pi^{-1}(P)$ then $\alpha(P) = \widetilde{P}$.

**Theorem 2.1.** *Let $X$ a Klein surface, $\pi : \widehat{X} \to X$ the universal covering of $X$ and $\mathcal{G}$ the covering transformations group of $\pi$. If $\mathcal{O}_2 = \widehat{X}/\mathcal{G}_1$, then $\mathcal{O}_2$ has a Riemann surface structure.*

Proof. Because $\mathbf{k}(P) \neq P$, for every $P \in \mathcal{O}_2$, then exists $\widetilde{U}$a parametric disk of $X$, small enough such that $q^{-1}(\widetilde{U})= U_1 \cup U_2$, where $\{U_i\}_{i\in\{1,2\}}$are open sets of $\mathcal{O}_2$, $U_1 \cap U_2 =\emptyset$, because $\{S(\widehat{P})| \ S \in \mathcal{G} \setminus \mathcal{G}_1\} \cap \{T(\widehat{P}) \mid T \in \mathcal{G}_1\} = \emptyset$ and such that the mappings $q/U_i : U_i \longrightarrow \widetilde{U}$, $i \in \{1, 2\}$are homeomorphisms and also $U_2 = \mathbf{k}(U_1)$, because $q = q \circ \mathbf{k}$.

We denote $U_1= U$, so $U_2 = \mathbf{k}(U)$.

Then $p^{-1}(U) = \{\widehat{U}_j\}_{j\in J}$, where $\widehat{U}_j$are open sets of $\widehat{X}$, for every $j \in J$and the mappings $p/\widehat{U}_j : \widehat{U}_j \longrightarrow U$are homeomorphisms.

Let $\widehat{U}$be a fixed element of $\{\widehat{U}_j\}_{j\in J}$and $(T(\widehat{U}), \widehat{h}_{T,\widehat{U}}, V_{T,\widehat{U}})$, $(S(\widehat{U}), \widehat{h}_{S,\widehat{U}}, V_{S,\widehat{U}})$the corresponding maps on $\widehat{X}$, where $T \in \mathcal{G}_1$and $S \in \mathcal{G} \setminus \mathcal{G}_1$. We will use the following notations : $\mathbf{p}_{T,U} = (p/T(\widehat{U}))\circ\widehat{h}^{-1}_{T,\widehat{U}}$and $\mathbf{p}_{S,U}=(p/S(\widehat{U}))\circ\widehat{h}^{-1}_{S,\widehat{U}}$, with $T \in \mathcal{G}_1$and $S \in \mathcal{G}\setminus\mathcal{G}_1$. If $T \in \mathcal{G}_1$, then $(p/T(\widehat{U}))(T(\widehat{U}))= U$and if $S \in \mathcal{G}\setminus\mathcal{G}_1$, then $(p/S(\widehat{U}))(S(\widehat{U}))=\mathbf{k}(U)$.Therefore $(U,\ \mathbf{p}^{-1}_{T,U},\ V_{T,\widehat{U}})$and $(U,\ \mathbf{p}^{-1}_{S,U}\circ(\mathbf{k}/U),\ V_{S,\widehat{U}})$, where $T \in \mathcal{G}_1$and $S \in \mathcal{G} \setminus \mathcal{G}_1$, are maps on $\mathcal{O}_2$.

Thus

$$\mathcal{B}^*_1 = \left\{(U, \mathbf{p}^{-1}_{T,U}, V_{T,\widehat{U}}\ ) \mid T \in \mathcal{G}_1, U \in \mathcal{U}\right\} \quad \text{and}$$

$$\mathcal{B}^*_2 = \left\{(U, \mathbf{p}^{-1}_{S,U} \circ (\mathbf{k}/U), V_{S,\widehat{U}}) \mid S \in \mathcal{G} \setminus \mathcal{G}_1, U \in \mathcal{U}\right\},$$

where $U$belong to the family $\mathcal{U}$of sets $U$of the previous type, are analytic atlases on $\mathcal{O}_2$.

Indeed, let $U \in \mathcal{U}$and $\widehat{P} = T(\widehat{P}_1)$, with $T \in G_1$and $\widehat{P}_1 \in \widehat{U}$( a fixed element of the family $\{\widehat{U}_j\}_{j\in J}$). Then, by the definition of $P$, $P = P_1$and $(p/T(\widehat{U}))(\widehat{P})=(p/\widehat{U})(\widehat{P}_1)$,by the definition of $p$, for every $\widehat{P}_1 \in \widehat{U}$and $T \in \mathcal{G}_1$with $\widehat{P} = T(\widehat{P}_1)$. Therefore, $p/T(\widehat{U}) = (p/\widehat{U}) \circ (T/\widehat{U})^{-1}$, for every $T \in \mathcal{G}_1$.

For a fixed set $U$, it$'$is easy to check that the maps $(U, \mathbf{p}_{T,U}^{-1}, V_{T,\widehat{U}})$, where $T \in \mathcal{G}_1$,are analytically compatible..

For $U_1, U_2$of the previous type, with $U_1 \cap U_2 \neq \emptyset$and $T_1, T_2 \in \mathcal{G}_1$, the maps $(U_1, \mathbf{p}_{T_1,U_1}^{-1}, V_{T_1,\widehat{U}_1})$and $(U_2, \mathbf{p}_{T_2,U_2}^{-1}, V_{T_2,"\widehat{U}_2})$are analytically compatible, because if $U$is a connected component of $U_1 \cap U_2$, then we can apply, for $U$, the previous results. If for an initial $\widetilde{U}$we fixed $\widehat{U}$then we do all the other choices such that to remain on the same analytic atlas on $\widehat{X}$.

Therefore, $\mathcal{B}_1^*$is an analytic atlas on $\mathcal{O}_2$.

By analogy, $\mathcal{B}_2^*$is also an analytic atlas on $\mathcal{O}_2$.

In the same way, we can verify that maps $(U, \mathbf{p}_{T,U}^{-1}, V_{T,\widehat{U}}) \in \mathcal{B}_1^*$and $(U, \mathbf{p}_{S,U}^{-1} \circ (\mathbf{k}/U), V_{S,\widehat{U}}) \in \mathcal{B}_2^*$are antianalytic compatible.

Let $\mathcal{B}_1$(respectively $\mathcal{B}_2$) the maximal analytic atlases on $\mathcal{O}_2$, which contains $\mathcal{B}_1^*$( respectively $\mathcal{B}_2^*$). $\mathcal{B}_1$determine the analytic structure on $\mathcal{O}_2$and $\mathcal{B}_2$the analytic structure on $\mathbf{k}(\mathcal{O}_2)$. $\mathbf{k}(\mathcal{O}_2)$is then the surface $\mathcal{O}_2$provided with its second orientation. Then $\mathbf{k} : \mathcal{O}_2 \rightarrow \mathbf{k}(\mathcal{O}_2)$is an antianalytic isomorphism. Thus, $q : \mathcal{O}_2 \rightarrow X$is a dianalytic mapping, which mixed the structures of $\mathcal{O}_2$and $\mathbf{k}(\mathcal{O}_2)$. Then $\mathcal{O}_2 = (\mathcal{O}_2, \mathcal{B}_1^*)$, $\mathbf{k}(\mathcal{O}_2) = (\mathcal{O}_2, \mathcal{B}_2^*)$.

We denote with $\mathcal{H}$the group consists of $\mathbf{k}$and $Id$, with respect of the composition of functions. It is easy to see that

Theorem 2. *$\mathcal{O}_2/\mathcal{H}$ is dianalytic equivalent with $X$.*

For proof see[R].

Theorem 3. *Let $(\mathcal{O}_2, \mathbf{k})$ be a symmetric Riemann surface . Then the covering projection $q : \mathcal{O}_2 \rightarrow \mathcal{O}_2/\mathcal{H}$ induce o structură de suprafaţă Klein pe $\mathcal{O}_2/\mathcal{H}$.*

Proof. Let $(\mathcal{O}_2, \mathbf{k})$ be a symmetric Riemann surface, then $\mathcal{O}_2$ is a Riemann surface and $\mathbf{k} : \mathcal{O}_2 \longrightarrow \mathcal{O}_2$ is an homeomorphism for which we have:$\mathbf{k} \circ \mathbf{k} = Id$, $\mathbf{k}(P) \neq P$, for every $P \in \mathcal{O}_2$ and $\varphi_1 \circ \mathbf{k} \circ \varphi^{-1}$ is antianalytic, for every pair of maps $(U, \varphi, V) \in \mathcal{B}$, $(U_1, \varphi_1, V_1) \in \mathcal{B}$ of $\mathcal{O}_2$ which satisfy the condition $\mathbf{k}(U) \subseteq U_1$, where $\mathcal{B} = \{(U_i, \varphi_i, V_i) \mid i \in I\}$ is an analytic atlas on $\mathcal{O}_2$.Let $X = \mathcal{O}_2/\{Id, \mathbf{k}\}$, the orbits space of $\mathcal{O}_2$, with respect to the group $(\{Id, \mathbf{k}\}, \circ)$ and $q : \mathcal{O}_2 \rightarrow \mathcal{O}_2/\mathcal{H}$ the covering projection, such that $q(P) = \{P, \mathbf{k}(P)\}$, for every $P \in \mathcal{O}_2$.

Then $\mathcal{O}_2/\mathcal{H}$ has a dianalytic structure. We denote with $\widetilde{\mathcal{U}} = \{q(U_i) \mid i \in I\}$. Because $q \circ \mathbf{k} = q$, we can suppose that for every $\widetilde{U}_i = q(U_i) \in \widetilde{\mathcal{U}}$ , $q^{-1}(\widetilde{U}_i) = U_i \cup \mathbf{k}(U_i)$, with $U_i \cap \mathbf{k}(U_i) = \emptyset$, because $\mathbf{k}$ doesn$'$t have fixed points and $q/U_i : U_i \rightarrow \widetilde{U}_i$ and $q/\mathbf{k}(U_i) : \mathbf{k}(U_i) \rightarrow \widetilde{U}_i$ are homeomorphisms. We consider $(U_j, \varphi_j, V_j) \in \mathcal{B}$ and $(U_i, \varphi_i, V_i) \in \mathcal{B}$, $i$, $j \in I$ with $U_i$ enough small such that $\mathbf{k}(U_i) \subseteq U_j$ and we denote the restriction at $\mathbf{k}(U_i)$, corresponding to $\varphi_j$ with $\varphi_{ij}^k$ and on $\varphi_{ij}^k(\mathbf{k}(U_i))$ with $V_{ij}^k$ . We define $h_i = \varphi_i \circ (q/U_i)^{-1} : \widetilde{U}_i \rightarrow V_i$ and $g_i = \varphi_{ij}^k \circ (q/\mathbf{k}(U_i))^{-1} : \widetilde{U}_i \rightarrow V_{ij}^k$, for every $i \in I$ .

Let $\mathcal{A}_1 = \{ (\widetilde{U}_i, h_i, V_i) \mid i \in I\}$ and $\mathcal{A}_2 = \{ (\widetilde{U}_i, g_i, V_i^k) \mid i \in I\}$, where $V_i^k = V_{ij}^k$ , with $j \in I$ a fixed element. Then $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is a dianalytic atlas on $\mathcal{O}_2/\mathcal{H}$. Then the following diagram is commutative :

$$\begin{array}{ccccc}
U_i & & \stackrel{\varphi_i}{\longrightarrow} & & V_i \\
& \stackrel{q/U_i}{\searrow} & & \nearrow_{h_i} & \\
\mathbf{k}\downarrow & & \widetilde{U}_i & & \\
& \nearrow_{q/k(U_i)} & & \stackrel{g_i}{\searrow} & \\
\mathbf{k}(U_i) & & \underset{\varphi^k_{ij}}{\longrightarrow} & & V^k_{ij}
\end{array}$$

If we consider $\mathcal{B}_1^*, \mathcal{B}_2^*$ the analytic atlases on $\mathcal{O}_2$, we can define a dianalytic structure on $\mathcal{O}_2/\mathcal{H}$, dianalytical equivalent with $X$ :

Let $\widetilde{\mathcal{U}} = \{q(U) \mid U \in \mathcal{U}\}$. If $\widetilde{U} \in \widetilde{\mathcal{U}}$, $q^{-1}(\widetilde{U}) = U \cup \mathbf{k}(U)$, with $U \cap \mathbf{k}(U) = \emptyset$ then $q/U : U \to \widetilde{U}$ and $q/\mathbf{k}(U) : \mathbf{k}(U) \to \widetilde{U}$ are homeomorphisms.

For every $T \in \mathcal{G}_1$, we define $h_{T,U} = (q/U \circ \mathbf{p}_{T,U})^{-1} : \widetilde{U} \to V_{T,\widehat{U}}$ , where the set $\widehat{U}$ from the definition of $\mathbf{p}_{T,U}$ is a fixed component of $p^{-1}(U)$, such that $p/\widehat{U} : \widehat{U} \to U$ is an homeomorphism

For each $S \in \mathcal{G}\backslash\mathcal{G}_1$, we define $g_{S,U} = (q/\mathbf{k}(U) \circ \mathbf{p}_{S,U})^{-1} : \widetilde{U} \to V_{S,\widehat{U}}$. Let

$$\mathcal{A}_1' = \left\{(\widetilde{U}, h_{T,U}, V_{T,\widehat{U}}) \mid \widetilde{U} \in \widetilde{\mathcal{U}},\, T \in \mathcal{G}_1\right\}$$

and

$$\mathcal{A}_2' = \left\{(\widetilde{U}, g_{S,U}, V_{S,\widehat{U}}) \mid \widetilde{U} \in \widetilde{\mathcal{U}},\, S \in \mathcal{G}\backslash\mathcal{G}_1\right\}$$

Then, $\mathcal{A}' = \mathcal{A}_1' \cup \mathcal{A}_2'$ is an antianalytic atlas on $X$.

For $\widetilde{U}_i \in \widetilde{\mathcal{U}}$, $i \in \{1,2\}$ we used the notations $q^{-1}(\widetilde{U}_i) = U_i \cup \mathbf{k}(U_i)$ and $p(\widehat{U}_i) = U_i$, for every $i \in \{1,2\}$.

For the notations and definitions about meromorphic quadratic differential $\Phi$ on $\mathcal{O}_2$ see [R].

Let $X$ be a Klein surface and $\mathcal{A} = \left\{(\widetilde{U}_i, h_i, V_i) \mid i \in I\right\}$ the induce atlas on $X$ and $\widetilde{\Phi}$ is a $N$- meromorphic quadratic differential, respectively holomorphic, on $X$ .

Let $Q^2(X)$ be the vectorial space of $N$- meromorphic quadratic differentials on $X$, with respect with $\mathbf{C}$.

Let $V$ and $f(V)$ the images through the corresponding charts of the parametric disks $U$, respectively $k(U)$. Because $k$ is an antianalytic involution it results that $f$ is an antianalytic involution. We will use $z$, like local parameter on $U$ and $w$, like local parameter on $k(U)$.

Theorem 4. *There is an isomorphism $\widetilde{q}$, between $Q^s(\mathcal{O}_2)$ and $Q^2(X)$.*

Proof. Let $q : \mathcal{O}_2 \to \mathcal{O}_2/\mathcal{H}$ , $q(P) = \widetilde{P}$ , for every $P \in \mathcal{O}_2$ , the canonical projection. The mapping $\widetilde{q} : Q^s(\mathcal{O}_2) \to Q^2(X)$ is defined by :

$$\widetilde{q}(\Phi + \Phi \circ \mathbf{k}) = \widetilde{\Phi}, \text{ for every } \Phi \in Q^2(\mathcal{O}_2),$$

where $\widetilde{\Phi}^*/\widetilde{U} = \Phi^*/U + (\Phi \circ \mathbf{k})^*/U$ and $q^{-1}(\widetilde{U}) = \{U, \mathbf{k}(U)\}$, for every parametric disk $\widetilde{U}$ of $X$.

Let $\Phi \in Q^2(\mathcal{O}_2)$ with the local representation:

$$\Phi^*/U \cup \mathbf{k}(U) = \begin{cases} \varphi(z)dz^2, & z \in V \\ \widehat{\varphi}(w)dw^2, & w \in \mathbf{f}(V) \end{cases} \tag{1}$$

where $\varphi$ and $\widehat{\varphi}$ are meromorphic functions on $V$, respectively $\mathbf{f}(V)$. If $\varphi$ is not holomorphic, that is it has at least a pole, by $z \in V$ we means that $z$ is not a pole of $\varphi$.

Then, the symmetry $\mathbf{k}$ will induce the isomorphism $\mathbf{K} : Q^2(\mathcal{O}_2) \to \overline{Q^2(\mathcal{O}_2)}$ defined as (see [R]):

$$\mathbf{K}(\Phi)^*/U \cup \mathbf{k}(U) = (\Phi \circ \mathbf{k})^*/U \cup \mathbf{k}(U) = \begin{cases} \widehat{\varphi}(\mathbf{f}(z))d\mathbf{f}(z)^2, & \text{dacă } z \in V \\ \varphi(\mathbf{f}(w))d\mathbf{f}(w)^2, & \text{dacă } w \in \mathbf{f}(V) \end{cases} \tag{2}$$

Then

$$\widetilde{\Phi}^*/\widetilde{U} = \varphi(z)dz^2 + \widehat{\varphi}(\mathbf{f}(z))d\mathbf{f}(z)^2,$$

where we consider a map from $\mathcal{A}_1'$, therefore $\widetilde{z} = h(\widetilde{P}) = z$ is the local parameter corresponding to $\widetilde{U}$, $q^{-1}(\widetilde{P}) = \{P, \mathbf{k}(P)\}$ ; $\varphi$ and $\widehat{\varphi}$ are meromorphic functions in the local parameters $z$, respectively $\mathbf{f}(z)$.

*$\widetilde{q}$ is well defined.*

Because the maps on $X$ are of two types, we have to study the following maps :

a$_1$) Let $(\widetilde{U}, h_{T,U}, V_{T,\widehat{U}}) \in \mathcal{A}_1'$, then if $V = V_{T,\widehat{U}}$ we define

$$\widetilde{\varphi}_1/\widetilde{U} = \varphi/V \circ \mathbf{p}_{T,U}^{-1} \circ (q/U)^{-1} \text{ and } \widetilde{\varphi}_2/\widetilde{U} = \widehat{\varphi}/\mathbf{f}(V) \circ \mathbf{p}_{S,U}^{-1} \circ (q/\mathbf{k}(U))^{-1} \tag{3}$$

$$\widetilde{\Phi}_1^*/\widetilde{U} = (\widetilde{\varphi}_1 \circ h_{T,U}^{-1})(\widetilde{z})d\widetilde{z}^2 \quad \text{and} \quad \widetilde{\Phi}_2^*/\widetilde{U} = (\widetilde{\varphi}_2 \circ h_{T,U}^{-1})(\widetilde{z})d\mathbf{f}(\widetilde{z})^2 \ . \tag{4}$$

So $\widetilde{\Phi}^*/\widetilde{U} = \Phi^*/U + (\Phi \circ \mathbf{k})^*/U = \varphi(z)\ dz^2 + \widehat{\varphi}(\mathbf{f}(z))d\mathbf{f}(z)^2 = \widetilde{\Phi}_1^*/\widetilde{U} + \widetilde{\Phi}_2^*/\widetilde{U}$ , for every $\widetilde{P} \in \widetilde{U}$ with $\widetilde{z} = h_{T,U}(\widetilde{P}) = \mathbf{p}_{T,U}^{-1}(P) = z$, $T \in \mathcal{G}_1$. because $\Phi \in Q^2(\mathcal{O}_2)$, the function $\widetilde{\varphi}_1/\widetilde{U} \circ h_{T,U}^{-1} = \varphi/V$ is a meromorphic function. By the invariant condition for $\Phi$, if $\widetilde{z}_0$ şi $\widetilde{z}$ are parametric values corresponding to the same point of $X$ through two maps of $\mathcal{A}_1'$, we have $\varphi(z)dz^2 = \varphi_0(z_0)dz_0^2$, where $\varphi_0$ is the local representation of $\Phi$ with respect to the parameter $z_0$ and thus the parametric representations of $\widetilde{\Phi}_1$ with respect to $\widetilde{z}$ and $\widetilde{z}_0$ are the same on every connected component of the common region. So , $\widetilde{\Phi}_1 \in Q^2(\mathcal{O}_2)$. Analogous, $\widetilde{\varphi}_2/\widetilde{U} \circ h_{T,U}^{-1} = \widehat{\varphi}/\mathbf{f}(V) \circ \mathbf{p}_{S,U}^{-1} \circ \mathbf{k}/U \circ \mathbf{p}_{T,U}^{-1} = \widehat{\varphi} \circ \mathbf{f}$ is an antimeromorphic function, therefore $\widehat{\varphi}/\mathbf{f}(V) \circ \mathbf{p}_{S,U}^{-1} \circ \mathbf{k}/U$ is an antimeromorphic mapping on $(\mathcal{O}_2, \mathcal{B}_1^*)$. By theorem 2.1., if $\Phi \in Q^2(\mathcal{O}_2)$, then $\Phi \circ \mathbf{k}$ is an antimeromorphic quadratic differential on $(\mathcal{O}_2, \mathcal{B}_1^*)$, therefore $\widetilde{\Phi}_2 \in \overline{Q^2(\mathcal{O}_2)}$.

a$_2$) Let $(\widetilde{U}, g_{S,U}, V_{S,\widehat{U}}) \in \mathcal{A}_2'$, then if $\mathbf{f}(V) = V_{S,\widehat{U}}$ we define

$$\widetilde{\varphi}_1/\widetilde{U} = \widehat{\varphi}/\mathbf{f}(V) \circ \mathbf{p}_{S,U}^{-1} \circ (q/\mathbf{k}(U))^{-1} \text{and } \widetilde{\varphi}_2/\widetilde{U} = \varphi/V \circ \mathbf{p}_{T,U}^{-1} \circ (q/U)^{-1} \tag{5}$$

$$\widetilde{\Phi}_1^*/\widetilde{U} = (\widetilde{\varphi}_1 \circ g_{S,U}^{-1})(\widetilde{z})d\widetilde{z}^2 \quad \text{and} \quad \widetilde{\Phi}_2^*/\widetilde{U} = (\widetilde{\varphi}_2 \circ g_{S,U}^{-1})(\widetilde{z})d\mathbf{f}(\widetilde{z})^2 \tag{6}$$

Thus $\widetilde{\Phi}^*/\widetilde{U} = \Phi^*/\mathbf{k}(U) + (\Phi \circ \mathbf{k})^*/\mathbf{k}(U) = \widehat{\varphi}(w)dw^2 + \varphi(\mathbf{f}(w))\ d\mathbf{f}(w)^2 = \widetilde{\Phi}_1^*/\widetilde{U} + \widetilde{\Phi}_2^*/\widetilde{U}$, for every $\widetilde{P} \in \widetilde{U}$, where $\widetilde{z} = g_{S,U}(\widetilde{P}) = \mathbf{p}_{S,U}^{-1}(\mathbf{k}(P)) = \mathbf{f}(z) = w$, $S \in \mathcal{G} \backslash \mathcal{G}_1$. The function $\widetilde{\varphi}_1/\widetilde{U} \circ g_{S,U}^{-1} = \widehat{\varphi}/\mathbf{f}(V)$ is a meromorphic function and because $\Phi \in Q^2(\mathcal{O}_2)$, it results that if $\widetilde{w}_0$ and $\widetilde{w}$ are parametric values corresponding to the same point of $X$, through two maps of $\mathcal{A}_2'$ then $\widehat{\varphi}(w)dw^2 = \widehat{\varphi}_0(w_0)\ dw_0^2$, where $\widehat{\varphi}_0$ is the representation of $\Phi$ with respect to the parameter $w_0$ and thus the parametric representations of $\widetilde{\Phi}_1$ with respect to $\widetilde{w}$ and $\widetilde{w}_0$ are the same on every connected component of the common region. , So, $\widetilde{\Phi}_1$ doesn't change with the conform parameter, therefore $\widetilde{\Phi}_1 \in Q^2(\mathcal{O}_2)$ and we have $\widetilde{\Phi}_2 \in \overline{Q^2(\mathcal{O}_2)}$. Thus a) condition is satisfied.

To prove b) we study the two cases in which the transition function is analytical.

1) Let $(\widetilde{U}_1, h_{T_1,U_1}, V_{T_1,\widehat{U}_1})$, $(\widetilde{U}_2, h_{T_2,U_2}, V_{T_2,\widehat{U}_2})$ be two maps of $\mathcal{A}'_1$, such that $\widetilde{U}_1 \cap \widetilde{U}_2 \neq \emptyset$ and $\widetilde{U} \subseteq \widetilde{U}_1 \cap \widetilde{U}_2$, $\widetilde{U}$ is a connected component. Then, if $V_{T_1,\widehat{U}_1} = V_1$, $V_{T_2,\widehat{U}_2} = V_2$ and $\widetilde{\Phi}^*/\widetilde{U}_1 = (\widetilde{\Phi}'_1)^*/\widetilde{U}_1 + (\widetilde{\Phi}'_2)^*/\widetilde{U}_1$ and $\widetilde{\Phi}^*/\widetilde{U}_2 = (\ \widetilde{\Phi}''_1)^*/\widetilde{U}_2 + (\ \widetilde{\Phi}''_2)^*/\widetilde{U}_2$ are the local representations of $\widetilde{\Phi}$ corresponding to the two maps, by using the same kind of maps like in (3) and (4), we have $\widetilde{\varphi}'_1/\widetilde{U} = \varphi \circ \mathbf{p}^{-1}_{T,U} \circ (q/U)^{-1} = \widetilde{\varphi}''_1/\widetilde{U}$ and $\widetilde{\varphi}'_2/\widetilde{U} = \widehat{\varphi} \circ \mathbf{p}^{-1}_{S,U} \circ (q/\mathbf{k}(U))^{-1} = \widetilde{\varphi}''_2/\widetilde{U}$, where $U \subset \mathcal{O}_2$, $q(U) = \widetilde{U}$ , $\varphi$ and $\widehat{\varphi}$ are meromorphic functions in the parameter $z$, respectively $\mathbf{f}(z)$, so $\mathrm{b}_1$) is satisfied.

2) Let $(\widetilde{U}_1, g_{S_1,U_1}, V_{S_1,\widehat{U}_1})$, $(\widetilde{U}_2, g_{S_2,U_2}, V_{S_2,\widehat{U}_2})$ be two maps of $\mathcal{A}'_2$, such that $\widetilde{U}_1 \cap \widetilde{U}_2 \neq \emptyset$ and $\widetilde{U} \subseteq \widetilde{U}_1 \cap \widetilde{U}_2$, $\widetilde{U}$ a connected component. Then, if $V_{S_1,\widehat{U}_1} = V_1$, $V_{S_2,\widehat{U}_2} = V_2$ by using the same kind of maps like in (5) and (6), we have $\widetilde{\varphi}'_1/\widetilde{U} = \widehat{\varphi} \circ \mathbf{p}^{-1}_{S,U} \circ (q/\mathbf{k}(U))^{-1} = \widetilde{\varphi}''_1/\widetilde{U}$ and $\widetilde{\varphi}'_2/\widetilde{U} = \varphi \circ \mathbf{p}^{-1}_{T,U} \circ (q/U)^{-1} = \widetilde{\varphi}'_2/\widetilde{U}$, where $U \subset \mathcal{O}_2$ and $q(U) = \widetilde{U}$, so $\mathrm{b}_1$) is satisfied.

The transition mapping is antianalytical if we consider the maps $(\widetilde{U}_1,\, h_{T_1,U_1},\, V_{T,\widehat{U}})$ $\in \mathcal{A}'_1$ and $(\widetilde{U}_2,\, g_{S_2,U_2},\, V_{S,\widehat{U}}) \in \mathcal{A}''_2$, such that $\widetilde{U}_1 \cap \widetilde{U}_2 \neq \emptyset$. Let $\widetilde{U} \subseteq \widetilde{U}_1 \cap \widetilde{U}_2$, $\widetilde{U}$ is a connected component then, using the same notations like in (2), (3), (4) and (5) it results that $\widetilde{\varphi}'_1/\widetilde{U} = \varphi \circ \mathbf{p}^{-1}_{T,U} \circ (q/U)^{-1} = \widetilde{\varphi}''_2/\widetilde{U}$ and analogous $\widetilde{\varphi}'_2/\widetilde{U} = \widehat{\varphi} \circ \mathbf{p}^{-1}_{S,U} \circ (q/\mathbf{k}(U))^{-1} = \widetilde{\varphi}''_1/\widetilde{U}$. So $\mathrm{b}_2$) is satisfied.

*$\widetilde{q}$ is a morphism of vectorial spaces .*

By (3), (4), (5) and (6) it results that the local components of $\widetilde{q}[(\Phi + \Psi) + (\Phi + \Psi) \circ \mathbf{k}]$ and $\widetilde{q}(\Phi + \Phi \circ \mathbf{k}) + \widetilde{q}(\Psi + \Psi \circ \mathbf{k})$ are the same, for every $\Phi, \Psi \in Q^2(\mathcal{O}_2)$, so $\widetilde{q}$ $[(\Phi + \Phi \circ \mathbf{k}) + (\Psi + \Psi \circ \mathbf{k})] = \widetilde{q}(\Phi + \Phi \circ \mathbf{k}) + \widetilde{q}(\Psi + \Psi \circ \mathbf{k})$.

Analogous, $\widetilde{q}[\lambda(\Phi + \Phi \circ \mathbf{k})] = \lambda\ \widetilde{q}(\Phi + \Phi \circ \mathbf{k})$, for every $\lambda \in \mathbf{C}$ and $\Phi \in Q^2(\mathcal{O}_2)$.

*$\widetilde{q}$ is bijective.*

Let $\Phi + \Phi \circ \mathbf{k} \in Q^s(\mathcal{O}_2)$, such that $\widetilde{q}(\Phi + \Phi \circ \mathbf{k}) = \widetilde{\Phi} = 0$. By definition of $\widetilde{\Phi}$ it results $\widetilde{\Phi}_1/\widetilde{U} = \widetilde{\Phi}_2/\widetilde{U} = 0$, for every $(\widetilde{U}, h, V) \in \widetilde{\mathcal{A}}$. So by (3), (4), (5) and (6) it results that $\Phi/U = 0$ and $(\Phi \circ \mathbf{k})/U = 0$, for every parametric disk $U$ of $\mathcal{O}_2$. Thus $\Phi + \Phi \circ \mathbf{k} = 0$.

Let $\widetilde{\Phi} \in Q^2(X)$ and $U \subset \mathcal{O}_2$ be a parametric disk such that $q(U) = \widetilde{U}$. Let $\widetilde{\Phi}^*_1/\widetilde{U}$ $= \varphi_1(z)dz^2$ and $\widetilde{\Phi}^*_2/\widetilde{U} = \varphi_2(\mathbf{f}(z))d\mathbf{f}(z)^2$ the local components of $\widetilde{\Phi}$ with respect to the map $(\widetilde{U}, h_{T,U}, V_{T,\widehat{U}} = V) \in \mathcal{A}'_1$. We define

$$\Phi^*/U \cup \mathbf{k}(U) = \begin{cases} \varphi_1(z)dz^2, & z \in V \\ \varphi_2(w)dw^2, & w \in \mathbf{f}(V) \end{cases}, \tag{7}$$

for every $U \subset \mathcal{O}_2$. By condition a) of a $N$ -meromorphic quadratic differential definition, it results that $\Phi \in Q^2(\mathcal{O}_2)$ and by definitions of $\Phi$ and $\widetilde{q}$ we have $\widetilde{q}(\Phi + \Phi \circ \mathbf{k}) = \widetilde{\Phi}$.

Corollary. *By analogy exists an isomorphism between the vectoarial spaces $Q^a(\mathcal{O}_2)$ and $Q^2(\mathcal{O}_2)$.*

We can associate to a meromorphic quadratic differential on $\mathcal{O}_2$ a meromorphic quadratic differential on $\widehat{X}$.

Let $\Phi \in Q^2(\mathcal{O}_2)$. We denote with $\widehat{\Phi}$ , the meromorphic quadratic differential on $\widehat{X}$ associated with $\Phi$. By definition $\widehat{\Phi}$ is a family of meromorphic functions $(\widehat{\varphi}_i)_{i \in I}$ in local parameters $\widehat{z}_i$ of the analytical structure on $\widehat{X}$ such that if $P_i = p(\widehat{P}_i)$, then we have :

$$\widehat{\varphi}_i(\widehat{z}_i)d\widehat{z}_i^2 = \widehat{\varphi}_j(\widehat{z}_j)d\widehat{z}_j^2 = \varphi_i(z_i)dz_i^2, \tag{8}$$

for every parametric values $\widehat{z}_i$ and $\widehat{z}_j$ which correspond to the same point $\widehat{P}$ of $\widehat{X}$, where $\widehat{\varphi}_i$ and $\widehat{\varphi}_j$ are the local representations of $\widehat{\Phi}$ in parameter $\widehat{z}_i$, respectively $\widehat{z}_j$, and $\varphi_i$ is the local representation of $\Phi$ in $z_i$, the image of $P_i$ by the associated map.

Let $Q^2(\widehat{X})$ be the vectorial space of the meromorphic quadratic differentials on $\widehat{X}$, over $\mathbf{C}$.

Theorem 6.There is an isomorphism $\widetilde{p}$, *between the vectorial spaces* $Q^2(\mathcal{O}_2)$ *and* $Q^2(\widehat{X})$.

Proof. Let $\Phi \in Q^2(\mathcal{O}_2)$, $\widehat{P} \in \widehat{X}$ and $(\widehat{U}, \widehat{h}, V)$ a map of $\widehat{X}$ such that $\widehat{P} \in \widehat{U}$ and $\widehat{h}(\widehat{P}) = \widehat{z}$. We consider $U = p(\widehat{U})$. Then $p/\widehat{U} : \widehat{U} \longrightarrow U$ is an analytical homeomorphism and if $T \in \mathcal{G}_1$, $(p/T(\widehat{U}))(T(\widehat{U})) = U$ and if $S \in \mathcal{G}\backslash\mathcal{G}_1$, $(p/S(\widehat{U}))(S(\widehat{U})) = \mathbf{k}(U)$. Let $P = p(\widehat{P}) = p(T(\widehat{P}))$. Then we can define $\widehat{\Phi} \in Q^2(\widehat{X})$ with $\widehat{\Phi}^*/\widehat{U} = \widehat{\varphi}(\widehat{z})d\widehat{z}^2$ the local representation of $\widehat{\Phi}$ on $\widehat{U}$, such that $\widehat{\varphi}/V \circ \widehat{h} \circ (p/\widehat{U})^{-1} = \varphi/V \circ \mathbf{p}_{T,U}^{-1}$. Because $\Phi \in Q^2(\mathcal{O}_2)$, it results that $\widehat{\varphi}/V$ is a meromorphic function and by (6) we obtain that $\widehat{\varphi}(\widehat{z})d\widehat{z}^2 = \varphi(z)dz^2 = \widehat{\varphi}_0(\widehat{z}_0))d\widehat{z}_0^2$, for every $\widehat{z}_0$ and $\widehat{z}$ parametric values associated to the same point of $\widehat{X}$ , where $\widehat{\varphi}_0$ is the local representation of $\widehat{\Phi}$ in the parameter $\widehat{z}_0$. Therefore $\widetilde{p} : Q^2(\mathcal{O}_2) \to Q^2(\widehat{X})$, $\widetilde{p}(\Phi) = \widehat{\Phi}$ is well define. Evident $\widetilde{p}$ is an isomorphism.

Let $Q^2(\mathcal{G}) = \{\widehat{\Phi} + \widehat{\Phi} \circ S \mid \widehat{\Phi} \in Q^2(\widehat{X}),\ S \in \mathcal{G}\backslash\mathcal{G}_1\}$ the vectorial space of the meromorphic quadratic differential on $\widehat{X}$ associated to the group $\mathcal{G}$, with respect to addition and multiplication with scalars, where $\widehat{\Phi} \circ S$ is the meromorphic quadratic differential , that is the family $(\widehat{\varphi}_S)_{S\in\mathcal{G}\backslash\mathcal{G}_1}$ of meromorphic functions, where $\widehat{\varphi}_S$ is the representation of $\widehat{\Phi}$ in parameter $\widehat{z}_S = (\widehat{h}_{S,\widehat{U}} \circ S \circ \widehat{h}^{-1})(\widehat{z})$ and $S \in \mathcal{G}\backslash\mathcal{G}_1$.

Remark.*The definition of* $\widehat{\Phi} + \widehat{\Phi} \circ S$ *does not depend of* $S \in \mathcal{G}\backslash\mathcal{G}_1$.

Proof. Let $\widehat{\Phi} \in Q^2(\widehat{X})$, $\widehat{Q} = S(\widehat{P})$, $\widehat{P} \in \widehat{U}$, $\widehat{\varphi}(\widehat{w})d\widehat{w}^2$ be the local representation of $\widehat{\Phi}$ in $S(\widehat{U})$ and $S' \in \mathcal{G}\backslash\mathcal{G}_1$, $S' \neq S$ such that $\widehat{Q} = S'(\widehat{P})$. Then $\widehat{Q}' = (S' \circ S^{-1})\ (\widehat{Q})$ and because $S' \circ S^{-1} \in \mathcal{G}_1$ it results that

$$\widehat{\varphi}(\widehat{w})d\widehat{w}^2 = \widehat{\varphi}'(\widehat{h}'(\widehat{w}))d\widehat{h}'(\widehat{w})^2 = \widehat{\varphi}'(\widehat{w}')d\widehat{w}'^2,$$

where $\widehat{\varphi}'$ is the local representation of $\widehat{\Phi}$ in $S'(\widehat{U})$ and $\widehat{h}' = \widehat{h}_{S',\widehat{U}} \circ S' \circ S^{-1} \circ \widehat{h}_{S,\widehat{U}}^{-1}$.

**Proposition 2.1.** *The elements of* $Q^2(\mathcal{G})$ *are automorphic with respect to* $\mathcal{G}$.

*The elements of* $Q^2(\mathcal{G})$ *are automorphic with respect to* $\mathcal{G}$.

Proof. Let $\widehat{\Phi} + \widehat{\Phi} \circ S \in Q^2(\mathcal{G})$. By the previous remark $\widehat{\Phi} + \widehat{\Phi} \circ S$ doesn't depend on choosing $S$. If $T \in \mathcal{G}_1$, using analogue notations with the above one it results that $\widehat{\Phi}^*/\widehat{U} + (\widehat{\Phi} \circ S)^*/\widehat{U} = \widehat{\varphi}(\widehat{z})d\widehat{z}^2 + \widehat{\varphi}_S(\widehat{z}_S))d\widehat{z}_S^2 = \widehat{\varphi}_T(\widehat{z}_T)d\widehat{z}_T{}^2 + \widehat{\varphi}_{S\circ T}(\widehat{z}_{ST})d\widehat{z}_{ST}{}^2 = (\widehat{\Phi} \circ T)^*/\widehat{U} + (\widehat{\Phi} \circ S \circ T)^*/\widehat{U}$. If $S' \in \mathcal{G}\backslash\mathcal{G}_1$, then $\widehat{\Phi}^*/\widehat{U} + (\widehat{\Phi} \circ S)^*/\widehat{U} = \widehat{\varphi}(\widehat{z})d\widehat{z}^2 + \widehat{\varphi}_S(\widehat{z}_S)d\widehat{z}_S{}^2 = \widehat{\varphi}_{S\circ S'}(\widehat{z}_{S\circ S'})d\widehat{z}_{S\circ S'}^2 + \widehat{\varphi}_{S'}(\widehat{z}_{S'})d\widehat{z}_{S'}^2 = (\widehat{\Phi} \circ (S \circ S')^*/\widehat{U} + (\widehat{\Phi} \circ S')^*/\widehat{U}$. We used the local representations of $\widehat{\Phi}$ and the fact that $S' \circ S^{-1} \in \mathcal{G}_1$ and $S \circ S' \in \mathcal{G}_1$.

**Theorem 2.2.** *There is an isomorphism between the vectorial spaces $Q^2(\widehat{X})$ and $Q^2(\mathcal{G})$.*

*There is an isomorphism between the vectorial spaces $Q^2(\widehat{X})$ and $Q^2(\mathcal{G})$.*

*Proof.* et $\widetilde{F} : Q^2(\widehat{X}) \longrightarrow Q^2(\mathcal{G})$ be such that : $\widetilde{F}(\widehat{\Phi}) = \widehat{\Phi} + \widehat{\Phi} \circ S$, for every $\widehat{\Phi} \in Q^2(\widehat{X})$. By the above remark , we deduce that $\widetilde{F}$ is well defined. Evident $\widetilde{F}$ is an isomorphism. □

**Remark 2.1.** *By the previous isomorphisms, there is an isomorphism $\widetilde{\Pi}$ between the vectorial spaces $Q^2(\mathcal{G})$ and $Q^2(X)$ such that the following diagram is commutative..*

$$\begin{array}{ccccc} Q^2(\widehat{X}) & & \stackrel{\widetilde{F}}{\longrightarrow} & & Q^2(\mathcal{G}) \\ \widetilde{p}\uparrow & & & & \downarrow \widetilde{\Pi} \\ Q^2(\mathcal{O}_2) & \stackrel{\widetilde{\mathbf{K}}}{\longrightarrow} & Q^s(\mathcal{O}_2) & \stackrel{\widetilde{q}}{\longrightarrow} & Q^2(X) \end{array}$$

By the previous isomorphisms, there is an isomorphism $\widetilde{\Pi}$ between the vectorial spaces $Q^2(\mathcal{G})$ and $Q^2(X)$ such that the following diagram is comutative.

$$\begin{array}{ccccc} Q^2(\widehat{X}) & & \stackrel{\widetilde{F}}{\longrightarrow} & & Q^2(\mathcal{G}) \\ \widetilde{p}\uparrow & & & & \downarrow \widetilde{\Pi} \\ Q^2(\mathcal{O}_2) & \stackrel{\widetilde{\mathbf{K}}}{\longrightarrow} & Q^s(\mathcal{O}_2) & \stackrel{\widetilde{q}}{\longrightarrow} & Q^2(X) \end{array}$$

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(Monica Roşiu) Department of Mathematics, University of Craiova,
Al.I. Cuza Street, No. 13, Craiova RO-200585, Romania, Tel. & Fax: 40-251412673

(Peter Kessler) Department of Mathematics, University of Craiova,
Al.I. Cuza Street, No. 13, Craiova RO-200585, Romania, Tel. & Fax: 40-251412673