

# A Random Graph Representation for Green's Functions

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**ABSTRACT.** The method of Feynman graphons is applied to formulate random graph representations for divergent perturbative series of higher loop order Feynman diagrams which contribute to 1PI Green's functions and their corresponding fixed point equations in a quantum field theory.

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## 1. Introduction

A quantum field theory is a special relativistic platform for the study of interacting physical systems with infinite degrees of freedom where quantum fields describe fermions and bosons which can be created or annihilated. The method of Feynman path integral is applied to formulate a quantum field theory in terms of towers of a certain class of  $n$ -correlated functions such that the interaction part of the Lagrangian generates (divergent) perturbative series of powers of coupling constants together with momentum integrals as their coefficients [26]. These perturbative series are known as Green's functions [40, 41, 42] such that Feynman rules enable us to combinatorially represent them in terms of formal expansions of higher loop order Feynman diagrams [28, 29, 33]. The power of coupling constant in each terms of these expansions determines the loop number of 1PI Feynman diagrams as the coefficients of that term where the computational complexities of 1PI Green's functions increase in terms of growing the number of higher loop order Feynman diagrams in each term [35, 55]. The most difficult challenge is dealing with divergent perturbative series which contribute to 1PI Green's functions and solutions of their fixed point equations under strong coupling constants. In this situation, perturbation theory fails to generate any information and we need to apply non-perturbative methods.

The *large  $N$  method* modifies the validity domain of the approximations of reducing an interacting physical theory to a free fermion theory with self-consistently determined parameters where for fields with  $N$  components, in the large  $N$  limit composite fields could have small fluctuations. In this setting, an effective scalar theory should be constructed where by integrating out the degrees of freedom, an interacting theory with  $O(N)$  or  $U(N)$  symmetries on large  $N$  limit and in  $1/N$  expansion are expected to be solved. The *Borel resummation method* replaces divergent perturbative series with Borel sums which might have singularities on the positive real axis in the Borel

plane. The *lattice methods* describe passing from perturbation to non-perturbation regimes on the basis of the continuum limit of discrete approximations of space-time generated by finite lattices such that their number of nodes tend to infinity. The *AdS/CFT correspondence* applies the assumption of the zero beta function to describe non-perturbative quantum field theory in terms of perturbative string theory [17, 23, 24, 25, 26, 28, 31, 34, 37, 38, 39, 40, 41, 42, 55, 56].

Graphons, as a class of dense graphs, are combinatorial tools for the presentation of *graph limits* of sequences of finite weighted graphs with increasing vertex sets. The theory of graphons has been developed in terms of combinatorial and measure theoretic settings to search for random nature of models of complex networks in Computer Science and Applied Mathematics [1, 5, 6, 7, 10, 11, 12, 20, 36]. In this regard, the method of *Feynman graphons*, which formulates graphon representations for graph limits of sequences of Feynman diagrams with increasing loop orders [43, 48, 50], assigns a statistical model to non-perturbative domain of a quantum field theory [44]. The Banach space of stretched Feynman graphons led us to define the notion of *Kolmogorov complexity* for solutions of fixed point equations of (1PI) Green's functions in a quantum field theory [47, 51]. This research work proceeds the method of Feynman graphons to formulate processes of graphon / random graphs which restore information from divergent perturbative series which contribute to 1PI Green's functions and their corresponding fixed point equations in a quantum field theory.

**1.1. Quantum field theory via renormalization Hopf algebra.** Quantum fields are operator-valued distributions which cannot be evaluated at a single point and they do make sense over regions of space-time. Localized measurements in regions of space-time give information of quantum fields. The  $N$ -point correlation functions are given by

$$G_n(x_1, \dots, x_n) = \frac{\int e^{\frac{iS[\phi]}{\hbar}} \phi(x_1) \dots \phi(x_n) \mathcal{D}[\phi]}{\int e^{\frac{iS[\phi]}{\hbar}} \mathcal{D}[\phi]} \quad (1)$$

such that  $S[\phi] = S_0[\phi] + cS_{\text{int}}[\phi]$  is the time-integral of the Lagrangian with the interaction part  $S_{\text{int}}[\phi]$  and the (running) coupling constant  $c$ . The infinite dimensional Lebesgue–Feynman measure  $\mathcal{D}$  is a measure on the space of all possible trajectories, which can be traveled by an elementary particle, between the initial state at position  $x_1$  and time  $t_1$  and the final state at position  $x_n$  and time  $t_n$ . The Taylor expansions of  $e^{\frac{iS[\phi]}{\hbar}}$  in (1) generate perturbative series of powers of coupling constants together with iterated ill-defined momentum integrals, namely (1PI) *Green's functions*. Feynman rules replace these perturbative series of integrals with polynomials of powers of coupling constants with higher loop order Feynman diagrams as their coefficients where Feynman diagrams encode interactions of quantum fields in a bounded region of space-time [26, 40, 41].

**Definition 1.1.** • A Feynman diagram  $\Gamma$  is a space-time diagram. It is a finite decorated weighted graph with a vertex set  $\Gamma^{[0]}$  and an edge set  $\Gamma^{[1]}$ . Edges in  $\Gamma^{[1]}$  represent elementary particles such that vertices in  $\Gamma^{[0]}$  represent interactions. The subset  $\Gamma_{\text{ext}}^{[1]} \subseteq \Gamma^{[1]}$  of external edges, which contains edges with beginning or ending vertex, represents elementary particles with the assigned momenta. Each external edge is a half-edge. The subset  $\Gamma_{\text{int}}^{[1]} \subseteq \Gamma^{[1]}$  of internal edges, which contains edges with beginning and ending vertices, represents virtual particles.

Each internal edge is a pair of two half-edges. The law of the conservation of momenta is valid in each vertex of  $\Gamma$  and for the whole graph.

- $\Gamma$  is called 1PI, if it remains a connected graph after removing a single edge.
- A subgraph  $\gamma$  in  $\Gamma$  is a graph such that  $\gamma_{\text{int}}^{[1]} \subseteq \Gamma_{\text{int}}^{[1]}$ ,  $\gamma^{[0]} \subseteq \Gamma^{[0]}$ , and for each  $v \in \gamma^{[0]}$ ,

$$\text{res}_\gamma(v) := \{e \in \gamma^{[1]} : v \in e\} = \{e' \in \Gamma^{[1]} : v \in e'\} := \text{res}_\Gamma(v) \quad (2)$$

[29, 30, 31, 32].

The momentum integral corresponding to a Feynman diagram could have (nested) divergences in terms of the momentum variable which varies from zero to infinity. These divergences are represented by nested loops in Feynman diagram. The Bogoliouov–Zimmermann’s forest formula provides a recursive process to remove these loops where the regularization techniques such as dimensional regularization (i.e.  $D \mapsto D - z$ ) or momentum cut-off (i.e.  $0 \mapsto \Lambda$ ) enable us to replace divergent sub-integrals with some analytic series. Then a renormalization schemes such as minimal subtraction (i.e.  $\sum_{n \geq -m} a_n z^n \mapsto \sum_{n=-m}^{-1} a_n z^n$ ) is required to extract finite renormalized values [3, 21, 22, 62].

The BPHZ perturbative renormalization has been reformulated in terms of the theory of Hopf algebras and Riemann–Hilbert problem [15, 16] where the space of Feynman diagrams of a quantum field theory  $\Phi$  can be equipped with a graded commutative non-cocommutative Hopf algebra  $H_{\text{FG}}(\Phi) = \bigoplus_{n \geq 0} H_{(n)}(\Phi)$  over the field  $\mathbb{Q}$  or  $\mathbb{R}$  [3, 32]. This Hopf algebra, known as Connes–Kreimer renormalization Hopf algebra, is free generated by 1PI Feynman diagrams and graded by loop number. For each  $n$ ,  $H_{(n)}(\Phi)$  is the vector space of 1PI  $n$ -loop Feynman diagrams and products of 1PI Feynman diagrams with the overall loop number  $n$ . For each  $\Gamma \in H_{\text{FG}}(\Phi)$ , its coproduct is given by

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma} \gamma \otimes \Gamma/\gamma \quad (3)$$

such that the sum is taken over all disjoint unions of superficially divergent subgraphs of  $\Gamma$  called *Feynman subdiagrams*. The insertion operator  $\star$  on the space of Feynman diagrams recovers the gluing of factorized components generated by the coproduct (3). For 1PI Feynman diagrams  $\Gamma_1, \Gamma_2$ , the insertion of  $\Gamma_2$  into  $\Gamma_1$ , presented by  $\Gamma_1 \star \Gamma_2$ , is the sum over all possible bijections between sets

$$\text{res}_{\Gamma_1}(v) = \{e \in \Gamma_1^{[1]} : v \in e\}, v \in \Gamma_1^{[0]} \quad (4)$$

and  $\Gamma_{2,\text{ext}}^{[1]}$  such that topologically different graphs are generated once. The insertion operator, as a bilinear map from 1PI Feynman diagrams to 1PI Feynman diagrams, defines a pre-Lie algebra structure on the space of Feynman diagrams in  $\Phi$ . The commutator

$$[\Gamma_1, \Gamma_2] := \Gamma_1 \star \Gamma_2 - \Gamma_2 \star \Gamma_1 \quad (5)$$

defines a graded Lie algebra structure  $\mathcal{L}_\Phi$  such that the graded dual of the universal enveloping algebra of  $\mathcal{L}_\Phi$  reconstructs the renormalization Hopf algebra  $H_{\text{FG}}(\Phi)$  [3, 29, 30, 32].

Feynman rules of  $\Phi$  is encoded by a certain class of elements of the complex Lie group  $\mathbb{G}_\Phi(A_{\text{dr}}) = \text{Hom}(H_{\text{FG}}(\Phi), A_{\text{dr}})$  such that  $A_{\text{dr}}$  is the algebra of Laurent series with finite pole parts [4]. Let  $\Delta^*$  be an infinitesimal punctured disk around  $z = 0$  in

the complex plane. The dimensional regularization replaces a Feynman rules character  $\varphi$  to  $\varphi^z$  which sends each Feynman diagram  $\Gamma$  to  $\varphi^z(\Gamma) = I_\Gamma^z \in A_{\text{dr}}$ . Thanks to the renormalization coproduct (3), and the existence of the Birkhoff factorization  $(\varphi_-^z, \varphi_+^z)$  for  $\varphi^z$ , the BPHZ renormalization of  $\Gamma$  is reformulated by the recursive equations

$$\varphi_-^z(\Gamma) = -R_{\text{ms}} \left( \varphi^z(\Gamma) + \sum_{\gamma} \varphi_-^z(\gamma) \varphi^z(\Gamma/\gamma) \right), \quad (6)$$

$$\varphi_+^z(\Gamma) = \varphi^z(\Gamma) + \varphi_-^z(\Gamma) + \sum_{\gamma} \varphi_-^z(\gamma) \varphi^z(\Gamma/\gamma) \quad (7)$$

such that  $\varphi^z = (\varphi_-^z)^{*^{-1}} * \varphi_+^z$ ,  $\varphi_-^z : H_{\text{FG}}(\Phi) \rightarrow A_{\text{dr},-}$ ,  $\varphi_+^z : H_{\text{FG}}(\Phi) \rightarrow A_{\text{dr},+}$ ,  $A_{\text{dr}} = A_{\text{dr},-} \oplus A_{\text{dr},+}$ , and  $R_{\text{ms}} : A_{\text{dr}} \rightarrow A_{\text{dr}}$  is the minimal subtraction map which sends each Laurent series onto its pole parts. The convolution coproduct  $*$  is defined in terms of the renormalization coproduct (3), namely

$$\begin{aligned} (\varphi_-^z)^{*^{-1}} * \varphi_+^z(\Gamma) &= \\ &= (\varphi_-^z)^{*^{-1}}(\Gamma) \otimes \varphi_+^z(\mathbb{I}) + (\varphi_-^z)^{*^{-1}}(\mathbb{I}) \otimes \varphi_+^z(\Gamma) + \sum_{\gamma} (\varphi_-^z)^{*^{-1}}(\gamma) \varphi_+^z(\Gamma/\gamma) \end{aligned} \quad (8)$$

[3, 15, 16, 32].

A gauge field theory, as a quantum field theory with a Lagrangian invariant under a symmetry group, is formulated on the basis of a certain class of identities between its Feynman diagrams, namely Ward–Takahashi or Slavnov–Taylor identities for abelian or non-abelian gauge field theories, respectively [24, 61]. The Ward–Takahashi or Slavnov–Taylor elements generate Hopf ideals  $I_{\text{WT}}$  or  $I_{\text{ST}}$  of the renormalization Hopf algebra. The compatibility of quantum gauge symmetries with the Hopf algebraic renormalization is encoded by the quotient Hopf algebras  $H_{\text{FG}}(\text{QED})/I_{\text{WT}}$  or  $H_{\text{FG}}(\text{QCD})/I_{\text{ST}}$  [58, 59]. In addition, fixed point equations of (1PI) Green's functions in the physical theory, which recover quantum motions, can be reformulated in terms of a certain class of recursive Hochschild equations [3, 29, 31] such that their compatibility with the BPHZ renormalization have been recovered by the renormalization Hopf algebra [47, 52, 53, 54].

The combinatorial representation of the renormalization Hopf algebra has been formulated in terms of the Connes–Kreimer Hopf algebra of non-planar rooted trees  $H_{\text{CK}}$  defined on the polynomial algebra of non-planar rooted trees. This is a free commutative non-cocommutative Hopf algebra which is graded by the vertex number. Its coproduct is given by

$$\Delta_{\text{CK}}(t) = t \otimes 1 + 1 \otimes t + \sum_{c_t} R_{c_t} \otimes P_{c_t} \quad (9)$$

such that the sum is taken over all non-trivial admissible cuts in  $t$ . An *admissible cut*  $c_t$  is a collection of edges of  $t$  such that any path between the root of  $t$  and each of its (vertices or) leaves has at most one element of  $c_t$ . An admissible cut  $c_t$  divides  $t$  into two separated parts. The subtree  $R_{c_t}$  has the root of  $t$  and  $P_{c_t}$  is a forest of subtrees as the rest of  $t$ . The grafting operator  $B^+$  recovers the recursive nature of  $\Delta_{\text{CK}}$ . It sends a forest of trees  $t_1 \dots t_n$  to a new non-planar rooted tree  $t$  by adding a new vertex  $r_t$  as the root of  $t$  and  $n$  edges between  $r_t$  and roots  $r_{t_1}, \dots, r_{t_n}$ . The pair  $(H_{\text{CK}}, B^+)$  is the universal object in a category of pairs  $(H, L_H)$  of (a commutative

Hopf algebra, a Hochschild one-cocycle). This universal object recovers the BPHZ perturbative renormalization independent of physical theories [3, 21, 22, 30, 32].

**1.2. From topological completion of the space of graphs to Feynman graphons.** Graphons, as generalizations of graphs, are weighted graphs with the vertex set  $[0, 1]$  or  $\Omega \subseteq [0, \infty)$ . They have been introduced and developed to provide representations for the graph limits of sequences of finite weighted dense or sparse graphs in terms of a certain class of metrics [14, 19, 36]. For the Lebesgue measure space  $([0, 1], m)$ , a *graphon* is a Lebesgue measurable symmetric function  $W : [0, 1] \times [0, 1] \rightarrow [0, 1]$ . For any invertible  $m$ -measure preserving transformation  $\rho$  on  $[0, 1]$ ,  $W^\rho(x, y) = W(\rho(x), \rho(y))$  is a *labeled graphon*. Graphons  $W_1, W_2$  are called *weakly isomorphic* (i.e.  $W_1 \approx W_2$ ) iff there exist Lebesgue measure preserving transformations  $\rho_1, \rho_2$  on  $[0, 1]$  and a graphon  $W$  such that  $W_1 = W^{\rho_1}$  and  $W_2 = W^{\rho_2}$  almost everywhere with respect to the Lebesgue measure. The function

$$\|W\|_{\text{cut}} := \sup_{A, B \subset [0, 1]} \left| \int_{A \times B} W(x, y) dx dy \right| \quad (10)$$

defines a semi-norm on the space of graphons such that the supremum is taken over Lebesgue measurable subsets  $A, B$  of  $[0, 1]$ . It determines a pseudo-metric

$$d_{\text{cut}}(V, W) := \inf_{\tau, \psi} \|V^\tau - W^\psi\|_{\text{cut}} \quad (11)$$

such that the infimum is taken over invertible Lebesgue measure preserving transformations on  $[0, 1]$ . Up to the weakly isomorphic relation,  $d_{\text{cut}}$  is a metric on the space of weakly isomorphic classes

$$[W]_{\approx} := \{W^\rho : \rho \in \mathcal{A}_{[0, 1], m}\} \quad (12)$$

such that  $\mathcal{A}_{[0, 1], m}$  is the semi-group of Lebesgue measure preserving transformations on  $[0, 1]$ . The quotient space  $\mathcal{W}_{\approx}^{([0, 1], m)}([0, 1])$  of these weakly isomorphic classes is a compact complete Hausdorff metric space. The graph limits of Cauchy sequences of finite weighted dense graphs with respect to the metric (11) are represented by elements in  $\mathcal{W}_{\approx}^{([0, 1], m)}([0, 1])$  [14, 19, 36].

Graph limits of Cauchy sequences of sparse graphs with respect to the metric (11) converges to the zero graphon. It is possible to generate non-zero graph limits in terms of rescaling or stretching processes. One way is formulating stretched graphons on a  $\sigma$ -finite measure space  $(\Omega \subseteq [0, \infty), \mu)$  which are bounded symmetric  $\mu$ -measurable real valued functions defined on  $\Omega \times \Omega$ . The quotient space  $\mathcal{W}_{\approx}^{(\Omega \subseteq [0, \infty), \mu)}(\mathbb{R})$  is a complete Hausdorff metric space. For a given graphon  $W \in \mathcal{W}^{([0, 1], m)}([0, 1])$ , its corresponding stretched / rescaled versions are given by

$$W_s(x, y) = \begin{cases} W\left(\frac{x}{\|W\|_1^{1/2}}, \frac{y}{\|W\|_1^{1/2}}\right), & 0 \leq x, y \leq \|W\|_1^{1/2} \\ 0, & \text{otherwise} \end{cases}, \quad (13)$$

$W_r = \frac{W}{\|W\|_{\text{cut}}}$  or  $W_r = \frac{W}{\|W\|_p}$  such that

$$\|W\|_p = \left( \int_{\Omega \times \Omega} |W(x, y)|^p dx dy \right)^{1/p}, \quad p \geq 1 \quad (14)$$

[6, 8, 9, 12].

Consider the renormalization Hopf algebra of Feynman diagrams  $H_{\text{FG}}(\Phi)$  of a quantum field theory  $\Phi$ . There exists an injective homomorphism of Hopf algebras from  $H_{\text{FG}}(\Phi)$  to  $H_{\text{CK}}(\Phi)$  such that the vertex set of each non-planar rooted tree is decorated by (1PI) primitive Feynman diagrams. For each Feynman diagram  $\Gamma$  without overlapping loop and with the loop number  $|\Gamma| = n$ ,  $t_\Gamma$  is a decorated non-planar rooted tree with  $n$  vertices. If  $\Gamma$  has overlapping loops, then  $t_\Gamma$  is a linear combination of decorated non-planar rooted trees [3, 30, 32].

**Definition 1.2.** • For the Lebesgue measure space  $([0, 1], m)$ , the adjacency matrix of  $t_\Gamma$ ,  $|t_\Gamma| = n$ , defines the symmetric Lebesgue measurable function

$$P_{t_\Gamma}^\sigma : [0, 1] \times [0, 1] \rightarrow [0, 1], (x, y) \in I_i \times I_j \mapsto a_{ij} \in \text{Ad}(t_\Gamma) \quad (15)$$

such that  $\sigma = \{I_1, \dots, I_n\}$  is a partition of  $[0, 1]$ . If  $m(I_i) = \frac{1}{n}$  for  $1 \leq i \leq n$ , then  $P_{t_\Gamma}^\sigma$  is called the canonical Feynman graphon associated to  $\Gamma$ .

- For any invertible Lebesgue measure preserving transformation  $\rho$  on  $[0, 1]$ ,  $P_{t_\Gamma}^{\sigma, \rho}$  is called a labeled Feynman graphon associated to  $\Gamma$ . Up to the weakly isomorphic relation,  $W_\Gamma := [P_{t_\Gamma}^\sigma]_\approx$  is the unique Feynman graphon associated to  $\Gamma$  [50, 49].

Let there exist additional weighting information  $\{\alpha_i\}_{s=1}^n$ ,  $\alpha_s \in \mathbb{R}$ , on the vertex set of  $t_\Gamma$ , and additional weighting information  $\{\beta_{ij}\}$ ,  $\beta_{ij} \in \mathbb{R}$ ,  $\beta_{ij} = \beta_{ji}$ , on the edge set of  $t_\Gamma$ . We replace the partition  $\sigma$  with a partition  $\sigma' = \{J_1, \dots, J_n\}$  for  $[0, 1]$  with  $m(J_s) = \frac{|\alpha_s|}{\sum_{i=1}^n |\alpha_i|}$ , and define the function

$$P'_{t_\Gamma, \sigma'} : [0, 1] \times [0, 1] \rightarrow \mathbb{R}, (x, y) \in I_i \times I_j \mapsto \beta_{ij} a_{ij}. \quad (16)$$

Up to the weakly isomorphic relation,  $W_\Gamma := [P'_{t_\Gamma, \sigma'}]_\approx$  is the unique Feynman graphon associated to  $\Gamma$ . For a  $\sigma$ -finite measure space  $(\Omega \subseteq [0, \infty), \mu)$ ,  $W_\Gamma$  is called *stretched Feynman graphon* associated to  $\Gamma$  [43, 48, 50, 49].

**1.3. Achievement.** The topological Hopf algebra of renormalization is considered to associate a category of random graphs to a quantum field theory. It leads us to assign renormalized values to graph limits of sequences of higher loop order Feynman diagrams in terms of processes of random graphs. From the perspective of the topological Hopf algebra of renormalization, 1PI Green's functions and their corresponding fixed point equations can be described as the graph limits of Cauchy sequences of finite partial expansions of higher loop order 1PI Feynman diagrams with respect to the cut-distance topology or its  $L^p$ -modifications for  $p \geq 1$  [43, 45, 47, 49]. We proceed this setting to associate processes of random graphs to divergent perturbative series which contribute to 1PI Green's functions and their corresponding fixed point equations. The homomorphism density of Feynman graphons, as continuous functionals on the space of stretched Feynman graphons associated to Feynman diagrams of a quantum field theory, are applied to associate expectation values, in the context of Fisher information, to solutions of fixed point equations of 1PI Green's functions.

## 2. Topological Hopf algebra of renormalization

The renormalization Hopf algebra is not rich enough to recover the renormalization of 1PI Green's functions and solutions of their fixed point equations under strong (running) coupling constants. A certain class of graphons, namely (stretched) Feynman

graphons, enables us to study 1PI Green's functions as graph limits for sequences of higher loop order Feynman diagrams with increasing loop numbers with respect to the metric (11). This graph limit setting can recover a non-perturbative generalization of the BPHZ program [47, 48, 49, 52].

Let  $\mathcal{S}_{\text{graphon}, \approx}^{(\Omega \subseteq [0, \infty), \mu), \Phi}(\mathbb{R}) \subset \mathcal{W}_{\approx}^{(\Omega \subseteq [0, \infty), \mu)}(\mathbb{R})$  be the subspace of stretched Feynman graphons associated to Feynman diagrams in  $\Phi$ . Up to the weakly isomorphic relation with respect to  $\mu$ -measure preserving transformations, it is a complete Hausdorff topological space with respect to the metric structure

$$d_{\text{cut}}(W_{\Gamma_1}, W_{\Gamma_2}) = \inf_{\rho_1, \rho_2} \sup_{A, B \subseteq \Omega} \left| \int_{A \times B} \left( W_{\Gamma_1}^{\rho_1}(x, y) - W_{\Gamma_2}^{\rho_2}(x, y) \right) dx dy \right|. \quad (17)$$

The subspace  $\mathcal{S}_{\text{graphon}, \approx}^{([0, 1], m), \Phi}([0, 1]) \subset \mathcal{W}_{\approx}^{([0, 1], m)}([0, 1])$  is a compact complete Hausdorff metric space [48, 50].

Stretched Feynman graphons in  $\mathcal{S}_{\text{graphon}, \approx}^{(\Omega \subseteq [0, \infty), m), \Phi}(\mathbb{R})$  can be projected onto their corresponding Feynman graphons in  $\mathcal{S}_{\text{graphon}, \approx}^{([0, 1], m), \Phi}(\mathbb{R})$  by applying Lebesgue measure preserving and affine transformations on the real line. For  $p \geq 1$ , elements of  $\mathcal{S}_{\text{graphon}, \approx}^{([0, 1], m), \Phi}(\mathbb{R})$  are also  $L^p$ -Feynman graphons which means that

$$\|W_{\Gamma}\|_p = \left( \int_{[0, 1] \times [0, 1]} |W(x, y)|^p dx dy \right)^{1/p} < \infty, \quad (18)$$

and the distance function

$$d_{p, \text{cut}}(W_{\Gamma_1}, W_{\Gamma_2}) = \inf_{\rho_1, \rho_2} \|W_{\Gamma_1}^{\rho_1} - W_{\Gamma_2}^{\rho_2}\|_p \quad (19)$$

defines a complete Hausdorff metric space  $\mathcal{S}_{\text{graphon}, \approx, p}^{([0, 1], m), \Phi}(\mathbb{R})$ . It leads us to define a new distance function on the space of Feynman diagrams in  $\Phi$  given by

$$d_{\text{cut}}(\Gamma_1, \Gamma_2) := d_{\text{cut}}(W_{\Gamma_1}, W_{\Gamma_2}) \text{ or} \quad (20)$$

$$d_{p, \text{cut}}(\Gamma_1, \Gamma_2) := d_{p, \text{cut}}(W_{\Gamma_1}, W_{\Gamma_2}) \quad (21)$$

[8, 9, 48].

**Lemma 2.1.** *There exists a category of random graphs associated to the space of Feynman diagrams in a quantum field theory.*

*Proof.* Let  $\Gamma$  be a 1PI Feynman diagram without overlapping loops, the loop number  $|\Gamma| = n_{\Gamma}$ , the rooted tree representation  $t_{\Gamma}$  and Feynman graphon representation  $W_{\Gamma} \in \mathcal{S}_{\text{graphon}, \approx}^{([0, 1], m), \Phi}(\mathbb{R})$ . The vertex set  $V(t_{\Gamma})$  as an ordered set can be projected onto a subset  $S_{\Gamma}^{\theta} \subset [0, 1]$  with  $n_{\Gamma} = |V(t_{\Gamma})|$  nodes by an embedding  $\theta$ . Define a new random graph  $R_{n_{\Gamma}, W_{\Gamma}}$  with the vertex set  $S_{\Gamma}^{\theta}$  such that with the probability  $\frac{1}{\|W_{\Gamma}\|_{\text{cut}}} |W_{\Gamma}(x_i, x_j)|$ , there exists an edge between  $x_i$  and  $x_j$  in  $R_{n_{\Gamma}, W_{\Gamma}}$ .

Thanks to Sampling Lemmas for Graphs / Graphons [10, 11, 12, 36], with the probability  $1 - \exp\left(\frac{-n_{\Gamma}}{2 \log n_{\Gamma}}\right)$ , we have

$$d_{\text{cut}}(t_{\Gamma}, R_{n_{\Gamma}, W_{\Gamma}}) = d_{\text{cut}}(W_{\Gamma}, W_{R_{n_{\Gamma}, W_{\Gamma}}}) \leq \frac{20}{\sqrt{\log n_{\Gamma}}}. \quad (22)$$

Therefore if the loop number  $|\Gamma| = \infty$ , then  $d_{\text{cut}}(t_{\Gamma}, R_{\infty, W_{\Gamma}}) = 0$  which means that  $t_{\Gamma}$  and  $R_{\infty, W_{\Gamma}}$  are weakly isomorphic.

Define a new category  $\mathcal{C}_{\text{random}}^\Phi$  such that its objects are random graphs  $R_{n\Gamma, W_\Gamma}$ . Homomorphisms of Feynman diagrams in  $\text{hom}(\Gamma_1, \Gamma_2)$  determine homomorphisms of random graphs in  $\text{mor}(R_{n\Gamma_1, W_{\Gamma_1}}, R_{n\Gamma_2, W_{\Gamma_2}}) \in \mathcal{C}_{\text{random}}^\Phi$ . If  $\text{hom}(\Gamma_1, \Gamma_2) = \emptyset$ , then  $\text{mor}(R_{n\Gamma_1, W_{\Gamma_1}}, R_{n\Gamma_2, W_{\Gamma_2}}) = \emptyset$ .  $\square$

**Definition 2.1.** A sequence  $\{\Gamma_n\}_n$  of Feynman diagrams in  $\Phi$  is convergent with respect to the metric (20) when  $n$  tends to infinity iff the sequence  $\{\frac{W_{\Gamma_n}}{\|W_{\Gamma_n}\|_{\text{cut}}}\}_n$  in the metric space  $\mathcal{S}_{\text{graphon}, \approx}^{(\Omega \subseteq [0, \infty), \mu), \Phi}(\mathbb{R})$  is convergent to a non-trivial stretched Feynman graphon with respect to the metric (17) when  $n$  tends to infinity.

Following Definition 2.1, a convergent sequence  $\{\Gamma_n\}_n$  of Feynman diagrams can be described in terms of the asymptotic densities of the sequence  $\{t_{\Gamma_n}\}_n$  of rooted tree representations and its graph limit  $t_\infty$  where rooted trees with different densities might have a similar structure. Following Definition 1.2,  $W \in [P_{t_\infty}]_{\approx}$  recovers the graph limit  $X$  for  $\{\Gamma_n\}_n$  which is called *large Feynman diagram* such that  $W_X$  is weakly isomorphic to  $W$ . In other words,  $[W_X]_{\approx} = [P_{t_\infty}]_{\approx}$ .

**Lemma 2.2.** (1) *Any sequence  $\{\Gamma_n\}_n$  of Feynman diagrams in  $\Phi$  with increasing loop number has a convergent subsequence with respect to the metric (20).*  
 (2) *The graph limit  $X$  of any Cauchy sequence  $\{\Gamma_n\}_n$  of Feynman diagrams in  $\Phi$  is representable by a random graph.*

*Proof.* 1. Following [19, 36, 47, 48, 50] leads us to show that for the sequence  $\{W_{\Gamma_n}\}_n$  in  $\mathcal{S}_{\text{graphon}, \approx}^{(\Omega \subseteq [0, \infty), \mu), \Phi}(\mathbb{R})$ , there exists a subsequence  $\{W_{\Gamma_{n_i}}\}_{i \geq 1}$  and a stretched Feynman graphon  $W$  such that

$$\lim_{i \rightarrow \infty} d_{\text{cut}}(W_{\Gamma_{n_i}}, W) = 0. \tag{23}$$

2. Thanks to Definition 2.1, we know that

$$\lim_{n \rightarrow \infty} \Gamma_n = X \Leftrightarrow \lim_{n \rightarrow \infty} \frac{W_{\Gamma_n}}{\|W_{\Gamma_n}\|_{\text{cut}}} = W_X \tag{24}$$

with respect to the metrics (17) and (20), respectively. Following proof of Lemma 2.1, for each  $n$  consider the random graph  $R_{\alpha_{\Gamma_n}, W_{\Gamma_n}}$  corresponding to the Feynman diagram  $\Gamma_n$  with the loop number  $\alpha_{\Gamma_n}$ . The vertex set  $V(t_{\Gamma_n})$  as an ordered set can be projected onto a subset  $S_{\Gamma_n}^{\theta_n} \subset \Omega \subseteq [0, \infty)$  with  $\alpha_{\Gamma_n} = |V(t_{\Gamma_n})|$  nodes by an embedding  $\theta_n$ . The random graph  $R_{\alpha_{\Gamma_n}, W_{\Gamma_n}}$  is defined on the vertex set  $S_{\Gamma_n}^{\theta_n}$  such that with the probability  $\frac{1}{\|W_{\Gamma_n}\|_{\text{cut}}} |W_{\Gamma_n}(x_{s_i}, x_{s_j})|$ , there exists an edge between  $x_{s_i}$  and  $x_{s_j}$ .

By tending  $n$  to infinity, the vertex set  $V(t_{\Gamma_\infty})$  as an ordered set can be projected onto a subset  $S_X^{\theta_\infty} \subset \Omega \subseteq [0, \infty)$  with infinite countable nodes by an embedding  $\theta_\infty = \sqcup_n \theta_n$ . The random graph  $R_{\infty, W_X}$  is defined on the vertex set  $S_X^{\theta_\infty}$  such that with the probability  $\frac{1}{\|W_X\|_{\text{cut}}} |W_X(x_{s_i}, x_{s_j})|$ , there exists an edge between  $x_{s_i}$  and  $x_{s_j}$ . Thanks to the inequality (22), we have  $d_{\text{cut}}(t_X, R_{\infty, W_X}) = 0$  which means that  $t_X$  is representable by  $R_{\infty, W_X}$ .  $\square$

The space  $\mathcal{S}_{\text{graphon}, \approx}^{(\Omega \subseteq [0, \infty), \mu), \Phi}(\mathbb{R})$ , as a closed subspace of  $\mathcal{W}_{\approx}^{(\Omega \subseteq [0, \infty), \mu)}(\mathbb{R})$ , is a complete Hausdorff metric space, and  $\mathcal{S}_{\text{graphon}, \approx}^{([0, 1], m), \Phi}([0, 1])$  is a compact complete Hausdorff metric space with respect to the distance function (17).

**Lemma 2.3.**  $\mathcal{S}_{\text{graphon}, \approx}^{(\Omega \subseteq [0, \infty), \mu), \Phi}(\mathbb{R})$  topologically completes  $H_{\text{FG}}(\Phi)$ .

*Proof.* Let  $\mu(\Omega) < \infty$  and  $\Gamma$  be a Feynman diagram with overlapping loops, the loop number  $|\Gamma| = n$  and the rooted forest representation  $t_\Gamma = c_1 s_1 + \dots + c_n s_n$  as a linear combination of non-planar rooted trees such that  $|s_j| = n$  for each  $1 \leq j \leq n$ . The stretched Feynman graphon  $W_\Gamma$  is defined as a certain direct sum  $W_\Gamma = W_{s_1} + \dots + W_{s_n}$ ,

$$W_\Gamma : \bigsqcup_{j=1}^n C_j \times \bigsqcup_{j=1}^n C_j \rightarrow \mathbb{R}, \quad W_\Gamma|_{C_j} = W_{s_j} \quad (25)$$

such that (i) For  $1 \leq j \leq n$ ,  $W_{s_j} : C_j \times C_j \rightarrow \mathbb{R}$  is a stretched Feynman graphon with  $\mu(C_j) = \frac{c_j}{\sum_{k=1}^n c_k} \mu(\Omega)$  for  $c_j > 0$ , (ii) For  $c_j < 0$ ,  $-W_{s_j} : C_j \times C_j \rightarrow \mathbb{R}$  is considered with  $\mu(C_j) = \frac{|c_j|}{\sum_{k=1}^n |c_k|} \mu(\Omega)$ , and (iii)  $C_l \cap C_t = \emptyset$  for  $l \neq t$ .

Following Definition 2.1, let  $X$  be a large Feynman diagram as the graph limit of the sequence  $\{\Gamma_n\}_n$ . The renormalization coproduct (3) can be extended to  $X$  where  $\Delta(X)$  is defined as the limit of the sequence  $\{\Delta(\Gamma_n)\}_n$ . It leads us to topologically complete the renormalization Hopf algebra to achieve the topological Hopf algebra  $H_{\text{FG}}^{\text{cut}}(\Phi)$  with the coproduct  $\Delta : H_{\text{FG}}^{\text{cut}}(\Phi) \rightarrow H_{\text{FG}}^{\text{cut}}(\Phi) \otimes H_{\text{FG}}^{\text{cut}}(\Phi)$ .  $\Delta$ , as a linear bounded operator between Banach spaces with respect to the cross-norm on  $H_{\text{FG}}^{\text{cut}}(\Phi) \otimes H_{\text{FG}}^{\text{cut}}(\Phi)$ , is continuous.

Thanks to the recursive equations (6) and (7), the renormalized value corresponding to the renormalization of  $X$  is given by

$$\text{ren}(X) = \lim_{n \rightarrow \infty} \varphi_+^z(\Gamma_n). \quad (26)$$

□

**Proposition 2.4.** The space  $\mathcal{S}_{\text{graphon}, \approx}^{(\Omega \subseteq [0, \infty), \mu), \Phi}(\mathbb{R})$  can be equipped with a topological Hopf algebra structure.

*Proof.* Let  $H_{\text{gr}}(\Phi)$  be the free commutative algebra generated by  $W_\Gamma \in \mathcal{S}_{\text{graphon}, \approx}^{(\Omega \subseteq [0, \infty), \mu), \Phi}(\mathbb{R})$  corresponding to 1PI Feynman diagrams in  $\Phi$ . It is a graded vector space such that for each  $n$ ,  $H_{\text{gr}}^n$  is the vector space of stretched Feynman graphons corresponding to 1PI Feynman diagrams with the loop order  $n$  or products of 1PI Feynman diagrams with the overall loop number  $n$ . The algebra  $H_{\text{gr}}(\Phi) = \bigoplus_{n=0}^{\infty} H_{\text{gr}}^n$  is equipped with the coproduct

$$\Delta_{\text{gr}}(W_\Gamma) = W_\Gamma \otimes W_{\mathbb{I}} + W_{\mathbb{I}} \otimes W_\Gamma + \sum_{W_\gamma} W_\gamma \otimes W_{\Gamma/\gamma} \quad (27)$$

such that the sum is taken over all stretched Feynman graphons  $W_\gamma$  where  $\gamma$  contributes to the renormalization coproduct of  $\Gamma$ . The antipode is given by

$$S_{\text{gr}}(W_{\mathbb{I}}) = W_{\mathbb{I}}, \quad S_{\text{gr}}(W_\Gamma) = -W_\Gamma - \sum_{W_\gamma} S_{\text{gr}}(W_\gamma) W_{\Gamma/\gamma} \quad (28)$$

with respect to the coproduct (27). The map  $\Gamma \mapsto W_\Gamma$  provides an injective homomorphism of Hopf algebras from  $H_{\text{FG}}(\Phi)$  to  $H_{\text{gr}}(\Phi)$ .

This Hopf algebra can be topologically enriched. Let  $\{W_\Gamma_n\}_n$  be a sequence in  $H_{\text{gr}}(\Phi)$  which converges to  $W \in \mathcal{S}_{\text{graphon}, \approx}^{(\Omega \subseteq [0, \infty), \mu), \Phi}(\mathbb{R})$ .  $\Delta_{\text{gr}}(W)$  is defined as the limit of the sequence  $\{\Delta_{\text{gr}}(W_\Gamma_n)\}_n$  with respect to the metric (17). The resulting topological Hopf algebra is presented by  $H_{\text{gr}}^{\text{cut}}(\Phi)$ . □

**Corollary 2.5.** *Renormalized values associated to large Feynman diagrams can be recovered by processes of random graphs.*

*Proof.* We apply Lemmas 2.2, 2.3 and Proposition 2.4. Let  $X$  be a large Feynman diagram with  $W_X \in \mathcal{S}_{\text{graphon}, \approx}^{([0,1], m), \Phi}(\mathbb{R})$ . Consider the sequence  $\{R_{\alpha_{\Gamma_n}, W_{\Gamma_n}}\}_n$  of random graphs (generated by  $\{W_{\Gamma_n}\}_n$ ) which converges to  $R_{\infty, W_X}$ . For each  $n \geq 1$ , the homomorphism density of  $\Gamma_n$  with respect to  $W_X$  is given by

$$t(\Gamma_n, W_X) := t(t_{\Gamma_n}, W_X) \Leftrightarrow \quad (29)$$

$$t(\Gamma_n, W_X) = \int_{[0,1]^{|t_{\Gamma_n}|}} \prod_{e_{ij} \in E(t_{\Gamma_n})} \frac{1}{\|W_X\|_{\text{cut}}} |W_X(x_i, x_j)| \quad (30)$$

$$\prod_{e_{ij} \notin E(t_{\Gamma_n})} \left(1 - \frac{1}{\|W_X\|_{\text{cut}}} |W_X(x_i, x_j)|\right) dx_1 \dots dx_{|t_{\Gamma_n}|}$$

such that  $t(X, W_X) = 1$ . It determines the probability that the  $W_X$ -random graph of order  $|t_{\Gamma_n}|$  is isomorphic to  $t_{\Gamma_n}$ . Thanks to [36, 50], if  $t_{\Gamma} = s_1 + \dots + s_k$  be a forest of decorated non-planar rooted trees, then we have

$$t(t_{\Gamma}, W_X) = t(s_1, W_X) \dots t(s_k, W_X). \quad (31)$$

For any fixed Feynman diagram  $\Gamma \in H_{\text{FG}}(\Phi)$ , the functional  $W \mapsto t(\Gamma, W)$  is a continuous map on  $\mathcal{S}_{\text{graphon}, \approx}^{([0,1], m), \Phi}(\mathbb{R})$  with respect to the metric (17). For a dimensionally regularized Feynman rules character  $\varphi^z \in \text{Hom}(H_{\text{FG}}(\Phi), A_{\text{dr}})$ , consider its corresponding character  $\tilde{\varphi}^z \in \text{Hom}(H_{\text{gr}}(\Phi), A_{\text{dr}})$ . We have

$$\begin{aligned} \text{ren}(X) &= \lim_{n \rightarrow \infty} \varphi_+^z(\Gamma_n) \Leftrightarrow \text{ren}(W_X) = \lim_{n \rightarrow \infty} \tilde{\varphi}_+^z(W_{\Gamma_n}) \\ &\Leftrightarrow \text{ren}(R_{\infty, W_X}) = \lim_{n \rightarrow \infty} \tilde{\varphi}_+^z(R_{\alpha_{\Gamma_n}, W_{\Gamma_n}}). \end{aligned} \quad (32)$$

□

**Corollary 2.6.** • *For 1PI Feynman diagrams  $\Gamma, \Gamma_1, \Gamma_2$  with nested loops without overlapping loops and  $|\Gamma| = n$ , we have*

$$|t(\Gamma, W_{\Gamma_1}) - t(\Gamma, W_{\Gamma_2})| \leq (n-1) d_{\text{cut}}(\Gamma_1, \Gamma_2). \quad (33)$$

- *Consider large Feynman diagrams  $X, Y$  with  $W_X, W_Y \in \mathcal{S}_{\text{graphon}, \approx}^{([0,1], m), \Phi}(\mathbb{R})$  and a 1PI Feynman diagram  $\Gamma$  with nested loops without overlapping loops and  $|\Gamma| = n$ , we have*

$$|t(\Gamma, W_X) - t(\Gamma, W_Y)| \leq (n-1) d_{\text{cut}}(W_X, W_Y). \quad (34)$$

- *For  $n \geq 2$ , let  $R_{n, W_X}$  and  $R_{n, W_Y}$  be finite random graphs with the vertex sets  $V_{n, X}, V_{n, Y}$  uniformly independent selected from  $[0, 1]$  such that  $|V_{n, X}| = |V_{n, Y}|$ . Define*

$$d_{\text{var}}(R_{n, W_X}, R_{n, W_Y}) := \frac{1}{2} \sum_{\Gamma, |\Gamma|=n} |t(\Gamma, W_X) - t(\Gamma, W_Y)| \quad (35)$$

*such that the sum is taken over all 1PI Feynman diagrams with nested loops without overlapping loops with the loop order  $n$ . Then we have*

$$d_{\text{var}}(R_{n, W_X}, R_{n, W_Y}) \leq 2^{n^2} d_{\text{cut}}(W_X, W_Y). \quad (36)$$

*Proof.* This is the result of Lemma 2.2, the homomorphism density (30) and Counting Lemmas for Graphs / Graphons [10, 11, 12, 36]. □

### 3. Applications to Green's functions

Let  $\mathcal{A}_\Phi = \{v_i, e_j : i, j\}$  be the collection of types of elementary particles  $e_j$  and types of interactions  $v_i$  between them in a quantum field theory  $\Phi$ . For a Feynman diagram  $\Gamma$ ,  $\text{res}(\Gamma)$  is a graph generated by shrinking all internal edges of  $\Gamma$  into a vertex or edge such that external edges are attached to it. For a primitive 1PI Feynman diagram  $\gamma$  with  $\text{res}(\gamma) = r \in \mathcal{A}_\Phi$  and with the associated Feynman integral  $\int I_\gamma$ , its corresponding 1PI Green's function  $G^r$  is given by

$$\begin{aligned} G^r &= 1 \pm c \int I_\gamma + c^2 \int I_\gamma \int I_\gamma + c^3 \int I_\gamma \int I_\gamma \int I_\gamma + \dots \\ &= 1 \pm c \int I_\gamma \left( 1 \pm c \int I_\gamma + c^2 \int I_\gamma \int I_\gamma + \dots \right) \end{aligned} \quad (37)$$

such that 1 is a Feynman diagram without loop,  $\text{res}(1) = r$  and  $G^r = 1 \pm c \int I_\gamma G^r$ . By setting  $\mathcal{F}(G) = 1 \pm c \int I_\gamma G^r$ , we have the fixed point equation  $\mathcal{F}(G^r) = G^r$  which is known as *Dyson–Schwinger equation*, quantum equation of motion or quantum motion. Solutions of Dyson–Schwinger equations under strong coupling constants, which encode non-perturbative aspects of a gauge field theory, have been studied by certain types of analytic and numerical techniques [2, 13, 17, 23, 24, 25, 29, 28, 33, 39, 51, 53, 54]. Feynman rules of  $\Phi$  replaces the integral presentation (37) with its combinatorial version

$$G^r = \mathbb{I} \pm \sum_{\Gamma, \text{res}(\Gamma)=r} c^{|\Gamma|} \frac{\Gamma}{\text{Sym}(\Gamma)}, \quad (38)$$

such that  $|\cdot|$  is the loop number value. The renormalization Hopf algebra describes fixed point equations associated to (38) in terms of a certain class of recursive Hochschild equations with the general form

$$X = \mathbb{I} + \sum_{n \geq 1} c^n \omega_n B_{\gamma_n}^+(X^{n+1}) \quad (39)$$

with respect to a family  $\{\gamma_n\}_{n \geq 1}$  of primitive (1PI) Feynman diagrams with  $\text{res}(\gamma_n) = r$ . The linear operator  $B_{\gamma_n}^+ : H_{\text{FG}}(\Phi) \rightarrow H_{\text{FG}}(\Phi)$  inserts  $\Gamma$  into  $\gamma_n$  in terms of types of the vertices of  $\gamma_n$  and types of external edges of  $\Gamma$ . It is observed that

$$\Delta B_{\gamma_n}^+ = B_{\gamma_n}^+ \otimes \mathbb{I} + (\text{id} \otimes B_{\gamma_n}^+) \Delta. \quad (40)$$

The equation (39) is called *combinatorial Dyson–Schwinger equation* such that its solution  $X = \sum_{n \geq 1} c^n X_n$  is given by the recursive relations

$$X_n = \sum_{j=1}^n \omega_j B_{\gamma_j}^+ \left( \sum_{k_1 + \dots + k_{j+1} = n-j, k_i \geq 0} X_{k_1} \dots X_{k_{j+1}} \right) \quad (41)$$

such that  $X_0 = \mathbb{I}$ . Each  $X_n$  is built by applying the grafting operators on the lower order components  $X_j$ ,  $j < n$ , which contributes to the order  $n$  of the equation [3, 24, 28, 29, 30, 31, 33].

For sufficiently small values of the coupling constants  $c \ll 1$ , 1PI Green's functions  $G^r$  are convergent perturbative series such that solutions of their corresponding fixed point equations are handled by perturbation theory. However, when  $c \geq 1$ , perturbative series which contribute to 1PI Green's functions and their corresponding fixed point equations are divergent. Extracting some information from

these divergent perturbative series is the main scope of non-perturbative methods [2, 13, 17, 23, 24, 25, 28, 29, 33, 38, 39, 40, 43, 44, 45, 46, 47, 49, 50, 51].

The space  $\mathcal{S}_{\text{graphon}, \approx}^{(\Omega \subseteq [0, \infty), m), \Phi}(\mathbb{R})$  of stretched Feynman graphons, which topologically completes the renormalization Hopf algebra, enables us to study divergent 1PI Green's functions in terms of processes of random graphs.

**Theorem 3.1.** *Each 1PI Green's function  $G^r$  and its corresponding fixed point equations have well-defined Feynman graphon representations.*

*Proof.* We apply Definition 2.1, Lemmas 2.2, 2.3 and Proposition 2.4 together with [43, 49, 51].

**1PI Green's functions.** For the 1PI Green's function (38), consider its partial sums

$$G_{\leq \ell}^r = G_1^r + \dots + G_\ell^r, \quad \ell \geq 1 \quad (42)$$

such that for each  $1 \leq k \leq \ell$ ,

$$G_k^r = \mathbb{I} \pm \sum_{|\Gamma|=k, \text{res}(\Gamma)=r} c^{|\Gamma|} \frac{\Gamma}{\text{Sym}(\Gamma)}. \quad (43)$$

The stretched Feynman graphon  $W_{G_k^r} : \sqcup C_\Gamma \times \sqcup C_\Gamma \rightarrow \mathbb{R}$  corresponding to  $G_k^r$  is a direct sum of stretched Feynman graphons  $W_\Gamma : C_\Gamma \times C_\Gamma \rightarrow \mathbb{R}$  with  $m(C_\Gamma) = \frac{c^k}{\text{Sym}(\Gamma)}$  such that  $C_{\Gamma_1} \cap C_{\Gamma_2} = \emptyset$  for  $\Gamma_1 \neq \Gamma_2$  and  $W_{G_k^r}|_\Gamma = W_\Gamma$ ,  $m$ -almost everywhere. The stretched Feynman graphon  $W_{G_{\leq \ell}^r} : \sqcup E_\Gamma \times \sqcup E_\Gamma \rightarrow \mathbb{R}$  corresponding to  $G_{\leq \ell}^r$  is a direct sum of stretched Feynman graphons  $W_\Gamma : E_\Gamma \times E_\Gamma \rightarrow \mathbb{R}$  with  $m(E_\Gamma) = \frac{c^{|\Gamma|}}{\text{Sym}(\Gamma)}$  such that  $E_{\Gamma_1} \cap E_{\Gamma_2} = \emptyset$  for  $\Gamma_1 \neq \Gamma_2$  and  $W_{G_{\leq \ell}^r}|_{\Gamma, |\Gamma|=k} = W_{G_k^r}|_\Gamma = W_\Gamma$ ,  $m$ -almost everywhere. When  $\ell$  tends to infinity, the sequence  $\{W_{G_{\leq \ell}^r}\}_\ell$  is convergent, with respect to the metric (17), to a new stretched Feynman graphon  $W_{G^r} : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ . It is a direct sum of stretched Feynman graphons  $W_\Gamma : J_\Gamma \times J_\Gamma \rightarrow \mathbb{R}$  with  $m(J_\Gamma) = \frac{c^{|\Gamma|}}{\text{Sym}(\Gamma)}$  such that  $\{J_\Gamma\}_{\Gamma \in G^r}$  is a partition for  $[0, \infty)$ , and  $W_{G^r}|_{\Gamma, |\Gamma|=k} = W_{G_k^r}|_\Gamma = W_\Gamma$ ,  $m$ -almost everywhere.

Thanks to the Cauchy property of the sequence  $\{W_{G_{\leq \ell}^r}\}_\ell$ , for each  $\epsilon > 0$  there exists some order  $N_\epsilon$  such that for any  $\ell \geq N_\epsilon$ ,

$$d_{\text{cut}}(W_{G_{\leq \ell+1}^r}, W_{G_{\leq \ell}^r}) < \epsilon. \quad (44)$$

In other words, for enough large orders  $M \geq N_\epsilon$ ,  $W_{G_{\leq M+1}^r}$  and  $W_{G_{\leq M}^r}$  are weakly isomorphic. Therefore, when  $\ell$  tends to infinity, sequences  $\{W_{G_{\leq \ell}^r}\}_\ell$  and  $\{W_{G_\ell^r}\}_\ell$  behave similar in convergent to  $W_{G^r}$  with respect to the metric (17).

**Fixed point equations.** For the equation (39) with the solution  $X = \sum_{n \geq 1} c^n X_n$ , consider the partial sums of  $X$  given by

$$Y_m := \mathbb{I} + cX_1 + \dots + c^m X_m, \quad m \geq 1, \quad (45)$$

such that the sequence  $\{Y_m\}_m$  is convergent to  $X$  with respect to the metric (20) or (21). The stretched Feynman graphon  $\tilde{W}_{Y_m} : \sqcup \tilde{A}_i \times \sqcup \tilde{A}_i \rightarrow \mathbb{R}$  is the direct sum of stretched Feynman graphons  $\tilde{W}_{X_i} : \tilde{A}_i \times \tilde{A}_i \rightarrow \mathbb{R}$  with  $m(\tilde{A}_i) = c^i$  such that  $\tilde{A}_i \cap \tilde{A}_j = \emptyset$  for  $i \neq j$  and  $\tilde{W}_{Y_m}|_{\tilde{A}_i} = \tilde{W}_{X_i}$ ,  $m$ -almost everywhere. This leads us to define a new stretched Feynman graphon  $\tilde{W}_X : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  as the direct

sum of stretched Feynman graphons  $\tilde{W}_{X_i} : \tilde{B}_i \times \tilde{B}_i \rightarrow \mathbb{R}$  with  $m(\tilde{B}_i) = c^i$  such that  $\{\tilde{B}_i\}_{i=1}^\infty$  is a partition of  $[0, \infty)$  and  $\tilde{W}_X|_{\tilde{B}_i} = \tilde{W}_{X_i}$ ,  $m$ -almost everywhere.

For each  $n \geq 1$ , and

$$\text{val}(\Gamma) := \max \{t \geq 1 : \Gamma \in \bigoplus_{l \geq t} H_{\text{FG}}^{(l)}(\Phi)\}, \quad (46)$$

consider the  $n$ -adic distance

$$d_{\text{adic}}(X_n, \mathbb{I}) = 2^{-\text{val}(X_n - \mathbb{I})} = 2^{-n} \quad (47)$$

as the weight of  $X_n$ . If we apply the Lebesgue measure preserving transformation  $\tau_{2^n} : x \mapsto 2^n x, (\text{mod } 1)$ , then  $\tilde{W}_{X_n}^{\tau_{2^n}}$  is weakly isomorphic to  $\tilde{W}_{X_n}$  such that  $\tilde{W}_{X_n}^{\tau_{2^n}}$  is a stretched Feynman graphon of  $2^n \times 2^n$  copies of  $\tilde{W}_{X_n}$  with weights  $2^{-n}$ . If we apply affine transformations of the real line and Lebesgue measure preserving transformations  $x \mapsto nx, (\text{mod } 1)$ , then  $\tilde{W}_X \in \mathcal{S}_{\text{graphon}, \approx}^{([0, \infty), m), \Phi}(\mathbb{R})$  can be projected onto  $W_X \in \mathcal{S}_{\text{graphon}, \approx}^{([0, 1), m), \Phi}(\mathbb{R})$  where the partition  $\{\tilde{B}_i\}_{i=1}^\infty$  is replaced with the partition  $\{B_i\}_{i=1}^\infty$  for  $[0, 1)$  such that for  $c \leq 1$ ,  $m(B_i) = c^i 2^{-i}$  and for  $c > 1$ ,  $m(B_i) = (\frac{1}{c})^i 2^{-i}$ .  $\square$

Theorem 3.1 shows that for any amplitude  $r$ ,  $W_{G^r}$  associates an infinite set  $A_x$ , which has infinitesimally small measure, to each  $x \in \Omega \subseteq [0, \infty)$  such that there exists a random graph between  $A_x$  and  $A_y$  with the density  $\frac{1}{\|W_{G^r}\|_{\text{cut}}} |W_{G^r}(x, y)|$ .

**Corollary 3.2.** *There exists processes of random graphs which recover the divergent perturbative series of Feynman diagrams which contribute to fixed point equations of 1PI Green's functions in  $\Phi$  under coupling constants  $c \geq 1$ .*

*Proof.* We apply Lemma 2.2, Corollary 2.6 and Theorem 3.1.

For the equation (39) with the solution  $X = \sum_{n \geq 1} c^n X_n$ , consider the partial sums of  $X$  given by

$$Y_m := \mathbb{I} + cX_1 + \dots + c^m X_m, \quad m \geq 1 \quad (48)$$

such that the sequence  $\{Y_m\}_m$  is convergent to  $X$  with respect to the metric (20) or (21). For each  $k$ , let  $t_{X_k}$  be the rooted tree or forest representation for  $X_k$  such that its vertex set  $V(t_{X_k})$ , decorated in terms of (1PI) primitive Feynman subgraphs of  $X_k$ , as an ordered set, can be projected onto a subset  $S^{\theta_k} \subset [0, 1)$  with  $|V(t_{X_k})|$  nodes by an embedding  $\theta_k$ . For  $m \geq 1$ , the vertex set  $V(t_{Y_m}) = \sqcup_{k=1}^m V(t_{X_k})$  of the rooted forest representation of  $Y_m$ , as an ordered set, can be projected onto a subset  $S^{\oplus_{k=1}^m \theta_k} = \sqcup_{k=1}^m S^{\theta_k} \subset [0, 1)$  with  $a_m := \sum_{k=1}^m |V(t_{X_k})|$  nodes.

Define a random graph  $R_{a_m, W_{Y_m}}$  with the vertex set  $S^{\oplus_{k=1}^m \theta_k}$  such that with the probability  $\frac{1}{\|W_{Y_m}\|_{\text{cut}}} |W_{Y_m}(x_i, x_j)|$ , there exists an edge between  $x_i$  and  $x_j$  in  $R_{a_m, W_{Y_m}}$ . When  $m$  tends to infinity, the vertex  $V(t_X)$  of the rooted forest representation of  $X$ , as an ordered set, can be projected onto a subset  $S^{\oplus_{k=1}^\infty \theta_k} = \sqcup_{k=1}^\infty S^{\theta_k} \subset [0, 1)$  with infinite countable nodes. The sequence  $\{R_{a_m, W_{Y_m}}\}_m$  of random graphs converges to the random graph  $R_{\infty, W_X}$  such that with the probability  $\frac{1}{\|W_X\|_{\text{cut}}} |W_X(x_i, x_j)|$ , there exists an edge between  $x_i$  and  $x_j$  in  $R_{\infty, W_X}$ . Therefore we have

$$\lim_{m \rightarrow \infty} R_{a_m, W_{Y_m}} = R_{\infty, W_X} \Leftrightarrow \lim_{m \rightarrow \infty} W_{Y_m} = W_X \Leftrightarrow \lim_{m \rightarrow \infty} Y_m = X. \quad (49)$$

Thanks to the homomorphism density (30) and [50], the probability distribution of the partial sum  $Y_m$  in  $W_X$  is given by integrating over all possible choices of  $x_1, \dots, x_{a_m}$  with respect to  $W_X$ , namely

$$t(Y_m, W_X) := t(t_{Y_m}, W_X) \Leftrightarrow \quad (50)$$

$$t(Y_m, W_X) = \int_{[0,1]^{a_m}} \prod_{e_{ij} \in E(t_{Y_m})} \frac{1}{\|W_X\|_{\text{cut}}} |W_X(x_i, x_j)| \prod_{e_{ij} \notin E(t_{Y_m})} \left(1 - \frac{1}{\|W_X\|_{\text{cut}}} |W_X(x_i, x_j)|\right) dx_1 \dots dx_{a_m}. \quad (51)$$

Let  $\Gamma$  be a Feynman diagram with nested loops but without overlapping loops and  $\text{res}(\Gamma) = r$ . The homomorphism density (30) together with Sample Concentration for Graphs/Graphons [5, 7, 10, 11, 36] lead us to show that for each  $0 < \epsilon < 1$ ,

$$\mathbb{P}\left(|t(\Gamma, R_{a_m, W_X}) - t(\Gamma, W_X)| > \epsilon\right) \leq 2\exp\left(-\frac{\epsilon^2 a_m}{8|t_\Gamma|^2}\right). \quad (52)$$

Here  $R_{a_m, W_X}$  is a random graph with the vertex set  $S_{a_m}$  defined by choosing uniformly  $a_m$  nodes from  $[0, 1]$  such that for  $u_i, u_j \in S_{a_m}$  with the probability  $\frac{1}{\|W_X\|_{\text{cut}}} |W_X(u_i, u_j)|$  there exists an edge between  $u_i$  and  $u_j$  in  $R_{a_m, W_X}$ .  $\square$

**Corollary 3.3.** *Let  $\{X^{r_n}\}_n$  be a sequence of large Feynman diagrams which contribute to solutions of combinatorial Dyson–Schwinger equations  $\text{DSE}_n$ ,  $n \geq 1$ . Let  $W_{X^{r_n}}, W \in \mathcal{S}_{\text{graphon}, \approx}^{([0,1], m), \Phi}([0,1])$ . For any 1PI Feynman diagram  $\Gamma$  with nested loops without overlapping loops, when  $n$  tends to infinity, the sequence  $\{t(\Gamma, W_{X^{r_n}})\}_n$  converges to  $t(\Gamma, W)$  iff  $\{d_{\text{cut}}(W_{X^{r_n}}, W)\}_n$  converges to zero.*

*Proof.* If  $\{d_{\text{cut}}(W_{X^{r_n}}, W)\}_n$  converges to zero, then for each  $\epsilon > 0$  there exists some order  $N_\epsilon$  such that for any  $m \geq N_\epsilon$ , we have  $d_{\text{cut}}(W_{X^{r_m}}, W) < \epsilon$ . Thanks to Corollary 2.6, for any  $m \geq N_\epsilon$ , we have

$$|t(\Gamma, W_{X^{r_m}}) - t(\Gamma, W)| \leq (|\Gamma| - 1)d_{\text{cut}}(W_{X^{r_m}}, W) < (|\Gamma| - 1)\epsilon \quad (53)$$

which means that the sequence  $\{t(\Gamma, W_{X^{r_n}})\}_n$  converges to  $t(\Gamma, W)$ .

If the sequence  $\{t(\Gamma, W_{X^{r_n}})\}_n$  converges to  $t(\Gamma, W)$ , then for each  $\epsilon > 0$  there exists some order  $M_\epsilon$  such that for each  $m \geq M_\epsilon$ , we have

$$|t(\Gamma, W_{X^{r_m}}) - t(\Gamma, W)| < \epsilon. \quad (54)$$

Thanks to Inverse Counting Lemma [10, 11, 36], by setting  $\epsilon = 2^{-|\Gamma|^2}$ , we can get the inequality

$$d_{\text{cut}}(W_{X^{r_m}}, W) \leq \frac{50}{\sqrt{\ln|\Gamma|}}, \quad m \geq M_\epsilon. \quad (55)$$

Therefore, by choosing those  $\Gamma$  with the loop numbers larger than  $\exp((\frac{50}{\epsilon})^2)$ , we have

$$d_{\text{cut}}(W_{X^{r_m}}, W) < \epsilon, \quad (56)$$

which means that the sequence  $\{d_{\text{cut}}(W_{X^{r_n}}, W)\}_n$  converges to zero.  $\square$

**Corollary 3.4.** *The homomorphism density (30) characterizes quantum motions in  $\Phi$ .*

*Proof.* For combinatorial Dyson–Schwinger equations  $\text{DSE}_{r_1}, \text{DSE}_{r_2}$  with the corresponding solutions  $X^{r_1}, X^{r_2}$  and  $W_{X^{r_1}}, W_{X^{r_2}} \in \mathcal{S}_{\text{graphon}, \approx}^{([0,1], m), \Phi}(\mathbb{R})$ , Corollary 2.6 and Sampling Distance between Graphs / Graphons [10, 11, 36] show that if for  $\Gamma \in H_{\text{FG}}(\Phi)$ ,

$$\sum_{\Gamma} 2^{-|\text{tr}|\Gamma|} |t(\Gamma, W_{X^{r_1}}) - t(\Gamma, W_{X^{r_2}})| = \sum_{k=1}^{\infty} \frac{1}{2^k} d_{\text{var}}(R_{k, W_{X^{r_1}}}, R_{k, W_{X^{r_2}}}) = 0, \quad (57)$$

then

$$d_{\text{var}}(R_{\infty, W_{X^{r_1}}}, R_{\infty, W_{X^{r_2}}}) = 0 \Leftrightarrow X^{r_1} \approx X^{r_2}. \quad (58)$$

□

Homomorphism densities of Feynman graphons are applied to describe the structure of the Gibbs probability measure on the space of 1PI Green’s functions to associate a statistical manifold to a quantum field theory [46]. Geodesics in this manifold are candidates to recover the notion of information entropy between quantum motions.

**Corollary 3.5.** *The homomorphism density of Feynman graphons associates expectation values to solutions of quantum motions as observed information of a quantum field theory.*

*Proof.* For a combinatorial Dyson–Schwinger equation DSE with the solution  $X = \sum_{n=0}^{\infty} c^n X_n$  and the sequence  $\{Y_m\}_m$  of its partial sums, and a random variable  $u$  chosen from a compact interval  $J \subset [1, \infty)$ , define

$$X^{\lceil u \rceil} := X - Y_{\lceil u \rceil} = \sum_{n=\lceil u \rceil+1}^{\infty} c^n X_n \quad (59)$$

such that  $u \mapsto \lceil u \rceil$  is the ceiling function. The homomorphism density of the stretched Feynman graphon  $W_X \in \mathcal{S}_{\text{graphon}, \approx}^{([0,1], m), \Phi}(\mathbb{R})$  determines a distribution

$$(u, X^{\lceil u \rceil}) \mapsto T_{\text{DSE}}(u) := 1 - t(Y_{\lceil u \rceil}, W_X), \quad (60)$$

, conditioned on  $u$ , with the corresponding Gibbs measure

$$\mathbf{P}_{T_{\text{DSE}}}(u) = \frac{1}{Z[\mathbf{j}]} \exp\left(-\mathbf{j}T_{\text{DSE}}(u)\right), \quad (61)$$

such that

$$Z[\mathbf{j}] = \int \exp\left(iS[\phi] + i \int \mathbf{j}(v)\phi(v)d^D v\right) \mathcal{D}[\phi] \quad (62)$$

is the partition function with respect to the source field  $\mathbf{j}$ . The probability measure  $\mathbf{P}_{T_{\text{DSE}}}(u)$  describes the probability that we observe a (non-)perturbative solution  $X^{\lceil u \rceil}$  of the equation DSE such that these probability values are conditioned on the random variable  $u$ .

The probability distribution  $\mathbf{P} : (W_X, u) \mapsto \mathbf{P}_{T_{\text{DSE}}}(u)$  on  $\mathcal{S}_{\text{graphon}, \approx}^{([0,1], m), \Phi}(\mathbb{R})$  determines the Fisher information

$$\int_J \left( \frac{\partial}{\partial u} \log \mathbf{P}(W_X, u) \right)^2 \mathbf{P}(W_X, u) du \quad (63)$$

as the expectation value of  $X$  generated by the distribution  $t(X^{\lceil u \rceil}, W_X)$ . □

## 4. Conclusion

This research work presented a new application of the method of Feynman graphons to quantum field theory. It is explained that perturbative series of higher loop order Feynman diagrams which contribute to 1PI Green's functions or solutions of their corresponding fixed point equations can be described by graph limits generated by processes of random graphs. Thanks to the homomorphism density of Feynman graphons together with the Gibbs measure mechanism, expectation values are assigned to solutions of combinatorial Dyson–Schwinger equations.

**4.1. A comparison.** The method of Feynman graphons has some advantages in comparison to other non-perturbative methods. In this methods, non-perturbative series of 1PI Green's functions and their corresponding fixed point equations are replaced with stretched Feynman graphons as some bounded symmetric (Lebesgue) measurable functions in contrast to the Borel resummations which deal with the challenge of singularities and summability [27, 34]. As it is shown in this research work, these Feynman graphon representations are reliable to show the importance of randomness in the core-stone of non-perturbative aspects of a quantum field theory. In addition, (non-)perturbative solutions of quantum motions in a quantum field theory can be recovered by the geometry of a separable Banach space of stretched Feynman graphons without any discretization of space-time [49, 50]. This is in contrast to lattice models or large  $N$  limits [18, 37, 39, 57, 60]. Furthermore, the method of Feynman graphons addresses the existence of intermediate phases in non-perturbative domain a quantum field theory such that phase transitions and equilibrium states can be described in terms of a statistical model [44, 49].

## References

- [1] T. Austin, On exchangeable random variables and the statistics of large graphs and hypergraphs, *Probability Surveys* **5** (2008), 80–145.
- [2] A.C. Aguilar, M.N. Ferreira, B.M. Oliveira, et al, Schwinger–Dyson truncations in the all-soft limit: a case study, *Eur. Phys. J. C* **82** (2022), paper no. 1068.
- [3] C. Bergbauer, D. Kreimer, New algebraic aspects of perturbative and non-perturbative quantum field theory, In: *New Trends in Mathematical Physics*, 2009, 45–58.
- [4] F. Brown, D. Kreimer, Angles, scales and parametric renormalization, *Lett. Math. Phys.* **103** (2013), 933–1007.
- [5] C. Borgs, J. Chayes, L. Lovasz, V. Sos, K. Vesztergombi, Limits of randomly grown graph sequences, *Eur. J. Comb.* **32** (2011), no. 7, 985–999.
- [6] C. Borgs, J.T. Chayes, H. Cohn, N. Holden, Sparse exchangeable graphs and their limits via graphon processes, *J. Mach. Learn. Res.* **18** (2017), paper no. 210.
- [7] C. Borgs, J. Chayes, L. Lovasz, Moments of two-variable functions and the uniqueness of graph limits, *Geom. Func. Anal.* **19** (2010), 1597–1619.
- [8] C. Borgs, J.T. Chayes, H. Cohn, Y. Zhao, An  $L^p$  theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions, *Trans. Am. Math. Soc.* **372** (2019), no. 5, 3019–3062.
- [9] C. Borgs, J.T. Chayes, H. Cohn, Y. Zhao, An  $L^p$  theory of sparse graph convergence II: LD convergence, quotients, and right convergence, *Ann. Probab.* **46** (2018), no. 1, 337–396.
- [10] C. Borgs, J. Chayes, L. Lovasz, V.T. Sos, K. Vesztergombi, Convergent sequences of dense graphs I: subgraph frequencies, metric properties and testing, *Adv. Math.* **219** (2008), no. 6, 1801–1851.

- [11] C. Borgs, J. Chayes, L. Lovasz, V.T. Sos, K. Vesztergombi, Convergent sequences of dense graphs II: multiway cuts and statistical physics, *Annals Math.* **176** (2012), 151–219.
- [12] C. Borgs, J. Chayes, L. Lovasz, V.T. Sos, K. Vesztergombi, Limits of randomly grown graph sequences, *Europ. J. Combin.* **32** (2011), 985–999.
- [13] C.M. Bender, C. Karapoulitidis, S.P. Klevansky, Underdetermined Dyson-Schwinger Equations, *Phys. Rev. Lett.* **130** (2023), no. 10, paper no. 101602, 5 pp.
- [14] B. Bollobas, S. Janson, O. Riordan, The cut metric, random graphs, and branching processes, *J. Stat. Phys.* **140** (2010), 289–335.
- [15] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem I: the Hopf algebra structure of graphs and the main theorem, *Comm. Math. Phys.* **210** (2000), no. 1, 249–273.
- [16] A. Connes, D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem II: the beta-function, diffeomorphisms and the renormalization group, *Comm. Math. Phys.* **216** (2001), no. 1, 215–241.
- [17] J. E. Davis, Schwinger-Dyson analysis of an exactly solvable model, *Phys. Rev. D* **46** (1992), no. 4, 1750–1760.
- [18] T. DeGrand, C. DeTar, *Lattice methods for Quantum Chromodynamics*, World Scientific, 2006.
- [19] P. Diaconis, S. Janson, Graph limits and exchangeable random graphs, *Rend. Mat. Appl.* (7) **28** (2008), no. 1, 33–61.
- [20] G. Elek, On limits of finite graphs, *Combinatorica* **27** (2007), 503–507.
- [21] K. Ebrahimi-Fard, L. Guo, D. Kreimer, Integrable renormalization. I. The ladder case, *J. Math. Phys.* **45** (2004), no. 10, 3758–3769.
- [22] K. Ebrahimi-Fard, L. Guo, D. Kreimer, Integrable renormalization. II. The general case, *Ann. Henri Poincaré* **6** (2005), no. 2, 369–395.
- [23] M. Frasca, Differential Dyson–Schwinger equations for quantum chromodynamics, *Eur. Phys. J. C* **80** (2020), paper no. 707.
- [24] J.A. Gracey, H. Kissler, D. Kreimer, Self-consistency of off-shell Slavnov-Taylor identities in QCD, *Phys. Rev. D* **100** (2019), no. 8, paper no. 085001, 40 pp.
- [25] E. Guadagnini, V. Urso, Renormalized Schwinger–Dyson functional, *Eur. Phys. J. C* **81** (2021), paper no. 103.
- [26] Z. Haba, *Lectures on quantum field theory and functional integration*, Springer, 2023.
- [27] Y. Hatsuda, K. Okuyama, Resummations and non-perturbative corrections, *J. High Energ. Phys.* **2015** (2015), paper no. 51.
- [28] O. Kruger, Log expansions from combinatorial Dyson–Schwinger equations, *Lett. Math. Phys.* **110** (2020), 2175–2202.
- [29] D. Kreimer, Factorization in quantum field theory: an exercise in Hopf algebras and local singularities, In: *Frontiers in number theory, physics, and geometry. II*, 2007, 715–736.
- [30] D. Kreimer, Combinatorics of (perturbative) quantum field theory, *Phys. Rep.* **363** (2002), no. 4–6, 387–424.
- [31] D. Kreimer, Anatomy of a gauge theory, *Ann. Phys.* **321** (2006), no. 12, 2757–2781.
- [32] D. Kreimer, R. Delbourgo, Using the Hopf algebra structure of QFT in calculations, *Phys. Rev. D* **60** (1999), no. 10, paper no. 105025, 14 pp.
- [33] O. Kruger, D. Kreimer, Filtrations in Dyson-Schwinger equations: next-to-j-leading log expansions systematically, *Ann. Phys.* **360** (2015), 293–340.
- [34] T. Lee, Nonperturbative effects from the resummation of perturbation theory, *Phys. Rev. D* **66** (2002), no. 3, paper no. 034027.
- [35] L.N. Lipatov, Divergence of the perturbation-theory series pseudoparticles, *JETP Lett.* **25** (1977), no. 2, 104–107.
- [36] L. Lovasz, Large networks and graph limits, *AMS Colloquium Publications* **60**, 2012.
- [37] M. Moshe, J. Zinn-Justin, Quantum field theory in the large N limit: a review, *Phys. Rep.* **385** (2003), no. 3–6, 69–228.
- [38] J.P. Nery, F. Mauri, Nonperturbative Green’s function method to determine the electronic spectral function due to electron-phonon interactions: Application to a graphene model from weak to strong coupling, *Phys. Rev. B* **105** (2022), paper no. 245120.
- [39] H. Pagels, Nonperturbative approach to quantum chromodynamics, *Phys. Rev. D* **15** (1977), no. 10, paper no. 2991.

- [40] V.E. Rochev, A non-perturbative method of calculation of Green functions, *J.Phys. A* **30** (1997), 3671–3680.
- [41] S.V. Shabanov, Non-perturbative Green functions in quantum gauge theories, *Modern Phys. Lett. A* **06**(1991), no. 10, 909–921.
- [42] S. Schenk, The breakdown of resummed perturbation theory at high energies, *J. High Energ. Phys.* **2022** (2022), no. 3, paper no. 100, 27 pp.
- [43] A. Shojaei-Fard, Subsystems via quantum motions, *Anal. Math. Phys.* **14** (2024), no. 3, paper no. 61, 36 pp.
- [44] A. Shojaei-Fard, statistical mechanical model for non-perturbative regimes, *Nuclear Phys. B* **991** (2023), paper no. 116220, 24 pp.
- [45] A. Shojaei-Fard, Graph polynomials associated with Dyson–Schwinger equations, *Mathematica Moravica* **27** (2023), no. 2, 91–114.
- [46] A. Shojaei-Fard, From Dyson–Schwinger equations to quantum entanglement, *J. Math. Sci.* **266** (2022), no. 6, 892–916.
- [47] A. Shojaei-Fard, Non-perturbative graph languages, halting problem and complexity, *Forum Math.* **34** (2022), no. 5, 1159–1185.
- [48] A. Shojaei-Fard, Halting problem in Feynman graphon processes derived from the renormalization Hopf algebra, *Bull. Transilv. Univ. Brasov, Ser. III: Math. Comput. Sci. (2)* **64** (2022), no. 1, 139–157.
- [49] A. Shojaei-Fard, The dynamics of non-perturbative phases via Banach bundles, *Nuclear Phys. B* **969** (2021), paper no. 115478, 39 pp.
- [50] A. Shojaei-Fard, The analytic evolution of Dyson–Schwinger equations via homomorphism densities, *Math. Phys. Anal. Geom.* **24** (2021), no. 2, paper no. 18, 28 pp.
- [51] A. Shojaei-Fard, The complexities of nonperturbative computations, *Russ. J. Math. Phys.* **28** (2021), no. 3, 358–376.
- [52] A. Shojaei-Fard, Non-perturbative  $\beta$ -functions via Feynman graphons, *Modern Phys. Lett. A* **34** (2019), no. 14, paper no. 1950109, 10 pp.
- [53] A. Shojaei-Fard, The global  $\beta$ -functions from solutions of Dyson-Schwinger equations, *Mod. Phys. Lett. A* **28** (2013), no. 34, paper no. 1350152, 12 pp.
- [54] A. Shojaei-Fard, A geometric perspective on counterterms related to Dyson-Schwinger equations, *Intern. J. Mod. Phys. A* **28** (2013), no. 32, paper no. 1350170, 15 pp.
- [55] I.M. Suslov, Summing divergent perturbative series in a strong coupling limit. The Gell-Mann-Low function of the  $\phi^4$  theory, *J. Exp. Theor. Phys.* **93** (2001), 1–23.
- [56] I.M. Suslov, Divergent perturbation serie, *J. Exp. Theor. Phys.* **100** (2005), 1188–1233.
- [57] A.W. Schreiber, A.G. Williams, A.W. Thomas (Eds), *Non-perturbative methods in Quantum Field Theory, Proceedings of the Workshop*, Adelaide 1998, World Scientific, 1999, 348 pp.
- [58] W.D. van Suijlekom, The Hopf algebra of Feynman graphs in quantum electrodynamics, *Lett. Math. Phys.* **77** (2006), no. 3, 265–281.
- [59] W.D. van Suijlekom, Renormalization of gauge fields: a Hopf algebra approach, *Commun. Math. Phys.* **276** (2007), 773–798.
- [60] H. Wittig, QCD on the Lattice, In: H. Schopper (ed) *Particle Physics Reference Library*, Springer, Cham, 2020, 137–262.
- [61] J. Zinn-Justin, From Slavnov–Taylor identities to the renormalization of gauge theories, *Proc. Steklov Inst. Math.* **309** (2020), 317–324.
- [62] W. Zimmermann, Convergence of Bogoliubov’s method of renormalization in momentum space, *Commun. Math. Phys.* **15** (1969), 208–234.

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