Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 52(2), 2025, Pages 632–645, DOI: 10.52846/ami.v52i2.2053
ISSN: 1223-6934

New Results for Some Differential Equations with Reflection and Application

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Abstract. In this paper we prove the existence and the uniqueness of pseudo almost periodic (pap) solutions for integro-differential equations with involution. We will use the features of exponential dichotomy and the Banach fixed point theorem. We close with an example illustrated by the model of Markus and Yamabe.

2020 *Mathematics Subject Classification.* Primary 34K30; Secondary 35B15.
Key words and phrases. pseudo almost periodic solutions, involution, exponential dichotomy.

1. Introduction

The study of systems with perturbations progressively become one of the centers of interest in the field of dynamic, physical sciences, mathematical biology, control theory and others. During the last years there have been a number of works related to the study of differential equations with involution (see [2, 3, 13]).
The principal objective of this paper is to inquire the existence and uniqueness of pseudo almost periodic solutions for integro-differential with involution with connection of doubly measures and the theory of exponential dichotomy which the last one has played a main role in this context. The notion of pseudo almost periodicity was introduced in the literature by Zhang and Diagana (see [8, 9, 16]) as a generalisation of the classical almost periodicity in the sense of Bohr (see [5]).
Ait Dads et al (see [1, 2]) impart the concept of generalised pap functions by prolongation the ergodic perturbation, and Veech (see [15]) researched the characteristic of this type of functions. For fundamental results on pap functions we can see the references: ([1, 6, 8, 14, 16]).
More lately, Ait Dads, Toka Diagana, Khalil Ezzimbi and Mohsen Miraoui put the concept of (μ,ν)-pap functions as a generalisation of μ-pap functions (see [10, 11, 12]).
In the last period, several researchers who specialize their studies in the search for solutions of certain equation with reflecion (see Xin, Piao, Gupta, Ait Dads...) and who have studied the existence of the solution of the following equation:

$$u^{'}(t) = f(t, u(t), u(-t)), t \in \mathbb{R}. \quad (1)$$

The research of almost periodic and pseudo almost periodic solutions of the equations:

$$u^{'}(t) = au(t) + bu(-t) + g(t), b \neq 0, t \in \mathbb{R}, \quad (2)$$

$$u^{'}(t) = au(t) + bu(-t) + f(t, u(t), u(-t)), b \neq 0, t \in \mathbb{R}, \quad (3)$$

Received November 17, 2024. Accepted August 9, 2025.

were investigated in [13].
More generally knowing that A et B are two square matrix, Gupta, Ait Dads et al. (see [2]) has studied the equations:

$$u'(t) = Au(t) + Bu(-t) + g(t), b \neq 0, t \in \mathbb{R}, \tag{4}$$

and

$$u'(t) = Au(t) + Bu(-t) + f(t, u(t), u(-t)), b \neq 0, t \in \mathbb{R}. \tag{5}$$

Recently, in [2, 13], M. Miraoui showed the existence of paa solutions to the following semi linear equation:

$$\begin{aligned} u'(t) &= A(t)u(t) + B(t)u(-t) + f(t, u(t), u(-t)) + \int_t^{+\infty} L(y-t)g(y, u(y), u(-y))dy \\ &+ \int_{-t}^{+\infty} L(y+t)g(y, u(y), u(-y))dy, \end{aligned} \tag{6}$$

where A et B are square matrix of order n in $\mathbb{N}^*$, $L : \mathbb{R}^+ \to \mathbb{R}^+$ and $f, g : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ are continuous functions.

In this paper, we will study the existence and the uniquence of pap solutions of such equation with two measures. That's why throughout the research we choose to consider the following hypothesis as valid:

(**H0**): There exist continuous and strictly increasing functions $\gamma_i : \mathbb{R} \to \mathbb{R}$ such that for all $u \in \mathrm{AP}(\mathbb{R}, \mathbb{R})$ we have $u \circ \gamma_i \in \mathrm{AP}(\mathbb{R}, \mathbb{R}), i = 1, 2$.

The primary objective of our paper is to examine equation that more comprehensive than equation given by the following expression:

$$\begin{aligned} &\frac{d}{dt}[u(t) - h(t, u(\gamma_1(t)), u(\gamma_1(-t)))] \\ &= A(t)[u(t) - h(t, u(\gamma_1(t)), u(\gamma_1(-t)))] + B(t)[u(-t) - h(-t, u(\gamma_1(-t)), u(\gamma_1(t)))] \\ &\quad + f(t, u(\gamma_2(t)), u(\gamma_2(-t))) + \int_t^{+\infty} L(y-t)g(y, u(\gamma_2(y)), u(\gamma_2(-y)))dy \\ &\quad + \int_{-t}^{+\infty} L(y+t)g(y, u(\gamma_2(y)), u(\gamma_2(-y)))dy. \end{aligned} \tag{7}$$

The planning of this paper is as follows: In section 2 we recall some definitions of (μ, ν)-pap functions and properties of exponential dichotomy. In section 3 we study the existence and uniqueness of pap solution to equation (7) with doubly measures. We close (section 4) by an example to illustrate our abstract results.

2. Preliminaires

2.1. Exponential dichotomy. In the sequel, A denotes a continuous mapping from $\mathbb{R}$ to $\mathcal{M}_n(\mathbb{R})$, where $\mathcal{M}_n(\mathbb{R})$ is the space of square matrices with real coefficients.

Definition 2.1. Let $A(t)$ be a continuous square matrix on an interval I and let $X(t)$ be a fundamental matrix of the following system:

$$x'(t) = A(t)x(t), \tag{8}$$

satisfying $x(0) = \mathrm{I}_n$, where I_n is the unit matrix.
The system of differential equations is said to process an exponential on the interval I,

if there exists a projection matrix P(that's to say $P^2 = P$) and constants $k > 1, \alpha > 0$, such that:

$$\|X(t)PX^{-1}(s)\| \leq k\exp(-\alpha(t-s)), \text{ for } s \leq t \text{ with } s,t \in \mathrm{I},$$
$$\|X(t)(I-P)X^{-1}(s)\| \leq k\exp(-\alpha(s-t)), \text{ for } t \leq s \text{ with } s,t \in \mathrm{I}.$$

We note by (P,k,α) the three associated coefficients with exponential dichotomy.

Theorem 2.1. [7] *Suppose that the differential equation (8) has an exponential dichotomy on $\mathbb{R}$ with parameters (P,k,α). Let $B : \mathbb{R} \to \mathcal{M}_n(\mathbb{R})$ be a bounded continuous function such that $\sigma = \sup_{t\in\mathbb{R}}\|B(s)\| < \frac{\alpha}{4k^2}$. Then the perturbed equation*

$$x^{'}(t) = (A(t) + B(t))x(t), \tag{9}$$

has an exponential dichotomy on $\mathbb{R}$ with parameters($\mathrm{Q}, \alpha - 2k\sigma, \frac{5}{2}k^2$), where Q is a projection with the same kernel as the one of P. Moreover, if $Y(t)$ is the fundamental matrix of (9) satisfying $Y(0) = I_n$ then

$$\|Y(t)QY^{-1}(t) - X(t)PX^{-1}(t)\| \leq \tfrac{4}{\alpha}\sigma k^3 \quad \forall t \in \mathbb{R}.$$

Let U be a solution of equation (9) and $W(t) = \begin{pmatrix} U(t) - h(t, U\circ\gamma_1(t), U\circ\gamma_1(-t)) \\ U(-t) - h(-t, U\circ\gamma_1(-t), U\circ\gamma_1(t)) \end{pmatrix}$.
Then the function W checks the equation

$$W^{'}(t) = D(t)W(t) + R(t, U(t)),$$

where

$$D(t) = \begin{pmatrix} A(t) & B(t) \\ -B(-t) & -A(-t) \end{pmatrix},$$

$$R(t, U(t)) = \begin{pmatrix} f(t, U\circ\gamma_2(t), U\circ\gamma_2(-t)) \\ -f(-t, U\circ\gamma_2(-t), U\circ\gamma_2(t)) \end{pmatrix} + G(t, U(t)),$$

such that

$$G(t, U(t)) =$$
$$\begin{pmatrix} \displaystyle\int_t^{+\infty} L(y-t)g(y, U\circ\gamma_2(y), U\circ\gamma_2(-y))dy + \int_{-t}^{+\infty} L(y+t)g(y, U\circ\gamma_2(y), U\circ\gamma_2(-y))dy \\ \displaystyle -\int_{-t}^{+\infty} L(y+t)g(y, U\circ\gamma_2(y), U\circ\gamma_2(-y))dy - \int_t^{+\infty} L(y-t)g(y, U\circ\gamma_2(y), U\circ\gamma_2(-y))dy \end{pmatrix}.$$

2.2. (μ,ν)-Pseudo almost periodic functions : In this work, F is a Banach space and $\mathcal{BC}(\mathbb{R}, F)$ denotes the Banach space of bounded continuous functions from $\mathbb{R}$ to F, equipped with the supremum norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}}\|f(t)\|.$$

Definition 2.2. A continuous function $f : \mathbb{R} \to \mathrm{F}$ is called almost periodic, if: $\forall\epsilon > 0, \exists l(\epsilon) > 0$, such that $\forall a \in \mathbb{R}, \exists\tau \in [a, a + l(\epsilon)]$ with:

$$\|f(x+\tau) - f(x)\|_\infty < \epsilon \; \forall x \in \mathbb{R}.$$

The sequel AP($\mathbb{R}$, F) denotes the space of almost periodic functions.

Example 2.1. Let $f : \mathbb{R} \to \mathbb{C}$ be such that

$$f(t) = e^{it} + e^{i\sqrt{2}t}.$$

Then f is almost periodic, but it is not periodic.

Definition 2.3. A function $f : \mathbb{R} \times F \to F$ is said to be be almost periodic in t uniformly with respect to x in F, if the following two conditions hold:

(1) $\forall x \in F, f(\cdot, x) \in \mathrm{AP}(\mathbb{R}, \mathrm{F})$;

(2) f is uniformly continuous on each compact C in F with respect to second variable x namely:
$\forall C \in F$, C compact set, $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x_1, x_2 \in C$, we have:
$$\|x_1 - x_2\| \leq \delta \Rightarrow \sup_{t\in\mathbb{R}} \|f(t, x_1) - f(t, x_2)\| \leq \epsilon.$$

Let $\mathrm{APU}(\mathbb{R}, F)$ denote the collection of all such functions.

Let $\mathbf{B}$ denote the Lebesque σ-filed of $\mathbb{R}$, then we have the following definition:

Definition 2.4. We define the set $\mathbf{M}$ by the set of all positive measures μ on $\mathbf{B}$ satisfying :

- $\mu([a, b]) < \infty, \forall a, b \in \mathbb{R}(a \leq b)$;
- $\mu(\mathbb{R}) = +\infty$.

Definition 2.5. Let $\mu, \nu \in \mathbf{M}$ a bounded continuous function $f : \mathbb{R} \to \mathrm{F}$ is said to be (μ, ν) ergodic, if:
$$\lim_{z\to+\infty} \frac{1}{\nu([-z, z])} \int_{[-z,z]} \|f(t)\| d\mu(t) = 0.$$

The set $\xi(\mathbb{R}, \mathrm{F}, \mu, \nu)$ denotes the space of the (μ, ν)-ergodic functions.

Definition 2.6. Let $\mu, \nu \in \mathbf{M}$. $f : \mathbb{R} \times \mathrm{F} \to \mathrm{F}$ is said to be (μ, ν)-ergodic in t uniformly with respect to $x \in \mathrm{F}$ if and only if the following two conditions are true:

- (i) $\forall x \in \mathrm{F}, [t \to f(t, x)] \in \xi(\mathbb{R}, \mathrm{F}, \mu, \nu)$,
- (ii) f is uniformly continuous on all C compact in F with respect to the second variable x.

Let $\xi U(\mathbb{R} \times \mathrm{F}, \mathrm{F}, \mu, \nu)$ denote the collection of all such functions.

Definition 2.7. Let $\mu, \nu \in \mathbf{M}$. A continuous function $f : \mathbb{R} \to \mathrm{F}$ is (μ, ν)-pseudo almost periodic if it can be written in this form: $f = f_1 + f_2$, where $f_1 \in \mathrm{AP}(\mathbb{R}, \mathrm{F})$ and $f_2 \in \xi(\mathbb{R}, \mathrm{F}, \mu, \nu)$.
The set $\mathrm{PAP}(\mathbb{R}, \mathrm{F}, \mu, \nu)$ denotes the space of (μ, ν)-pseudo almost periodic functions.

Definition 2.8. Let $\mu, \nu \in \mathbf{M}, f : \mathbb{R} \times \mathrm{F} \to \mathrm{F}$ is (μ, ν)-pseudo almost periodic(or in $\mathrm{PAPU}(\mathbb{R} \times \mathrm{F}, \mathrm{F}, \mu, \nu)$ if we have $f = f_1 + f_2$, where $f_1 \in \mathrm{APU}(\mathbb{R}, \mathrm{F})$ and $f_2 \in \xi U(\mathbb{R}, \mathrm{F}, \mu, \nu)$.

We formulate and assume the following hypotheses:

(**H1**) : Let $\mu, \nu \in \mathbf{M}$ such that:
$$\lim_{z\to\infty} sup(\frac{\mu([-z, z])}{\nu([-z, z])}) < +\infty,$$

(**H2**) : for all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that :
$\mu(a + \tau : a \in A) \leq \beta\mu(A)$ when $A \in \mathbf{B}$ satisfies $A \cap I = \emptyset$.

Lemma 2.2. [1] *Let $\mu, \nu \in \mathbf{M}$, then we have:*

(1) *The decomposition in* $\mathrm{PAP}(\mathbb{R}, \mathrm{F}, \mu, \nu)$ *is unique,*

(2) $(\mathrm{PAP}(\mathbb{R}, \mathrm{F}, \mu, \nu), \|.\|_\infty)$*is a Banach space,*

(3) *The space* $\mathrm{PAP}(\mathbb{R}, \mathrm{F}, \mu, \nu)$ *is translation invariant.*

2.3. (μ,ν)-Pseudo almost periodic solutions: In this paper, we shall also need the following hypotheses:

(**H3**) : $\exists m,n>0$, such that $\forall A\in\mathbf{B}$, we have $m+n\mu(A)-\mu(-A)\geq 0$,

(**H4**) : $f,g,h:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ are (μ,ν)-pseudo almost periodic in t,

(**H5**) : There exists a continuous, strictly increasing function $\lambda:\mathbb{R}\to\mathbb{R}_+$ such that: $d\mu_\lambda(t)\leq\lambda(t)d\mu(t)$, where $\mu\in\mathbf{M}, \mu_\lambda(O)=\mu(\lambda^{-1}(O))$ for all $O\in\mathbf{B}(\mathbb{R})$ and

$$\limsup_{z\to+\infty}\frac{\mu([-T(z),T(z)])}{\mu([-z,z])}S(T(z))\leq+\infty,$$

where $T(z)=|\lambda(z)|+|\lambda(-z)|$ and $S(T(z))=sup(\lambda(t))_{t\in[-T(z),T(z)]}$.

(**H6**) :

(1) There exists $L_f^1,L_f^2>0$, such that for all $x_1,x_2,y_1,y_2\in\mathbb{R}^n$, we have : $\|f(t,x_1,x_2)-f(t,y_1,y_2)\|\leq L_f^1\|x_1-y_1\|+L_f^2\|x_2-y_2\|$, $\forall t\in\mathbb{R}$,

(2) There exists $L_g^1,L_g^2>0$, such that for all $x_1,x_2,y_1,y_2\in\mathbb{R}^n$, we have : $\|g(t,x_1,x_2)-g(t,y_1,y_2)\|\leq L_g^1\|x_1-y_1|+L_g^2\|x_2-y_2\|$, $\forall t\in\mathbb{R}$,

(3) There exists $L_h^1,L_h^2>0$, such that for all $x_1,x_2,y_1,y_2\in\mathbb{R}^n$, we have : $\|h(t,x_1,x_2)-h(t,y_1,y_2)\|\leq L_h^1\|x_1-y_1|+L_h^2\|x_2-y_2\|$, $\forall t\in\mathbb{R}$,

(**H7**) : $L:\mathbb{R}\to\mathbb{R}_+, \displaystyle\int_0^{+\infty}L(y)dy=c_1\in\mathbb{R}_+$,

(**H8**) : The equation $x^{'}(t)=A(t)x(t)$ has an exponential dichotomy with coefficients (P,k,α),

(**H9**) : The operator $B:\mathbb{R}\to\mathcal{M}_n(\mathbb{R})$ is uniformly bounded in $t\in\mathbb{R}$ and continuous. In addition, one of the following two conditions is assumed

$$\lim_{z\to+\infty}\frac{1}{2z}\int_{-z}^{z}\|B(t)\|dt=0 \text{ or } sup\|B(t)\|_{t\in\mathbb{R}}<\infty.$$

Lemma 2.3. [13, 14] *Let $f\in\mathrm{PAP}(\mathbb{R},\mathrm{F},\mu,\nu)$ and $\mu\in\mathcal{M}$ satisfies (**H3**). Then*

$$[t\mapsto f(-t)]\in\mathrm{PAP}(\mathbb{R},\mathrm{F},\mu,\nu).$$

Lemma 2.4. [3] *Suppose that hypotheses (**H0**), (**H1**) and (**H5**) are true. Then*

$$[t\mapsto u\circ\gamma_i(t)]\in\mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu), i=1,2.$$

Lemma 2.5. *If the conditions (**H3**),(**H4**) and (**H6**) are true and $u\in\mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu)$, then*

$$[t\mapsto f(t,u(t),u(-t))]\in\mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu).$$

Proof. By using (**H4**), we have $f\in\mathrm{PAPU}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n,\mu,\nu)$, then $f=\phi_1+\phi_2$, where $\phi_1\in\mathrm{APU}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$ and $\phi_2\in\xi\mathrm{U}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n,\mu,\nu)$. Since $u\in\mathrm{PAP}(\mathbb{R},\mu,\nu)$, then $u=u_1+u_2$ where $u_1\in\mathrm{AP}(\mathbb{R},\mathbb{R}^n)$ and $u_2\in\xi(\mathbb{R},\mathbb{R}^n,\mu,\nu)$. We can write:

$$\begin{aligned}
f(t,u(t),u(-t)) &= \phi_1(t,u_1(t),u_1(-t))+f(t,u(t),u(-t))-\phi_1(t,u_1(t),u_1(-t))\\
&= \phi_1(t,u_1(t),u_1(-t))+f(t,u(t),u(-t))-f(t,u_1(t),u_1(-t))\\
&+ \phi_2(t,u_1(t),u_1(-t)).
\end{aligned}$$

We have:

- $t\mapsto\phi_1(t,u_1(t),u_1(-t))\in\mathrm{AP}(\mathbb{R},\mathbb{R}^n)$ according to [4].
- $t\mapsto\phi_2(t,u_1(t),u_1(-t))\in\xi(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ according to [4].

We pose the function $\psi(t) = f(t, u(t), u(-t)) - f(t, u_1(t), u_1(-t))$. It is easy to see that $\psi \in \mathcal{BC}(\mathbb{R}, \mathbb{R}^n)$, also we get:

$$\begin{aligned}
&\frac{1}{\nu([-z,z])}\int_{-z}^{z}\|\psi(t)\|d\mu(t)\\
&= \frac{1}{\nu([-z,z])}\int_{-z}^{z}\|f(t,u(t),u(-t)) - f(t,u_1(t),u_1(-t))\|d\mu(t)\\
&\leq \frac{1}{\nu([-z,z])}\int_{-z}^{z}(L_f^1\|u(t)-u_1(t)\| + L_f^2\|u(-t)-u_1(-t)\|)d\mu(t)\\
&\leq \frac{L_f^1}{\nu([-z,z])}\int_{-z}^{z}\|u_2(t)\|d\mu(t) + \frac{L_f^2}{\nu([-z,z])}\int_{-z}^{z}\|u_2(-t)\|d\mu(t).
\end{aligned}$$

From Lemma 2.3, we get

$$\lim_{z\to+\infty}\frac{1}{\nu([-z,z])}\int_{-z}^{z}\|\psi(t)\|d\mu(t) = 0 \ \ \textit{then } \psi \in \xi(\mathbb{R},\mathbb{R}^n,\nu,\mu).$$

So $[t \mapsto f(t, u(t), u(-t))] \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$.

Lemma 2.6. *Suppose that hypotheses **(H2)**, **(H3)**,**(H4)**,**(H6)** and **(H7)** hold. If $u \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ then $[s \mapsto \int_s^{+\infty} L(y-s)g(y, u(y), u(-y))dy] \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$.*

Proof. From Lemma 2.5, $[y \mapsto g(y, u(y), u(-y))] \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$. Then $g(y, u(y), u(-y)) = g_1(y) + g_2(y)$, where $g_1 \in \mathrm{AP}(\mathbb{R}, \mathbb{R}^n)$ and $g_2 \in \xi(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$.
We pose $K(t) = \displaystyle\int_t^{+\infty} L(y-t)g(y, u(y), u(-y))dy$, then

$$K(t) = \int_t^{+\infty} L(y-t)g_1(y)dy + \int_t^{+\infty} L(y-t)g_2(y)dy = K_1(t) + K_2(t),$$

with $K_1(t) = \displaystyle\int_t^{+\infty} L(y-t)g_1(y)dy$ and $K_2(t) = \displaystyle\int_t^{+\infty} L(y-t)g_2(y)dy$.
Firstly, we prove that $[t \mapsto K_1(t)] \in \mathrm{AP}(\mathbb{R}, \mathbb{R}^n)$. With a change of variable, we obtain:

$$K_1(t) = \int_0^{+\infty} L(y)g_1(y+t)dy.$$

Since $g_1 \in \mathrm{AP}(\mathbb{R}, \mathbb{R}^n)$, so $\forall \epsilon > 0, \exists l(\epsilon) > 0$, such that $\forall a \in \mathbb{R}, \exists \tau \in [a, a + l(\epsilon)]$ with:

$$\|g_1(t+\tau) - g_1(t)\|_\infty < \epsilon \ \forall t \in \mathbb{R}.$$

Now, we have

$$\begin{aligned}
\|K_1(t+\tau) - K_1(t)\| &= \|\int_0^{+\infty} L(y)g_1(y+t+\tau)dy - \int_0^{+\infty} L(y)g_1(y+t)dy\|\\
&\leq \int_0^{+\infty} L(y)\|g_1(y+t+\tau) - g_1(y+t)\|dy\\
&< c_1\epsilon = \epsilon'.
\end{aligned}$$

Therefore $K_1 \in \mathrm{AP}(\mathbb{R},\mathbb{R}^n)$. On the other hand, we prove that $K_2 \in \xi(\mathbb{R},\mathbb{R}^n,\mu,\nu)$. We have :

$$\begin{aligned}
&\lim_{z\to+\infty}\frac{1}{\nu([-z,z])}\int_{[-z,z]}\|K_2(t)\|d\mu(t)\\
&\le \lim_{z\to+\infty}\frac{1}{\nu([-z,z])}\int_{-z}^{z}\int_{t}^{+\infty}L(y-t)\|g_2(y)\|dyd\mu(t)\\
&\le \lim_{z\to+\infty}\frac{1}{\nu([-z,z])}\int_{-z}^{z}\int_{0}^{+\infty}L(y)\|g_2(y+t)\|dyd\mu(t)\\
&= \lim_{z\to+\infty}\int_{0}^{+\infty}\frac{L(y)}{\nu([-z,z])}\int_{-z}^{z}\|g_2(y+t)\|d\mu(t)dy.\ (Tonelli)
\end{aligned}$$

Using the L.C.D Theorem, we have

$$\lim_{z\to+\infty}\frac{1}{\nu([-z,z])}\int_{[-z,z]}\|K_2(t)\|d\mu(t)=0.$$

We obtain so $K_2 \in \xi(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ as a result $K \in \mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu)$.

Theorem 2.7. *Assume that $M:\mathbb{R}\to\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ defined by*

$$D(t)=\begin{pmatrix} A(t) & B(t)\\ -B(-t) & -A(-t)\end{pmatrix}$$

*be continuous, non-singular and almost periodic function such that $(D^{-1}(s))_{s\in\mathbb{R}}$ is bounded. If (**H0**),... (**H9**) are true, then equation (7) has a unique solution*

$$x(t)=h(t,x(\gamma_1(t)),x(\gamma_1(-t)))+\int_{-\infty}^{+\infty}G(s,t)H(s,x(\gamma_2(s)),x(\gamma_2(-s)))ds$$

$\in \mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ if and only if we have:

$$2k(L_f^1+L_f^2+2c_1(L_g^1+L_g^2))+\alpha(L_h^1+L_h^2)<\alpha.$$

Proof. Let ζ be the operator defined in $\mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ by

$$\zeta u(t)=h(t,u(\gamma_1(t)),u(\gamma_1(-t)))+\int_{-\infty}^{+\infty}G(s,t)H(s,u(\gamma_2(s)),u(\gamma_2(-s)))ds,$$

where

$$\text{G(s,t)}=\begin{cases} X(s)PX^{-1}(t) & \text{if } s\le t\\ -X(s)(I_n-P)X^{-1}(t) & \text{if } t\le s\end{cases}$$

and

$$\begin{aligned}
&H(s,u(\gamma_2(s)),u(\gamma_2(-s)))\\
&=f(s,u(\gamma_2(s)),u(\gamma_2(-s)))+\int_{s}^{+\infty}L(y-s)g(y,u(\gamma_2(y)),u(\gamma_2(-y)))dy\\
&\quad+\int_{-s}^{+\infty}L(y+s)g(y,u(\gamma_2(y)),u(\gamma_2(-y)))dy.
\end{aligned}$$

From Lemmas 2.3 and 2.4, $[y\mapsto u(\gamma_2(y)), y\mapsto(\gamma_2(-y))] \in \mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu)$, also according to Lemmas 2.5 and 2.6, we obtain that:

$$\begin{cases} \int_{s}^{+\infty}L(y-s)g(y,u(\gamma_2(y)),u(\gamma_2(-y)))dy\in\mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu),\\ \int_{-s}^{+\infty}L(y+s)g(y,u(\gamma_2(y)),u(\gamma_2(-y)))dy\in\mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu).\end{cases}$$

So $s \mapsto H(s, u(\gamma_2(s)), u(\gamma_2(-s))) \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$, then there exists $\phi_1 \in \mathrm{AP}(\mathbb{R}, \mathbb{R}^n)$ and $\phi_2 \in \xi(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ such that

$$H(s, u(\gamma_2(s)), u(\gamma_2(-s))) = \phi_1(s) + \phi_2(s).$$

Then we obtain

$$\begin{aligned}&\int_{-\infty}^{+\infty} G(s,t)H(s, u(\gamma_2(s)), u(\gamma_2(-s)))ds = \int_{-\infty}^{+\infty} G(s,t)(\phi_1(s) + \phi_2(s))ds\\ &= \int_{-\infty}^{+\infty} G(s,t)\phi_1(s)ds + \int_{-\infty}^{+\infty} G(s,t)\phi_2(s)ds\\ &= \zeta_1(t) + \zeta_2(t),\end{aligned}$$

From [13], we have $\zeta_1 \in \mathrm{AP}(\mathbb{R}, \mathbb{R}^n)$. Now, we prove that $\zeta_2 \in \xi(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$:

$$\begin{aligned}&\lim_{z\to+\infty} \tfrac{1}{\nu([-z,z])} \int_{[-z,z]} \|\zeta_2(t)\| d\mu(t)\\ &\leq \lim_{z\to+\infty} \tfrac{1}{\nu([-z,z])} \int_{[-z,z]} \int_{-\infty}^{+\infty} \|G(s,t)\| \|\phi_2(s)\| d\mu(t)\\ &\leq \lim_{z\to+\infty} \tfrac{k}{\nu([-z,z])} \int_{-z}^{z} [\int_{-\infty}^{t} e^{-\alpha(t-s)} \|\phi_2(s)\| ds + \int_{t}^{+\infty} e^{-\alpha(s-t)} \|\phi_2(s)\| ds] d\mu(t)\\ &\leq \lim_{z\to+\infty} \tfrac{k}{\nu([-z,z])} \int_{-z}^{z} [\int_{0}^{+\infty} e^{-\alpha s} \|\phi_2(t-s)\| ds + \int_{0}^{+\infty} e^{-\alpha s} \|\phi_2(s+t)\| ds] d\mu(t)\\ &\leq \lim_{z\to+\infty} k \int_{0}^{+\infty} e^{-\alpha s} [\frac{1}{\nu([-z,z])} \int_{-z}^{z} \|\phi_2(t-s)\| d\mu(t)\\ &+ \frac{1}{\nu([-z,z])} \int_{-z}^{z} \|\phi_2(s+t)\| d\mu(t)] ds,\end{aligned}$$

then in view of the Lebesgue dominated convergence Theorem, we obtain:

$$\begin{aligned}\lim_{z\to+\infty} \frac{1}{\nu([-z,z])} \int_{[-z,z]} \|\zeta_2(t)\| d\mu(t) &\leq k \int_{0}^{+\infty} e^{-\alpha s} [\lim_{z\to+\infty} \frac{1}{\nu([-z,z])} \int_{-z}^{z} \|\phi_2(t-s)\| d\mu(t)\\ &+ \lim_{z\to+\infty} \frac{1}{\nu([-z,z])} \int_{-z}^{z} \|\phi_2(t+s)\| d\mu(t)] ds = 0.\end{aligned}$$

So $\zeta_2 \in \xi(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$, as a result :

$$\zeta : \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu) \to \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu).$$

After we seek a condition so that ζ is a contraction in order to apply Banach's fixed point theorem.

Let $u, v \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$, then we have:

$$\begin{aligned}&\|H(t, v(\gamma_2(t)), v(\gamma_2(-t))) - H(t, u(\gamma_2(t)), u(\gamma_2(-t)))\|\\ &\leq \|f(t, v(\gamma_2(t)), v(\gamma_2(-t))) - f(t, u(\gamma_2(t)), u(\gamma_2(-t)))\|\\ &+ \int_{t}^{+\infty} L(s-t) \|g(s, v(\gamma_2(s)), v(\gamma_2(-s))) - g(s, u(\gamma_2(s)), u(\gamma_2(-s)))\| ds\\ &+ \int_{-t}^{+\infty} L(s+t) \|g(s, v(\gamma_2(s)), v(\gamma_2(-s))) - g(s, u(\gamma_2(s)), u(\gamma_2(-s)))\| ds\\ &\leq \|f(t, v(\gamma_2(t)), v(\gamma_2(-t))) - f(t, u(\gamma_2(t)), u(\gamma_2(-t)))\|\\ &+ \int_{0}^{+\infty} L(s) \|g(s+t, v(\gamma_2(s+t)), v(\gamma_2(-(s+t))) - g(s+t, u(\gamma_2(s+t)), u(\gamma_2(-(s+t)))\| ds\\ &+ \int_{0}^{+\infty} L(s) \|g(s-t, v(\gamma_2(s-t)), v(\gamma_2(-(s-t))) - g(s-t, u(\gamma_2(s-t)), u(\gamma_2(-(s-t)))\| ds\\ &\leq L_f^1 \|v \circ \gamma_2(t) - u \circ \gamma_2(t)\| + L_f^2 \|v \circ \gamma_2(-t) - u \circ \gamma_2(-t)\|\end{aligned}$$

$+\int_0^{+\infty} L(s)[L_g^1\|v\circ\gamma_2(s+t)-u\circ\gamma_2(s+t)\|+L_g^2\|v\circ\gamma_2(-(s+t))-u\circ\gamma_2(-(s+t))\|]ds$
$+\int_0^{+\infty} L(s)[L_g^1\|v\circ\gamma_1(s-t)-u\circ\gamma_2(s-t)\|+L_g^2\|v\circ\gamma_1(-(s-t))-u\circ\gamma_2(-(s-t))\|]ds$
$\leq (L_f^1+L_f^2)\|u\circ\gamma_2-v\circ\gamma_2\|_\infty+2c_1(L_g^1+L_g^2)\|u\circ\gamma_2-v\circ\gamma_2\|_\infty$
$\leq (L_f^1+L_f^2+2c_1(L_g^1+L_g^2))\|u-v\|_\infty.$
$= c_3\|u-v\|_\infty,$
with $c_3=L_f^1+L_f^2+2c_1(L_g^1+L_g^2)$. Then

$$\begin{aligned}
&\|\zeta v(t)-\zeta u(t)\|\\
&=\|h(t,v(\gamma_1(t)),v(\gamma_1(-t)))+\int_{-\infty}^{+\infty}G(s,t)H(s,v(\gamma_2(s)),v(\gamma_2(-s)))ds\\
&-h(t,u(\gamma_1(t)),u(\gamma_1(-t)))+\int_{-\infty}^{+\infty}G(s,t)H(s,u(\gamma_2(s)),u(\gamma_2(-s)))ds\|\\
&\leq\int_{-\infty}^{+\infty}\|G(s,t)\|\|H(s,v(\gamma_2(s)),v(\gamma_2(-s)))-H(s,u(\gamma_2(s)),u(\gamma_2(-s)))\|ds\\
&+\|h(t,v(\gamma_1(t)),v(\gamma_1(-t)))-h(t,u(\gamma_1(t)),u(\gamma_1(-t)))\|\\
&\leq c_3\|u-v\|_\infty\int_{-\infty}^{+\infty}\|G(s,t)\|ds+L_h^1\|v(\gamma_1(t))-u(\gamma_1(t))\|+L_h^2\|v(\gamma_1(-t))-u(\gamma_1(-t))\|\\
&\leq c_3\|u-v\|_\infty\int_{-\infty}^{+\infty}\|G(s,t)\|ds+(L_h^1+L_h^2)\|u-v\|_\infty\\
&\leq c_3\|u-v\|_\infty(\int_{-\infty}^{t}ke^{-\alpha(t-s)}ds+\int_t^{+\infty}ke^{-\alpha(s-t)}ds)+(L_h^1+L_h^2)\|u-v\|_\infty\\
&\leq 2kc_3\|u-v\|_\infty(\int_0^{+\infty}e^{-\alpha s)}ds+(L_h^1+L_h^2)\|u-v\|_\infty\\
&\leq(\frac{2kc_3}{\alpha}+L_h^1+L_h^2)\|u-v\|_\infty.
\end{aligned}$$

So the operator ζ is contractive if and only if

$$2k(L_f^1+L_f^2+2c_1(L_g^1+L_g^2))+\alpha(L_h^1+L_h^2)<\alpha.$$

According to Banach fixed point theorem ζ admits a unique fixed point in $\mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ and the Equation (7) has a unique (μ,ν) pseudo almost periodic solution if and only if

$$2k(L_f^1+L_f^2+2c_1(L_g^1+L_g^2))+\alpha(L_h^1+L_h^2)<\alpha.$$

2.4. The lipschitz coefficients of the functions are variable. In this section we consider the following conditions :

(**H10**) :

(1) There exists $L_f^1,L_f^2\in L^p(\mathbb{R},\mathbb{R}_+,ds)\cap L^p(\mathbb{R},\mathbb{R}_+,d\mu(s)),p>1$ and $\frac{1}{p}+\frac{1}{q}=1$ such that $\forall s\in\mathbb{R},x_1,x_2,y_1,y_2\in\mathbb{R}^n$ we have :
$\|f(s,x_1,x_2)-f(s,y_1,y_2)\|\leq L_f^1(s)\|x_1-y_1\|+L_f^2(s)\|x_2-y_2\|,$

(2) There exists $L_g^1,L_g^2\in L^p(\mathbb{R},\mathbb{R}_+,ds)\cap L^p(\mathbb{R},\mathbb{R}_+,d\mu(s)),p>1$ and $\frac{1}{p}+\frac{1}{q}=1$ such that $\forall s\in\mathbb{R},x_1,x_2,y_1,y_2\in\mathbb{R}^n$ we have :
$\|g(s,x_1,x_2)-g(s,y_1,y_2)\|\leq L_g^1(s)\|x_1-y_1\|+L_g^2(s)\|x_2-y_2\|,$

(3) There exists $L_h^1, L_h^2 : \mathbb{R} \to \mathbb{R}_+$ bounded functions such that $\forall s \in \mathbb{R}, x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ we have :

$$\begin{aligned} \|h(s,x_1,x_2) - h(s,y_1,y_2)\| &\leq L_h^1(s)\|x_1 - y_1\| + L_h^2(s)\|x_2 - y_2\| \\ &\leq c_4(\|x_1 - y_1\| + \|x_2 - y_2\|), \end{aligned}$$

with $c_4 = \max_{s\in\mathbb{R}}(L_h^1(s), L_h^2(s))$.

(**H11**) : $L : \mathbb{R} \to \mathbb{R}_+$, such that for all $\tau > 1$, $(\int_0^{+\infty} L(y)^\tau dy)^{\frac{1}{\tau}} = c_2 < +\infty$.

Lemma 2.8. *If (**H3**),(**H4**)and (**H10**) are true. If $u \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ then $[t \to f(t, u(t), u(-t))] \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$.*

Proof. We repeat the same things done for the proof of Lemma 2.5, we just have to prove that $[t \mapsto f(t, u(t), u(-t)) - f(t, u_1(t), u_1(-t))] \in \xi(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$.
We have:

$\frac{1}{\nu([-z,z])}\int_{-z}^{z} \|f(t,u(t),u(-t)) - f(t,u_1(t),u_1(-t))\|d\mu(t)$

$\leq \frac{1}{\nu([-z,z])}\int_{-z}^{z}(L_f^1(t)\|u(t) - u_1(t)\| + L_f^2(t)\|u(-t) - u_1(-t)\|)d\mu(t)$ (according to (**H10**))

$\leq \frac{\|u_2\|_\infty}{\nu([-z,z])}[\int_{-z}^{z} L_f^1(t)d\mu(t) + \int_{-z}^{z} L_f^2(t)d\mu(t)]$

$\leq \frac{\|u_2\|_\infty}{\nu([-z,z])}[\int_{-z}^{z}(L_f^1(t))^p d\mu(t))^{\frac{1}{p}}(\int_{-z}^{z} d\mu(t))^{\frac{1}{q}} + \int_{-z}^{z}(L_f^2(t))^p d\mu(t))^{\frac{1}{p}}(\int_{-z}^{z} d\mu(t))^{\frac{1}{q}}]$ (according to Holder's inequality)

$\leq \frac{\|u_2\|_\infty(\mu([-z,z]))^{\frac{1}{q}}}{\nu([-z,z])}(\|L_f^1\|_p + \|L_f^2\|_p)$

$\leq \frac{cste}{(\mu([-z,z]))^{\frac{1}{p}}}\frac{\mu([-z,z])}{\nu([-z,z])}$.

We deduce from hypothese (**H1**) that

$$\lim_{z\to+\infty}\frac{cste}{(\mu([-z,z]))^{\frac{1}{p}}}\frac{\mu([-z,z])}{\nu([-z,z])} = 0.$$

So

$$\lim_{z\to+\infty}\frac{1}{\nu([-z,z])}\int_{-z}^{z}\|f(t,u(t),u(-t)) - f(t,u_1(t),u_1(-t))\|d\mu(t) = 0.$$

Therefore $[t \mapsto f(t, u(t), u(-t)) - f(t, u_1(t), u_1(-t))] \in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$.

Theorem 2.9. *Assume that $D : \mathbb{R} \to \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ defined by*

$$D(t) = \begin{pmatrix} A(t) & B(t) \\ -B(-t) & -A(-t) \end{pmatrix}$$

*be continuous, non-singular and almost periodic function such that $(D^{-1}(s))_{s\in\mathbb{R}}$ is bounded. If (**H0**),..,(**H5**), (**H8**),..,(**H11**) are true, then equation (7) has a unique solution in $\mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ if and only if we have:*

$$c_4 + k(\frac{\|L_f^1\|_p + \|L_f^2\|_p}{(\alpha q)^{\frac{1}{q}}} + \frac{2c_2(\|L_g^1\|_p + \|L_g^2\|_p)}{\alpha}) < \frac{1}{2}.$$

Proof. Let u,v $\in \mathrm{PAP}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$, then we have:
$\|H(t, v(\gamma_2(t)), v(\gamma_2(-t))) - H(t, u(\gamma_2(t)), u(\gamma_2(-t)))\|$

$\leq \|f(t,v(\gamma_2(t)),v(\gamma_2(-t))) - f(t,u(\gamma_2(t)),u(\gamma_2(-t)))\|$
$+ \int_t^{+\infty} L(y-t)\|g(y,v(\gamma_2(y)),v(\gamma_2(-y))) - g(y,u(\gamma_2(y)),u(\gamma_2(-y)))\|dy$
$+ \int_{-t}^{+\infty} L(y+t)\|g(y,v(\gamma_2(y)),v(\gamma_2(-y))) - g(y,u(\gamma_2(y)),u(\gamma_2(-y)))\|dy$
$\leq L_f^1(t)\|v\circ\gamma_2(t) - u\circ\gamma_2(t)\| + L_f^2(t)\|v\circ\gamma_2(-t) - u\circ\gamma_2(-t)\|$
$+ \int_0^{+\infty} L(y)[L_g^1(y+t)\|v\circ\gamma_2(y+t) - u\circ\gamma_2(y+t)\| + L_g^2(y+t)\|v\circ\gamma_2(-(y+t)) - u\circ$ $\gamma_2(-(y+t))\|]dy$
$+ \int_0^{+\infty} L(y)[L_g^1(y-t)\|v\circ\gamma_2(y-t) - u\circ\gamma_2(y-t)\| + L_g^2(y-t)\|v\circ\gamma_2(-(y-t)) - u\circ$ $\gamma_2(-(y-t))\|]dy$
$\leq (L_f^1(t) + L_f^2(t))\|u-v\|_\infty$
$+ \int_0^{+\infty} L(y)(L_g^1(y+t) + L_g^2(y+t))dy)\|u-v\|_\infty$
$+ \int_0^{+\infty} L(y)(L_g^1(y-t) + L_g^2(y-t))dy)\|u-v\|_\infty$
$\leq (L_f^1(t)+L_f^2(t))\|u-v\|_\infty + 2[(\int_0^{+\infty} L(y)^q dy)^{\frac{1}{q}}\|L_g^1\|_p + (\int_0^{+\infty} L(y)^q dy)^{\frac{1}{q}}\|L_g^2\|_p]\|u-v\|_\infty$
$\leq [L_f^1(t) + L_f^2(t) + 2(\|L_g^1\|_p + \|L_g^2\|_p)(\int_0^{+\infty} L(y)^q dy)^{\frac{1}{q}})]\|u-v\|_\infty.$

So, we have

$$\|H(t,v(\gamma_2(t)),v(\gamma_2(-t) - H(t,u(\gamma_2(t)),u(\gamma_2(-t)\| \\ \leq [L_f^1(t) + L_f^2(t) + 2c_2(\|L_g^1\|_p + \|L_g^2\|_p)]\|u-v\|_\infty.$$

Then
$\|\zeta v(t) - \zeta u(t)\|$
$= \|h(t,v(\gamma_1(t)),v(\gamma_1(-t))) - h(t,u(\gamma_1(t)),u(\gamma_1(-t)))$
$+ \int_{-\infty}^{+\infty} G(s,t)H(s,v(\gamma_2(s)),v(\gamma_2(-s)))ds - \int_{-\infty}^{+\infty} G(s,t)H(s,u(\gamma_2(s)),u(\gamma_2(-s)))ds\|$
$\leq L_h^1(t)\|v(\gamma_1(t)) - u(\gamma_1(t))\| + L_h^2(t)\|v(\gamma_1(-t)) - u(\gamma_1(-t))\|$
$+ \int_{-\infty}^{+\infty} \|G(s,t)\|\|H(s,v(\gamma_2(s)),v(\gamma_2(-s)) - H(s,u(\gamma_2(s)),u(\gamma_2(-s))\|ds$
$\leq (L_h^1(t) + L_h^2(t))\|u-v\|_\infty$
$+ \int_{-\infty}^{+\infty} \|G(s,t)\|[L_f^1(s) + L_f^2(s) + 2c_2(\|L_g^1\|_p + \|L_g^2\|_p)]\|_p)]ds\|u-v\|_\infty$
$\leq 2c_4\|u-v\|_\infty + \int_{-\infty}^{t} [L_f^1(s) + L_f^2(s) + 2c_2(\|L_g^1\|_p + \|L_g^2\|_p)]ke^{-\alpha(t-s)}ds\|u-v\|_\infty$
$+ \int_t^{+\infty} [L_f^1(s) + L_f^2(s) + 2c_2(\|L_g^1\|_p + \|L_g^2\|_p)]ke^{-\alpha(s-t)}ds\|u-v\|_\infty$
$\leq 2c_4\|u-v\|_\infty + \int_0^{+\infty} [L_f^1(t-s) + L_f^2(t-s) + 2c_2(\|L_g^1\|_p + \|L_g^2\|_p)]ke^{-\alpha s)}\|u-v\|_\infty ds$
$+ \int_0^{+\infty} [L_f^1(s+t) + L_f^2(s+t) + 2c_2(\|L_g^1\|_p + \|L_g^2\|_p)]ke^{-\alpha s}ds\|u-v\|_\infty$

$$\leq 2c_4\|u-v\|_\infty + k\int_0^{+\infty}(L_f^1(s+t)+L_f^1(t-s))e^{-\alpha s}ds\|u-v\|_\infty$$
$$+k\int_0^{+\infty}(L_f^2(s+t)+L_f^2(t-s))e^{-\alpha s}ds\|u-v\|_\infty + 4kc_2(\|L_g^1\|_p+\|L_g^2\|_p)\int_0^{+\infty}e^{-\alpha s}ds\|u-v\|_\infty$$
$$\leq [2c_4 + 2k[(\|L_f^1\|_p + L_f^2\|_p)(\int_0^{+\infty} e^{-\alpha q s}ds)^{\frac{1}{q}} + \frac{2c_2}{\alpha}(\|L_g^1\|_p + \|L_g^2\|_p)]]\|u-v\|_\infty$$
$$\leq [2c_4 + 2k(\frac{\|L_f^1\|_p + \|L_f^2\|_p}{(\alpha q)^{\frac{1}{q}}} + \frac{2c_2(\|L_g^1\|_p + \|L_g^2\|_p)}{\alpha})]\|u-v\|_\infty.$$

Since $[c_4 + k(\frac{\|L_f^1\|_p+\|L_f^2\|_p}{(\alpha q)^{\frac{1}{q}}} + \frac{2c_2(\|L_g^1\|_p+\|L_g^2\|_p)}{\alpha})] < \frac{1}{2}$.

Then the operator ζ is a strict contraction. Using the Banach fixed point theorem, there exists a unique $x \in \mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ such that $\zeta x = x$ and Equation (7) has a unique solution in $\mathrm{PAP}(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ defined by :

$$x(t) = h(t, x(\gamma_1(s)), x(\gamma_1(-s))) + \int_{-\infty}^{+\infty} G(s,t)H(s, x(\gamma_2(s)), x(\gamma_2(-s)))ds.$$

Proposition 2.10. *If $sp(A+B)(A-B)\cap\mathbb{R}^- = \emptyset$ and g is pap, then the equation (4) has a unique pap solution.*

Proof. If A and B are constant, the system (4) has an exponential dichotomy if and only if $sp(D)\cap i\mathbb{R} = \emptyset$, with $D = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}$.

In effect, $sp(D)\cap i\mathbb{R} = \emptyset \Leftrightarrow sp(D^2)\cap\mathbb{R}^- = \emptyset$.

We have $D^2 = \begin{pmatrix} A^2-B^2 & AB-BA \\ AB-BA & A^2-B^2 \end{pmatrix} = \begin{pmatrix} M & N \\ N & M \end{pmatrix}$ with $\begin{cases} M = A^2-B^2, \\ N = AB-BA. \end{cases}$

We take $P = \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}$. Then, we have $P^{-1} = \frac{1}{2}\begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}$ and

$$D^2 = P\begin{pmatrix} M+N & M-N \\ M-N & M+N \end{pmatrix}P^{-1}.$$

Then $sp(D)\cap i\mathbb{R} = \emptyset \Leftrightarrow sp(D^2)\cap\mathbb{R}^- = \emptyset \Leftrightarrow sp(M+N)\cap\mathbb{R}^- = \emptyset$ and $sp(M-N)\cap\mathbb{R}^- = \emptyset$ or M+N=(A+B)(A-B), M-N=(A-B)(A+B) and sp(A+B)(A-B)=sp(A-B)(A+B)

So $sp(D)\cap i\mathbb{R} = \emptyset \Leftrightarrow sp(A+B)(A-B)\cap\mathbb{R}^- = \emptyset$.

We then assume that $sp(A+B)(A-B)\cap\mathbb{R}^- = \emptyset$:

Then we pose y(t)=x(-t) and $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, we obtain

$$\begin{cases} x'(t) = Ax(t) + By(t) + g(t), \\ y'(t) = -Bx(t) - Ay(t) - g(-t). \end{cases}$$

Putting $Z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$ with $\begin{cases} z_1(t) = y(t), \\ z_2(t) = x(-t). \end{cases}$

Then we have:

$$\begin{cases} z_1'(t) = Bx(-t) + Ay(-t) + g(t) = Az_1(t) + Bz_2(t) + g(t), \\ z_2'(t) = -Ax(-t) - By(-t) - g(t) = -Bz_1(t) - Az_2(t) - g(-t). \end{cases}$$

We note that Z(t), X(t) are bounded solutions of the previous system, by the uniqueness of bounded solution we have that Z(t)=X(t) consequently, x is a unique bounded solution of (4).

3. Example

We consider the two measures μ and ν, defined by $d\mu(t) = \rho_1(t)dt$ and $d\nu(t) = \rho_2(t)dt$ where

$$\rho_1(t) = e^{\sin(t)}, t \in \mathbb{R} \quad \text{and} \quad \rho_2(t) = \begin{cases} e^t & \text{if} \quad t \leq 0 \\ 1 & \text{if} \quad t > 0 \end{cases}.$$

We have

$$\lim_{z\to\infty} \sup(\frac{\mu([-z,z])}{\nu([-z,z])}) < +\infty,$$

then (H.1) is true.
Since $2 + \sin(t) \geq \sin(-t)\forall t \in \mathbb{R}$, so for all $I = [a, b]$, we have $1 + e^2\mu(I) \geq \mu(-I)$. Consequently (**H3**) is hold.

$$A(s) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2(s) & 1 - \frac{3}{2}\cos(s)\sin(s) \\ -1 - \frac{3}{2}\cos(s)\sin(s) & -1 + \frac{3}{2}\sin^2(s) \end{pmatrix}$$

A is periodic of period π and the eigenvalues of $A(s)$ are

$$\lambda_1 = \tfrac{-1+i\sqrt{7}}{4} \text{ and } \lambda_2 = \tfrac{-1-i\sqrt{7}}{4}.$$

We consider:

$$\begin{aligned} \frac{d}{dt}[u(t) - h(t, u(\gamma_1(t)), u(\gamma_1(-t)))] &= A(t)[u(t) - h(t, u(\gamma_1(t)), u(\gamma_1(-t)))] \\ &+ B(t)[u(-t) - h(-t, u(\gamma_1(-t)), u(\gamma_1(t)))] + f(t, u(\gamma_2(t)), u(\gamma_2(-t))) \quad (10) \\ &+ \int_t^{+\infty} L(y-t)g(y, u(\gamma_2(y)), u(\gamma_2(-y)))dy \quad (11) \\ &+ \int_{-t}^{+\infty} L(y+t)g(y, u(\gamma_2(y)), u(\gamma_2(-y)))dy, \end{aligned}$$

where:

$$\begin{cases} f(s,x,y) = ae^{-|s|}(\sin(x) + \sin(\sqrt{2}x) + \cos(y)), & \forall s,x,y \in \mathbb{R}, a \in \mathbb{R}, \\ g(s,x,y) = be^{-|s|}(\sin(x) + \cos(\sqrt{2}y) + \cos(y)), & \forall s,x,y \in \mathbb{R}, b \in \mathbb{R}, \\ h(s,y,z) = ce^{-|s|}(\sin(x) + \sin(\sqrt{2}y)), & \forall s,x,y \in \mathbb{R}, c \in \mathbb{R}, \\ L(s) = \frac{e^{-s}}{2}, & \\ B(s) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \\ \gamma_1(s) = \gamma_2(s) = s - p, p > 0. & \end{cases}$$

Then we have :

- the system $u^{'}(t) = A(t)u(t)$ has an exponential dichotomy on $\mathbb{R}$ with parameters (P,k,α)(according to Markus and Yamabe) then ((H.8)is true),
- $f, g \in \text{PAPU}(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}, \mu, \nu)$ ((H.4)is true),
- $L_f^1(t) = (1+\sqrt{2})|a|e^{-|t|}, L_f^2(t) = |a|e^{-|t|}, L_g^1(t) = |b|e^{-|t|}, L_g^2(t) = (1+\sqrt{2})|b|e^{-|t|}$ $L_f^1(t), L_f^2(t), L_g^1(t), L_g^2(t) \in L^2(\mathbb{R}, \mathbb{R}_+, ds) \cap L^2(\mathbb{R}, \mathbb{R}_+, d\mu(s)), p = q = 2,$
- $\|L_f^1\|_2 = \sqrt{2}(1+\sqrt{2})|a|, \|L_f^2\|_2 = \sqrt{2}|a|, \|L_g^1\|_2 = \sqrt{2}|b|, \|L_g^2\|_2 = \sqrt{2}(1+\sqrt{2})|b|,$

- $c_2 = \frac{1}{2\sqrt{2}}$,
- $\sup_{s\in\mathbb{R}} \|B(s)\| = 0 < \frac{\alpha}{4k^2}$ ((**H9**) is true),
- $c_4 = \sqrt{2}|c|$.

We deduce that the hypotheses (**H0**),..,(**H5**),(**H8**),..,(**H11**) of the Theorem 2.7 are hold and we have $D : \mathbb{R} \to \mathbb{R}^4 \times \mathbb{R}^4$ defined by

$$D(t) = \begin{pmatrix} A(t) & B(t) \\ -B(-t) & -A(-t) \end{pmatrix}$$

be continuous, non-singular and almost periodic function that $(M^{-1}(s))_{s\in\mathbb{R}}$ is bounded. Then, the equation (10) has a unique solution if and only if

$$|c| + (2+\sqrt{2})\alpha k(\sqrt{\alpha}|a| + |b|) < \frac{1}{2\sqrt{2}}.$$

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