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On Metric Dimension of Tridecagonal Circular Ladder

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Abstract. In n-dimensional space, convex polytopes are geometric objects characterised by straight edges and flat faces. They are a fascinating and significant challenge in many branches of mathematics and its applications because of their convexity, simplicity, and rich mathematical features. Let $\mathbb{G} = (V, E)$ be a simple, connected and undirected graph of order h. Let B be an ordered subset of the set of vertices $V(\mathbb{G})$. If vector of distances of disticnt vertices of $\mathbb{G}$ with respect to the set B are distinct, then the set B is referred as resolving set or vertex resolving set for the graph $\mathbb{G}$. A resolving set for $\mathbb{G}$ with least possible cardinality is termed as metric basis for $\mathbb{G}$ and the number of elements in a metric basis for $\mathbb{G}$ is known as the metric dimension of the graph $\mathbb{G}$. In this manuscript, we prove that the metric dimension is three for two closely related families of convex polytopes.

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1. Introduction

Metric dimension, a fundamental concept within graph theory, relates to the minimum number of vertices in a graph that uniquely identifies individual vertices. In other words, it seeks to capture the essentials of location and direction within the graph by determining the set size needed to "resolve" or uniquely label the vertices. This notion has applications in network design, facility location and navigation, making it a significant area of study within graph theory. Metric dimension and resolving set are concepts with numerous applications in both computer technology and mathematics. Let $\mathbb{G} = (V, E)$ represent a simple, undirected and connected graph, with V and E denoting the corresponding sets of vertices and edges. Let $d_{\mathbb{G}}(p, q)$, or $d(p, q)$ for short, be the distance between the vertices p and q. $\delta(\mathbb{G})$ and $\Delta(\mathbb{G})$, respectively, represent the minimum and highest degrees of $\mathbb{G}$.

A vertex v is said to resolve two vertices $l, m \in V(\mathbb{G})$ if $d(v, l) \neq d(v, m)$. Let B be an ordered subset of vertices in $V(\mathbb{G})$, then B is referred to as the resolving set (or metric generator) for $\mathbb{G}$ if every pair of vertices $p, q \in V(\mathbb{G})$ with $p \neq q$ is resolved by some vertex $v \in B$. The lowest cardinality resolving set for graph $\mathbb{G}$ is referred to as its metric basis or minimum resolving set for $\mathbb{G}$, and this minimal cardinality is also termed its metric dimension. It is represented by $dim(\mathbb{G})$.
With respect to an ordered subset of vertices, $B = \{c_1, c_2, c_3, ..., c_p\}$ and a vertex x in $V(\mathbb{G})$, $d(x|B) = (d(c_1, x), d(c_2, x), d(c_3, x), \ldots, d(c_p, x)$ is its p-code, co-ordinate

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representation with respect to B. If $d(m|B) \neq d(n|B)$ for each pair of vertices $m, n \in V(\mathbb{G})$ with $m \neq n$, then the set B is the metric generator for $\mathbb{G}$.

Slater[13] introduced the concept of the metric dimension and termed the metric generators as locating sets. However, the same notion of metric dimension was also introduced by Harary and Melter [4], who called metric generators resolving sets. In this paper, we shall adopt resolving sets for metric generators and locating sets. Apart from the aforementioned groundbreaking articles [4, 11], there are further research works in the literature [1, 3, 7, 8, 9] that address the theoretical and practical aspects of this invariant.

A polytope is a geometric object with flat sides in basic geometry. A covnex polytope is a particular kind of polytope that possesses an extra characteristic of being a convex set contained in the n-dimensional space R^n. Convex polytopes have found applications in fields such as operations research, computer science, architecture and materials science. They are employed in optimization problems, as they often represent feasible solution spaces with desirable properties. Convex polytopes have been extensively investigated due to their elegance and relevance in various mathematical domains, ranging from combinatorics and optimization to geometry and topology. The metric dimensions of certain types of convex polytopes have been considered in [5, 6, 10]. Plane graphs with constant metric dimension were presented by Sharma and Bhat [11, 12]. Next, we give a brief overview of a new family of convex polytopes and some of its basic properties.

Tridecagonal Circular Ladder: T_r denotes the tridecagonal circular ladder (TCL), a convex polytope with radial symmetry that has $7r$ vertices and $8r$ edges. It has $5r$ vertices of degree 2 and $2r$ vertices of degree 3. As shown in Figure 1, the tridecagonal circular ladder is composed of r faces with 13 sides each, one face with r edges, and a second face with $2r$ edges.

Imran et al. [5] presented the following open problem:

Problem: *Characterize the classes of radially symmetrical plane graphs H obtained from $\mathbb{G}$ by adding new edges in $\mathbb{G}$ such that $dim(H) = dim(\mathbb{G})$ and $V(H) = V(\mathbb{G})$.*

By building a planar graph family, T_r, as previously stated, we try to partially solve this problem. Next we construct a new family of convex polytopes named T_r^* with a similar set of vertices by adding additional edges to T_r at different places. For two classes of convex polytopes that have a close relationship and have an identical set of vertices, we calculated the metric dimension in this research paper.

2. Preliminaries

In this section, we discuss some basic concepts about the metric dimension of graphs.

Independent Set:[2] An independent set of vertices is a collection of vertices where no two vertices are adjacent.

Independent resolving set:[2] An independent resolving set is a subset B of vertices that is both resolving and independent.

We consider two convex polytopes in this research, for which we have $V(T_r) =$

$V(T_r^*) = \{a_i, b_i, c_i, d_i, e_i, f_i, g_i : 1 \leq i \leq r\}$. A, B, C, D, E, F, and G, respectively, represent the set of coordinates or metric codes for the convex polytopes T_r and T_r^*, corresponding to the vertices $a_i, b_i, c_i, d_i, e_i, f_i$, and g_i.

Khuller et al. [7] proved that for any connected graphs with metric dimension two, the following result holds: **Theorem.** [7] For the connected graph $\mathbb{G}$ of cardinality two, that is, $\mid A \mid = 2$, let $A \subset V(\mathbb{G})$ be the metric basis. Let $A = \{w, e\}$. The following thus holds true:
(i) There is only a single unique shortest path, P, between the vertices w and e.
(ii) The valencies of the vertices w and e can never exceed 3.
(iii) The valency of any other vertex on P can never exceed 5.

3. Metric Dimension of Tridecagonal Circular Ladder T_r

The structure of the new family of T_r is discussed in this section. We examine some of its fundamental attributes and determine its metric dimension.

The Graph of T_r:

The TCL T_r can be obtained from from the Heptagonal Circular Ladder Γ_r by placing $4n$ new vertices between the vertices a_i and b_i $(1 \leq i \leq r)$ in Γ_n. It has $8r$ number of edges and $7r$ number of vertices (see Figure 1). The TCL's T_r set of vertices and edges are separately portrayed by $V(T_r)$ and $E(T_r)$, where $V(T_r) = \{a_i, b_i, c_i, d_i, e_i, f_i, g_i : 1 \leq i \leq r\}$ and $E(T_r) = \{a_i b_i, b_i c_i, c_i d_i, d_i e_i, e_i f_i, f_i g_i, f_i g_{i-1} : 1 \leq i \leq r\}$.

We call the cycle that the vertices $\{a_i : 1 \leq i \leq r\}$ induced in the graph T_r as a-cycle, the cycle generated by the vertices $\{f_i, g_i : 1 \leq i \leq r\}$ in the graph, T_r as the fg-cycle, the vertices $\{f_i : 1 \leq i \leq r\}$ in the graph T_r as inner vertices, the vertices $\{g_i : 1 \leq i \leq r\}$ in the graph T_r as outer vertices. In the following result, we look into the TCL T_r graph's metric dimension.

Theorem 1. $dim(T_r) = 3$, where $r \geq 6$ is even integer.

Proof. Since, r is even so $r = 2m$, where $m \geq 3$ is an integer. Let $F = \{a_2, a_{m+1}, a_r\} \subset V(T_r)$. Now, to show that F is a resolving set for TCL T_r, we give metric codes for each vertex of T_r with respect to the set F.
The metric coordinate of $a_l (1 \leq l \leq r)$ vertices are

$$d(a_l|F) = \left\{ \begin{array}{ll} (l, m, l), & ; l = 1 \\ (l-2, m-l+1, l), & ; 2 \leq l \leq m+1 \\ (2m-l+2, l-m-1, 2m-l), & ; m+2 \leq l \leq 2m \end{array} \right\}$$

The metric coordinate of $b_l (1 \leq l \leq k)$ vertices are:

$$d(b_l|F) = d(a_l|F) + (1, 1, 1).$$

The metric coordinates of $c_l (1 \leq l \leq k)$ vertices are:

$$d(c_l|F) = d(a_l|F) + (2, 2, 2).$$

The metric coordinate of $d_l (1 \leq l \leq k)$ vertices are:

$$d(d_l|F) = d(a_l|F) + (3, 3, 3).$$

The metric coordinate of $e_l (1 \leq l \leq k)$ vertices are:

$$d(e_l|F) = d(a_l|F) + (4, 4, 4).$$

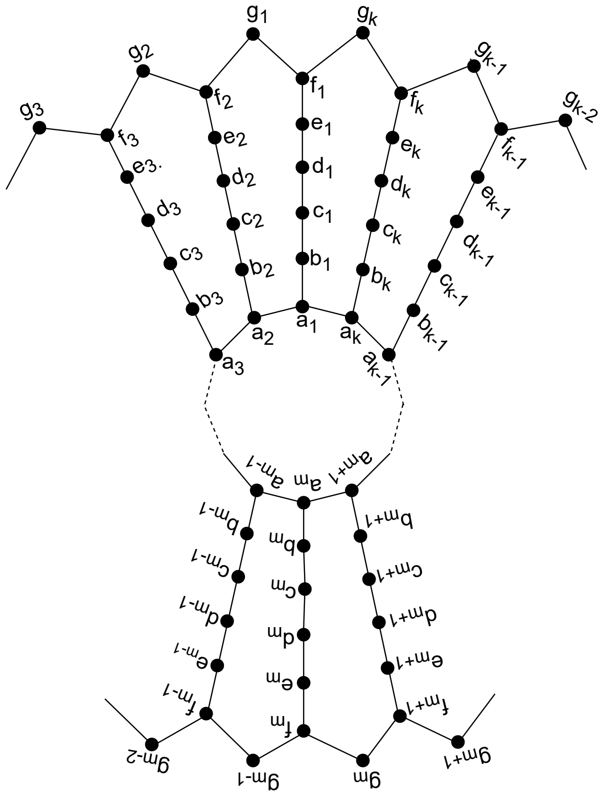

FIGURE 1. Tridecagonal Circular Ladder T_r.

The metric coordinate of $f_l (1 \leq l \leq k)$ vertices are:

$$d(f_l|F) = d(a_l|F) + (5, 5, 5).$$

$$d(g_l|F) \quad = \quad \left\{ \begin{array}{ll} (6, m+5, 7), & ; l = 1 \\ (l+4, m-l+6, l+6), & ; 2 \leq l \leq m-1 \\ (l+4, m-l+6, 2m-l+5), & ; l = m \\ (l+4, m-l+5, 2m-l+5), & ; l = m+1 \\ (2m-l+7, l-m+5, 2m-l+5), & ; m+2 \leq l \leq 2m-1 \\ (2m-l+7, l-m+5, 2m-l+6), & ; l = 2m \end{array} \right\}$$

From these codes, we have $A \cap B \cap C \cap D \cap E \cap F \cap G = \phi$, so we find that no two vertices in T_r have the same metric co-ordinates, implying that $dim(T_r) \leq 3$. Now to finish the proof, we prove that $dim(T_r) \geq 3$.

We prove that $dim(T_r) \geq 3$ by showing that there does not exist a resolving set F with $\mid F \mid = 2$. Then, we have the following:

TABLE 1. Possible resolving sets with cardinality two and their contradictions.

Possible resolving set with two elements	Contradictions
$F = \{a_1, a_j\},\ 2 \leq j \leq k$	$d(b_1\|F) = d(a_k\|F)$ for $2 \leq j \leq m$ $d(a_2\|F) = d(a_k\|F)$ for $j = m+1$, a contradiction
$F = \{b_1, b_j\},\ 2 \leq j \leq k$	$d(b_k\|F) = d(a_{k-1}\|F)$ for $2 \leq j \leq m-1$ $d(c_2\|F) = d(b_{k-1}\|F)$ for $j = m$ $d(a_2\|F) = d(a_k\|F)$ for $j = m+1$, a contradiction
$F = \{c_1, c_j\},\ 2 \leq j \leq k$	$d(b_k\|F) = d(a_{k-1}\|F)$ for $2 \leq j \leq m-1$ $d(c_2\|F) = d(b_{k-1}\|F)$ for $j = m$ $d(a_2\|F) = d(a_k\|F)$ for $j = m+1$
$F = \{d_1, d_j\},\ 2 \leq j \leq k$	$d(b_k\|F) = d(a_{k-1}\|F)$ for $2 \leq j \leq m-1$ $d(c_2\|F) = d(b_{k-1}\|F)$ for $j = m$ $d(a_2\|F) = d(a_k\|F)$ for $j = m+1$ a contradiction
$F = \{e_1, e_j\},\ 2 \leq j \leq k$	$d(b_{k-1}\|F) = d(a_{k-2}\|F)$ for $2 \leq j \leq m-2$ $d(e_3\|F) = d(b_k\|F)$ for $m-1 \leq j \leq m$ $d(f_2\|F) = d(f_k\|F)$ for $j = m+1)$ a contradiction
$F = \{f_1, f_j\},\ 2 \leq j \leq k$	$d(c_{k-1}\|F) = d(b_{k-2}\|F)$ for $2 \leq j \leq m-2$ $d(d_3\|F) = d(a_k\|F)$ for $m-1 \leq j \leq m$ $d(b_3\|F) = d(b_{k-1}\|F)$ for $j = m+1$ a contradiction
$F = \{g_1, g_j\},\ 2 \leq j \leq k$	$d(b_{k-1}\|F) = d(a_{k-2}\|F)$ for $2 \leq j \leq m-2$ $d(c_3\|F) = d(a_1\|F)$ for $m-1 \leq j \leq m$ $d(f_3\|F) = d(f_k\|F)$ for $j = m+1$ a contradiction
$F = \{a_1, b_j\},\ 1 \leq j \leq k$	$d(b_k\|F) = d(a_{k-1}\|F)$ for $1 \leq j \leq m-1$ $d(f_2\|F) = d(e_{k-1}\|F)$ for $j = m$ $d(g_1\|F) = d(g_k\|F)$ for $j = m+1$ a contradiction
$F = \{a_1, c_j\},\ 1 \leq j \leq k$	$d(b_k\|F) = d(a_{k-1}\|F)$ for $1 \leq j \leq m-1$ $d(f_2\|F) = d(d_{k-1}\|F)$ for $j = m$ $d(g_1\|F) = d(g_k\|F)$ for $j = m+1$ a contradiction
$F = \{a_1, d_j\},\ 1 \leq j \leq k$	$d(b_k\|F) = d(a_{k-1}\|F)$ for $1 \leq j \leq m-1$ $d(e_1\|F) = d(d_k\|F)$ for $j = m$ $d(g_1\|F) = d(g_k\|F)$ for $j = m+1$ a contradiction
$F = \{a_1, e_j\},\ 1 \leq j \leq k$	$d(b_{k-1}\|F) = d(a_{k-2}\|F)$ for $1 \leq j \leq m-2$ $d(b_k\|F) = d(a_{k-1}\|F)$ for $j = m-1$ $d(g_k\|F) = d(f_k\|F)$ for $j = m$ $d(g_1\|F) = d(g_k\|F)$ for $j = m+1$ a contradiction
$F = \{a_1, f_j\},\ 1 \leq j \leq k$	$d(b_{k-1}\|F) = d(a_{k-2}\|F)$ for $1 \leq j \leq m-2$ $d(b_k\|F) = d(a_{k-1}\|F)$ for $j = m-1$ $d(g_k\|F) = d(f_k\|F)$ for $j = m$ $d(g_1\|F) = d(g_k\|F)$ for $j = m+1$ a contradiction
$F = \{a_1, g_j\},\ 1 \leq j \leq k$	$d(b_{k-1}\|F) = d(a_{k-2}\|F)$ for $j = 1$ $d(b_k\|F) = d(a_{k-1}\|F)$ for $2 \leq j \leq m-2$ $d(b_1\|F) = d(a_k\|F)$ for $j = m-1$ $d(g_1\|F) = d(f_k\|F)$ for $j = m$ $d(f_3\|F) = d(g_{k-1}\|F)$ for $j = m+1$ a contradiction
$F = \{b_1, c_j\},\ 1 \leq j \leq k$	$d(b_k\|F) = d(a_{k-1}\|F)$ for $1 \leq j \leq m-1$ $d(e_2\|F) = d(d_{k-1}\|F)$ for $(j = m)$ $d(e_2\|F) = d(e_k\|F)$ for $(j = m+1)$ a contradiction
$F = \{b_1, d_j\},\ 1 \leq j \leq k$	$d(b_k\|F) = d(a_{k-1}\|F)$ for $1 \leq j \leq m-1$ $d(e_2\|F) = d(d_{k-1}\|F)$ for $j = m$ $d(e_2\|F) = d(e_k\|F)$ for $j = m+1$ a contradiction
$F = \{b_1, e_j\},\ 2 \leq j \leq k$	$d(b_{k-1}\|F) = d(a_{k-2}\|F)$ for $j = 1$ $d(b_k\|F) = d(a_{k-1}\|F)$ for $2 \leq j \leq m-2$ $d(d_2\|F) = d(c_{k-1}\|F)$ for $j = m$ $d(e_2\|F) = d(e_k\|F)$ for $j = m+1$ a contradiction

Table 2. Possible resolving sets with cardinality two and their contradictions

Possible resolving set with two elements	Contradictions
$F=\{b_1,f_l\},\ 1\le l\le k$	$d(b_{k-1}\|F)=d(a_{k-2}\|F)$ for $j=1$ $d(b_k\|F)=d(a_{k-1}\|F)$ for $2\le j\le m-2$ $d(d_2\|F)=d(c_{k-1}\|F)$ for $j=m$ $d(e_2\|F)=d(e_k\|F)$ for $j=m+1$ a contradiction
$F=\{b_1,g_j\},\ 1\le j\le k$	$d(b_{k-1}\|F)=d(a_{k-2}\|F)$ for $j=1$ $d(b_k\|F)=d(a_{k-1}\|F)$ for $2\le j\le m-2$ $d(f_2\|F)=d(d_{k-1}\|F)$ for $j=m-1$ $d(f_2\|F)=d(e_k\|F)$ for $j=m$ $d(e_2\|F)=d(f_k\|F)$ for $j=m+1$ a contradiction
$F=\{c_1,d_j\},\ 1\le j\le k$	$d(b_k\|F)=d(a_{k-1}\|F)$ for $1\le j\le m-1$ $d(f_5\|F)=d(e_{k-4}\|F)$ for $j=m$ $d(d_2\|F)=d(d_k\|F)$ for $j=m+1$ a contradiction
$F=\{c_1,e_j\},\ 1\le j\le k$	$d(b_k\|F)=d(a_{k-1}\|F)$ for $1\le j\le m-1$ $d(e_3\|F)=d(d_{k-2}\|F)$ for $j=m$ $d(d_2\|F)=d(d_k\|F)$ for $j=m+1$ a contradiction
$F=\{c_1,f_j\},\ 1\le j\le k$	$d(b_{k-1}\|F)=d(a_{k-2}\|F)$ for $j=1$ $d(b_k\|F)=d(a_{k-1}\|F)$ for $2\le j\le m-1$ $d(d_2\|F)=d(c_{k-1}\|F)$ for $j=m$ $d(c_2\|F)=d(c_k\|F)$ for $j=m+1$ a contradiction
$F=\{c_1,g_j\},\ 1\le j\le k$	$d(b_{k-1}\|F)=d(a_{k-2}\|F)$ for $j=1$ $d(b_k\|F)=d(a_{k-1}\|F)$ for $2\le j\le m-2$ $d(f_4\|F)=d(d_{k-3}\|F)$ for $j=m-1$ $d(f_4\|F)=d(e_{k-2}\|F)$ for $j=m$ $d(d_2\|F)=d(e_k\|F)$ for $j=m+1$ a contradiction
$F=\{d_1,e_j\},\ 1\le j\le k$	$d(b_k\|F)=d(a_{k-1}\|F)$ for $1\le j\le m-1$ $d(e_5\|F)=d(d_{k-4}\|F)$ for $j=m$ $d(d_2\|F)=d(d_k\|F)$ for $j=m+1$ a contradiction
$F=\{d_1,f_j\},\ 1\le j\le k$	$d(b_{k-1}\|F)=d(a_{k-2}\|F)$ for $j=1$ $d(b_k\|F)=d(a_{k-1}\|F)$ for $2\le j\le m-1$ $d(e_5\|F)=d(d_{k-4}\|F)$ for $j=m$ $d(d_2\|F)=d(d_k\|F)$ for $j=m+1$ a contradiction
$F=\{d_1,g_j\},\ 1\le j\le k$	$d(b_{k-1}\|F)=d(a_{k-2}\|F)$ for $j=1$ $d(b_k\|F)=d(a_{k-1}\|F)$ for $2\le j\le m-2$ $d(g_5\|F)=d(d_{k-4}\|F)$ for $j=m-1$ $d(g_2\|F)=d(e_k\|F)$ for $j=m$ $d(e_6\|F)=d(f_{k-4}\|F)$ for $j=m+1$ a contradiction
$F=\{e_1,f_j\},\ 1\le j\le k$	$d(b_{k-1}\|F)=d(a_{k-2}\|F)$ for $j=1$ $d(b_k\|F)=d(a_{k-1}\|F)$ for $2\le j\le m-1$ $d(f_{10}\|F)=d(e_{k-9}\|F)$ for $j=m$ $d(e_2\|F)=d(e_k\|F)$ for $j=m+1$ a contradiction
$F=\{e_1,g_j\},\ 1\le j\le k$	$d(b_{k-1}\|F)=d(a_{k-2}\|F)$ for $j=1$ $d(b_k\|F)=d(a_{k-1}\|F)$ for $2\le j\le m-2$ $d(g_9\|F)=d(e_{k-9}\|F)$ for $j=m-1$ $d(g_3\|F)=d(e_{k-1}\|F)$ for $j=m$ $d(e_2\|F)=d(g_{k-1}\|F)$ for $j=m+1$ a contradiction
$F=\{f_1,g_j\},\ 1\le j\le k$	$d(b_{k-1}\|F)=d(a_{k-2}\|F)$ for $j=1$ $d(b_k\|F)=d(a_{k-1}\|F)$ for $2\le j\le m-3$ $d(g_3\|F)=d(c_k\|F)$ for $m-2\le j\le m-1$ $d(f_3\|F)=d(d_k\|F)$ for $j=m$ $d(e_2\|F)=d(g_{k-1}\|F)$ for $j=m+1$ a contradiction

Consequently, the explanation in Table 1 and Table 2, leads us to the conclusion that $V(T_r)$ does not include a resolving set of cardinality two, implying that $dim(T_r) = 3$. □

Corollary 1. *The independent resolving number for convex polytope T_r for an even positive integer $r \geq 6$ is* 3.

4. Metric Dimension of the Convex Polytope T_r^*

We begin by discussing the construction of a new family of T_r^* that we managed to generate from an TCL T_r. We find its metric dimension and explore some of its basic properties.

The graph of T_r^*: Taking the TCL T_r and extending r additional edges between the vertices f_i and f_{i+1} in graph T_r for $1 \leq i \leq r$ yields the convex polytope T_r^*. There are $7r$ vertices and $9r$ edges in it. It has r faces with 12 edges (see Fig. 2). $V(T_r^*)$ and $E(T_r^*)$ represent the set of vertices and set of edges of T_r^*, respectively, where $V(T_r^*) = V(T_r)$ and $E(T_r^*) = E(T_r) \cup \{f_i f_{i+1} : 1 \leq i \leq r\}$.

The cycle generated by the vertices set $\{a_i : 1 \leq i \leq r\}$ in the graph T_r^* as the a-cycle; the cycle originated by the vertices $\{f_i, g_i : 1 \leq i \leq r\}$ in the graph, T_r^* as the fg-cycle; the vertices $\{b_i : 1 \leq i \leq r\}$ in the graph T_r^* as inner vertices, and the vertices $\{e_i : 1 \leq i \leq r\}$ in the graph T_r^* as outer vertices. In the next result, we study about the vertex set of the graph T_r^*.

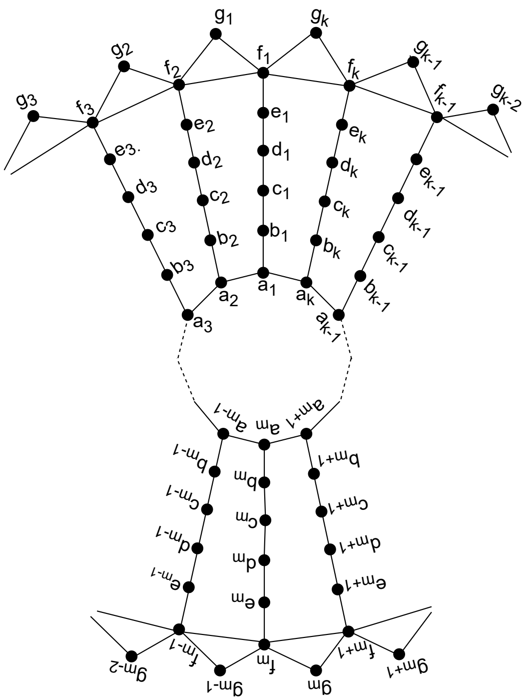

FIGURE 2. The Convex Polytope T_r^*.

Theorem 2. *For $r \geq 6$ an even integer, $dim(T_r^*) = 3$.*

Proof. Since, n is even so $r = 2m$, where $m \geq 3$ is an integer. Let $F = \{a_2, a_{m+1}, a_r\} \subset V(T_r^*)$. Now, to prove that F serves as resolving set for the convex polytope T_r^*, we assign metric codes for all vertices of T_r^* with respect to the set F.
The metric codes for the vertices $\{a_l : 1 \leq l \leq r\}$ are

$$d(a_l|F) = \left\{ \begin{array}{ll} (l, m, l), & ; l = 1 \\ (l-2, m-l+1, l), & ; 2 \leq l \leq m+1 \\ (2m-l+2, l-m-1, 2m-l), & ; m+2 \leq l \leq 2m \end{array} \right\}$$

The metric co-ordinates of $b_l (1 \leq l \leq r)$ vertices are:

$$d(b_l|F) = d(a_l|F) + (1, 1, 1).$$

The metric co-ordinates of $c_l (1 \leq l \leq r)$ vertices are:

$$d(c_l|F) = d(a_l|F) + (2, 2, 2).$$

The metric co-ordinates of $d_l (1 \leq l \leq r)$ vertices are:

$$d(d_l|F) = d(a_l|F) + (3, 3, 3).$$

The metric co-ordinates of $e_l (1 \leq l \leq r)$ vertices are:

$$d(e_l|F) = d(a_l|F) + (4, 4, 4).$$

The metric co-ordinates of $f_l (1 \leq l \leq r)$ vertices are:

$$d(f_l|F) = d(a_l|F) + (5, 5, 5).$$

$$d(g_l|F) = \left\{ \begin{array}{ll} (6, m+5, 7), & ; l = 1 \\ (l+4, m-l+6, l+6) & ; 2 \leq l \leq m-1 \\ (l+4, m-l+6, 2m-l+5), & ; l = m \\ (l+4, m-l+5, 2m-l+5), & ; l = m+1 \\ (2m-l+7, l-m+5, 2m-l+5), & ; m+2 \leq l \leq 2m-1 \\ (2m-l+7, l-m+5, 2m-l+6), & ; l = 2m \end{array} \right\}$$

From these codes, we have $A \cap B \cap C \cap D \cap E \cap F \cap G = \phi$, so we find that no two vertices in T_r^* have the same metric co-ordinates, it follows that $dim(T_r^*) \leq 3$. Now to finish the proof for this case, we prove that $dim(T_r^*) \geq 3$.
We prove that $dim(T_r^*) \geq 3$ by showing that there exist no resolving set F with $\mid F \mid = 2$. Then, we have the following:

TABLE 3. Possible resolving sets with cardinality two and their contradictions.

Possible resolving set with two elements	Contradictions
$F = \{a_1, a_j\},\ 2 \leq j \leq k$	$d(b_1|F) = d(a_k|F)$ for $2 \leq j \leq m$ $d(a_2|F) = d(a_k|F)$ for $j = m+1$, a contradiction
$F = \{b_1, b_j\},\ 2 \leq j \leq k$	$d(b_k|F) = d(a_{k-1}|F)$ for $2 \leq j \leq m-1$ $d(e_2|F) = d(d_{k-1}|F)$ for $j = m$ $d(e_2|F) = d(e_k|F)$ for $j = m+1$, a contradiction
$F = \{c_1, c_j\},\ 2 \leq j \leq k$	$d(b_k|F) = d(a_{k-1}|F)$ for $2 \leq j \leq m-1$ $d(f_k|F) = d(g_k|F)$ for $j = m$ $d(f_2|F) = d(f_k|F)$ for $j = m+1$ a contradiction

TABLE 4. Possible resolving sets with cardinality two and their contradictions

Possible resolving set with two elements	Contradictions
$F=\{d_1,d_j\},\ 2\le j\le k$	$d(b_k\|F)=d(a_{k-1}\|F)$ for $2\le j\le m-1$ $d(c_2\|F)=d(b_{k-1}\|F)$ for $j=m$ $d(f_2\|F)=d(f_k\|F)$ for $j=m+1$ a contradiction
$F=\{e_1,e_j\},\ 2\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $2\le j\le m$ $d(f_2\|F)=d(f_k\|F)$ for $j=m+1$ a contradiction
$F=\{g_1,g_j\},\ 2\le j\le k$	$d(d_k\|F)=d(e_{k-1}\|F)$ for $2\le j\le m-2$ $d(d_2\|F)=d(f_{k-1}\|F)$ for $j=m-1$ $d(d_3\|F)=d(e_{k-1}\|F)$ for $j=m$ $d(d_3\|F)=d(d_k\|F)$ for $j=m+1$ a contradiction
$F=\{a_1,b_j\},\ 1\le j\le k$	$d(b_k\|F)=d(a_{k-1}\|F)$ for $1\le j\le m-1$ $d(f_k\|F)=d(g_k\|F)$ for $j=m$ $d(f_1\|F)=d(e_k\|F)$ for $j=m+1$ a contradiction
$F=\{a_1,c_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m$ $d(f_1\|F)=d(e_k\|F)$ for $j=m+1$ a contradiction
$F=\{a_1,d_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m$ $d(f_1\|F)=d(e_k\|F)$ for $j=m+1$ a contradiction
$F=\{a_1,e_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m$ $d(f_1\|F)=d(e_k\|F)$ for $j=m+1$ a contradiction
$F=\{a_1,g_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m-1$ $d(g_1\|F)=d(f_k\|F)$ for $j=m$ $d(f_1\|F)=d(e_2\|F)$ for $j=m+1$ a contradiction
$F=\{b_1,c_j\},\ 1\le j\le k)$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m$ $d(f_2\|F)=d(g_k\|F)$ for $j=m+1$ a contradiction
$F=\{b_1,d_j\},1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m$ $d(f_2\|F)=d(g_k\|F)$ for $j=m+1$ a contradiction
$F=\{b_1,e_j\},\ 2\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m$ $d(f_2\|F)=d(g_k\|F)$ for $j=m+1$ a contradiction
$F=\{b_1,g_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m-1$ $d(g_1\|F)=d(f_k\|F)$ for $j=m$ $d(f_2\|F)=d(g_k\|F)$ for $j=m+1$ a contradiction
$F=\{c_1,d_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m$ $d(f_2\|F)=d(g_k\|F)$ for $j=m+1$ a contradiction
$F=\{c_1,e_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m$ $d(f_2\|F)=d(g_k\|F)$ for $j=m+1$ a contradiction
$F=\{c_1,g_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m-1$ $d(g_1\|F)=d(f_k\|F)$ for $j=m$ $d(f_2\|F)=d(g_k\|F)$ for $j=m+1$ a contradiction
$F=\{d_1,e_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m$ $d(f_2\|F)=d(g_k\|F)$ for $j=m+1$ a contradiction
$F=\{d_1,g_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m-1$ $d(g_1\|F)=d(f_k\|F)$ for $j=m$ $d(f_2\|F)=d(g_k\|F)$ for $j=m+1$ a contradiction
$F=\{e_1,g_j\},\ 1\le j\le k$	$d(f_k\|F)=d(g_k\|F)$ for $1\le j\le m-1$ $d(g_1\|F)=d(f_k\|F)$ for $j=m$ $d(f_2\|F)=d(g_k\|F)$ for $j=m+1$ a contradiction

Consequently, we deduce from the previous explanation in Table 3 and Table 4 that for T_r^*, no resolving set of cardinality two exists, that is, no resolving set with two vertices, implying that $dim(T_r^*)=3$.

Corollary 2. *The independent resolving number for the convex polytope T_r^* is 3 for even positive integers $r\ge 6$.*

5. Conclusion

The metric dimension of TCL T_r and the resulting convex polytope T_r^* are examined in this work. For even $r \geq 6$, we have demonstrated that $V(T_r) = V(T_r^*)$ and $dim(T_r) = dim(T_r^*) = 3$ for these two structures (a partial solution to the problem pointed out in [5]). Furthermore, we demonstrated the independence of the resolving sets for each of these families.

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