Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 53(1), 2026, Pages 90–102, DOI: 10.52846/ami.v53i1.2065
ISSN: 1223-6934

On the Generalization of Right Fractional Order Derivatives and the Darboux Problem for Partial Differential Equations

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Abstract. In this work, we explore the existence and uniqueness of solutions to the Darboux problem for partial differential equations (DPPDEs) involving the right partial Caputo ϑ-fractional derivative. The Ulam-Hyers stability (UHS) of DPPDE is also studied by using a generalized Gronwall inequality. Finally, we illustrate our results by an example.

2020 *Mathematics Subject Classification.* Primary 35R11, 26A33; Secondary 34A08, 35A01, 35A02, 34D20.
Key words and phrases. fractional partial differential equations; Darboux problem; Right partial Caputo ϑ-fractional derivative; Ulam–Hyers stability.

1. Introduction

Fractional calculus is a powerful tool used to investigate a wide range of fields in science and engineering. It has numerous applications across various disciplines, including mathematical physics, biology, economics, statistical mechanics, biophysics, control theory, and more [1, 2, 3, 4, 5].

In recent years, significant work has been done on both ordinary and partial Fractional Differential Equations (FDEs). For instance, the case of ordinary FDEs with Caputo derivative is explored in the works [6, 7]. Additionally, Almeida in [8] has studied the existence of solutions for FDE with respect to another function. Regarding Caputo fractional-order partial differential equations, the research by authors in [9, 10, 11, 12] provides valuable insights. Furthermore, Arfaoui and Ben Makhlouf in [13] investigated Caputo-Hadamard partial fractional differential equations. In this study, our focus centers on investigating the existence, uniqueness, and stability of solutions associated with fractional partial differential systems given by the following form:

$$ {}^{C}D_{b^-}^{\delta,\vartheta}\mathcal{U}(\upsilon,\varpi)=\mathcal{F}(\upsilon,\varpi,\mathcal{U}(\upsilon,\varpi)), \qquad (\upsilon,\varpi)\in J=[a_1,b_1]\times[a_2,b_2], \tag{1} $$

$$ \begin{aligned} \mathcal{U}(\upsilon,b_2) &= \varphi(\upsilon), \quad \upsilon\in[a_1,b_1],\\ \mathcal{U}(b_1,\varpi) &= \psi(\varpi), \quad \varpi\in[a_2,b_2],\\ \varphi(b_1) &= \psi(b_2), \end{aligned} \tag{2} $$

where $\delta=(\delta_1,\delta_2)\in(0,1)^2$, $b=(b_1,b_2)\in\mathbb{R}^2$, $\vartheta=(\vartheta_1,\vartheta_2)$ with $\vartheta_1\in C^1([a_1,b_1],\mathbb{R})$ and $\vartheta_2\in C^1([a_2,b_2],\mathbb{R})$ are positive, increasing functions, ${}^{C}D_{b^-}^{\delta,\vartheta}$ is the right partial Caputo ϑ-fractional derivative of order δ and $\mathcal{F}: J\times\mathbb{R}\to\mathbb{R}$, $\varphi:[a_1,b_1]\to\mathbb{R}$ and

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Received December 7, 2024. Accepted December 27, 2025.

$\psi : [a_2, b_2] \to \mathbb{R}$ are given continuous functions.
The remainder of the paper involves Section 2, presenting various definitions and preliminaries, followed by Section 3, where the proofs for existence, uniqueness, and stability results are demonstrated.

2. Preliminarily

Definition 2.1. Let $\delta = (\delta_1, \delta_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$. The right partial Riemann-Liouville ϑ-fractional integral of order δ for $\mathcal{V}(\upsilon, \varpi) \in L^1(J)$ with respect to ϑ_1 and ϑ_2 is defined as

$$\begin{aligned}\left(I_{b^-}^{\delta,\vartheta}\mathcal{V}\right)(\upsilon,\varpi) &= \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1} \\ &\qquad \times \left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\mathcal{V}(s,t)dtds.\end{aligned}$$

Definition 2.2. Let $\delta = (\delta_1, \delta_2) \in (0,1] \times (0,1]$. The right partial Riemann-Liouville ϑ-fractional derivative of order δ for $\mathcal{V}(\upsilon, \varpi) \in L^1(J)$ with respect to ϑ_1 and ϑ_2 is defined as

$$\begin{aligned}\left(D_{b^-}^{\delta,\vartheta}\mathcal{V}\right)(\upsilon,\varpi) &= \left(D_{\upsilon\varpi}^{\vartheta}I_{b^-}^{1-\delta,\vartheta}\mathcal{V}\right)(\upsilon,\varpi) \\ &= \frac{1}{\Gamma(1-\delta_1)\Gamma(1-\delta_2)}D_{\upsilon\varpi}^{\vartheta}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{-\delta_1} \\ &\qquad \times \left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{-\delta_2}\mathcal{V}(s,t)dtds,\end{aligned}$$

where

$$\left(D_{\upsilon\varpi}^{\vartheta}\mathcal{V}\right)(\upsilon,\varpi) = \left(\frac{1}{\vartheta_1'(\upsilon)\vartheta_2'(\varpi)}\frac{\partial^2}{\partial\upsilon\partial\varpi}\mathcal{V}\right)(\upsilon,\varpi).$$

Definition 2.3. Let $\delta = (\delta_1, \delta_2) \in (0,1] \times (0,1]$. The right partial Caputo ϑ-fractional derivative of order δ for $\mathcal{V}(\upsilon, \varpi) \in L^1(J)$ with respect to ϑ_1 and ϑ_2 is defined as

$$\left({}^{C}D_{b^-}^{\delta,\vartheta}\mathcal{V}\right)(\upsilon,\varpi) = D_{b^-}^{\delta,\vartheta}\Big(\mathcal{V}(\upsilon,\varpi)-\mathcal{V}(\upsilon,b_2)-\mathcal{V}(b_1,\varpi)+\mathcal{V}(b_1,b_2)\Big).$$

Lemma 2.1. *If $\mu_1, \mu_2 \in (-1, \infty)$ and $\delta = (\delta_1, \delta_2) \in (0,\infty) \times (0,\infty)$, then for all $(\upsilon, \varpi) \in J$, we have*

$$\begin{aligned}&I_{b^-}^{\delta,\vartheta}\left(\left(\vartheta_1(b_1)-\vartheta_1(\upsilon)\right)^{\mu_1}\left(\vartheta_2(b_2)-\vartheta_2(\varpi)\right)^{\mu_2}\right) \\ &= \frac{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}{\Gamma(\delta_1+\mu_1+1)\Gamma(\delta_2+\mu_1+1)}\left(\vartheta_1(b_1)-\vartheta_1(\upsilon)\right)^{\delta_1+\mu_1}\left(\vartheta_2(b_2)-\vartheta_2(\varpi)\right)^{\delta_2+\mu_2}.\end{aligned}$$

Lemma 2.2. *Let $\delta = (\delta_1, \delta_2), \sigma = (\sigma_1, \sigma_2) \in (0,\infty) \times (0,\infty)$ and $\mathcal{V}(\upsilon, \varpi) \in L^1(J)$. Then, the following property is satisfied*

$$\left(I_{b^-}^{\delta,\vartheta}I_{b^-}^{\sigma,\vartheta}\mathcal{V}\right)(\upsilon,\varpi) = \left(I_{b^-}^{\sigma,\vartheta}I_{b^-}^{\delta,\vartheta}\mathcal{V}\right)(\upsilon,\varpi) = \left(I_{b^-}^{\delta+\sigma,\vartheta}\mathcal{V}\right)(\upsilon,\varpi). \quad \textit{(Semi group property)}$$

Proof. We have

$$\begin{aligned}
\left(I_{b^-}^{\delta,\vartheta} I_{b^-}^{\sigma,\vartheta}\mathcal{V}\right)(\upsilon,\varpi) &= \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\
&\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\left(I_{b^-}^{\sigma,\vartheta}\mathcal{V}\right)(s,t)dtds\\
&= \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\int_{s}^{b_1}\int_{t}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\vartheta_1'(\tau)\vartheta_2'(\zeta)\\
&\times\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\\
&\times\left(\vartheta_1(\tau)-\vartheta_1(s)\right)^{\sigma_1-1}\left(\vartheta_2(\zeta)-\vartheta_2(t)\right)^{\sigma_2-1}\mathcal{V}(\tau,\zeta)d\zeta d\tau dtds.
\end{aligned}$$

By using Fubini's theorem, we obtain

$$\begin{aligned}
\left(I_{b^-}^{\delta,\vartheta} I_{b^-}^{\sigma,\vartheta}\mathcal{V}\right)(\upsilon,\varpi) &= \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(\tau)\vartheta_2'(\zeta)\mathcal{V}(\tau,\zeta)\\
&\times\int_{\upsilon}^{\tau}\int_{\varpi}^{\zeta}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\\
&\times\left(\vartheta_1(\tau)-\vartheta_1(s)\right)^{\sigma_1-1}\left(\vartheta_2(\zeta)-\vartheta_2(t)\right)^{\sigma_2-1}dtdsd\zeta d\tau\\
&= \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\Big[\vartheta_1'(\tau)\vartheta_2'(\zeta)\mathcal{V}(\tau,\zeta)\\
&\times\int_{\upsilon}^{\tau}\vartheta_1'(s)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\left(\vartheta_1(\tau)-\vartheta_1(s)\right)^{\sigma_1-1}ds\\
&\times\int_{\varpi}^{\zeta}\vartheta_2'(t)\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\left(\vartheta_2(\zeta)-\vartheta_2(t)\right)^{\sigma_2-1}dt\Big]d\zeta d\tau.
\end{aligned}$$

By using the following change of variables $\vartheta_1(s)=\varrho, \vartheta_2(t)=\nu, \xi=\frac{\varrho-\vartheta_1(\upsilon)}{\vartheta_1(\tau)-\vartheta_1(\upsilon)}$ and $\eta=\frac{\nu-\vartheta_2(\varpi)}{\vartheta_2(\zeta)-\vartheta_2(\varpi)}$ and by using the fact that $\int_0^1(1-r)^{\alpha-1}r^{\beta-1}dr=B(\alpha,\beta)$ and $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, we get

$$\begin{aligned}
\left(I_{b^-}^{\delta,\vartheta} I_{b^-}^{\sigma,\vartheta}\mathcal{V}\right)(\upsilon,\varpi) &= \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\Big[\vartheta_1'(\tau)\vartheta_2'(\zeta)\mathcal{V}(\tau,\zeta)\\
&\times\int_{\vartheta_1(\upsilon)}^{\vartheta_1(\tau)}\left(\varrho-\vartheta_1(\upsilon)\right)^{\delta_1-1}\left(\vartheta_1(\tau)-\varrho\right)^{\sigma_1-1}d\varrho\\
&\times\int_{\vartheta_2(\varpi)}^{\vartheta_2(\zeta)}\left(\nu-\vartheta_2(\varpi)\right)^{\delta_2-1}\left(\vartheta_2(\zeta)-\nu\right)^{\sigma_2-1}d\nu\Big]d\zeta d\tau\\
&= \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\Big[\vartheta_1'(\tau)\vartheta_2'(\zeta)\\
&\times\left(\vartheta_1(\tau)-\vartheta_1(\upsilon)\right)^{\delta_1+\sigma_1-1}\left(\vartheta_2(\zeta)-\vartheta_1(\varpi)\right)^{\delta_2+\sigma_2-1}\mathcal{V}(\tau,\zeta)\\
&\times\int_0^1\xi^{\delta_1-1}(1-\xi)^{\sigma_1-1}d\xi\int_0^1\eta^{\delta_2-1}(1-\eta)^{\sigma_2-1}d\eta\Big]d\zeta d\tau\\
&= \frac{1}{\Gamma(\delta_1+\sigma_1+1)\Gamma(\delta_2+\sigma_2+1)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\Big[\vartheta_1'(\tau)\vartheta_2'(\zeta)\\
&\times\left(\vartheta_1(\tau)-\vartheta_1(\upsilon)\right)^{\delta_1+\sigma_1-1}\left(\vartheta_2(\zeta)-\vartheta_1(\varpi)\right)^{\delta_2+\sigma_2-1}\mathcal{V}(\tau,\zeta)\Big]d\zeta d\tau\\
&= \left(I_{b^-}^{\delta+\sigma,\vartheta}\mathcal{V}\right)(\upsilon,\varpi).
\end{aligned}$$

□

Lemma 2.3. *Let $\delta = (\delta_1, \delta_2) \in (0,1] \times (0,1]$ and $\mathcal{V}(\upsilon, \varpi) \in C(J)$. Then, the following property is satisfied*

$$\left(D_{b^-}^{\delta,\vartheta} I_{b^-}^{\delta,\vartheta} \mathcal{V}\right)(\upsilon, \varpi) = \mathcal{V}(\upsilon, \varpi).$$

Proof. From Definition 2.2 and Lemma 2.2 we get

$$\begin{aligned}\left(D_{b^-}^{\delta,\vartheta} I_{b^-}^{\delta,\vartheta} \mathcal{V}\right)(\upsilon, \varpi) &= \left(D_{\upsilon\varpi}^{\vartheta} I_{b^-}^{1-\delta,\vartheta} I_{b^-}^{\delta,\vartheta} \mathcal{V}\right)(\upsilon, \varpi) \\ &= \left(D_{\upsilon\varpi}^{\vartheta} I_{b^-}^{1,\vartheta} \mathcal{V}\right)(\upsilon, \varpi) \\ &= \mathcal{V}(\upsilon, \varpi).\end{aligned}$$

□

Proposition 2.4. *Let $\delta = (\delta_1, \delta_2) \in (0,1] \times (0,1]$. The right partial Caputo ϑ-fractional derivative of order δ for $\mathcal{V}(\upsilon, \varpi) \in AC(J)$ with respect to ϑ_1 and ϑ_2 is given by*

$$\begin{aligned}\left({}^{C}D_{b^-}^{\delta,\vartheta} \mathcal{V}\right)(\upsilon, \varpi) &= \left(I_{b^-}^{1-\delta,\vartheta} D_{\upsilon\varpi}^{\vartheta} \mathcal{V}\right)(\upsilon, \varpi) \\ &= \frac{1}{\Gamma(1-\delta_1)\Gamma(1-\delta_2)} \int_{\upsilon}^{b_1} \int_{\varpi}^{b_2} \vartheta_1'(s)\vartheta_2'(t) \left(\vartheta_1(s) - \vartheta_1(\upsilon)\right)^{-\delta_1} \\ &\quad \times \left(\vartheta_2(t) - \vartheta_2(\varpi)\right)^{-\delta_2} \left(D_{st}^{\vartheta} \mathcal{V}\right)(s,t)dtds,\end{aligned}$$

where $D_{\upsilon\varpi}^{\vartheta}$ is defined as in definition 2.2.

Lemma 2.5. *Let $\delta = (\delta_1, \delta_2) \in (0,1] \times (0,1]$ and $\mathcal{V}(\upsilon, \varpi) \in AC(J)$. Then, the following property is satisfied*

$$\left(I_{b^-}^{\delta,\vartheta}\, {}^{C}D_{b^-}^{\delta,\vartheta} \mathcal{V}\right)(\upsilon, \varpi) = \mathcal{V}(\upsilon, \varpi) - \mathcal{V}(\upsilon, b_2) - \mathcal{V}(b_1, \varpi) + \mathcal{V}(b_1, b_2).$$

Definition 2.4. [15] Let $m \in \mathbb{N}^*$, $\alpha_j, \beta_j, z, \omega \in \mathbb{C}$, such that $\Re(\alpha_j), \Re(\beta_j) > 0$ for $j \in \{1, 2, \dots, m\}$. The generalized Mittag-Leffler function is defined by

$$\mathbb{E}_{\omega}\left(\left(\alpha_j, \beta_j\right)_{j=1,m}; (z)\right) = \sum_{\kappa=0}^{+\infty} \frac{(\omega)_{\kappa}}{\prod_{j=1}^{m} \Gamma(\kappa\alpha_j + \beta_j)} \frac{z^{\kappa}}{\kappa!},$$

where

$$(\omega)_{\kappa} = \omega\,(\omega+1)\dots(\omega+\kappa-1) = \frac{\Gamma(\omega+\kappa)}{\Gamma(\omega)}.$$

In this specific instance, when $m = 2$ and $\omega = 1$, we get

$$\mathbb{E}_1\left(\left(\alpha_j, \beta_j\right)_{j=1,2}; (z)\right) = \mathbb{E}\left(\left(\alpha_j, \beta_j\right)_{j=1,2}; (z)\right) = \sum_{\kappa=0}^{+\infty} \frac{z^{\kappa}}{\Gamma(\kappa\alpha_1 + \beta_1)\Gamma(\kappa\alpha_2 + \beta_2)}.$$

The motivation behind the Lemma below stems from Theorem 1 in [14].

Lemma 2.6. *Suppose $\mathcal{U}(\upsilon, \varpi)$ and $\mathcal{H}(\upsilon, \varpi)$ are two non-negative integrable functions, and $\mathcal{G}(\upsilon, \varpi)$ is a continuous function on J. Additionally, assume that $\mathcal{G}(\upsilon, \varpi)$ is a non-negative and non-increasing on J. If*

$$\begin{aligned}\mathcal{U}(\upsilon, \varpi) &\le \mathcal{H}(\upsilon, \varpi) + \mathcal{G}(\upsilon, \varpi) \int_{\upsilon}^{b_1} \int_{\varpi}^{b_2} \vartheta_1'(s)\vartheta_2'(t) \left(\vartheta_1(s) - \vartheta_1(\upsilon)\right)^{\delta_1 - 1} \\ &\quad \times \left(\vartheta_2(t) - \vartheta_2(\varpi)\right)^{\delta_2 - 1} \mathcal{U}(s,t)dtds,\end{aligned} \tag{3}$$

then

$$\begin{aligned}\mathcal{U}(\upsilon,\varpi) &\leq \mathcal{H}(\upsilon,\varpi)+\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\sum_{k=1}^{\infty}\frac{\left(\mathcal{G}(\upsilon,\varpi)\Gamma(\delta_1)\Gamma(\delta_2)\right)^k}{\Gamma(k\delta_1)\Gamma(k\delta_2)}\\ &\quad\times\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{k\delta_1-1}\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{k\delta_2-1}\mathcal{H}(s,t)dtds,\end{aligned}\tag{4}$$

where $(\delta_1,\delta_2)\in(0,1)^2$. *Furthermore, if* $\mathcal{H}$ *exhibits non-increasing with respect its variables, therefore:*

$$\begin{aligned}\mathcal{U}(\upsilon,\varpi) &\leq \mathcal{H}(\upsilon,\varpi)\mathbb{E}\Big((\delta_1,1),(\delta_2,1);\mathcal{G}(\upsilon,\varpi)\\ &\quad\times\Gamma(\delta_1)\Gamma(\delta_2)\left(\vartheta_1(b_1)-\vartheta_1(\upsilon)\right)^{\delta_1}\left(\vartheta_2(b_2)-\vartheta_2(\varpi)\right)^{\delta_2}\Big).\end{aligned}\tag{5}$$

Proof. Define the operator $\mathcal{A}$ by

$$\begin{aligned}\mathcal{A}\mathcal{V}(\upsilon,\varpi) &= \mathcal{G}(\upsilon,\varpi)\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ &\qquad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\mathcal{V}(s,t)dtds.\end{aligned}$$

So,

$$\mathcal{U}(\upsilon,\varpi)\leq\mathcal{H}(\upsilon,\varpi)+\mathcal{A}\mathcal{U}(\upsilon,\varpi),$$

moreover,

$$\mathcal{U}(\upsilon,\varpi)\leq\sum_{k=0}^{n-1}\mathcal{A}^k\mathcal{H}(\upsilon,\varpi)+\mathcal{A}^n\mathcal{U}(\upsilon,\varpi),$$

Now, we use the mathematical induction method to prove the following inequality

$$\begin{aligned}\mathcal{A}^n\mathcal{H}(\upsilon,\varpi) &\leq \frac{\left(\Gamma(\delta_1)\Gamma(\delta_2)\right)^n}{\Gamma(n\delta_1)\Gamma(n\delta_2)}\mathcal{G}^n(\upsilon,\varpi)\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{n\delta_1-1}\\ &\qquad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{n\delta_2-1}\mathcal{H}(s,t)dtds.\end{aligned}\tag{6}$$

for $n=1$ the inequality (6) is holds. Now we suppose that (6) is holds for $n=k$ and we prove that (6) is holds for $n=k+1$. We have

$$\begin{aligned}\mathcal{A}^{k+1}\mathcal{H}(\upsilon,\varpi) &= \mathcal{A}\left(\mathcal{A}^k\mathcal{H}(\upsilon,\varpi)\right)\\ &\leq \mathcal{G}(\upsilon,\varpi)\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ &\quad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\left(\mathcal{A}^k\mathcal{H}(s,t)\right)dtds\\ &\leq \frac{\left(\Gamma(\delta_1)\Gamma(\delta_2)\right)^k}{\Gamma(k\delta_1)\Gamma(k\delta_2)}\mathcal{G}(\upsilon,\varpi)\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ &\quad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\mathcal{G}^k(s,t)\\ &\quad\times\int_{s}^{b_1}\int_{t}^{b_2}\vartheta_1'(\tau)\vartheta_2'(\zeta)\left(\vartheta_1(\tau)-\vartheta_1(s)\right)^{k\delta_1-1}\\ &\quad\times\left(\vartheta_2(\zeta)-\vartheta_2(t)\right)^{k\delta_2-1}\mathcal{H}(\tau,\zeta)d\zeta d\tau dtds\\ &\leq \frac{\left(\Gamma(\delta_1)\Gamma(\delta_2)\right)^k}{\Gamma(k\delta_1)\Gamma(k\delta_2)}\mathcal{G}^{k+1}(\upsilon,\varpi)\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\int_{s}^{b_1}\int_{t}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\vartheta_1'(\tau)\vartheta_2'(\zeta)\\ &\quad\times\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\\ &\quad\times\left(\vartheta_1(\tau)-\vartheta_1(s)\right)^{k\delta_1-1}\left(\vartheta_2(\zeta)-\vartheta_2(t)\right)^{k\delta_2-1}\mathcal{H}(\tau,\zeta)d\zeta d\tau dtds.\end{aligned}$$

By using Fubini's theorem, we obtain

$$\begin{aligned}
\mathcal{A}^{k+1}\mathcal{H}(\upsilon,\varpi) \leq\ & \frac{(\Gamma(\delta_1)\Gamma(\delta_2))^k}{\Gamma(k\delta_1)\Gamma(k\delta_2)}\mathcal{G}^{k+1}(\upsilon,\varpi)\int_\upsilon^{b_1}\int_\varpi^{b_2}\vartheta_1'(\tau)\vartheta_2'(\zeta)\mathcal{H}(\tau,\zeta) \\
& \times\int_\upsilon^\tau\int_\varpi^\zeta \vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1} \\
& \times\left(\vartheta_1(\tau)-\vartheta_1(s)\right)^{k\delta_1-1}\left(\vartheta_2(\zeta)-\vartheta_2(t)\right)^{k\delta_2-1}dtdsd\zeta d\tau \\
\leq\ & \frac{(\Gamma(\delta_1)\Gamma(\delta_2))^k}{\Gamma(k\delta_1)\Gamma(k\delta_2)}\mathcal{G}^{k+1}(\upsilon,\varpi)\int_\upsilon^{b_1}\int_\varpi^{b_2}\Big[\vartheta_1'(\tau)\vartheta_2'(\zeta)\mathcal{H}(\tau,\zeta) \\
& \times\int_\upsilon^\tau \vartheta_1'(s)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\left(\vartheta_1(\tau)-\vartheta_1(s)\right)^{k\delta_1-1}ds \\
& \times\int_\varpi^\zeta \vartheta_2'(t)\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\left(\vartheta_2(\zeta)-\vartheta_2(t)\right)^{k\delta_2-1}dt\Big]d\zeta d\tau.
\end{aligned}$$

By using the following change of variables $\vartheta_1(s)=\varrho,\vartheta_2(t)=\nu,\xi=\frac{\varrho-\vartheta_1(\upsilon)}{\vartheta_1(\tau)-\vartheta_1(\upsilon)}$ and $\eta=\frac{\nu-\vartheta_2(\varpi)}{\vartheta_2(\zeta)-\vartheta_2(\varpi)}$ and by using the fact that $\int_0^1(1-r)^{\alpha-1}r^{\beta-1}dr=B(\alpha,\beta)$ and $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, we get

$$\begin{aligned}
\mathcal{A}^{k+1}\mathcal{H}(\upsilon,\varpi) \leq\ & \frac{(\Gamma(\delta_1)\Gamma(\delta_2))^k}{\Gamma(k\delta_1)\Gamma(k\delta_2)}\mathcal{G}^{k+1}(\upsilon,\varpi)\int_\upsilon^{b_1}\int_\varpi^{b_2}\Big[\vartheta_1'(\tau)\vartheta_2'(\zeta)\mathcal{H}(\tau,\zeta) \\
& \times\int_{\vartheta_1(\upsilon)}^{\vartheta_1(\tau)}\left(\varrho-\vartheta_1(\upsilon)\right)^{\delta_1-1}\left(\vartheta_1(\tau)-\varrho\right)^{k\delta_1-1}d\varrho \\
& \times\int_{\vartheta_2(\varpi)}^{\vartheta_2(\zeta)}\left(\nu-\vartheta_2(\varpi)\right)^{\delta_2-1}\left(\vartheta_2(\zeta)-\nu\right)^{k\delta_2-1}d\nu\Big]d\zeta d\tau \\
\leq\ & \frac{(\Gamma(\delta_1)\Gamma(\delta_2))^k}{\Gamma(k\delta_1)\Gamma(k\delta_2)}\mathcal{G}^{k+1}(\upsilon,\varpi)\int_\upsilon^{b_1}\int_\varpi^{b_2}\Big[\vartheta_1'(\tau)\vartheta_2'(\zeta) \\
& \times\Big(\vartheta_1(\tau)-\vartheta_1(\upsilon)\Big)^{(k+1)\delta_1-1}\Big(\vartheta_2(\zeta)-\vartheta_1(\varpi)\Big)^{(k+1)\delta_2-1}\mathcal{H}(\tau,\zeta) \\
& \times\int_0^1\xi^{\delta_1-1}(1-\xi)^{k\delta_1-1}d\xi\int_0^1\eta^{\delta_2-1}(1-\eta)^{k\delta_2-1}d\eta\Big]d\zeta d\tau \\
\leq\ & \frac{(\Gamma(\delta_1)\Gamma(\delta_2))^{k+1}}{\Gamma((k+1)\delta_1)\Gamma((k+1)\delta_2)}\mathcal{G}^{k+1}(\upsilon,\varpi)\int_\upsilon^{b_1}\int_\varpi^{b_2}\Big[\vartheta_1'(\tau)\vartheta_2'(\zeta) \\
& \times\Big(\vartheta_1(\tau)-\vartheta_1(\upsilon)\Big)^{(k+1)\delta_1-1}\Big(\vartheta_2(\zeta)-\vartheta_1(\varpi)\Big)^{(k+1)\delta_2-1}\mathcal{H}(\tau,\zeta)\Big]d\zeta d\tau.
\end{aligned}$$

Then, (6) is holds for $n=k+1$. Hence (6) is proved. Moreover

$$\begin{aligned}
\mathcal{A}^n\mathcal{H}(\upsilon,\varpi) \leq\ & M^n\frac{(\Gamma(\delta_1)\Gamma(\delta_2))^n}{\Gamma(n\delta_1)\Gamma(n\delta_2)}\int_\upsilon^{b_1}\int_\varpi^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{n\delta_1-1} \\
& \times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{n\delta_2-1}\mathcal{H}(s,t)dtds.
\end{aligned} \tag{7}$$

Since $\mathcal{G}^n(\upsilon,\varpi)$ is continuous on $J=[a_1,b_1]\times[a_2,b_2]$. Then, we can see $\lim\limits_{n\to\infty}\mathcal{A}^n\mathcal{H}(\upsilon,\varpi)=0$. Therefore, the inequality (4) is proved. Furthermore, if $\mathcal{H}$

exhibits non-increasing with respect its variables, we get

$$\begin{aligned}
\mathcal{U}(\upsilon,\varpi) &\leq \mathcal{H}(\upsilon,\varpi)+\mathcal{H}(\upsilon,\varpi)\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\sum_{k=1}^{\infty}\frac{\left(\mathcal{G}(\upsilon,\varpi)\Gamma(\delta_1)\Gamma(\delta_2)\right)^k}{\Gamma(k\delta_1)\Gamma(k\delta_2)} \\
&\quad \times\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{k\delta_1-1}\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{k\delta_2-1}dtds \\
&\leq \mathcal{H}(\upsilon,\varpi)\Bigg(1+\sum_{k=1}^{\infty}\frac{\left(\mathcal{G}(\upsilon,\varpi)\Gamma(\delta_1)\Gamma(\delta_2)\right)^k}{\Gamma(k\delta_1+1)\Gamma(k\delta_2+1)} \\
&\quad \times\left(\vartheta_1(b_1)-\vartheta_1(\upsilon)\right)^{k\delta_1}\left(\vartheta_2(b_1)-\vartheta_2(\varpi)\right)^{k\delta_2}\Bigg) \\
&:= \mathcal{H}(\upsilon,\varpi)\mathbb{E}\Big((\delta_1,1),(\delta_2,1);\mathcal{G}(\upsilon,\varpi)\Gamma(\delta_1)\Gamma(\delta_2) \\
&\qquad\qquad \times\left(\vartheta_1(b_1)-\vartheta_1(\upsilon)\right)^{\delta_1}\left(\vartheta_2(b_2)-\vartheta_2(\varpi)\right)^{\delta_2}\Big).
\end{aligned}$$

Therefore, the inequality (5) is proved. ☐

3. Main results

To begin, we will first establish the following:

Lemma 3.1. *$\mathcal{U}\in C(J)$ is a solution of (1)-(2) if and only if*

$$\mathcal{U}(\upsilon,\varpi) = \varphi(\upsilon)+\psi(\varpi)-\varphi(b_1)+I_{b^-}^{\delta,\vartheta}\mathcal{F}(\upsilon,\varpi,\mathcal{U}(\upsilon,\varpi)). \tag{8}$$

Proof. Suppose $\mathcal{U}$ solves the integral equation (8). Applying ${}^{C}D_{b^-}^{\delta,\vartheta}$ and utilizing Lemma 2.3, we deduce that $\mathcal{U}$ satisfies (1). Considering that the integral vanishes when $\upsilon=b_1$ or $\varpi=b_2$, the initial conditions in (2) are fulfilled. Therefore, $\mathcal{U}$ satisfies (1)-(2).

Conversely, if $\mathcal{U}$ satisfies (1)-(2), and let

$$\begin{aligned}
h(\upsilon,\varpi) &= \mathcal{F}(\upsilon,\varpi,\mathcal{U}(\upsilon,\varpi)) \\
&= D_{b^-}^{\delta,\vartheta}\Big[\mathcal{U}(\upsilon,\varpi)-\mathcal{U}(\upsilon,b_2)-\mathcal{U}(b_1,\varpi)+\mathcal{U}(b_1,b_2)\Big] \\
&= D_{\upsilon\varpi}^{\vartheta}I_{b^-}^{1-\delta,\vartheta}\Big[\mathcal{U}(\upsilon,\varpi)-\mathcal{U}(\upsilon,b_2)-\mathcal{U}(b_1,\varpi)+\mathcal{U}(b_1,b_2)\Big].
\end{aligned} \tag{9}$$

Applying the operator $I_{b^-}^{1,\vartheta}$ to (9), we get

$$I_{b^-}^{1,\vartheta}h(\upsilon,\varpi)=I_{b^-}^{1-\delta,\vartheta}\Big[\mathcal{U}(\upsilon,\varpi)-\mathcal{U}(\upsilon,b_2)-\mathcal{U}(b_1,\varpi)+\mathcal{U}(b_1,b_2)\Big].$$

Applying the operator $D_{b^-}^{1-\delta,\vartheta}$ to the last equation, we obtain

$$\begin{aligned}
\mathcal{U}(\upsilon,\varpi) &= \mathcal{U}(\upsilon,b_2)+\mathcal{U}(b_1,\varpi)-\mathcal{U}(b_1,b_2)+D_{b^-}^{1-\delta,\vartheta}I_{b^-}^{1,\vartheta}h(\upsilon,\varpi) \\
&= \mathcal{U}(\upsilon,b_2)+\mathcal{U}(b_1,\varpi)-\mathcal{U}(b_1,b_2)+D_{\upsilon\varpi}^{\vartheta}I_{b^-}^{\delta,\vartheta}I_{b^-}^{1,\vartheta}h(\upsilon,\varpi) \\
&= \mathcal{U}(\upsilon,b_2)+\mathcal{U}(b_1,\varpi)-\mathcal{U}(b_1,b_2)+I_{b^-}^{\delta,\vartheta}h(\upsilon,\varpi).
\end{aligned}$$

☐

Let's consider the hypothesis:

(H1) There exist $\mathcal{J}$ and $\mathcal{N}$ in $C(J,\mathbb{R}_+)$ such that:

$$|\mathcal{F}(\upsilon,\varpi,\mathcal{U})|\leq\mathcal{J}(\upsilon,\varpi)+\mathcal{N}(\upsilon,\varpi)|\mathcal{U}|,\ \forall(\upsilon,\varpi)\in J,\ \mathcal{U}\in\mathbb{R}.$$

(H2) There is $L_{\mathcal{F}} > 0$ such that

$$\left|\mathcal{F}\left(\upsilon,\varpi,\mathcal{U}\right)-\mathcal{F}\left(\upsilon,\varpi,\mathcal{V}\right)\right| \leq L_{\mathcal{F}}\left|\mathcal{U}-\mathcal{V}\right|,\ \forall\left(\upsilon,\varpi\right)\in J,\ \mathcal{U},\mathcal{V}\in\mathbb{R}.$$

Theorem 3.2. *If (H1) is assumed to hold true, then the problem (1)-(2) possesses at least one solution.*

Proof. Define $\mathcal{A}: C\left(J,\mathbb{R}\right)\to C\left(J,\mathbb{R}\right)$ by

$$\begin{aligned}\left(\mathcal{A}\mathcal{U}\right)\left(\upsilon,\varpi\right) &= \mathcal{T}\left(\upsilon,\varpi\right)+\frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ &\qquad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\mathcal{F}\left(s,t,\mathcal{U}\left(s,t\right)\right)dtds,\end{aligned} \tag{10}$$

where the function $\mathcal{T}$ is defined for every $\left(\upsilon,\varpi\right)\in J$ as follows:

$$\mathcal{T}\left(\upsilon,\varpi\right)=\varphi\left(\upsilon\right)+\psi\left(\varpi\right)-\varphi\left(b_1\right). \tag{11}$$

Obviously, $\mathcal{A}$ is continuous. Now, we show that bounded sets in $C\left(J,\mathbb{R}\right)$ are mapped by $\mathcal{A}$ into bounded sets of $C\left(J,\mathbb{R}\right)$. To do so, we prove that for any $R>0$, there is $L>0$ such that for every $\mathcal{U}\in\mathcal{B}_R=\left\{\mathcal{U}\in C\left(J,\mathbb{R}\right),\left\|\mathcal{U}\right\|_\infty\leq R\right\}$, we obtain $\left\|\mathcal{A}\mathcal{U}\right\|_\infty\leq L$. Let $\mathcal{U}\in\mathcal{B}_R$ and $\left(\upsilon,\varpi\right)\in J$, we get

$$\begin{aligned}\left|\left(\mathcal{A}\mathcal{U}\right)\left(\upsilon,\varpi\right)\right| &\leq \left|\mathcal{T}\left(\upsilon,\varpi\right)\right|+\frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ &\qquad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\left|\mathcal{F}\left(s,t,\mathcal{U}\left(s,t\right)\right)\right|dtds,\end{aligned}$$

From (H1), we find

$$\begin{aligned}\left|\left(\mathcal{A}\mathcal{U}\right)\left(\upsilon,\varpi\right)\right| &\leq \left\|\mathcal{T}\right\|_\infty+\frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_{\upsilon}^{b_1}\int_{\varpi}^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ &\qquad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\Big(\mathcal{J}(s,t)+\mathcal{N}(s,t)\left|\mathcal{U}(s,t)\right|\Big)dtds\\ &\leq \left\|\mathcal{T}\right\|_\infty+\mathcal{M}\int_{\vartheta_1(\upsilon)}^{\vartheta_1(b_1)}\int_{\vartheta_2(\varpi)}^{\vartheta_2(b_2)}\left(s-\vartheta_1(\upsilon)\right)^{\delta_1-1}\left(t-\vartheta_2(\varpi)\right)^{\delta_2-1}dtds\\ &\leq \left\|\mathcal{T}\right\|_\infty+\frac{\mathcal{M}}{\delta_1\delta_2}\Big(\vartheta_1(b_1)-\vartheta_1(\upsilon)\Big)^{\delta_1}\Big(\vartheta_2(b_2)-\vartheta_2(\varpi)\Big)^{\delta_2},\end{aligned}$$

where $\mathcal{M}=\dfrac{\left(\left\|\mathcal{J}\right\|_\infty+\left\|\mathcal{N}\right\|_\infty R\right)}{\Gamma(\delta_1)\Gamma(\delta_2)}$. Then, for any $\mathcal{U}\in\mathcal{B}_R$ there exists

$$L=\left\|\mathcal{T}\right\|_\infty+\frac{\mathcal{M}}{\delta_1\delta_2}\Big(\vartheta_1(b_1)-\vartheta_1(a_1)\Big)^{\delta_1}\Big(\vartheta_2(b_2)-\vartheta_2(a_2)\Big)^{\delta_2},$$

such that $\left\|\mathcal{A}\mathcal{U}\right\|_\infty\leq L$.
Secondly, we prove that bounded sets in $C\left(J,\mathbb{R}\right)$ are mapped by $\mathcal{A}$ into equicontinuous sets in $C\left(J,\mathbb{R}\right)$. Let $(\upsilon_1,\varpi_1),(\upsilon_2,\varpi_2)\in J$ such that $\upsilon_1<\upsilon_2$ and $\varpi_1<\varpi_2$ and

$\mathcal{U} \in \mathcal{B}_R$. Then, from (H1) we get

$$\begin{aligned}
&\left|(\mathcal{A}\mathcal{U})\left(\upsilon_1, \varpi_1\right) - (\mathcal{A}\mathcal{U})\left(\upsilon_2, \varpi_2\right)\right| \\
&\quad \leq \left|T\left(\upsilon_1, \varpi_1\right) - T\left(\upsilon_2, \varpi_2\right)\right| \\
&\qquad + \mathcal{M} \int_{\upsilon_2}^{b_1} \int_{\varpi_2}^{b_2} \vartheta_1'(s)\vartheta_2'(t)\Big(\big(\vartheta_1(s) - \vartheta_1(\upsilon_2)\big)^{\delta_1 - 1}\big(\vartheta_2(t) - \vartheta_2(\varpi_2)\big)^{\delta_2 - 1} \\
&\qquad - \big(\vartheta_1(s) - \vartheta_1(\upsilon_1)\big)^{\delta_1 - 1}\big(\vartheta_2(t) - \vartheta_2(\varpi_1)\big)^{\delta_2 - 1}\Big) dtds \\
&\qquad + \mathcal{M} \int_{\upsilon_2}^{b_1} \int_{\varpi_1}^{\varpi_2} \vartheta_1'(s)\vartheta_2'(t)\big(\vartheta_1(s) - \vartheta_1(\upsilon_1)\big)^{\delta_1 - 1}\big(\vartheta_2(t) - \vartheta_2(\varpi_1)\big)^{\delta_2 - 1} dtds \\
&\qquad + \mathcal{M} \int_{\upsilon_1}^{\upsilon_2} \int_{\varpi_2}^{b_2} \vartheta_1'(s)\vartheta_2'(t)\big(\vartheta_1(s) - \vartheta_1(\upsilon_1)\big)^{\delta_1 - 1}\big(\vartheta_2(t) - \vartheta_2(\varpi_1)\big)^{\delta_2 - 1} dtds \\
&\qquad + \mathcal{M} \int_{\upsilon_1}^{\upsilon_2} \int_{\varpi_1}^{\varpi_2} \vartheta_1'(s)\vartheta_2'(t)\big(\vartheta_1(s) - \vartheta_1(\upsilon_1)\big)^{\delta_1 - 1}\big(\vartheta_2(t) - \vartheta_2(\varpi_1)\big)^{\delta_2 - 1} dtds \\
&\quad \leq \left|T\left(\upsilon_2, \varpi_2\right) - T\left(\upsilon_1, \varpi_1\right)\right| + \frac{\mathcal{M}}{\delta_1\delta_2}\bigg[2\Big(\big(\vartheta_1(b_1) - \vartheta_1(\upsilon_1)\big)^{\delta_1}\big(\vartheta_2(\varpi_2) - \vartheta_2(\varpi_1)\big)^{\delta_2} \\
&\qquad + \big(\vartheta_1(\upsilon_2) - \vartheta_1(\upsilon_1)\big)^{\delta_1}\big(\vartheta_2(b_2) - \vartheta_2(\varpi_1)\big)^{\delta_2}\Big)\bigg].
\end{aligned}$$

As $\upsilon_1 \to \upsilon_2$ and $\varpi_1 \to \varpi_2$, the RHS of the inequality obtained herein above goes to zero.

Thirdly, we prove a priori bounds. Let $\mathcal{U} \in C\left(J, \mathbb{R}\right)$ such that $\mathcal{U} = \lambda\mathcal{A}\left(\mathcal{U}\right)$ for some $\lambda \in (0, 1)$. Then, for any $(\upsilon, \varpi) \in J$, we have

$$\begin{aligned}
\left|\mathcal{U}\left(\upsilon, \varpi\right)\right| \leq{}& \left|\mathcal{T}\left(\upsilon, \varpi\right)\right| + \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)} \int_{\upsilon}^{b_1} \int_{\varpi}^{b_2} \vartheta_1'(s)\vartheta_2'(t)\big(\vartheta_1(s) - \vartheta_1(\upsilon)\big)^{\delta_1 - 1} \\
& \times \big(\vartheta_2(t) - \vartheta_2(\varpi)\big)^{\delta_2 - 1}\left|\mathcal{F}\left(s, t, \mathcal{U}\left(s, t\right)\right)\right| dtds,
\end{aligned}$$

From (H1), we find

$$\begin{aligned}
&\left|\mathcal{U}\left(\upsilon, \varpi\right)\right| \\
&\leq \|\mathcal{T}\|_\infty + \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)} \int_{\upsilon}^{b_1} \int_{\varpi}^{b_2} \vartheta_1'(s)\vartheta_2'(t)\big(\vartheta_1(s) - \vartheta_1(\upsilon)\big)^{\delta_1 - 1} \\
&\times \big(\vartheta_2(t) - \vartheta_2(\varpi)\big)^{\delta_2 - 1}\Big(\mathcal{J}(s, t) + \mathcal{N}(s, t)\left|\mathcal{U}(s, t)\right|\Big) dtds \\
&\leq \|\mathcal{T}\|_\infty + \frac{\|\mathcal{J}\|_\infty}{\Gamma(\delta_1)\Gamma(\delta_2)} \int_{\upsilon}^{b_1}\int_{\varpi}^{b_2} \vartheta_1'(s)\vartheta_2'(t)\big(\vartheta_1(s) - \vartheta_1(\upsilon)\big)^{\delta_1 - 1}\big(\vartheta_2(t) - \vartheta_2(\varpi)\big)^{\delta_2 - 1} dtds \\
&+ \frac{\|\mathcal{N}\|_\infty}{\Gamma(\delta_1)\Gamma(\delta_2)} \int_{\upsilon}^{b_1}\int_{\varpi}^{b_2} \vartheta_1'(s)\vartheta_2'(t)\big(\vartheta_1(s) - \vartheta_1(\upsilon)\big)^{\delta_1 - 1}\big(\vartheta_2(t) - \vartheta_2(\varpi)\big)^{\delta_2 - 1}\left|\mathcal{U}(s, t)\right| dtds \\
&\leq \|\mathcal{T}\|_\infty + \frac{\|\mathcal{J}\|_\infty}{\Gamma(\delta_1 + 1)\Gamma(\delta_2 + 1)}\big(\vartheta_1(b_1) - \vartheta_1(\upsilon)\big)^{\delta_1}\big(\vartheta_2(b_2) - \vartheta_2(\varpi)\big)^{\delta_2} \\
&+ \frac{\|\mathcal{N}\|_\infty}{\Gamma(\delta_1)\Gamma(\delta_2)} \int_{\upsilon}^{b_1}\int_{\varpi}^{b_2} \vartheta_1'(s)\vartheta_2'(t)\big(\vartheta_1(s) - \vartheta_1(\upsilon)\big)^{\delta_1 - 1}\big(\vartheta_2(t) - \vartheta_2(\varpi)\big)^{\delta_2 - 1}\left|\mathcal{U}(s, t)\right| dtds.
\end{aligned}$$

In view of Lemma 2.6 we get,

$$\begin{aligned}
|\mathcal{U}(\upsilon,\varpi)| &\leq \left(\|\mathcal{T}\|_\infty + \frac{\|\mathcal{J}\|_\infty}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)}(\vartheta_1(b_1)-\vartheta_1(\upsilon))^{\delta_1}(\vartheta_2(b_2)-\vartheta_2(\varpi))^{\delta_2}\right) \\
&\quad\times \mathbb{E}\Big((\delta_1,1),(\delta_2,1);\|\mathcal{N}\|_\infty(\vartheta_1(b_1)-\vartheta_1(\upsilon))^{\delta_1}(\vartheta_2(b_2)-\vartheta_2(\varpi))^{\delta_2}\Big) \\
&\leq \left(\|\mathcal{T}\|_\infty + \frac{\|\mathcal{J}\|_\infty}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)}(\vartheta_1(b_1)-\vartheta_1(a_1))^{\delta_1}(\vartheta_2(b_2)-\vartheta_2(a_2))^{\delta_2}\right) \\
&\quad\times \mathbb{E}\Big((\delta_1,1),(\delta_2,1);\|\mathcal{N}\|_\infty(\vartheta_1(b_1)-\vartheta_1(a_1))^{\delta_1}(\vartheta_2(b_2)-\vartheta_2(a_2))^{\delta_2}\Big),
\end{aligned}$$

Take

$$\begin{aligned}
\overline{M} &= \left(\|\mathcal{T}\|_\infty + \frac{\|\mathcal{J}\|_\infty}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)}(\vartheta_1(b_1)-\vartheta_1(a_1))^{\delta_1}(\vartheta_2(b_2)-\vartheta_2(a_2))^{\delta_2}\right) \\
&\quad\times \mathbb{E}\Big((\delta_1,1),(\delta_2,1);\|\mathcal{N}\|_\infty(\vartheta_1(b_1)-\vartheta_1(a_1))^{\delta_1}(\vartheta_2(b_2)-\vartheta_2(a_2))^{\delta_2}\Big),
\end{aligned}$$

and let

$$\mathcal{W} = \left\{\mathcal{U} \in C(J,\mathbb{R}),\ \|\mathcal{U}\|_\infty < \overline{M}+1\right\},$$

then, one can not find $\mathcal{U} \in \partial\mathcal{W}$ s.t. $\mathcal{U} = \lambda\mathcal{A}(\mathcal{U})$, for any $\lambda \in (0,1)$. It yields from Theorem 2.8 in [9] that $\mathcal{A}$ has a fixed point which is solution of (1)-(2). ☐

Theorem 3.3. *If (H2) is assumed to hold true, then the problem (1)-(2) possesses a unique solution.*

Proof. We know that if (H2) is holds then, (H1) is holds so, the problem (1)-(2) has at least one solution. It remains to prove the uniqueness of solution, to this end we suppose that (1)-(2) has tow solutions $\mathcal{U}(\upsilon,\varpi)$ and $\mathcal{V}(\upsilon,\varpi)$ then,

$$\begin{aligned}
&|\mathcal{U}(\upsilon,\varpi)-\mathcal{V}(\upsilon,\varpi)| \\
&\quad\leq \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_\upsilon^{b_1}\int_\varpi^{b_2}\vartheta_1'(s)\vartheta_2'(t)(\vartheta_1(s)-\vartheta_1(\upsilon))^{\delta_1-1} \\
&\qquad\times(\vartheta_2(t)-\vartheta_2(\varpi))^{\delta_2-1}|\mathcal{F}(s,t,\mathcal{U}(s,t))-\mathcal{F}(s,t,\mathcal{V}(s,t))|\,dtds \\
&\quad\leq \frac{L_{\mathcal{F}}}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_\upsilon^{b_1}\int_\varpi^{b_2}\vartheta_1'(s)\vartheta_2'(t)(\vartheta_1(s)-\vartheta_1(\upsilon))^{\delta_1-1} \\
&\qquad\times(\vartheta_2(t)-\vartheta_2(\varpi))^{\delta_2-1}|\mathcal{U}(s,t)-\mathcal{V}(s,t)|\,dtds,
\end{aligned}$$

it follows from Lemma 2.6 that $\mathcal{U}(\upsilon,\varpi)=\mathcal{V}(\upsilon,\varpi)$ Hence, the problem (1)-(2) has a unique solution. ☐

4. Ulam stability

We consider the following inequality

$$\left|{}^{C}D_{b^-}^{\delta,\vartheta}\mathcal{V}(\upsilon,\varpi)-\mathcal{F}(\upsilon,\varpi,\mathcal{V}(\upsilon,\varpi))\right| < \epsilon,\ \text{for } \epsilon>0,\ \text{and } (\upsilon,\varpi)\in J. \tag{12}$$

Definition 4.1. Eq. (1) is Ulam-Hyers stable if there is $C>0$ such that for every $\epsilon>0$, and for all $\mathcal{V}\in C(J,\mathbb{R})$ a solution of the inequality (12), there is $\mathcal{U}\in C(J,\mathbb{R})$ a solution of (1) such that

$$|\mathcal{V}(\upsilon,\varpi)-\mathcal{U}(\upsilon,\varpi)| \leq C\epsilon,\ (\upsilon,\varpi)\in J. \tag{13}$$

We have the following remark:

Remark 4.1. Let $\mathcal{V}$ be a solution of the inequality (12) then $\mathcal{V}$ is a solution of the following integral inequality

$$\begin{aligned}\Bigg|\mathcal{V}(\upsilon,\varpi)-\mathcal{K}(\upsilon,\varpi)-\frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_\upsilon^{b_1}\int_\varpi^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ \times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\mathcal{F}\left(s,t,\mathcal{U}(s,t)\right)dtds\Bigg|\\ \le\frac{\epsilon}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)}\left(\vartheta_1(b_1)-\vartheta_1(\upsilon)\right)^{\delta_1}\left(\vartheta_2(b_2)-\vartheta_2(\varpi)\right)^{\delta_2},\end{aligned}$$

where $\mathcal{K}(\upsilon,\varpi)$ is given by

$$\mathcal{K}(\upsilon,\varpi)=\mathcal{V}(\upsilon,b_2)+\mathcal{V}(b_1,\varpi)-\mathcal{V}(b_1,b_2).$$

Theorem 4.1. *Assume that (H2) is satisfied. Then (1) is Ulam-Hyers stable.*

Proof. Let $\mathcal{V}$ be a solution of (12) and $\mathcal{U}$ the unique solution of the following Cauchy problem

$$\begin{cases}{}^C D_{b^-}^{\delta,\vartheta}\mathcal{U}(\upsilon,\varpi) &= \mathcal{F}\left(\upsilon,\varpi,\mathcal{U}(\upsilon,\varpi)\right),(\upsilon,\varpi)\in J=[a_1,b_1]\times[a_2,b_2],\\ \mathcal{U}(\upsilon,b_2) &= \mathcal{V}(\upsilon,b_2),\upsilon\in[a_1,b_1],\\ \mathcal{U}(b_1,\varpi) &= \mathcal{V}(b_1,\varpi),\varpi\in[a_2,b_2],\end{cases}$$

Therefore,

$$\begin{aligned}\mathcal{U}(\upsilon,\varpi) &= \mathcal{K}(\upsilon,\varpi)+\frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_\upsilon^{b_1}\int_\varpi^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ &\qquad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\mathcal{F}\left(s,t,\mathcal{U}(s,t)\right)dtds,\end{aligned}$$

From Remark 4.1 and (H2) we find,

$$\begin{aligned}&|\mathcal{V}(\upsilon,\varpi)-\mathcal{U}(\upsilon,\varpi)|\\ &\le\Bigg|\mathcal{V}(\upsilon,\varpi)-\mathcal{K}(\upsilon,\varpi)-\frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_\upsilon^{b_1}\int_\varpi^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ &\qquad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\mathcal{F}\left(s,t,\mathcal{V}(s,t)\right)dtds\Bigg|\\ &\quad+\frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_\upsilon^{b_1}\int_\varpi^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ &\qquad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}\left|\mathcal{F}\left(s,t,\mathcal{V}(s,t)\right)-\mathcal{F}\left(s,t,\mathcal{U}(s,t)\right)\right|dtds\\ &\le\frac{\epsilon}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)}\left(\vartheta_1(b_1)-\vartheta_1(\upsilon)\right)^{\delta_1}\left(\vartheta_2(b_2)-\vartheta_2(\varpi)\right)^{\delta_2}\\ &\quad+\frac{L_{\mathcal{F}}}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_\upsilon^{b_1}\int_\varpi^{b_2}\vartheta_1'(s)\vartheta_2'(t)\left(\vartheta_1(s)-\vartheta_1(\upsilon)\right)^{\delta_1-1}\\ &\qquad\times\left(\vartheta_2(t)-\vartheta_2(\varpi)\right)^{\delta_2-1}|\mathcal{V}(s,t)-\mathcal{U}(s,t)|\,dtds.\end{aligned}$$

By using Lemma 2.6 we obtain

$$\begin{aligned}
&\left|\mathcal{V}(\upsilon,\varpi)-\mathcal{U}(\upsilon,\varpi)\right| \\
&\quad\leq \frac{\epsilon}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)}\left(\vartheta_1(b_1)-\vartheta_1(\upsilon)\right)^{\delta_1}\left(\vartheta_2(b_2)-\vartheta_2(\varpi)\right)^{\delta_2} \\
&\qquad\times\mathbb{E}\Big(\left(\delta_1,1\right),\left(\delta_2,1\right);L_{\mathcal{F}}\left(\vartheta_1(b_1)-\vartheta_1(\upsilon)\right)^{\delta_1}\left(\vartheta_2(b_2)-\vartheta_2(\varpi)\right)^{\delta_2}\Big) \\
&\quad\leq \frac{\epsilon}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)}\left(\vartheta_1(b_1)-\vartheta_1(a_1)\right)^{\delta_1}\left(\vartheta_2(b_2)-\vartheta_2(a_2)\right)^{\delta_2} \\
&\qquad\times\mathbb{E}\Big(\left(\delta_1,1\right),\left(\delta_2,1\right);L_{\mathcal{F}}\left(\vartheta_1(b_1)-\vartheta_1(a_1)\right)^{\delta_1}\left(\vartheta_2(b_2)-\vartheta_2(a_2)\right)^{\delta_2}\Big) \\
&\quad:= C\epsilon,
\end{aligned}$$

where

$$\begin{aligned}
C &= \frac{1}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)}\left(\vartheta_1(b_1)-\vartheta_1(a_1)\right)^{\delta_1}\left(\vartheta_2(b_2)-\vartheta_2(a_2)\right)^{\delta_2} \\
&\quad\times\mathbb{E}\Big(\left(\delta_1,1\right),\left(\delta_2,1\right);L_{\mathcal{F}}\left(\vartheta_1(b_1)-\vartheta_1(a_1)\right)^{\delta_1}\left(\vartheta_2(b_2)-\vartheta_2(a_2)\right)^{\delta_2}\Big),
\end{aligned}$$

so, the equation (1) is Ulam-Hyers stable. □

5. Example

Example 5.1. Let us consider the following problem

$$\begin{cases}
{}^{C}D_{b^-}^{\delta,\vartheta}\mathcal{U}(\upsilon,\varpi) = \mathcal{F}\left(\upsilon,\varpi,\mathcal{U}(\upsilon,\varpi)\right),(\upsilon,\varpi)\in J=\left[0,\frac{\pi}{4}\right]\times\left[0,\frac{\pi}{4}\right], \\
\mathcal{U}\left(\upsilon,\frac{\pi}{4}\right) = \sin(\upsilon),\upsilon\in\left[0,\frac{\pi}{4}\right], \\
\mathcal{U}\left(\frac{\pi}{4},\varpi\right) = \exp(\upsilon)-1,\varpi\in\left[0,\frac{\pi}{4}\right],
\end{cases} \tag{14}$$

where $b=\left(\frac{\pi}{4},\frac{\pi}{4}\right)$, $\delta=\left(\frac{1}{2},\frac{1}{2}\right)$, $\vartheta=(\vartheta_1,\vartheta_2)$ with $\left(\vartheta_1(\upsilon),\vartheta_2(\varpi)\right)=(\exp(\upsilon),\exp(\varpi))$ and $\mathcal{F}(\upsilon,\varpi,\mathcal{U})=\sin(\upsilon+\varpi)+\tan(\mathcal{U})$.

For all $\mathcal{U},\mathcal{V}\in\mathbb{R}$ and $(\upsilon,\varpi)\in\left[0,\frac{\pi}{4}\right]\times\left[0,\frac{\pi}{4}\right]$, we have

$$\left|\mathcal{F}(\upsilon,\varpi,\mathcal{U})\right|\leq\sin(\upsilon+\varpi)+2\left|\mathcal{U}\right|,$$

$$\left|\mathcal{F}(\upsilon,\varpi,\mathcal{U})-\mathcal{F}(\upsilon,\varpi,\mathcal{V})\right|\leq 2\left|\mathcal{U}-\mathcal{V}\right|.$$

Hence, the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied. As a consequence of Theorem 3.2 and Theorem 4.1 we deduce that the **(IVP)** (14) has a unique solution on $\left[0,\frac{\pi}{4}\right]\times\left[0,\frac{\pi}{4}\right]$ and it is Ulam-Hyers stable.

6. Conclusion

In conclusion, this work provides a comprehensive analysis of the existence and uniqueness of solutions to the DPPDEs involving the right partial Caputo ϑ-fractional derivative. Additionally, we investigate the UHS of DPPDEs using a generalized Gronwall inequality, offering insights into the stability of the solutions. To demonstrate the applicability of our theoretical results, we present an example that illustrates the key findings of our study. The combination of these approaches contributes to a deeper understanding of DPPDEs with fractional derivatives and their stability properties.

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