Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 52(2), 2025, Pages 656–667, DOI: 10.52846/ami.v52i2.2066
ISSN: 1223-6934

Fading Random Evolution on a Complex Plane

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Abstract. The article explores a generalization of the Goldstein-Kac model, specifically a model of random evolution on a complex plane, with the velocity that decreases over time. This process simulates the motion of a particle in a force field, among other phenomena. Limit theorems describing the distribution of the absorbing point for this process have been derived. Additionally, nonlinear integral equations for functionals of the process have been obtained, and the existence and uniqueness of their solutions have been proven.

2020 *Mathematics Subject Classification.* Primary 60J25; Secondary 35L10.
Key words and phrases. Goldstein-Kac model, fading random evolution, absorbing point, nonlinear integral equation.

1. Introduction

In our recent work [19] we discussed a random evolution (random flight) on a complex plane

$$\gamma_{r,z}^{\lambda,v}(t) = x + iy + v\int_0^t (-1)^{\zeta_r^\lambda(s)}ds + iv\int_0^t (-1)^{\zeta_r^\lambda(s)}ds$$
$$= z + (i+1)v\int_0^t (-1)^{\zeta_r^\lambda(s)}ds,$$

where $x+iy$ is the starting point, $v>0$ is the constant velocity of movement, $\zeta_r^\lambda(s)$ is the Markov chain that takes values in $\{0,1\}$ and has the infinitesimal matrix

$$Q_\lambda = \lambda \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and initial distribution $P\{\zeta_r^\lambda(0)=0\}=p, P\{\zeta_r^\lambda(0)=1\}=q, r=p-q$. We showed that $U_j(t,z)$ – the functionals from the process $\gamma_{r,z}^{\lambda,v}(t)$ of the form

$$U_j(t,z) = E_j f\left(z + (i+1)v\int_0^t (-1)^{\zeta_r^\lambda(s)}ds\right),$$

Received December 5, 2024. Accepted August 9, 2025.
The present work was supported by the National Research Foundation of Ukraine (project 2023.03/0059 ‘Contribution to modern theory of random series’).

$j \in \{0,1\}$ is the state of the process $\zeta_r^\lambda(s)$ at the moment of time $s=0$ satisfy a Schrödinger-type equation

$$\frac{\partial^2 U}{\partial t^2} + 2\lambda \frac{\partial U}{\partial t} = 2iv^2 \frac{\partial^2 U}{\partial z^2},$$
$$U(0,z) = f(z), \frac{\partial}{\partial t} U(t,z)|_{t=0} = rv(1+i)f'(z),$$

and proposed a method for solving such an equation for complex-analytic initial conditions.

This model is a generalization of original model described in [4] by M.Kac and is just one of many models exploring similar processes in multidimensional spaces. Corresponding results are mostly devoted to discussion of convergence of the process studied to the Wiener process, also description of corresponding equations and solving them for some well posed models ([2], [5] – [8], [10] – [18] and many others).

The main problem is that the methods for solving equations proposed there can not be applied for any model as soon as they are, as a rule, strictly connected with the structure of the corresponding equation. In the article [20] we proposed to change the approach, namely to solve a well posed Cauchy problem instead of a well posed equation.

Moreover, we aim to focus on models grounded in physics. Specifically, the aforementioned approach provides a pathway for creating simulations of physical processes defined by the Schrödinger equation based on the Goldstein-Kac model, which is relatively simple to implement. Additionally, our results make it possible to search for solutions to Schrödinger-type equations in the form of series with any desired accuracy. Possible applications of such models in physics, including the methods presented in [20], may be found in a recent article [3], published in Annals of Physics.

The model presented in this work builds on these ideas and specifically addresses the behaviour of random evolution on a complex plane in the case where a physical particle moves in a force field that "attracts" the particle, reducing its speed at each step. We can define the fading evolution as follows:

$$\zeta_{r,z}^{\lambda;v}(t) = z + vi^r \int_0^t (ia)^{N^\lambda(s)} ds, z \in C, a \in (0,1).$$

Here $N^\lambda(s)$ is the Poisson process with intensity λ that takes values $\{0,1,2,...\}$, parameter $r \in \{0,1,2,3\}$ defines initial direction of the process. If $r=0$ we start to the positive direction of real line ($Re+$); $r=1$ – positive direction of imaginary line ($Im+$); $r=2$ – negative direction of real line ($Re-$); $r=3$ – negative direction of imaginary line ($Im-$).

A key distinction of this model from all those mentioned above is the ability to examine its behavior as time approaches infinity. The other models mentioned, with probability 1, remain within a region (the shape of which is determined by the process structure; see, for example, [2], [10] – [14]) whose boundary linearly depends on t and therefore also tends toward infinity. In the case of our model, there exists a point where the process "freezes" and the position distribution of this point can be explicitly calculated if the process is represented as an infinite random series:

$$\zeta_{r,z}^{\lambda,v}(\infty) = z + i^r v \sum_{k=0}^{\infty} (ia)^k \tau_k,$$

where τ_k are random time intervals with identical exponential distribution. The second section of the article is dedicated to this question.

The complexity of this model lies in the fact that we cannot apply the classical approach to analysing functionals of the process, namely writing the backward Kolmogorov equations, deriving the corresponding higher-order equation, and finding its solutions. Instead, it is proposed to use the relevant non-linear integral equations, presented in the third section of the article.

This raises the question of the existence and uniqueness of their solution, which is examined in the last section. It is shown that the obtained equations can indeed be solved, for instance, by using the method of successive approximations.

2. Distribution of absorbing point

First, we will prove an auxiliary result which, however, is of independent interest, as it provides information about convolutions of functions that often arise in various problems as distributions of random variables.

Lemma 2.1. *If a distribution function satisfies the equation*

$$F(x) = \lambda \int_0^x e^{-\lambda u} F\left(\tfrac{x-u}{a^n}\right) du, x \geq 0 \tag{1}$$

then the following expansion holds true:

$$F(x) = 1 - s^{-1} \left[e^{-\lambda x} + \sum_{j=1}^{\infty} (-1)^j e^{-\frac{\lambda x}{a^{nj}}} \prod_{k=1}^{j} \frac{a^{n^k}}{1 - a^{n^k}} \right], \tag{2}$$

here $s = 1 + \sum_{j=1}^{\infty} (-1)^j \prod_{k=1}^{j} \frac{a^{n^k}}{1-a^{n^k}}$.

Remark 2.1. If we put $c = \frac{1}{a^n}$, then

$$s = 1 - \tfrac{1}{c-1} + \tfrac{1}{(c-1)(c^2-1)} - ...,$$

thus $s = \sum_{j=0}^{\infty} (-1)^n \frac{(0)_{c,n}}{(c)_{c,n}} = {}_1\Phi_0(0; -1; c)$, where $(c)_{q,n} := (1-c)(1-cq)\dots(1-cq^n)$; ${}_r\Phi_s\left(\begin{smallmatrix}\alpha_1,\dots,\alpha_r;z \\ \beta_1,\dots,\beta_s;q\end{smallmatrix}\right)$ is a basic hypergeometric series [1].

Proof. Obviously, $F(0) = 0$. We are looking for $F(x)$ in the form:

$$F(x) = 1 + a_1 e^{-\lambda x} + a_2 e^{-\frac{\lambda x}{a^n}} + a_3 e^{-\frac{\lambda x}{a^{n^2}}} + \cdots + a_n e^{-\frac{\lambda x}{a^{n^{n-1}}}} + \dots \tag{3}$$

Substituting this expression into the right-hand side of equation (1), we get

$$F(x) = \lambda \int_0^x e^{-\lambda u} \left[1 + a_1 e^{-\frac{\lambda(x-u)}{a^n}} + a_2 e^{-\frac{\lambda(x-u)}{a^{n^2}}} + \dots + a_n e^{-\frac{\lambda x}{a^{n^{n-1}}}} + \dots\right] du$$

$$= 1 - e^{-\lambda x} + \frac{a_1 a^n e^{-\frac{\lambda x}{a^n}}}{1-a^n}\left(e^{\frac{(1-a^n)\lambda x}{a^n}} - 1\right) + \frac{a_2 a^{n^2} e^{-\frac{\lambda x}{a^{n^2}}}}{1-a^{n^2}}\left(e^{\frac{(1-a^{n^2})\lambda x}{a^{n^2}}} - 1\right)$$

$$+ \dots + \frac{a_n a^{n^n} e^{-\frac{\lambda x}{a^{n^n}}}}{1-a^{n^n}}\left(e^{\frac{(1-a^{n^n})\lambda x}{a^{n^n}}} - 1\right) + \dots$$

Let's replace $F(x)$ in the left part with its expression (3) and equate the coefficients:
at $e^{-\lambda x}$: $a_1 = -1 + \frac{a_1 a^n}{1-a^n} + \frac{a_2 a^{n^2}}{1-a^{n^2}} + \dots,$
at $e^{-\frac{\lambda x}{a^n}}$: $a_2 = -\frac{a_1 a^n}{1-a^n},$
at $e^{-\frac{\lambda x}{a^{n^2}}}$: $a_3 = -\frac{a_2 a^{n^2}}{1-a^{n^2}},$
$\dots,$
at $e^{-\frac{\lambda x}{a^{n^{n-1}}}}$: $a_n = -\frac{a_{n-1} a^{n^{n-1}}}{1-a^{n^{n-1}}}$
$\dots$
Substituting $a_i, i > 1$ into the first equality we get:

$$a_1 = -1 + \frac{a_1 a^n}{1-a^n} - \frac{a_1 a^{n+n^2}}{(1-a^n)(1-a^{n^2})} + \dots,$$

thus

$$a_1 \left[1 - \frac{a^n}{1-a^n} + \frac{a^{n+n^2}}{(1-a^n)(1-a^{n^2})} - \dots\right] = -1.$$

We denote the sum of the convergent series in square brackets by s. Then

$$a_1 = -\frac{1}{s};$$

$$a_2 = \frac{a^n}{s(1-a^n)};$$

$$a_3 = -\frac{a^{n+n^2}}{s(1-a^n)(1-a^{n^2})};$$

$$\dots;$$

$$a_n = -\frac{a^{n+n^2+\dots+n^{n-1}}}{s(1-a^n)(1-a^{n^2})\dots(1-a^{n^{n-1}})}.$$

Thus, the distribution function satisfying equation (1) has the form

$$F(x) = 1 - s^{-1}\left[e^{-\lambda x} - \frac{a^n}{1-a^n} e^{-\frac{\lambda x}{a^n}} + \frac{a^{n+n^2}}{(1-a^n)(1-a^{n^2})} e^{-\frac{\lambda x}{a^{n^2}}} - \dots\right],$$

$x \geq 0$. □

Now let us examine fading evolution $\zeta_{r,z}^{\lambda,v}(\infty)$ for the case $r = 0$, namely

$$\zeta(\infty) := z + v \sum_{k=0}^{\infty} (ia)^k \tau_k.$$

We may present $\zeta(\infty) = x + iy + v\tau_0 + aiv\tau_1 - a^2 v\tau_2 - a^3 iv\tau_3 + \ldots$ in the form $\zeta(\infty) = Re\zeta(\infty) + iIm\zeta(\infty)$, where

$$Re\zeta(\infty) = x + v\sum_{k=0}^{\infty}(-1)^k a^{2k}\tau_{2k},$$

$$Im\zeta(\infty) = y + v\sum_{k=0}^{\infty}(-1)^k a^{2k+1}\tau_{2k+1}.$$

Theorem 2.2.

$$F_{Re\zeta(\infty)}(X) := P\{Re\zeta(\infty) \le X\} = \int_0^{\infty} F_\zeta\left(\frac{X-x}{v} + a^2 u\right) dF_\zeta(u),$$

$$F_{Im\zeta(\infty)}(Y) := P\{Im\zeta(\infty) \le Y\} = \int_0^{\infty} F_\zeta\left(\frac{Y-y}{av} + a^2 u\right) dF_\zeta(u),$$

where

$$F_\zeta(u) = 1 - s^{-1}\left[e^{-\lambda u} + \sum_{j=1}^{\infty}(-1)^j e^{-\frac{\lambda u}{a^{4j}}} \prod_{k=1}^{j} \frac{a^{4^k}}{1-a^{4^k}}\right], u \ge 0,$$

and equals 0 otherwise. Here $s = 1 + \sum_{j=1}^{\infty}(-1)^j \prod_{k=1}^{j} \frac{a^{4^k}}{1-a^{4^k}}$.

Proof. We are looking for the distribution of

$$P\{Re\zeta(\infty) \le X\} = P\left\{\sum_{k=0}^{\infty} a^{4k}\tau_{4k} - a^2\sum_{k=0}^{\infty} a^{4k}\tau_{4k+2} \le \frac{X-x}{v}\right\}.$$

Let us denote the following random series by ζ:

$$\zeta := \sum_{k=0}^{\infty} a^{4k}\tau_{4k}.$$

Then, $F_\zeta(t) = P\{\zeta \le t\} = P\{\tau_0 + a^4\sum_{k=0}^{\infty} a^{4k}\tau_{4k+4} \le t\} = P\{\tau_0 + a^4\zeta^* \le t\}$, where ζ^* has the same distribution function $F_\zeta(t)$.

Given that τ_i has an exponential distribution, we have

$$F_\zeta(t) = \lambda\int_0^{\infty} e^{-\lambda u} P\{u + a^4\zeta^* \le t\} du = \lambda\int_0^{t} e^{-\lambda u} P\{\zeta^* \le \frac{t-u}{a^4}\} du,$$

and finally

$$F_\zeta(t) = \lambda\int_0^{t} e^{-\lambda u} F_\zeta\left(\frac{t-u}{a^4}\right) du.$$

Note, that by the Lemma 2.1 $F_\zeta(t)$ may be presented obviously as a series:

$$F_\zeta(t) = 1 - s^{-1}\left[e^{-\lambda t} + \sum_{j=1}^{\infty}(-1)^j e^{-\frac{\lambda t}{a^{4j}}} \prod_{k=1}^{j} \frac{a^{4^k}}{1-a^{4^k}}\right].$$

Having this in hand, we may now regard $P\{Re\zeta(\infty) \le X\} = P\{\zeta - a^2\zeta^* \le \frac{X-x}{v}\}$, which means that

$$F_{Re\zeta(\infty)}(X) = \int_0^{\infty} F_\zeta\left(\frac{X-x}{v} + a^2 u\right) dF_\zeta(u).$$

Similar considerations regarding $P\{Im\zeta(\infty) \le X\} = P\{a\zeta - a^3\zeta^* \le \frac{Y-y}{v}\}$ lead to the equality

$$F_{Im\zeta(\infty)}(Y) = \int_0^\infty F_\zeta\left(\frac{Y-y}{av} + a^2 u\right) dF_\zeta(u).$$

□

Remark 2.2. We proved the previous result for the case when the initial direction of the evolution is the positive direction of the real line. Easy to see, that if the evolution starts, for instance, at the negative direction of the real line, then

$$F_{-Re\zeta(\infty)}(X) := P\{2x - Re\zeta(\infty) \le X\} = 1 - F_{Re\zeta(\infty)}(2x - X).$$

Common rules are the following: if the evolution starts at the negative direction of the real line, then we should change $Re\zeta(\infty)$ by $2x - Re\zeta(\infty)$ and $Im\zeta(\infty)$ by $2y - Im\zeta(\infty)$ to obtain corresponding distribution functions; if the evolution starts at the positive direction of the complex line, then we should change $Re\zeta(\infty)$ by $2x - Im\zeta(\infty)$ and $Im\zeta(\infty)$ by $Re\zeta(\infty)$; if the evolution starts at the negative direction of the complex line, then we should change $Re\zeta(\infty)$ by $Im\zeta(\infty)$ and $Im\zeta(\infty)$ by $2y - Re\zeta(\infty)$.

Thus, having the distribution functions for all initial directions, we may find the distribution of the general process in the form

$$\begin{aligned} F_{Re\zeta(\infty)}(X) &= p_1 P\{Re\zeta(\infty) \le X\} + p_2 P\{2x - Re\zeta(\infty) \le X\} \\ &\quad + p_3 P\{2x - Im\zeta(\infty) \le X\} + p_4 P\{Im\zeta(\infty) \le X\}, \end{aligned}$$

where $p = (p_1, p_2, p_3, p_4)$ is the distribution of initial directions in the following order: $(Re+, Re-, Im+, Im-)$.

The same for the distribution of $Im\zeta(\infty)$.

The obtained results regarding the distribution of the absorption point allow us to evaluate the probabilities of process attenuation in various regions of a complex plane. As an example, the following theorem presents probability that the absorption point lies within a circle of radius R centred at the starting point.

Theorem 2.3. *The probability that $\zeta(\infty)$ is inside a circle of radius R and center $z = x + iy$ equals*

$$\int_{-R}^{R} \left[F_{Re\zeta(\infty)}(x + \sqrt{R^2 - (u-y)^2}) - F_{Re\zeta(\infty)}(x - \sqrt{R^2 - (u-y)^2})\right] dF_{Im\zeta(\infty)}(u)$$

and does not depend on initial direction.

Proof. The following condition provides the Theorem:

$$(Re\zeta(\infty) - x)^2 + (Im\zeta(\infty) - y)^2 < R^2, \tag{4}$$

thus we have to find

$$\begin{aligned} &P\{(Re\zeta(\infty) - x)^2 + (Im\zeta(\infty) - y)^2 < R^2\} \\ &= P\{(Re\zeta(\infty) - x)^2 < R^2 - (Im\zeta(\infty) - y)^2\} \\ &= P\{x - \sqrt{R^2 - (Im\zeta(\infty) - y)^2} < Re\zeta(\infty) < x + \sqrt{R^2 - (Im\zeta(\infty) - y)^2}\} \\ &= \int_{-R}^{R}\Big[F_{Re\zeta(\infty)}(x + \sqrt{R^2 - (u-y)^2}) - F_{Re\zeta(\infty)}(x - \sqrt{R^2 - (u-y)^2})\Big] dF_{Im\zeta(\infty)}(u). \end{aligned}$$

As soon as inequality (4) is invariant with respect to substitutions of processes described in Remark 2.2 (for example, $(Re\zeta(\infty)-x)^2=(2x-Re\zeta(\infty)-x)^2$), the probability found does not depend on the initial direction. □

3. Integral equation for a function from fading random evolution on a complex plane

In the case of the fading random evolution, it is not possible to obtain a system of backward Kolmogorov differential equations and use the classical technique, presented, for example, in [10] – [18]. However, it turns out that it is possible to write down corresponding integral equations. Thus, we'll have nonlinear integral equations that describe the movement of a particle under the action of an external force when the speed of movement decreases over time.

Let $\pounds$ be the space of functions of the form

$$\phi(v,z,t)=\phi_0(v,z,t)+c,$$

$$c=\mathrm{const},\ \ \phi_0(v,z,t)\to 0,\quad v,z,t\to 0. \tag{5}$$

In the book [9], it was proven that this space is a Banach space with respect to the sup-norm

$$||\phi(v,z,t)||_{\pounds}=\sup_{v,z,t}\phi(v,z,t).$$

Let $f\in\pounds$. Consider the functionals that describe the studied process on a complex plane in the following form:

$$u_{Re+}(v,z,t)=Ef(z+v\int_0^t(ia)^{N^\lambda(s)}ds),u_{Im+}(v,x,t)=Ef(z+iv\int_0^t(ia)^{N^\lambda(s)}ds),$$
$$u_{Re-}(v,x,t)=Ef(z-v\int_0^t(ia)^{N^\lambda(s)}ds)=u_{Re+}(-v,z,t),$$
$$u_{Im-}(v,x,t)=Ef(z-iv\int_0^t(ia)^{N^\lambda(s)}ds)=u_{Im+}(-v,x,t)$$

the first of which is a function from the evolution that starts at the point z in positive direction of real line; the second – from evolution starting at the same point in positive direction of imaginary line, etc.

Let us derive the integral equations for these functionals. Denote by τ the time at which the first jump of the Poisson process N^λ occurs, that is, $N^\lambda(\tau)=1$ and $N^\lambda(s)=0$ for $s\in[0,\tau)$ almost surely. Note that τ follows an exponential distribution with a mean of $1/\lambda$. The random variable τ is a stopping time with respect to the filtration generated by the process N^λ.

Due to the strong Markov property of the Poisson process, the process $(N^\lambda(u+\tau)-N^\lambda(\tau))_{u\geq 0}$ is itself a Poisson process with intensity λ, independent of τ. Therefore,

$$
\begin{aligned}
&u_{Re+}(v,z,t) = Ef\left(z + v\int_0^t (ia)^{N^\lambda(u)}du\right)\\
&= Ef\left(z + v\int_0^t (ia)^{N^\lambda(u)}du\right)\mathbf{1}_{\{\tau>t\}} + f\left(z + v\int_0^\tau (ia)^{N^\lambda(u)}du + v\int_\tau^t (ia)^{N^\lambda(u)}du\right)\mathbf{1}_{\{\tau\le t\}}\\
&= P(N^\lambda(t)=0)Ef(z+vt) + f\left(z + v\tau + v(ia)^{N^\lambda(\tau)}\int_0^{t-\tau}(ia)^{N^\lambda(u+\tau)-N^\lambda(\tau)}du\right)\mathbf{1}_{\{\tau\le t\}}\\
&= e^{-\lambda t}f(z+vt) + \int_0^t \lambda e^{-\lambda s}Ef\left(z + vs + iav\int_0^{t-s}(ia)^{N^\lambda(u)}du\right)ds\\
&= e^{-\lambda t}f(z+vt) + \int_0^t \lambda e^{-\lambda s}u_{Im+}(av, z+vs, t-s)ds.
\end{aligned}
$$

The same for other functions:

$$
\begin{aligned}
&u_{Im+}(v,z,t) = Ef\left(z + iv\int_0^t (ia)^{N^\lambda(u)}du\right)\\
&= P(N^\lambda(t)=0)Ef(z+ivt) + \int_0^t \lambda e^{-\lambda s}Ef\left(z + ivs - av\int_0^{t-s}(ia)^{N^\lambda(u)}du\right)ds\\
&= e^{-\lambda t}f(z+ivt) + \int_0^t \lambda e^{-\lambda s}u_{Re-}(av, z+ivs, t-s)ds\\
&= e^{-\lambda t}f(z+ivt) + \int_0^t \lambda e^{-\lambda s}u_{Re+}(-av, z+ivs, t-s)ds.
\end{aligned}
$$

$$
u_{Re-}(v,z,t) = e^{-\lambda t}f(z-vt) + \int_0^t \lambda e^{-\lambda s}u_{Im-}(av, z-vs, t-s)ds.
$$

$$
u_{Im-}(v,z,t) = e^{-\lambda t}f(z-ivt) + \int_0^t \lambda e^{-\lambda s}u_{Re+}(av, z-ivs, t-s)ds.
$$

After substitution of $u_{Im+}(v,z,t)$ (with u_{Re+} in the right-hand side) into $u_{Re+}(v,z,t)$ we have:

$$
\begin{aligned}
u_{Re+}(v,x,t) &= e^{-\lambda t}f(z+vt) + \int_0^t \lambda e^{-\lambda s}e^{-\lambda(t-s)}f(z+vs+i(av)(t-s))ds +\\
&\int_0^t \lambda e^{-\lambda s}\int_0^{t-s}\lambda e^{-\lambda\tau}u_{Re+}(-a(av), z+vs+i(av)\tau, t-s-\tau)d\tau ds\\
&= (s+\tau=\theta) = e^{-\lambda t}f(z+vt) + \lambda e^{-\lambda t}\int_0^t f(z+vs+iav(t-s))ds\\
&+ \lambda^2\int_0^t\int_0^{t-s} e^{-\lambda\theta}u_{Re+}(-a^2v, z+vs+iav(\theta-s), t-\theta)d\tau ds \qquad (6)
\end{aligned}
$$

Another, more complicated way also exists. We may substitute $u_{Im+}(v,z,t)$ with u_{Re-} in the right-hand side:

$$
\begin{aligned}
u_{Re+}(v,x,t) &= e^{-\lambda t}f(z+vt) + \int_0^t \lambda e^{-\lambda s}e^{-\lambda(t-s)}f(z+vs+i(av)(t-s))ds\\
&+ \int_0^t \lambda e^{-\lambda s}\int_0^{t-s}\lambda e^{-\lambda\tau}u_{Re-}(a(av), z+vs+i(av)\tau, t-s-\tau)d\tau ds
\end{aligned}
$$

$$= (s+\tau=\theta) = e^{-\lambda t} f(z+vt) + \lambda e^{-\lambda t} \int_0^t f(z+vs+iav(t-s))ds$$
$$+ \lambda^2 \int_0^t \int_s^t e^{-\lambda\theta} u_{Re-}(a^2 v, z+vs-iav\theta+iavs, t-\theta) d\theta ds. \quad (7)$$

Similar equality for $u_{Re-}(v,z,t)$ may be found after substitution of $u_{Im-}(v,z,t)$:

$$u_{Re-}(v,x,t) = e^{-\lambda t} f(z-vt) + \lambda e^{-\lambda t} \int_0^t f(z-vs-iav(t-s))ds$$
$$+ \lambda^2 \int_0^t \int_s^t e^{-\lambda\theta} u_{Re+}(a^2 v, z-vs-iav\theta+iavs, t-\theta) d\theta ds. \quad (8)$$

Combining equations (7) and (8) we finally have integral equation for $u_{Re+}(v,z,t)$:

$$u_{Re+}(v,x,t) = e^{-\lambda t} f(z+vt) + \lambda e^{-\lambda t} \int_0^t f(z+vs+iav(t-s))ds+$$
$$\lambda^2 e^{-\lambda t} \int_0^t \int_s^t f(z+vs-iav(\theta-s)-a^2 v(t-\theta)) d\theta ds + \lambda^3 e^{-\lambda t} \int_0^t \int_s^t \int_\theta^t f(z+$$
$$vs-iav(\theta-s)-a^2 v(\tau-\theta)-ia^3 v(t-\tau)) d\tau d\theta ds + \lambda^4 \int_0^t \int_s^t \int_\theta^t \int_\tau^t e^{-\lambda\zeta} \times$$
$$u_{Re+}(a^4 v, z+vs-iav(\theta-s)-a^2 v(\tau-\theta)-ia^3 v(\zeta-\tau), t-\zeta) d\zeta d\tau d\theta ds. \quad (9)$$

Thus, we see that equations (6) and (9) are equivalent.

Equations of similar forms may be found for the functions $u_{Im+}(v,z,t)$, $u_{Re-}(v,z,t)$, $u_{Im-}(v,z,t)$ in the same way.

4. Existing and uniqueness of solution

Let us consider the question of the existence and uniqueness of a solution to nonlinear integral equations (6), (9) in the space $\pounds$, defined by formula (5). Since the space is a Banach space, the contraction mapping principle holds in this space. We illustrate the proof for the more complicated equation (9).

Let's rewrite equation (9) in the form $u(v,z,t) = Au(v,z,t)$, where $Au(v,z,t)$ – the right-hand part of equation (9). Let $f(z), \phi(v,z,t) \in \pounds$. For the functions from the space $\pounds$ the action of A is the following:

$$A\phi(v,z,t) = e^{-\lambda t} f(z+vt) + \lambda e^{-\lambda t} \int_0^t f(z+vs+iav(t-s))ds+$$
$$\lambda^2 e^{-\lambda t} \int_0^t \int_s^t f(z+vs-iav(\theta-s)-a^2 v(t-\theta)) d\theta ds + \lambda^3 e^{-\lambda t} \int_0^t \int_s^t \int_\theta^t f(z+$$
$$vs-iav(\theta-s)-a^2 v(\tau-\theta)-ia^3 v(t-\tau)) d\tau d\theta ds + \lambda^4 \int_0^t \int_s^t \int_\theta^t \int_\tau^t e^{-\lambda\zeta} \times$$
$$\phi(a^4 v, z+vs-iav(\theta-s)-a^2 v(\tau-\theta)-ia^3 v(\zeta-\tau), t-\zeta) d\zeta d\tau d\theta ds \le$$
$$\sup_z f(z+vt) e^{-\lambda t} + \lambda e^{-\lambda t} \int_0^t \sup_z f(z+vs+iav(t-s))ds + \lambda^2 e^{-\lambda t} \int_0^t \int_s^t \sup_z f(z+$$
$$vs-iav(\theta-s)-ia^2 v(t-\theta)) d\theta ds + \lambda^3 e^{-\lambda t} \int_0^t \int_s^t \int_\theta^t \sup_z f(z+$$

$$vs - iav(\theta - s) - a^2v(\tau - \theta) - ia^3v(t - \tau))d\tau d\theta ds + \lambda^4 \int_0^t \int_s^t \int_\theta^t \int_\tau^t e^{-\lambda\zeta} \times$$

$$\sup_{v,z,t} \phi(a^4v, z + vs - iav(\theta - s) - a^2v(\tau - \theta) - ia^3v(\zeta - \tau), t - \zeta)d\zeta d\tau d\theta ds \leq$$

$$K\left(e^{-\lambda t} + \lambda te^{-\lambda t} + \frac{(\lambda t)^2}{2!}e^{-\lambda t} + \frac{(\lambda t)^3}{3!}e^{-\lambda t} + \lambda^4 \int_0^t \int_s^t \int_\theta^t \int_\tau^t e^{-\lambda\zeta} d\zeta d\tau d\theta ds\right) =$$

$$K\left\{e^{-\lambda t} + \lambda te^{-\lambda t} + \frac{(\lambda t)^2}{2!}e^{-\lambda t} + \frac{(\lambda t)^3}{3!}e^{-\lambda t} + \lambda^4\left(-\frac{1}{\lambda}e^{-\lambda t}\frac{t^3}{3!} - \frac{1}{\lambda^2}e^{-\lambda t}\frac{t^2}{2!} - \right.\right.$$

$$\left.\left.\frac{1}{\lambda^3}e^{-\lambda t}t - \frac{1}{\lambda^4}(e^{-\lambda t} - 1)\right)\right\} = K$$

where $K = \max\{\sup_z f(z), \sup_{v,z,t} \phi(v, z, t)\}$, thus $A\phi(v, z, t)$ is a bounded function.

Let us show that A is a compression. We have

$$\rho(A\phi_1, A\phi_2) = \sup_{v,z,t} |\lambda^4 \int_0^t \int_s^t \int_\theta^t \int_\tau^t e^{-\lambda\zeta}\phi_1(a^4v, z + vs - iav(\theta - s) -$$

$$a^2v(\tau - \theta) - ia^3v(\zeta - \tau), t - \zeta)d\zeta d\tau d\theta ds - \lambda^4 \int_0^t \int_s^t \int_\theta^t \int_\tau^t e^{-\lambda\zeta} \times$$

$$\phi_2(a^4v, z + vs - iav(\theta - s) - a^2v(\tau - \theta) - ia^3v(\zeta - \tau), t - \zeta)d\zeta d\tau d\theta ds| \leq$$

$$\lambda^4 \int_0^t \int_s^t \int_\theta^t \int_\tau^t e^{-\lambda\zeta} \sup_{v,z,t} |\phi_1(a^4v, z + vs - iav(\theta - s) - a^2v(\tau - \theta) - ia^3v(\zeta -$$

$$\tau), t - \zeta) - \phi_2(a^4v, z + vs - iav(\theta - s) - a^2v(\tau - \theta) - ia^3v(\zeta - \tau), t - \zeta)|dlds =$$

$$\rho(\phi_1, \phi_2)[-\frac{(\lambda t)^3}{3!}e^{-\lambda t} - \frac{(\lambda t)^2}{2!}e^{-\lambda t} - \lambda te^{-\lambda t} - e^{-\lambda t} + 1],$$

where $-\frac{(\lambda t)^3}{3!}e^{-\lambda t} - \frac{(\lambda t)^2}{2!}e^{-\lambda t} - \lambda te^{-\lambda t} - e^{-\lambda t} + 1 < 1$. Thus, the solution of equation (9) exists and is unique.

Similarly, we may obtain the same result for the equations of the form (6) and for the functions $u_{Im+}(v, z, t)$, $u_{Re-}(v, z, t)$, $u_{Im-}(v, z, t)$, thus we have:

Theorem 4.1. *For any $f(z)$ from the space $\pounds$, equations for the functions $u_{Re+}(v, z, t)$, $u_{Im+}(v, z, t)$, $u_{Re-}(v, z, t)$, $u_{Im-}(v, z, t)$ of the forms (6), (9) have a unique solution in this space.*

This theorem enables us to find solutions to the obtained integral equations of the forms (6), (9) using the method of successive approximations, as a limit $\lim_{n\to\infty} u_n(v, z, t)$, where $u_0(v, z, t)$ – any function from $\pounds$.

5. Conclusion

By employing random series in the second section of the article, we successfully identified the absorbing point for a model that generalizes the Goldstein-Kac model to the case where the velocity is not constant but decreases with each change in the particle's direction of motion. Combined with the space where the evolution takes place (complex plane), this model appears intriguing from the perspective of physical applications, as it is associated with Schrödinger-type equations and processes related to particle attraction or "freezing."

The main result of the second section is presented in Theorem 2.2, which provides a method for determining the distribution of the absorbing point for the studied model and calculating the probability of this point falling into regions of various shapes and complexities. It should be noted that Theorem 2.3 merely demonstrates one possible variant of such a problem. By applying similar calculations, it is possible to obtain results for other regions, depending on the physical basis of the model.

The last two sections of the work also present only general methods that allow deriving nonlinear integral equations, as their form is quite extensive, and providing variants of the equations for all functions would significantly increase the length of the article. Results for other functions can be obtained by applying the described methods.

Unfortunately, it is challenging to compare the equations describing non-fading evolution (Schrödinger-type differential equations from the work [19]) with the nonlinear integral equations derived here. Nevertheless, certain high-order nonlinear differential equations, equivalent, for instance, to equation (6), may be obtained by differentiating the latter with respect to z and t.

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