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New Variants of the Hermite-Hadamard Inequality via Modified (h, m)-convex Functions of the Second Type

Juan Gabriel Galeano Delgado, Juan E. Nápoles Valdés, and Abel Alvarez Bustos

Abstract. In this study, we derive novel variants of the Hermite-Hadamard inequality using modified (h, m)-convex functions of the second type. Our approach leverages the generalized fractional integral operator to establish these variants, grounded in the theoretical underpinnings of (h, m)-convex functions that extend classical convex functions. By integrating the generalized fractional integral operator with modified (h, m)-convex functions, we formulate hypotheses that extend the Hermite-Hadamard inequality, exploring fresh boundaries and relationships within this mathematical framework. Several findings from the existing literature are extrapolated from our research, underscoring the robustness and practical relevance of the newly introduced Hermite-Hadamard inequality versions in this paper.

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1. Introduction

Mathematics has devoted much attention to the development of new Hermite-Hadamard type integral inequalities for modified (h, m)-convex functions of the second type. This is crucial because convex functions have numerous applications and are interconnected with various fields, including finance, numerical analysis, industrial optimization, and probability theory, in addition to others. This definition of a convex function is due by

Definition 1.1. A function $\varphi : I \to \mathbb{R}$ is said to be **convex** on an interval $I \subset \mathbb{R}$, if the inequality $\varphi(\tau u + (1-\tau)v) \leq \tau\varphi(u) + (1-\tau)\varphi(v)$, for, $u, v \in I$ is fulfilled with $\tau \in [0, 1]$.

In addition we say that φ is concave if $-\varphi$ is convex. For all of the above, in recent years various extensions and generalizations of this classic notion have proliferated (interested readers can consult [42], where a fairly complete overview of the aforementioned development is presented).

A similar expansion has occurred in the field of mathematical sciences regarding the investigation of integral inequalities, which now includes both theoretical and applied research on the inequality of convex functions. This increase is reflected in the growing number of researchers and notable discoveries in recent years. Among these, there exists an inequality that may be viewed as fundamental and that offers bounds in a

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particular class of convex functions over the integral mean value; this fundamental result is the so-called Hermite-Hadamard inequality (see, e.g., [13, ?, 30]).

Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined in the interval I of real numbers, and $a_1, a_2 \in I$ with $a_1 < a_2$. The following inequality arises.

$$\varphi(\frac{a_1+a_2}{2}) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2}\varphi(u)du \leq \frac{\varphi(a_1)+\varphi(a_2)}{2}, \tag{1}$$

holds. Since its discovery, this inequality has garnered significant attention. Various extensions and expansions of this inequality, including different fractional and generalized operators and using various convexity operators, may be found in [2, 6, 22, 24, 25, 29, 44, 45, 42, 43].

The concept of convex functions has had a different extension, this has resulted in a highly intricate development of the notion. In [42] presents An extensive categorization of the majority of the existing definitions has been established.

Toader in [55] defined the m-convexity in the following way:

Definition 1.2. The function $\varphi : [0,b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0,1]$, if

$$\varphi\left(\tau u + (1-\tau)v\right) \leq \tau\varphi(u) + m(1-\tau)\varphi(v) \tag{2}$$

holds for all $u, v \in [0,b]$ and $\tau \in [0,1]$.

If the above inequality holds in reverse, then we say that the function φ is m-concave.

Definition 1.3. [8, 31] Let $s \in (0,1]$ be a real number. A function $\varphi : [0,b] \to [0,+\infty)$ with $b > 0$ is said to be s-convex (in the first sense) if

$$\varphi(\tau u + m(1-\tau)v) \leq \tau^s\varphi(u) + (1-\tau^s)\varphi(v), \tag{3}$$

for all $u, v \in [0,b]$ and $\tau \in (0,1)$.

Definition 1.4. Let $h : [0,1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$ and $\psi : I = [0,+\infty) \to [0,+\infty)$. If inequality

$$\psi\left(\tau\xi + m(1-\tau)\varsigma\right) \leq h^s(\tau)\psi(\xi) + m(1-h^s(\tau))\psi\left(\varsigma\right) \tag{4}$$

is fulfilled for all $\xi, \varsigma \in I$ and $\tau \in [0,1]$, where $m \in [0,1]$, $s \in [-1,1]$. Then a function ψ is called a (h,m)-convex modified of the first type on I.

Definition 1.5. Let $h : [0,1] \to \mathbb{R}$ nonnegative functions, $h \neq 0$ and $\psi : I = [0,+\infty) \to [0,+\infty)$. If inequality

$$\psi\left(\tau\xi + m(1-\tau)\varsigma\right) \leq h^s(\tau)\psi(\xi) + m(1-h(\tau))^s\psi\left(\varsigma\right) \tag{5}$$

is fulfilled for all $\xi, \varsigma \in I$ and $\tau \in [0,1]$, where $m \in [0,1]$, $s \in [-1,1]$. Then a function ψ is called a (h,m)-convex modified of the second type on I.

Remark 1.1. From Definitions 1.4 and 1.5 we can define $N^s_{h,m}[a,b]$,where $a, b \in [0,+\infty)$, as the set of functions $(h,m)-$convex modified, for which $\psi(a) \geq 0$, characterized by the triple $(h(\tau), m, s)$. Note that if:

(1) $(h(\tau), 0, 0)$ we have the increasing functions ([9]).
(2) $(\tau, 0, s)$ we have the $s-$starshaped functions ([9]).
(3) $(\tau, 0, 1)$ we have the starshaped functions ([9]).
(4) $(\tau, 1, 1)$ then ψ is a convex function on $[0,+\infty)$ ([9]).

(5) $(\tau, m, 1)$ then ψ is a $m-$convex function on $[0, +\infty)$ ([?]).
(6) $(\tau, 1, s)$ $s \in (0, 1]$ then ψ is a $s-$convex function on $[0, +\infty)$ ([8, 31]).
(7) $(\tau, 1, s)$ $s \in [-1, 1]$ then ψ is a $s-$convex-extended function on $[0, +\infty)$ ([56]).
(8) (τ, m, s) $s \in (0, 1]$ then ψ is a (s, m)-convex extended function on $[0, +\infty)$ ([48]).
(9) $(\tau^a, 1, s)$ with $a \in (0, 1]$, then ψ is a (a, s)-convex function on $[0, +\infty)$ ([7]).
(10) $(\tau^a, m, 1)$ with $a \in (0, 1]$, then ψ is a (a, m)-convex function on $[0, +\infty)$ ([38]).
(11) (τ^a, m, s) with $a \in (0, 1]$, then ψ is a $s-(a, m)$-convex function on $[0, +\infty)$ ([57]).
(12) $(h(\tau), m, 1)$ then ψ is a variant of a $(h, m)-$convex function on $[0, +\infty)$ ([46]).

To encourage comprehension of the subject, we present the definition of Riemann-Liouville fractional integral (with $0 \leq a_1 < t < a_2 \leq \infty$). The first is the classic Riemann-Liouville fractional integrals.

Definition 1.6. Let $\phi \in L_1[a_1, a_2]$. Then the Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are defined by (right and left respectively):

$$ {}^{\alpha}I_{a_1^+}\phi(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x} (x-t)^{\alpha-1}\phi(t)\,dt, \quad x > a_1 \tag{6} $$

$$ {}^{\alpha}I_{a_2^-}\phi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{a_2} (t-x)^{\alpha-1}\phi(t)\,dt, \quad x < a_2. \tag{7} $$

Next, we present the weighted integral operators, which will be the basis of our work (see [5]).

Definition 1.7. Let $\varphi \in L([a_1, a_2])$ and let w be a continuous and positive function, $w : [0, 1] \to [0, +\infty)$, with first order derivatives piecewise continuous on I. Then the weighted fractional integrals are defined by (right and left respectively):

$$ J_{a_1^+}^{w}\varphi(u) = \int_{a_1}^{u} w'\left(\frac{\sigma - a_1}{a_2 - a_1}\right)\varphi(\sigma)d\sigma $$

and

$$ J_{a_2^-}^{w}\varphi(u) = \int_{u}^{a_2} w'\left(\frac{a_2 - \sigma}{a_2 - a_1}\right)\varphi(\sigma)d\sigma. $$

Remark 1.2. To have a clearer idea of the amplitude of the Definition 1.7, let's consider some particular cases of the kernel w':

(1) Putting $w'(t) \equiv 1$, we obtain the classical Riemann integral.
(2) If $w'(t) = \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}$, then we obtain the Riemann-Liouville fractional integral right and left can be obtained similarly.
(3) With convenient kernel choices w' we can get the k-Riemann-Liouville fractional integral right and left of ([41]), the right-sided fractional integrals of a function ψ for another function h on $[a, b]$ (see [1]), the right and left integral operator of [33], the right and left sided generalized fractional integral operators of [53] and the integral operators of [34] and [36], can also be obtained from above Definition by imposing similar conditions to w'.

The main purpose of this paper, use the weighted integral operator of Definition 1.7, is to establish several integral inequalities of Hermite-Hadamard type for modified (h, m)-convex functions of the second type, which contain as particular cases, several of those reported in the literature.

Lemma 1.1. *Let $\varphi : [a_1, a_2] \to \mathbb{R}$, with $0 < a_1 < a_2 < \infty$. Then for any $w \in [a_1, a_2]$,*

$$\varphi(a_1 + a_2 - w) \le (h(\tau) + h(1-\tau))[\varphi(a_1) + \varphi(a_2)] - \varphi(w); \tag{8}$$

with $\tau \in [a_1, a_2]$ depends on w.

2. Main Results

Theorem 2.1. *Let $m \in (0,1]$, $s \in [-1,1]$, $n \in \mathbb{N} \cup \{0\}$, $\varphi : [a_1, a_2] \to \mathbb{R}$, with $0 < a_1 < ma_2 < \infty$ and $\varphi \in L_1[a_1, a_2]$. If (h,m)-convex function then*

$$\begin{aligned}
&\varphi\left(\tfrac{a_1+ma_2}{2}\right)\left(w(1)-w(0)\right)\\
&\le \tfrac{n+1}{ma_2-a_1}\left[m^2\left(1-h\left(\tfrac{1}{2}\right)\right)^s J^w_{\left(\frac{na_1+ma_2}{m(n+1)}\right)^-}\varphi\left(\frac{a_1}{m}\right) + h^s\left(\tfrac{1}{2}\right) J^w_{\left(\frac{a_1+nma_2}{n+1}\right)^+}\varphi(ma_2)\right]\\
&\le \left[h^s\left(\tfrac{1}{2}\right)\varphi(a_1) + m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi(ma_2)\right]\int_0^1 w'(\tau)\, h^s\left(\frac{\tau}{n+1}\right)d\tau\\
&+ m\left[h^s\left(\tfrac{1}{2}\right)\varphi(a_2) + m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(\frac{a_1}{m^2}\right)\right]\int_0^1 w'(\tau)\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s d\tau.
\end{aligned} \tag{9}$$

Proof. Lets $u = \frac{\tau}{n+1}a_1 + m\left(\frac{n+1-\tau}{n+1}\right)a_2$, $v = \left(\frac{n+1-\tau}{n+1}\right)\frac{a_1}{m} + \frac{\tau}{n+1}a_2$, $m \in (0,1]$ and $\tau \in [0,1]$. We deduce that $\frac{u+mv}{2} = \frac{a_1+ma_2}{2}$. Since φ is a $(h,m,s)-$convex function, we have

$$\varphi\left(\tfrac{a_1+ma_2}{2}\right) \le h^s\left(\tfrac{1}{2}\right)\varphi(u) + m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi(v) \tag{10}$$

If we do $z = \frac{\tau}{n+1}a_1 + m\left(\frac{n+1-\tau}{n+1}\right)a_2$ and $r = \left(\frac{n+1-\tau}{n+1}\right)\frac{a_1}{m} + \frac{\tau}{n+1}a_2$, we get

$$\begin{aligned}
J^w_{\left(\frac{a_1+nma_2}{n+1}\right)^+}\varphi(ma_2) &= \int_{\frac{a_1+nma_2}{n+1}}^{ma_2} w'\left(\frac{ma_2-z}{\frac{nma_2-a_1}{n+1}}\right)\varphi(z)dz\\
&= \left(\tfrac{ma_2-a_1}{n+1}\right)\int_0^1 w'(\tau)\,\varphi\left(\frac{\tau}{n+1}a_1 + m\left(\frac{n+1-\tau}{n+1}\right)a_2\right)d\tau
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
J^w_{\left(\frac{na_1+ma_2}{m(n+1)}\right)^-}\varphi\left(\frac{a_1}{m}\right) &= \int_{\frac{a_1}{m}}^{\frac{na_1+ma_2}{m(n+1)}} w'\left(\frac{r-\frac{a_1}{m}}{\frac{ma_2-a_1}{m(n+1)}}\right)\varphi(r)dr\\
&= \left(\tfrac{ma_2-a_1}{m(n+1)}\right)\int_0^1 w'(\tau)\,\varphi\left(\left(\frac{n+1-\tau}{n+1}\right)\frac{a_1}{m} + \frac{\tau}{n+1}a_2\right)d\tau.
\end{aligned} \tag{12}$$

Multiplying both sides of (10) by $\left(\frac{ma_2-a_1}{m(n+1)}\right)w'(\tau)$ and integrating over $(0,1)$ with respect to τ, and using the equalities (11) and (12), we obtain

$$\begin{aligned}
\frac{\varphi\left(\frac{a_1+ma_2}{2}\right)(ma_2-a_1)\left(w(1)-w(0)\right)}{m(n+1)} &\le \frac{1}{m}h^s\left(\tfrac{1}{2}\right)\ J^w_{\left(\frac{a_1+nma_2}{n+1}\right)^+}\varphi(ma_2)\\
&+ m\left(1-h\left(\tfrac{1}{2}\right)\right)^s J^w_{\left(\frac{na_1+ma_2}{m(n+1)}\right)^-}\varphi\left(\frac{a_1}{m}\right)
\end{aligned} \tag{13}$$

On the other hand,

$$\varphi\left(\frac{\tau}{n+1}a_1+m\left(\frac{n+1-\tau}{n+1}\right)a_2\right)\leq h^s\left(\frac{\tau}{n+1}\right)\varphi\left(a_1\right)+m\left(1-h\left(\frac{\tau}{n+1}\right)\right)^s\varphi\left(a_2\right) \tag{14}$$

and

$$m\varphi\left(\left(\frac{n+1-\tau}{n+1}\right)\frac{a_1}{m}+\frac{\tau}{n+1}a_2\right)\leq mh^s\left(\frac{\tau}{n+1}\right)\varphi\left(a_2\right)+m^2\left(1-h\left(\frac{\tau}{n+1}\right)\right)^s\varphi\left(\frac{a_1}{m^2}\right). \tag{15}$$

Using the equalities (11), (12) and multiplying the inequalities (14) and (15) by $\left(\frac{ma_2-a_1}{m(n+1)}\right)w'(\tau)$ and integrating over $(0,1)$ with respect to τ, we get

$$\frac{1}{m}J^w_{\left(\frac{a_1+nma_2}{n+1}\right)^+}\varphi(ma_2)\leq\frac{(ma_2-a_1)}{m(n+1)}\left[\varphi\left(a_1\right)\int_0^1 w'\left(\tau\right)h^s\left(\frac{\tau}{n+1}\right)d\tau+m\varphi\left(a_2\right)\int_0^1 w'\left(\tau\right)\left(1-h\left(\frac{\tau}{n+1}\right)\right)^s d\tau\right], \tag{16}$$

and

$$mJ^w_{\left(\frac{na_1+ma_2}{m(n+1)}\right)^-}\varphi\left(\frac{a_1}{m}\right)\leq\frac{(ma_2-a_1)}{m(n+1)}\left[m\varphi\left(a_2\right)\int_0^1 w'\left(\tau\right)h^s\left(\frac{\tau}{n+1}\right)d\tau+m^2\varphi\left(\frac{a_1}{m^2}\right)\int_0^1 w'\left(\tau\right)\left(1-h\left(\frac{\tau}{n+1}\right)\right)^s d\tau\right]. \tag{17}$$

Now, adding (16) and (17) it follows

$$\begin{aligned}&\frac{1}{m}h^s\left(\tfrac{1}{2}\right)J^w_{\left(\frac{a_1+nma_2}{n+1}\right)^+}\varphi(ma_2)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s J^w_{\left(\frac{na_1+ma_2}{m(n+1)}\right)^-}\varphi\left(\frac{a_1}{m}\right)\\&\leq\frac{(ma_2-a_1)}{m(n+1)}\left\{\left[h^s\left(\tfrac{1}{2}\right)\varphi\left(a_1\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(a_2\right)\right]\int_0^1 w'\left(\tau\right)h^s\left(\frac{\tau}{n+1}\right)d\tau\right.\\&\left.+m\left[h^s\left(\tfrac{1}{2}\right)\varphi\left(a_2\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(\frac{a_1}{m^2}\right)\right]\int_0^1 w'\left(\tau\right)\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s d\tau\right\}.\end{aligned} \tag{18}$$

Thus, multiplying both sides of (13) and (18) by $\frac{n+1}{ma_2-a_1}$, then adding we conclude

$$\begin{aligned}&\varphi\left(\tfrac{a_1+ma_2}{2}\right)\left(w(1)-w(0)\right)\\&\leq\frac{n+1}{ma_2-a_1}\left[m^2\left(1-h\left(\tfrac{1}{2}\right)\right)^s J^w_{\left(\frac{na_1+ma_2}{m(n+1)}\right)^-}\varphi\left(\frac{a_1}{m}\right)+h^s\left(\tfrac{1}{2}\right)J^w_{\left(\frac{a_1+nma_2}{n+1}\right)^+}\varphi(ma_2)\right]\\&\leq\left[h^s\left(\tfrac{1}{2}\right)\varphi\left(a_1\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(a_2\right)\right]\int_0^1 w'\left(\tau\right)h^s\left(\frac{\tau}{n+1}\right)d\tau\\&+m\left[h^s\left(\tfrac{1}{2}\right)\varphi\left(a_2\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(\frac{a_1}{m^2}\right)\right]\int_0^1 w'\left(\tau\right)\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s d\tau.\end{aligned} \tag{19}$$

□

Remark 2.1. IF we put $w'(t) = t^{\frac{\alpha}{k}-1}$ we obtain the Theorem 2 of [52]. Corollaries 1 and 2 and Remark 2 of said work remain valid under the previous consideration.

Theorem 2.2. *Let $m \in (0,1]$, $s \in [-1,1]$, $n \in \mathbb{N} \cup \{0\}$, $\varphi : [a_1, a_2] \to \mathbb{R}$, with $0 < a_1 < ma_2 < \infty$ and $\varphi \in L_1[a_1, a_2]$. If $(h, m, s)-$convex function then*

$$\begin{aligned}
&\tfrac{n+1}{a_2-a_1}\left[J^{w}_{\left(\frac{a_1+na_2}{n+1}\right)^+}\varphi(a_2) + J^{w}_{\left(\frac{na_1+a_2}{n+1}\right)^-}\varphi(a_1)\right] \\
&\leq [\varphi(a_1) + \varphi(a_2)] \int_0^1 w'(\tau)\, h^s\left(\frac{\tau}{n+1}\right) d\tau \\
&+ m\left[\varphi\left(\frac{a_1}{m}\right) + \varphi\left(\frac{a_2}{m}\right)\right] \int_0^1 w'(\tau)\left[1 - h\left(\frac{\tau}{n+1}\right)\right]^s d\tau \\
&\leq \left(\int_0^1 |w'(\tau)|^p d\tau\right)^{\frac{1}{p}} \left[[\varphi(a_1) + \varphi(a_2)] \left(\int_0^1 \left|h^s\left(\frac{\tau}{n+1}\right)\right|^q d\tau\right)^{\frac{1}{q}} \right. \\
&\left. + m\left[\varphi\left(\frac{a_1}{m}\right) + \varphi\left(\frac{a_2}{m}\right)\right] \left(\int_0^1 \left|\left[1 - h\left(\frac{\tau}{n+1}\right)\right]^s\right|^q d\tau\right)^{\frac{1}{q}} \right].
\end{aligned} \tag{20}$$

Proof. Let $m \in (0,1]$, $s \in [-1,1]$, $n \in \mathbb{N} \cup \{0\}$, We know by hypothesis that φ is a $(h, m, s)-$convex function, then

$$\begin{aligned}
\varphi\left(\frac{\tau}{n+1}a_1 + m\left(\frac{n+1-\tau}{n+1}\right)\frac{a_2}{m}\right) &\leq h^s\left(\frac{\tau}{n+1}\right)\varphi(a_1) \\
&+ m\left(1 - h\left(\frac{\tau}{n+1}\right)\right)^s \varphi\left(\frac{a_2}{m}\right)
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\varphi\left(m\left(\frac{n+1-\tau}{n+1}\right)\frac{a_1}{m} + \frac{\tau}{n+1}a_2\right) &\leq h^s\left(\frac{\tau}{n+1}\right)\varphi(a_2) \\
&+ m\left(1 - h\left(\frac{\tau}{n+1}\right)\right)^s \varphi\left(\frac{a_1}{m}\right)
\end{aligned} \tag{22}$$

for all $\tau \in [0,1]$. From (11) and (12) it follows that

$$J^{w}_{\left(\frac{a_1+na_2}{n+1}\right)^+}\varphi(a_2) = \left(\tfrac{a_2-a_1}{n+1}\right) \int_0^1 w'(\tau)\, \varphi\left(\frac{\tau}{n+1}a_1 + m\left(\frac{n+1-\tau}{n+1}\right)\frac{a_2}{m}\right) d\tau \tag{23}$$

and

$$J^{w}_{\left(\frac{na_1+a_2}{n+1}\right)^-}\varphi(a_1) = \left(\tfrac{a_2-a_1}{n+1}\right) \int_0^1 w'(\tau)\, \varphi\left(m\left(\frac{n+1-\tau}{n+1}\right)\frac{a_1}{m} + \frac{\tau}{n+1}a_2\right) d\tau. \tag{24}$$

Multiplying both sides of (21) and (24) by $w'(\tau)$ and integrating over $(0,1)$ with respect to τ, if we add these resulting inequalities and we use (23) and (24), we get

$$\begin{aligned}
&\tfrac{n+1}{a_2-a_1}\left[J^w_{\left(\frac{a_1+na_2}{n+1}\right)^+}\varphi(a_2)+J^w_{\left(\frac{na_1+a_2}{n+1}\right)^-}\varphi(a_1)\right]\\
&\leq [\varphi(a_1)+\varphi(a_2)]\int_0^1 w'(\tau)\,h^s\left(\frac{\tau}{n+1}\right)d\tau\\
&+m\left[\varphi\left(\frac{a_1}{m}\right)+\varphi\left(\frac{a_2}{m}\right)\right]\int_0^1 w'(\tau)\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s d\tau. \qquad (25)
\end{aligned}$$

Now, by virtue of Holder's inequality

$$\int_0^1 w'(\tau)\,h^s\left(\frac{\tau}{n+1}\right)d\tau \leq \left(\int_0^1 |w'(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \left|h^s\left(\frac{\tau}{n+1}\right)\right|^q d\tau\right)^{\frac{1}{q}}. \qquad (26)$$

and

$$\int_0^1 w'(\tau)\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s d\tau \leq \left(\int_0^1 |w'(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1 \left|\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s\right|^q d\tau\right)^{\frac{1}{q}}. \qquad (27)$$

Therefore, of (25), (26) and (27), we deduce

$$\begin{aligned}
&\tfrac{n+1}{a_2-a_1}\left[J^w_{\left(\frac{a_1+na_2}{n+1}\right)^+}\varphi(a_2)+J^w_{\left(\frac{na_1+a_2}{n+1}\right)^-}\varphi(a_1)\right]\\
&\leq [\varphi(a_1)+\varphi(a_2)]\int_0^1 w'(\tau)\,h^s\left(\frac{\tau}{n+1}\right)d\tau\\
&+m\left[\varphi\left(\frac{a_1}{m}\right)+\varphi\left(\frac{a_2}{m}\right)\right]\int_0^1 w'(\tau)\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s d\tau\\
&\leq \left(\int_0^1 |w'(\tau)|^p d\tau\right)^{\frac{1}{p}}\left[[\varphi(a_1)+\varphi(a_2)]\left(\int_0^1 \left|h^s\left(\frac{\tau}{n+1}\right)\right|^q d\tau\right)^{\frac{1}{q}}\right.\\
&\left.+m\left[\varphi\left(\frac{a_1}{m}\right)+\varphi\left(\frac{a_2}{m}\right)\right]\left(\int_0^1 \left|\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s\right|^q d\tau\right)^{\frac{1}{q}}\right]. \qquad (28)
\end{aligned}$$

□

Remark 2.2. Taking into account the previous Remark, this result contains as a particular case the Theorem 3 of [52].

Lemma 2.3. *Let* $m \in (0,1]$, $s \in [-1,1]$, $n \in \mathbb{N}\cup\{0\}$, $\varphi : [a_1,a_2] \to \mathbb{R}$, *with* $0 < a_1 < ma_2 < \infty$ *and* $\varphi \in L_1[a_1,a_2]$. *If* $(h,m,s)-$*convex function, then*

$$\begin{aligned}
&\tfrac{\varphi(a_1)+\varphi(ma_2)}{2}-\tfrac{m(n+1)}{2(ma_2-a_1)}\left[J^w_{(a_1)^+}\varphi(ma_2)+J^w_{(ma_2)^-}\varphi(a_1)\right]\\
&=\tfrac{ma_2-a_1}{2}\int_0^1 [w(1-\tau)-w(\tau)]\,\varphi(\tau a_1+a_2(1-\tau))\,d\tau.
\end{aligned} \qquad (29)$$

Proof. The proof is very similar to that of Lemma 1.1 of [5]. □

Remark 2.3. Putting $w'(z)=\frac{1}{\Gamma(\alpha)}z^{\alpha-1}$, we obtain the Lemma 2 of [47], if $x=b$ the above result reduces to the Lemma 2 of [?]. With the kernel $w'(z)=\frac{1}{k\Gamma_k(\alpha)}z^{\frac{\alpha}{k}-1}$ if we put $x=b$ we get Lemma 2.3 of [19] (also see Lemma 2.1 of [60] and Lemma 3 of [52]).

Theorem 2.4. *Let* $m\in(0,1]$, $s\in[-1,1]$, $n\in\mathbb{N}\cup\{0\}$, $\varphi:[a_1,a_2]\to\mathbb{R}$, *with* $0<a_1<ma_2<\infty$ *and* $\varphi\in L_1[a_1,a_2]$. *If* $(h,m,s)-$*convex function then*

$$\begin{aligned}
&\varphi\left(\tfrac{a_1+a_2}{2}\right)\left(w(1)-w(0)\right)\\
&\leq \tfrac{n+1}{a_2-a_1}\left[\left(1-h\left(\tfrac{1}{2}\right)\right)^s J^w_{\left(\frac{na_1+a_2}{m(n+1)}\right)^-}\varphi\left(a_1\right)+h^s\left(\tfrac{1}{2}\right)J^w_{\left(\frac{a_1+na_2}{n+1}\right)^+}\varphi(a_2)\right]\\
&\leq\left[h^s\left(\tfrac{1}{2}\right)\varphi\left(a_1\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(a_2\right)\right]\int_0^1 w'\left(\tau\right)h^s\left(\frac{\tau}{n+1}\right)d\tau\\
&+m\left[h^s\left(\tfrac{1}{2}\right)\varphi\left(\frac{a_1}{m}\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(\frac{a_2}{m}\right)\right]\int_0^1 w'\left(\tau\right)\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s d\tau.
\end{aligned}\tag{30}$$

Proof. Lets $u=\frac{\tau}{n+1}a_1+\left(\frac{n+1-\tau}{n+1}\right)a_2$, $v=\left(\frac{n+1-\tau}{n+1}\right)a_1+\frac{\tau}{n+1}a_2$, $m\in(0,1]$ and $\tau\in[0,1]$. We deduce that $\frac{u+v}{2}=\frac{a_1+a_2}{2}$. Since φ is a $(h,m,s)-$convex function, we have

$$\varphi\left(\tfrac{a_1+a_2}{2}\right)\leq h^s\left(\tfrac{1}{2}\right)\varphi\left(u\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(v\right)\tag{31}$$

If we do $z=\frac{\tau}{n+1}a_1+\left(\frac{n+1-\tau}{n+1}\right)a_2$ and $r=\left(\frac{n+1-\tau}{n+1}\right)a_1+\frac{\tau}{n+1}a_2$, we get

$$\begin{aligned}
J^w_{\left(\frac{a_1+na_2}{n+1}\right)^+}\varphi(a_2)&=\int_{\frac{a_1+na_2}{n+1}}^{a_2}w'\left(\frac{a_2-z}{\frac{na_2-a_1}{n+1}}\right)\varphi(z)dz\\
&=\left(\tfrac{a_2-a_1}{n+1}\right)\int_0^1 w'\left(\tau\right)\varphi\left(\frac{\tau}{n+1}a_1+\left(\frac{n+1-\tau}{n+1}\right)a_2\right)d\tau
\end{aligned}\tag{32}$$

and

$$\begin{aligned}
J^w_{\left(\frac{na_1+a_2}{n+1}\right)^-}\varphi\left(a_1\right)&=\int_{\frac{a_1}{m}}^{\frac{na_1+ma_2}{m(n+1)}}w'\left(\frac{r-\frac{a_1}{m}}{\frac{ma_2-a_1}{m(n+1)}}\right)\varphi(r)dr\\
&=\left(\tfrac{a_2-a_1}{n+1}\right)\int_0^1 w'\left(\tau\right)\varphi\left(\left(\frac{n+1-\tau}{n+1}\right)\frac{a_1}{m}+\frac{\tau}{n+1}a_2\right)d\tau.
\end{aligned}\tag{33}$$

Multiplying both sides of (31) by $\left(\frac{a_2-a_1}{n+1}\right)w'\left(\tau\right)$ and integrating over $(0,1)$ with respect to τ, and using the equalities (11) and (33), we obtain

$$\begin{aligned}
\frac{\varphi\left(\frac{a_1+a_2}{2}\right)\left(a_2-a_1\right)\left(w(1)-w(0)\right)}{n+1}&\leq h^s\left(\tfrac{1}{2}\right)\ J^w_{\left(\frac{a_1+na_2}{n+1}\right)^+}\varphi\left(a_2\right)\\
&+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s J^w_{\left(\frac{na_1+a_2}{n+1}\right)^-}\varphi\left(a_1\right)
\end{aligned}\tag{34}$$

On the other hand,

$$\begin{aligned}
\varphi\left(\frac{\tau}{n+1}a_1+m\left(\frac{n+1-\tau}{n+1}\right)\frac{a_2}{m}\right)&\leq h^s\left(\frac{\tau}{n+1}\right)\varphi\left(a_1\right)\\
&+m\left(1-h\left(\frac{\tau}{n+1}\right)\right)^s\varphi\left(\frac{a_2}{m}\right)
\end{aligned}\tag{35}$$

and

$$m\varphi\left(m\left(\frac{n+1-\tau}{n+1}\right)\frac{a_1}{m}+\frac{\tau}{n+1}a_2\right)\leq mh^s\left(\frac{\tau}{n+1}\right)\varphi\left(a_2\right)+m^2\left(1-h\left(\frac{\tau}{n+1}\right)\right)^s\varphi\left(\frac{a_1}{m}\right) \quad (36)$$

Using the equalities (32), (33) and multiplying the inequalities (35) and (36) by $\left(\frac{a_2-a_1}{n+1}\right)w'(\tau)$ and integrating over $(0,1)$ with respect to τ, we get

$$J^w_{\left(\frac{a_1+nma_2}{n+1}\right)^+}\varphi(a_2)\leq\frac{(a_2-a_1)}{n+1}\left[\varphi\left(a_1\right)\int_0^1 w'\left(\tau\right)h^s\left(\frac{\tau}{n+1}\right)d\tau+m\varphi\left(\frac{a_2}{m}\right)\int_0^1 w'\left(\tau\right)\left(1-h\left(\frac{\tau}{n+1}\right)\right)^s d\tau\right] \quad (37)$$

and

$$mJ^w_{\left(\frac{na_1+a_2}{n+1}\right)^-}\varphi\left(a_1\right)\leq\frac{(a_2-a_1)}{n+1}\left[m\varphi\left(a_2\right)\int_0^1 w'\left(\tau\right)h^s\left(\frac{\tau}{n+1}\right)d\tau+m^2\varphi\left(\frac{a_1}{m}\right)\int_0^1 w'\left(\tau\right)\left(1-h\left(\frac{\tau}{n+1}\right)\right)^s d\tau\right]. \quad (38)$$

Now, adding (37) and (38) it follows

$$\begin{aligned}&h^s\left(\tfrac{1}{2}\right)J^w_{\left(\frac{a_1+na_2}{n+1}\right)^+}\varphi(a_2)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s J^w_{\left(\frac{na_1+a_2}{n+1}\right)^-}\varphi\left(a_1\right)\\&\leq\tfrac{(a_2-a_1)}{n+1}\left\{\left[h^s\left(\tfrac{1}{2}\right)\varphi\left(a_1\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(a_2\right)\right]\int_0^1 w'\left(\tau\right)h^s\left(\frac{\tau}{n+1}\right)d\tau\right.\\&\left.+m\left[h^s\left(\tfrac{1}{2}\right)\varphi\left(a_2\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(\frac{a_1}{m}\right)\right]\int_0^1 w'\left(\tau\right)\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s d\tau\right\}.\end{aligned} \quad (39)$$

Thus, multiplying both sides of (34) and (39) by $\frac{n+1}{a_2-a_1}$, then adding we conclude

$$\begin{aligned}&\varphi\left(\tfrac{a_1+a_2}{2}\right)\left(w(1)-w(0)\right)\\&\leq\tfrac{n+1}{a_2-a_1}\left[\left(1-h\left(\tfrac{1}{2}\right)\right)^s J^w_{\left(\frac{na_1+a_2}{n+1}\right)^-}\varphi\left(a_1\right)+h^s\left(\tfrac{1}{2}\right)J^w_{\left(\frac{a_1+na_2}{n+1}\right)^+}\varphi(a_2)\right]\\&\leq\left[h^s\left(\tfrac{1}{2}\right)\varphi\left(a_1\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(a_2\right)\right]\int_0^1 w'\left(\tau\right)h^s\left(\frac{\tau}{n+1}\right)d\tau\\&+m\left[h^s\left(\tfrac{1}{2}\right)\varphi\left(\frac{a_1}{m}\right)+m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\varphi\left(\frac{a_2}{m}\right)\right]\int_0^1 w'\left(\tau\right)\left[1-h\left(\frac{\tau}{n+1}\right)\right]^s d\tau.\end{aligned} \quad (40)$$

□

Remark 2.4. Starting from Lemma 2.3 we can obtain other extensions of the previous results that contain as particular cases, for example, Theorems 4, 5, 6 and 7 of [52] and their respective Corollaries.

3. Concluding Remarks

This paper gives new inequalities of the Hermite-Hadamard type, in the class of h-convex functions, some related inequalities (fractional or not) are also obtained as particular cases of our results.

From the results obtained, we can point out two open problems:

(1) Using the operators of the Definition 1.7, we can generalize different results already reported in the literature, selecting different kernels and even new operators can be used, for example [44].

(2) If we use other notions of convexity, (h, m)-convexity, s-convexity and others, the results obtained can be extended.

The study successfully establishes new versions of the Hermite-Hadamard inequality for modified (h, m)-convex functions of the second type, utilizing weighted integral operators. These new versions encompass several previously reported results in the literature as special cases, demonstrating the breadth and versatility of the proposed approach. The obtained results validate and extend existing theories on convexity and integral inequalities, providing a deeper understanding of these inequalities in more general contexts.

The findings of this research open new avenues for exploration in the theory of convexity and integral inequalities. The extended versions of the Hermite-Hadamard inequality can be applied in various fields such as optimization, probability theory, and numerical analysis. Additionally, these results suggest the potential for investigating other classes of convex functions and integral operators, which could lead to further theoretical developments and new practical applications in applied mathematics and related disciplines.

4. Conflict of Interest

The corresponding author, on behalf of all authors, affirms that there are no conflicts of interest.

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(Juan Gabriel Galeano Delgado) Departamento de matemáticas, Universidad de Cordoba, Carrera 6 N° 76 - 103, Montería Colombia
E-mail address: `jgabrielgaleano@correo.unicordoba.edu.co`

(Juan E., Nápoles Valdés) Universidad Nacional del Nordeste, Ave. Libertad 5450, Corrientes 3400, y Universidad Tecnológica Nacional, Ave. Libertad 5450, Corrientes 3400, Argentina
E-mail address: `jnapoles@exa.unne.edu.ar`

(Abel Alvarez Bustos) Pontificia Universidad Javeriana, Cl. 18 #118-250, Cali Colombia
E-mail address: `abel.alvarez@javerianacali.edu.co`