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Local Dynamics and Bifurcation for a Two-Dimensional Cubic Lotka-Volterra System (Part II)

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Abstract. The local bifurcation and dynamics for a two-dimensional cubic Kolmogorov system, depending on two small parameters, in certain hypotheses on the coefficients, are investigated. The paper continues the study performed in [4], by treating two non-generic cases, corresponding to the hypotheses that one of the significant coefficients vanishes. In the first non-generic case, the local dynamics is found to be similar to the one obtained in the generic case treated in [4]. In the second non-generic case new possibilities of behavior are found.

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1. Introduction

Consider the two-dimensional cubic Kolmogorov system

$$\begin{cases} \frac{dx}{dt} &= x\left(\mu_1 + p_{11}x + p_{12}y + p_{13}x^2 + p_{14}xy + p_{15}y^2\right) \\ \frac{dy}{dt} &= y\left(\mu_2 + p_{21}x + p_{22}y + p_{23}x^2 + p_{24}xy + p_{25}y^2\right) \end{cases} \tag{1.1}$$

where x, y are the state variables, $p_{ij} = p_{ij}(\mu)$ are smooth functions of the parameter variable $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$.

This class of Kolmogorov systems, which generalizes the Lotka-Volterra model, is often used to model various real-life phenomena in population modelling, biology, ecology, environment, engineering, economics or mechanics [1], [2], [5], [7], [8], [15], [16].

The present work is concerned with the study of the behavior of the system (1.1) when both $|\mu_1|$ and $|\mu_2|$ are infinitesimally small, i.e. $|\mu| \ll 1$. The coefficients $p_{ij} = p_{ij}(\mu)$ are assumed to be smooth functions on the open set $V_\varepsilon = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2, |\mu| = \sqrt{\mu_1^2 + \mu_2^2} < \varepsilon \ll 1\right\}$. Also, assume that $p_{12}(0)p_{21}(0) < 0$.

As only the behavior of the system with nonnegative variables presents relevance for practical applications, the study is restricted to the first quadrant of the phase plane

$$D = \left\{(x, y) \in \mathbb{R}^2, x \geq 0, y \geq 0\right\},$$

which is an invariant set for system (1.1).

The local dynamics of system (1.1) around the origin was analyzed in relation to the double Hopf bifurcation in [3], [6], in the hypotheses (HH.1) $p_{11}(0) \neq 0$, (HH.2) $p_{12}(0) \neq 0$, (HH.3) $p_{21}(0) \neq 0$, (HH.4) $p_{22}(0) \neq 0$, (HH.5) $(p_{11}p_{22} - p_{12}p_{21})(0) \neq 0$.

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Several papers considered some non-generic cases, when at least one of the above mentioned hypotheses is not satisfied. Thus, Tigan et al. analyzed the local dynamics of system (1.1), with the assumption $p_{12}(0)\,p_{22}(0) < 0$, in two different cases (i) $p_{11}(0)\,p_{21}(0) \neq 0$ [13] and (ii) either $p_{11}(0) = 0$ or $p_{21}(0) = 0$ [14]. In [10] the case $p_{12}(0)\,p_{22}(0) > 0$ was treated when one of the hypotheses (HH.1) or (HH.3) is not satisfied. In [9] the dynamics of system (1.1) was investigated when $p_{12}(0)p_{21}(0) > 0$, and one of the hypotheses (HH.1) or (HH.4) is not fulfilled.

In this paper we finalize the study of the local dynamics of the system (1.1) in the case when $p_{12}(0)p_{21}(0) < 0$ and one of the hypotheses (HH.1) or (HH.4) is not satisfied. The first part of this study can be found in [4].

As $p_{12}(0)p_{21}(0) < 0$, assuming $p_{12}(0) < 0$ and $p_{21}(0) > 0$, the change of variables

$$\xi_1 = -p_{12}(\mu)x, \xi_2 = p_{21}(\mu)y$$

leaves D invariant and maps system (1.1) into

$$\begin{cases} \frac{d\xi_1}{dt} = \xi_1\left(\mu_1 - \theta(\mu)\xi_1 + \gamma(\mu)\xi_2 + N(\mu)\xi_1^2 - M(\mu)\xi_1\xi_2 + L(\mu)\xi_2^2\right), \\ \frac{d\xi_2}{dt} = \xi_2\left(\mu_2 - \frac{1}{\gamma(\mu)}\xi_1 + \delta(\mu)\xi_2 + Q(\mu)\xi_1^2 - S(\mu)\xi_1\xi_2 + P(\mu)\xi_2^2\right). \end{cases} \tag{1.2}$$

Here the coefficients are given by $\theta(\mu) = \frac{p_{11}(\mu)}{p_{12}(\mu)}$, $\gamma(\mu) = \frac{p_{12}(\mu)}{p_{21}(\mu)}$, $N(\mu) = \frac{p_{13}(\mu)}{p_{12}^2(\mu)}$, $M(\mu) = \frac{p_{14}(\mu)}{p_{12}(\mu)p_{21}(\mu)}$, $L(\mu) = \frac{p_{15}(\mu)}{p_{21}^2(\mu)}$, $\delta(\mu) = \frac{p_{22}(\mu)}{p_{21}(\mu)}$, $Q(\mu) = \frac{p_{23}(\mu)}{p_{12}^2(\mu)}$, $S(\mu) = \frac{p_{24}(\mu)}{p_{21}(\mu)p_{12}(\mu)}$, and $P(\mu) = \frac{p_{25}}{p_{21}^2}(\mu)$.

As the coefficients are smooth functions of the $\mu \in V_\varepsilon$, in this study we use the asymptotic expansions of these coefficients at $\mu = 0$. Thus, one can write $P(\mu) = P(0) + \frac{\partial P}{\partial \mu_1}(0)\,\mu_1 + \frac{\partial P}{\partial \mu_2}(0) + O\left(|\mu|^2\right)$, and so on. Also, to simplify the wrinting, if in an expression the argument of a coefficient is not explicit then it is assumed to be its value in 0, that is $\theta = \theta(0)$, $\gamma = \gamma(0)$, $\delta = \delta(0)$, $P = P(0)$ and so on.

Remark 1.1. 1) As $p_{12}(0)p_{21}(0) < 0$ it follows $\gamma(0) < 0$; we may consider ε such that $\gamma(\mu) < 0$ for $\mu \in V_\varepsilon$.

2) With these notations, condition (HH.1) is equivalent to $\theta(0) \neq 0$, while condition (HH.4) with $\delta(0) \neq 0$.

The paper is organized as it follows. In Section 2 we analyze local dynamics and bifurcations of system (1.2) in the non-generic case $\delta = 0$ in V_ε, and $\theta(0) \neq 0$. We found four different cases, determined by $\theta(0)$ and P, each of them equivalent to one found in the nongeneric case in [4]. In Section 3 we investigate the non-generic case $\delta(0) = 0$, $\theta(0) \neq 0$, i.e the hypotheses (HH.1) is valid and (HH.4) is not satisfied. The existence of fold bifurcation of equilibria leads to new local phase portraits nonequivalent to the ones obtained in Section 2 and in [4], or in [9] for the case $\gamma > 0$. For each of the eight nonequivalent identified cases, detailed bifurcation diagrams are given. In Section 4 we prove that the local dynamics in the non-generic case $\delta(0) \neq 0$, $\theta(0) = 0$, i.e the hypotheses (HH.4) is valid and (HH.1) is not satisfied, can obtained from the non-generic case $\delta(0) = 0$, $\theta(0) \neq 0$. Finally, some conclusions are formulated.

2. Local dynamics and bifurcation when $\delta \equiv 0$, $\theta \neq 0$

In this section we analyze local dynamics and bifurcations of system (1.2) in the non-generic case $\delta = 0$ for all $\mu \in V_\varepsilon$, and $\theta(0) \neq 0$. Thus, the system (1.2) reads

$$\begin{cases} \frac{d\xi_1}{dt} = \xi_1 \left(\mu_1 - \theta(\mu)\xi_1 + \gamma(\mu)\xi_2 + N(\mu)\xi_1^2 - M(\mu)\xi_1\xi_2 + L(\mu)\xi_2^2\right) \\ \frac{d\xi_2}{dt} = \xi_2 \left(\mu_2 - \frac{1}{\gamma(\mu)}\xi_1 + Q(\mu)\xi_1^2 - S(\mu)\xi_1\xi_2 + P(\mu)\xi_2^2\right) \end{cases}. \tag{2.1}$$

System (2.1) has the trivial equilibrium $E_0 = (0,0)$, and two other equilibria $E_1 = \left(\hat{\xi}_1, 0\right)$ and $E_2 = \left(0, \hat{\xi}_2\right)$, close to E_0. The existence of these two equilibria is ensured by the Implicit Function Theorem (IFT) applied to equations $\mu_1 - \theta(\mu)\xi_1 + L(\mu)\xi_1^2 = 0$, and $\mu_2 + P(\mu)\xi_2^2 = 0$, respectively. As $\theta(0) \neq 0$, we find the solutions $\hat{\xi}_1 = \frac{1}{\theta}\mu_1 (1 + O(|\mu|))$, $\hat{\xi}_2 = \sqrt{-\frac{1}{P}\mu_2} (1 + O(|\mu|))$, close to 0, with $|\mu|$ sufficiently small, in the hypothesis $P(0) \neq 0$.

The existence of a third equilibrium $E_3 = (\xi_1^*, \xi_2^*)$ close to E_0 for $|\mu|$ small is also ensured by the IFT, applied to the system

$$\begin{cases} \mu_1 - \theta(\mu)\xi_1 + \gamma(\mu)\xi_2 + N(\mu)\xi_1^2 - M(\mu)\xi_1\xi_2 + L(\mu)\xi_2^2 = 0, \\ \mu_2 - \frac{1}{\gamma(\mu)}\xi_1 + Q(\mu)\xi_1^2 - S(\mu)\xi_1\xi_2 + P(\mu)\xi_2^2 = 0. \end{cases}$$

The coordinates of E_3 are found as

$$\xi_1^* = \left(\gamma\mu_2 + \frac{P\mu_1^2}{\gamma}\right)(1 + O(|\mu|)), \quad \xi_2^* = \left(-\frac{\mu_1}{\gamma} + \theta\mu_2\right)(1 + O(|\mu|)).$$

The equilibrium E_1 is in D only if $\theta\mu_1 \geq 0$, E_2 is in D if $\mu_2 P(0) \leq 0$, while E_3 is in D when the parameter (μ_1, μ_2) lies inside the region

$$R_1 = \left\{(\mu_1, \mu_2), \mu_1 - \theta\gamma\mu_2 < 0, \gamma\mu_2 + \frac{P\mu_1^2}{\gamma} > 0\right\}. \tag{2.2}$$

Denote by T_1, T_2 the parameter sets

$$T_1 = \left\{(\mu_1, \mu_2), \mu_1 = \theta\gamma\mu_2 + O\left(\mu_2^2\right),\ \mu_2 < 0\right\}, \tag{2.3}$$

$$T_2 = \left\{(\mu_1, \mu_2), \mu_2 = -\frac{P}{\gamma^2}\mu_1^2 + O\left(\mu_1^3\right), \mu_1 > 0\right\}, \tag{2.4}$$

If $(\mu_1, \mu_2) \in T_1$ then E_3 collides with E_1, while if $(\mu_1, \mu_2) \in T_2$ then E_3 collides with E_2. Note that only the lowest terms in (μ_1, μ_2) are used to describe the above parameter sets.

The following results concerning the topological type of equilibria E_0,E_1, E_2, E_3 can be easily obtained.

Lemma 2.1. *The trivial equlibrium point E_0 is:*
((i) a saddle if $\mu_1\mu_2 < 0$,
(ii) a repeller if $\mu_1 > 0, \mu_2 > 0$,
(iii) an attractor if $\mu_1 < 0, \mu_2 < 0$, or
(iv) nonhyperbolic of fold type if $\mu_1 = 0$ or $\mu_2 = 0$.

Lemma 2.2. *For $|\mu|$ sufficiently small, whenever E_1 lies in D, the equilibrium point E_1 is either:*
(i) a saddle if $\theta\mu_2 - \frac{1}{\gamma}\mu_1 > 0$,

(ii) a repeller if $\mu_2 - \frac{1}{\theta\gamma}\mu_1 > 0, \theta < 0,$
(iii) a stable node if $\mu_2 - \frac{1}{\theta\gamma}\mu_1 < 0, \theta > 0,$ *or*
(iv) nonhyperbolic of fold type if $\mu_1 = 0$ *or* $\theta\gamma\mu_2 - \mu_1 = 0$ $(\mu \in T_1)$.

Lemma 2.3. *For* $|\mu|$ *sufficiently small, if the equilibrium point* E_2 *lies in* D*, then* E_2 *is either:*
(i) a saddle if $\left(\mu_2 + \frac{P}{\gamma^2}\mu_1^2\right) P < 0,$
(ii) a stable node if $\mu_2 + \frac{P}{\gamma^2}\mu_1^2 < 0,\ P < 0,$
(iii) a repeller node if $\mu_2 + \frac{P}{\gamma^2}\mu_1^2 > 0,\ P > 0,$*or*
(iv) nonhyperbolic of fold type if $\mu_2 = 0$ *or* $\mu_2 + \frac{P}{\gamma^2}\mu_1^2 = 0.$

The topological type of the nontrivial equilibrium E_3 is established bellow.

Proposition 2.4. *If the equilibrium point* E_3 *is in* D*, then the following assertions are true.*
1) If $\mu \in R_1$*, then* E_3 *is*
(i) an attractor, if either $\theta > 0, P < 0$ *or* $\theta > 0, P > 0,\ \mu_2 < \frac{P(2-\theta\gamma)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^3\right),$ *or* $\theta < 0, P < 0,\ \mu_2 > \frac{P(2-\theta\gamma)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^3\right);$
(ii) a repeller if either $\theta < 0, P > 0$ *or* $\theta > 0, P > 0, \mu_2 > \frac{P(2-\theta\gamma)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^3\right),$ *or* $\theta < 0, P < 0,\ \mu_2 < \frac{P(2-\theta\gamma)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^3\right);$
(iii) nohyperbolic of Hopf type if $\mu_2 = \frac{P(2-\theta\gamma)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^3\right).$.
2) If $\mu \in T_1 \cup T_2$*, then* E_3 *is a nonhyperbolic equilibrium of fold type.*

Proof. First, note that E_3 collides with E_1 on T_1, respectively, with E_2 on T_2, for $|\mu|$ small, hence 2) is proved.

The eigenvalues $\lambda_{1,2}$ of E_3 satisfy the relations

$$\begin{aligned} \lambda_1\lambda_2 &= \xi_1^*\xi_2^*\left(1 + O(|\mu|)\right), \\ \lambda_1 + \lambda_2 &= \left[-\theta\left(\gamma\mu_2 + \frac{P}{\gamma}\mu_1^2\right) + \frac{2P}{\gamma}\mu_1^2\right]\left(1 + O(|\mu|)\right). \end{aligned}$$

As $\lambda_1\lambda_2 > 0$ in region R_1, E_3 cannot be a saddle.

Denote $p(\mu) = \lambda_1 + \lambda_2$ and $H = \{(\mu_1, \mu_2), p(\mu) = 0\}$. Applying IFT to the equation $p(\mu) = 0$, it follows

$$H = \left\{(\mu_1, \mu_2), \mu_2 = \frac{P(2-\theta\gamma)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^3\right)\right\}.$$

This curve intersect region R_1, iff $P\theta > 0$, while $R_1 \cap H = \varnothing$ if $P\theta < 0$. Thus, the topological type of E_3 does not change for $\mu \in R_1$ if $P\theta < 0$. Namely, E_3 is an attractor if $\theta > 0, P < 0$ and a repeller if $\theta < 0, P > 0$.

If $P > 0, \theta > 0$ and $\mu \in R_1$, the $p(\mu) < 0$ for $\mu_2 < \frac{P(2-\theta\gamma)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^3\right)$, and $p(\mu) > 0$ for $\mu_2 > \frac{P(2-\theta\gamma)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^3\right)$.

If $P < 0, \theta < 0$ and $\mu \in R_1$, then $p(\mu) < 0$ for $\mu_2 > \frac{P(2-\theta\gamma)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^3\right)$, and $p(\mu) > 0$ for $\mu_2 < \frac{P(2-\theta\gamma)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^3\right)$.

As E_3 is an attractor if $p(\mu) < 0$ and a repeller if $p(\mu) > 0$, the results are proved. ☐

Denote by

$$X_+ = \{(\mu_1,\mu_2), \mu_2 = 0,\ \mu_1 > 0\}, \quad X_- = \{(\mu_1,\mu_2), \mu_2 = 0,\ \mu_1 < 0\},$$
$$Y_+ = \{(\mu_1,\mu_2), \mu_1 = 0,\ \mu_2 > 0\}, \quad Y_- = \{(\mu_1,\mu_2), \mu_1 = 0,\ \mu_2 < 0\},$$

the four semiaxes of the (μ_1, μ_2) parameter plane.

Proposition 2.5. *The following transcritical bifurcations occur for system (2.1):*
(i) at the point E_0 as the parameter crosses the curves Y_+ or Y_- (when $E_0 = E_1$);
(ii) at the point E_0 as the parameter crosses the curves X_+ or X_- (when $E_0 = E_2$);
(iii) at the point E_1 as the parameter (μ_1, μ_2) crosses the curve T_1 (when $E_1 = E_3$);
(iv) at the point E_2 as the parameter (μ_1, μ_2) crosses the curve T_2 (when $E_2 = E_3$).

Proof. These statements are easily obtained by applying a Sotomayor Theorem ([11], p. 338). □

The above results determine four different cases with respect to θ and P, namely: C_1: $\theta > 0, P > 0$; C_2: $\theta < 0, P > 0$; C_3: $\theta < 0, P < 0$; C_4: $\theta > 0, P < 0$.

For each case, in the parametric portraits in the (μ_1, μ_2)- plane, the parameter strata are determined by the origin and the bifurcation curves X_-, X_+, Y_-, Y_+, T_1, T_2, and H. From Proposition 2.4 it follows that system (2.1) may exhibit a Hopf bifurcation only in the hypothesis $P\theta > 0$, thus the curve H is present only in cases C_1 and C_3.

Gathering all of the above information, we can formulate the following.

Theorem 2.6. *For all $\gamma < 0$, and θ, P in cases C_2 and C_4, the parameter portraits in the $(\mu_1, \mu_2)-$ plane consist of*

$$O \cup T_1 \cup T_2 \cup X_- \cup X_+ \cup Y_- \cup Y_+.$$

In addition, the parameter portraits and the corresponding generic phase portraits for case C_2 are equivalent to the ones in case A_4 obtained in part I of the study in [4], while those for case C_4 to the ones in the case A_3 in [4].

For cases C_1 and C_3, a Hopf bifurcation occurs when parameters cross H, if the first Lyapunov coefficient is nonzero. As the parameters move away from H, the limit cycle born through this bifurcation may disappear, either through saddle homoclinic bifurcation for parameters on a curve L, originating at $\mu = 0$, or it may exit the visible neighborhood of origin in D. The same phenomenon was also encountered in [4]. As a consequence, we can formulate the following result.

Theorem 2.7. *For all $\gamma < 0$, and θ, P in cases C_1 and C_3, the parameter portrait consists of*

$$O \cup T_1 \cup T_2 \cup X_- \cup X_+ \cup Y_- \cup Y_+ \cup H \cup L.$$

The parameter portraits and the corresponding generic phase portraits for case C_1 are equivalent to the ones in case A_5 obtained in part I of the study in [4], while those for case C_3 to the ones in the case A_6(iii) in [4].

3. Local dynamics and bifurcation when $\theta(0) \neq 0$ and $\delta(0) = 0$

In this section we analyze local dynamics and bifurcations of system (1.2) in the non-generic case $\delta(0) = 0$ and $\theta(0) \neq 0$. As $\delta(0) = 0$, near the value $(\mu_1, \mu_2) = 0$, the asympthotic expansion of δ as a function of μ reads: $\delta(\mu) = \delta_1\mu_1 + \delta_2\mu_2 + O\left(|\mu|^2\right)$, where $\delta_1 = \frac{\partial\delta}{\partial\mu_1}(0)$, $\delta_2 = \frac{\partial\delta}{\partial\mu_2}(0)$, thus system (1.2) has the form

$$\left\{\begin{array}{l} \frac{d\xi_1}{dt} = \xi_1\left(\mu_1 - \theta(\mu)\xi_1 + \gamma(\mu)\xi_2 + N(\mu)\xi_1^2 - M(\mu)\xi_1\xi_2 + L(\mu)\xi_2^2\right) \\ \frac{d\xi_2}{dt} = \xi_2\left(\mu_2 - \frac{1}{\gamma(\mu)}\xi_1 + \left(\delta_1\mu_1 + \delta_2\mu_2 + O\left(|\mu|^2\right)\right)\xi_2 \right. \\ \qquad\qquad \left. + Q(\mu)\xi_1^2 - S(\mu)\xi_1\xi_2 + P(\mu)\xi_2^2\right) \end{array}\right. . \tag{3.1}$$

where, $\theta = \theta(0) \neq 0$, $\gamma = \gamma(0) < 0$, and so on.

System (3.1) has the equilibria $E_0 = (0,0)$, $E_1 = \left(\frac{1}{\theta}\mu_1(1 + O(|\mu|), 0\right)$, as in the case $\theta\delta \neq 0$ in part I [4]. In contrast to the $\theta\delta \neq 0$ case, system (3.1) may possess at most two equilibria on the $O\xi_2$ axis, both close to E_0 for small $|\mu|$, namely $E_{21} = (0, \xi_{21})$, $E_{22} = (0, \xi_{22})$, where

$$\xi_{21} = \frac{-(\delta_1\mu_1 + \delta_2\mu_2) - \sqrt{\Delta(\mu)}}{2P}, \xi_{22} = \frac{-(\delta_1\mu_1 + \delta_2\mu_2) + \sqrt{\Delta(\mu)}}{2P}, \text{ with } \xi_{21} \leq \xi_{22},$$

if $P \neq 0$, and $\Delta(\mu) = (\delta_1\mu_1 + \delta_2\mu_2)^2 - 4P\mu_2 \geq 0$. As $\Delta(\mu) < 0$, there are no equilibria on the $O\xi_2$ axis.

Both equilibria E_{21}, E_{22} are in D, for $|\mu|$ sufficiently small, in the region

$$R_{21} = \{(\mu_1, \mu_2) \in V_\varepsilon \,|\Delta(\mu) > 0, \delta_1\mu_1 P < 0, \mu_2 P > 0\},$$

while only E_{22} is in D for parameters in

$$R_{22} = \{(\mu_1, \mu_2) \in V_\varepsilon \,|\Delta(\mu) > 0, \mu_2 P < 0\}.$$

Obviously, as $\Delta(\mu) = 0$, we have $\xi_{21} = \xi_{22}$.

Remark 3.1. As $\mu_1 = 0$, we have $E_{21} = E_0$ if $\mu_2 P > 0$, while $E_{22} = E_0$ if $\mu_2 P < 0$.

System (3.1) possesses also the equilibrium $E_3 = (\xi_1^*, \xi_2^*)$, close to the origin O for $|\mu|$ small, with

$$\xi_1^* = \left(\gamma\mu_2 + \frac{P - \delta_1\gamma}{\gamma}\mu_1^2\right)(1 + O(|\mu|), \quad \xi_2^* = \frac{-\mu_1 + \theta\gamma\mu_2}{\gamma}(1 + O(|\mu|),$$

obtained using the Implicit Functions Theorem applied to the system

$$\left\{\begin{array}{l} \mu_1 - \theta\xi_1 + \gamma\xi_2 + L\xi_2^2 - M\xi_1\xi_2 + N\xi_1^2 = 0, \\ \mu_2 - \frac{1}{\gamma}\xi_1 + (\delta_1\mu_1 + \delta_2\mu_2 + O\left(|\mu|^2\right))\xi_2 + Q\xi_1^2 - S\xi_1\xi_2 + P\xi_2^2 = 0. \end{array}\right. \tag{3.2}$$

The equilibrium E_3 is in D when the parameter (μ_1, μ_2) lies inside the region

$$R_3 = \left\{(\mu_1, \mu_2), \mu_1 - \theta\gamma\mu_2 > 0, \gamma\mu_2 + \frac{P - \delta_1\gamma}{\gamma}\mu_1^2 > 0\right\} \tag{3.3}$$

This equilibrium exits D when (μ_1, μ_2) crosses the bifurcation curves

$$T_1 = \left\{(\mu_1, \mu_2), \mu_1 = \theta\gamma\mu_2 + O\left(\mu_2^2\right),\ \mu_2 < 0\right\} \tag{3.4}$$

or

$$T_3 = \left\{(\mu_1, \mu_2), \mu_2 = \frac{\delta_1\gamma - P}{\gamma^2}\mu_1^2 + O\left(\mu_1^2\right),\ \mu_1 > 0\right\}. \tag{3.5}$$

Remark 3.2. Note that E_3 collides with E_1 for parameters in T_1, respectively, with E_{21} or E_{22} for parameters in T_3.

The topological type of equilibria E_0 and E_1 are the same as in the case $\delta \equiv 0$, given in Lemmas 2.1, 2.2.

Lemma 3.1. *Assume $P > 0$ and $\Delta > 0$. For $|\mu|$ sufficiently small, the following hold:*
(i) E_{21} and E_{22} are saddles if $\mu_2 < \frac{\delta_1\gamma - P}{\gamma^2}\mu_1^2$;
(ii) E_{21} is a saddle, and E_{22} is a repeller if $\mu_2 > \frac{\delta_1\gamma - P}{\gamma^2}\mu_1^2$ and $(2P - \delta_1\gamma)\mu_1 > 0$;
(iii) E_{21} is an attractor, and E_{22} is a saddle if $\mu_2 > \frac{\delta_1\gamma - P}{\gamma^2}\mu_1^2$ and $(2P - \delta_1\gamma)\mu_1 < 0$.

Proof. The eigenvalues of E_{21} are $\lambda_1^{E_{21}} = -\xi_{21}\sqrt{\Delta} < 0$, and $\lambda_2^{E_{21}} = L\xi_{21}^2 + \gamma\xi_{21} + \mu_1$, while for E_{22} we have $\lambda_1^{E_{22}} = \xi_{22}\sqrt{\Delta} > 0$, and $\lambda_2^{E_{22}} = L\xi_{22}^2 + \gamma\xi_{22} + \mu_1$. Taking into account that

$$\lambda_2^{E_{21}}\lambda_2^{E_{22}} = \frac{\gamma^2}{P}\left(\mu_2 - \frac{\delta_1\gamma - P}{\gamma^2}\mu_1^2\right),$$

if $\mu \in T_3$, there are two cases:

(i) if $(2P - \delta_1\gamma)\mu_1 > 0$, then $E_3 = E_{22} = \left(0, -\frac{\mu_1}{\gamma}\right)$, with $\lambda_2^{E_{22}} = 0$, and $E_{21} = \left(0, -\frac{\delta_1\gamma - P}{\gamma^2}\mu_1^2\right)$ is a saddle node;

(ii) if $(2P - \delta_1\gamma)\mu_1 < 0$, then $E_3 = E_{21} = \left(0, -\frac{\mu_1}{\gamma}\right)$, with $\lambda_2^{E_{21}} = 0$, and $E_{22} = \left(0, -\frac{\delta_1\gamma - P}{\gamma^2}\mu_1^2\right)$ is a saddle node;

As $\lambda_2^{E_{21}}\lambda_2^{E_{22}} = 0$ iff $\mu \in T_3$, the eigenvalues $\lambda_2^{E_{21}}, \lambda_2^{E_{22}}$ do not change the sign if μ is outside T_3. Consequently, the results. □

For $|\mu|$ sufficiently small, denote by

$$\Delta_+ = \left\{(\mu_1, \mu_2), \mu_2 = \frac{\delta_1^2}{4P}\mu_1^2, (2P - \delta_1\gamma)\mu_1 > 0\right\},$$
$$\Delta_- = \left\{(\mu_1, \mu_2), \mu_2 = \frac{\delta_1^2}{4P}\mu_1^2, (2P - \delta_1\gamma)\mu_1 < 0\right\}.$$

the parameter strata where $\Delta(\mu) = 0$.

Proposition 3.2. *Assume $P > 0$. As the parameter μ crosses Δ_+ or Δ_- a saddle-node bifurcation takes place, involving equilibria E_{21} and E_{22}.*

Proof. Consider $\mu_0 \in \Delta_+ \cup \Delta_-$. Then $\xi_{21} = \xi_{22}$, and the eigenvalues of equilibrium E_{11} are $\lambda_1 = 0$, $\lambda_2 = \frac{2P - \delta_1\gamma}{2P}\mu_1$. The Jacobian matrix $Df(E_{21}, \mu_0)$, has for the zero eigenvalue the right eigenvector $v = (0,1)^T$ and the left eigenvector $w = (\delta_1, \gamma(\delta_1\gamma - 2P))^T$, in its lowest terms. It follows

$$w^T f_{\mu_2}(E_{21}, \mu_0) = -\frac{\delta_1\mu_1\gamma(\delta_1\gamma - 2P)}{2P}(1 + O(|\mu|)) \neq 0,$$
$$w^T[D^2 f(E_{21}, \mu_0)(v, v)] = -\delta_1\mu_1\gamma(\delta_1\gamma - 2P)(1 + O(|\mu|)) \neq 0,$$

for sufficiently small $\|\mu\|$. Thus, according to Sotomayor Theorem, a saddle-node bifurcation takes place. □

The topological type of the nontrivial equilibrium E_3 is established bellow.

Proposition 3.3. *Assume E_3 is in D. The following assertions are true.*

(i) If $\theta > 0, \delta_1\gamma - 2P > 0$ then E_3 is an attractor;

(ii) If $\theta < 0, \delta_1\gamma - 2P < 0$ then E_3 is a repellor;

(iii) If $\theta > 0, \delta_1\gamma - 2P < 0$ then E_3 is an attractor as $\mu_2 < \frac{\theta\gamma(\delta_1\gamma-P)-(\delta_1\gamma-2P)}{\theta\gamma^3}\mu_1^2$ and a repellor as $\mu_2 > \frac{\theta\gamma(\delta_1\gamma-P)-(\delta_1\gamma-2P)}{\theta\gamma^3}\mu_1^2$;

(iv) If $\theta < 0, \delta_1\gamma - 2P > 0$ then E_3 is an attractor as $\mu_2 > \frac{\theta\gamma(\delta_1\gamma-P)-(\delta_1\gamma-2P)}{\theta\gamma^3}\mu_1^2$ and a repellor as $\mu_2 < \frac{\theta\gamma(\delta_1\gamma-P)-(\delta_1\gamma-2P)}{\theta\gamma^3}\mu_1^2$;

(v) If $\mu_2 = \frac{\theta\gamma(\delta_1\gamma-P)-(\delta_1\gamma-2P)}{\theta\gamma^3}\mu_1^2$ then E_3 is a a nonhyperbolic equilibrium of Hopf type.

Proof. The eigenvalues $\lambda_{1,2}$ of E_3 satisfy the relations

$$\begin{aligned}
\lambda_1\lambda_2 &= \xi_1^*\xi_2^*\left(1+O(|\mu|)\right), \\
\lambda_1+\lambda_2 &= -\theta\xi_1^* - (\delta_1\gamma-2P)\left(\xi_2^*\right)^2 + O(\|\mu\|^2) \\
&= \left(-\gamma\theta\mu_2 + \frac{\theta\gamma\left(\delta_1\gamma-P\right)-\left(\delta_1\gamma-2P\right)}{\gamma^2}\mu_1^2\right)\left(1+O(|\mu|)\right).
\end{aligned}$$

Denote by $p(\mu) = \lambda_1+\lambda_2$, for small $|\mu|$, and by $H_1 = \{(\mu_1,\mu_2), p(\mu) = 0\}$. Using IFT applied to the equation $p(\mu_1,\mu_2) = 0$, it follows

$$H_1 = \left\{(\mu_1,\mu_2), \mu_2 = \frac{\theta\gamma\left(\delta_1\gamma-P\right)-\left(\delta_1\gamma-2P\right)}{\theta\gamma^3}\mu_1^2 + O\left(\mu_1^2\right)\right\} \tag{3.6}$$

provided that $\theta\gamma \neq 0$. This curve intersect region R_3 only if $\theta(\delta_1\gamma - 2P) < 0$. It is easy to see that $p(\mu) < 0$ if $\theta > 0, \delta_1\gamma - 2P > 0$ (thus E_3 is an attractor) and that $p(\mu) > 0$ if $\theta < 0, \delta_1\gamma - 2P < 0$ (thus, E_3 is a repellor).

As $\theta(\delta_1\gamma - 2P) > 0$, the topological type of the equilibrium E_3 does not change for parameters inside region R_3; thus, E_3 is an attractor if $\theta > 0, \delta_1\gamma - 2P > 0$; and E_3 is a repellor if $\theta < 0, \delta_1\gamma - 2P < 0$. □

Several transcritical bifurcations take place when two equilibria collide.

Proposition 3.4. *The following transcritical bifurcations occur for system (3.1):*
(i) at the point E_0 as the parameter crosses the curves Y_+ or Y_- (when $E_0 = E_1$);
(ii) at the point E_0 as the parameter crosses the curves X_+ (when $E_0 = E_{22}$) or X_- (when $E_0 = E_{21}$);
(iii) at the point E_1 as the parameter (μ_1,μ_2) crosses the curve T_1 (when $E_1 = E_3$);
(iv) at the point E_{22} as the parameter (μ_1,μ_2) crosses the curve T_3 (when $E_3 = E_{22}$), if, in addition $\frac{\partial\gamma}{\partial\mu_2} \neq 0$.

Proof. A Sotomayor Theorem ([11], p. 338) is used in order to prove these statements.

(i) The Jacobian matrix $Df(E_0,\mu_0)$ at $\mu_0 = (0,\mu_2)$, $\mu_2 \neq 0$, has a zero eigenvalue with the right eigenvector $v = (1,0)^T$ and the left eigenvector $w = (1,0)^T$. It follows

$$w^T f_{\mu_1}(E_0,\mu_0) = 0, \quad w^T Df_{\mu_1}(E_0,\mu_0) = 1 \neq 0, \quad w^T[D^2 f(E_0,\mu_0)(v,v)] = -2\theta \neq 0,$$

thus the transcritical bifurcation conditions are satisfied.

(ii) The Jacobian matrix $Df(E_0,\mu_0)$ at $\mu_0 = (\mu_1,0)$, $\mu_1 \neq 0$, has a zero eigenvalue with the right eigenvector $v = (0,1)^T$ and the left eigenvector $w = (0,1)^T$. It follows

$$w^T f_{\mu_2}(E_0,\mu_0) = 0, \quad w^T Df_{\mu_2}(E_0,\mu_0) = 1 \neq 0, \quad w^T[D^2 f(E_0,\mu_0)(v,v)] = 2\delta_1\mu_1 \neq 0,$$

ensuring the existence of a transcritical bifurcation.

(iii) Consider $\mu_0 = (\mu_1, \mu_2) \in T_1$, $\mu_1 \neq 0$, and μ_2 as a bifurcation parameter, $\mu_0 = \left(\mu_1, \frac{\mu_1}{\theta\gamma}\right)$. Then $v = (\gamma, \theta)^T$, in its lowest terms, and $w = (0,1)^T$ are right and left eigenvectors of the Jacobian matrix $Df(E_1, \mu_0)$, respectively, corresponding to the zero eigenvalue, and

$$w^T f_{\mu_2}(E_1, \mu_0) = 0, \quad w^T Df_{\mu_2}(E_1, \mu_0) = \theta + O(\mu_1) \neq 0,$$
$$w^T[D^2 f(E_1, \mu_0)(v, v)] = -2\theta + O(\mu_1) \neq 0,$$

consequently, for sufficiently small $|\mu|$, the conditions are satisfied.

(iv) Finally, consider $\mu_0 = (\mu_1, \mu_2) \in T_3$, $\mu_1 \neq 0$, and μ_2 as a bifurcation parameter, thus $\mu_0 = (\mu_1, \frac{\delta_1\gamma - P}{\gamma^2}\mu_1^2 + O(\mu_1^2))$. We find the eigenvectors $v = ((\delta_1\gamma - 2P)\mu_1, 1)^T$, in its lowest terms, and $w = (1,0)^T$, and

$$w^T f_{\mu_2}(E_3, \mu_0) = 0, \quad w^T Df_{\mu_2}(E_3, \mu_0) = -\frac{\partial\gamma}{\partial\mu_2}\frac{(\delta_1\gamma - 2P)}{\gamma}\mu_1^2 + O(\mu_1^3) \neq 0,$$
$$w^T[D^2 f(E_3, \mu_0)(v, v)] = 2\gamma(\delta_1\gamma - 2P)\mu_1 + O(\mu_1^2) \neq 0,$$

for sufficiently small $|\mu|$. □

From Proposition 3.3 it follows that system (3.1) may exhibit a Hopf bifurcation only in the hypothesis $\theta(\delta_1\gamma - 2P) < 0$.

Theorem 3.5. *For all $\gamma < 0$, and $\theta(\delta_1\gamma - 2P) < 0$, a nondegenerated Hopf bifurcation takes place at E_3, when the parameters (μ_1, μ_2) transversally cross the curve H_1, for sufficiently small $|\mu|$, if the the following condition is satisfied:*

$$V(\mu) := \mu_1 (5P + L - \delta_1\gamma - \gamma\theta(\delta_1\gamma - 2P)(1 + \gamma\theta)) \neq 0, \tag{3.7}$$

for $\mu \in H_1$. In addition,

1) if $V(\mu) < 0$ for $\mu \in H_1$, then the Hopf bifurcation is supercritical;

2) if $V(\mu) > 0$ for $\mu \in H_1$, then the Hopf bifurcation is subcritical.

Proof. To simplify the computation, we chose to cross curve H_1 in the direction of the $O\mu_2$ axis, thus $\mu_1 \neq 0$, is fixed, and μ_2 is the bifurcation parameter. Similar computations can be performed for other transversal directions.

The first condition for the Hopf bifurcation is satisfied, as $\frac{Re(\lambda_1)}{d\mu_2}_{|H_1} = -\frac{\theta\gamma}{2}(1 + O(\mu_1)) \neq 0$, for sufficiently small $|\mu|$. Applying the usual algorithm to compute the Lyapunov coefficient L_1 (see [6]), we obtain $sign(L_1(\mu)) = sign(V(\mu))$, for μ in H_1, hence the result follows from the Andronov-Hopf Theorem. □

We may now combine all the above results in order to derive the bifurcation diagrams.

For a fixed $\gamma < 0$, and $P > 0$, the curves $\delta_1\gamma - 2P = 0, \delta_1\gamma - P = 0$, $\theta = 0$, $\delta_1 = 0$, determine eight regions in the $(\theta, \delta_1)-$ plane, corresponding to the following cases:

B_1: $\theta > 0, \delta_1 > 0$;

B_2: $\delta_1\gamma - P < 0, \theta > 0, \delta_1 < 0$;

B_3: $\delta_1\gamma - P > 0, \delta_1\gamma - 2P < 0, \theta > 0, \delta_1 < 0$;

B_4: $\delta_1\gamma - 2P > 0, \theta > 0, \delta_1 < 0$;

B_5: $\theta < 0, \delta_1 > 0$;

B_6: $\delta_1\gamma - P < 0, \theta < 0, \delta_1 < 0$;
B_7: $\delta_1\gamma - P > 0, \delta_1\gamma - 2P < 0, \theta < 0, \delta_1 < 0$;
B_8: $\delta_1\gamma - 2P > 0, \theta < 0, \delta_1 < 0$;

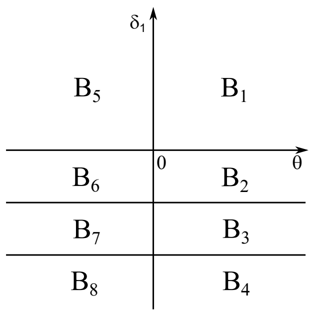

FIGURE 1. Eight regions in the (θ, δ_1) plane, $\gamma < 0$, for system(3.1).

For each region (see Fig. 1), in the parametric portraits in the (μ_1, μ_2)- plane, the parameter strata are determined by the origin and the bifurcation curves X_-, X_+, Y_-, Y_+, T_1, T_2, and H_1. As a consequence of Proposition 3.3, the curve H_1 is present only in regions B_1, B_2, B_3 and B_8.

Theorem 3.6. *For all $\gamma < 0$, and θ, δ, $\theta(\delta_1\gamma - 2P) > 0$ (in regions B_4, B_5, B_6, B_7 of the $(\theta, \delta_1) -$ plane), the parameter portraits consist of*

$$O \cup T_1 \cup T_3 \cup X_- \cup X_+ \cup Y_- \cup Y_+.$$

The four parameter portraits and the corresponding generic phase portraits are shown in Fig. 2, 3, 4, 5.

Remark 3.3. In regions B_1, B_2, B_3, and B_8 a Hopf bifurcation occurs when parameter cross H_1 and the first Lyapunov coefficient is nonzero. As well as in the nondegenerate case in [4], as the parameters move away from H_1, the limit cycle born through this bifurcation may encounter a saddle equilibrium, transforming into a homoclinic loop, or it may exist the visible neighborhood of origin in D, thus it disappears. In such cases there should exist o bifurcation curve L originating at $\mu = 0$, along which system (3.1) exhibits either a saddle homoclinic bifurcation or the limit cycle "blows up".

The following result is obtained.

Theorem 3.7. *For all $\gamma < 0$, and θ, δ, with $\theta(\delta_1\gamma - 2P) < 0$ (in regions B_1, B_2, B_3, and B_8 of the $(\theta, \delta_1) -$ plane), the parameter portrait consists of*

$$O \cup T_1 \cup T_3 \cup X_- \cup X_+ \cup Y_- \cup Y_+ \cup H_1 \cup L.$$

The parameter portraits and the generic phase portraits are shown in Fig. 6, 7, 8, 9.

Remark that in cases B_1 and B_2 we found only one possible position for the Hopf bifurcation curve. Fig. 6, 7 we represented both cases when the Hopf bifurcation is supercririal or subcritical. In cases B_3 and B_8 (fig. 8, 9) only the situations when the Hopf bifurcation is subcritical are considered. We found two different positions for the curve H_1, in each of the cases B_3 and B_8, represented in Fig. 8, 9.

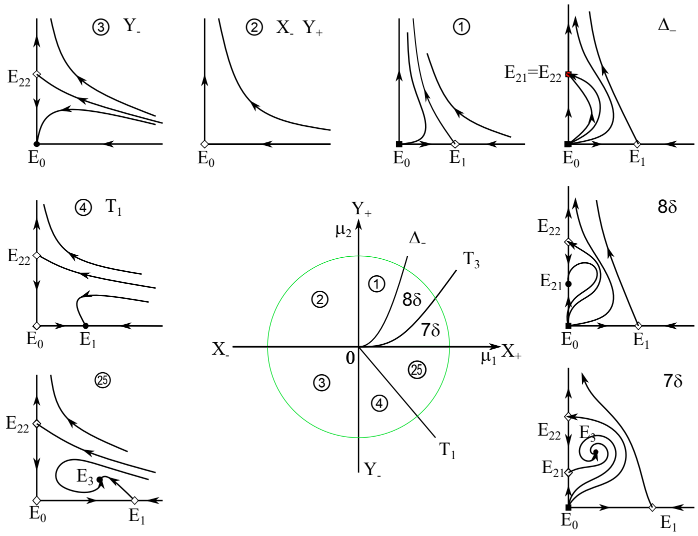

FIGURE 2. Parametric portrait and generic phase portraits in the case $\gamma < 0$, region B_4.

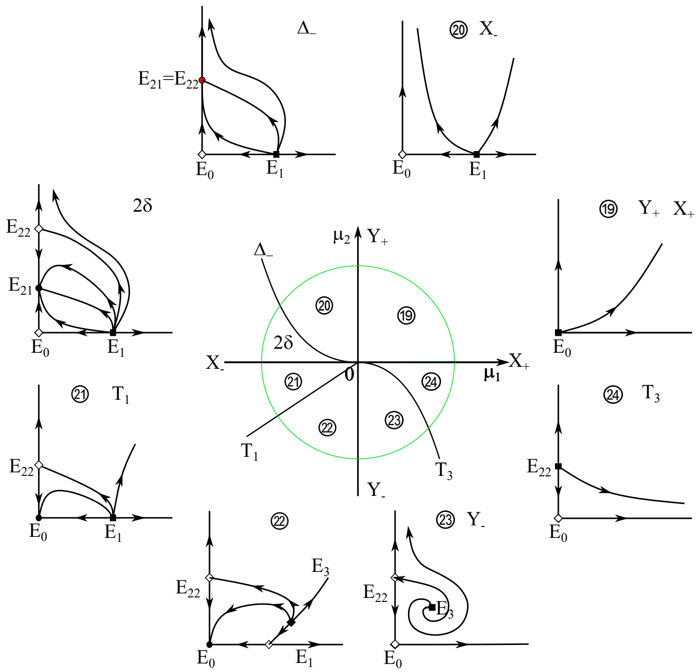

FIGURE 3. Parametric portrait and generic phase portraits in the case $\gamma < 0$, region B_5.

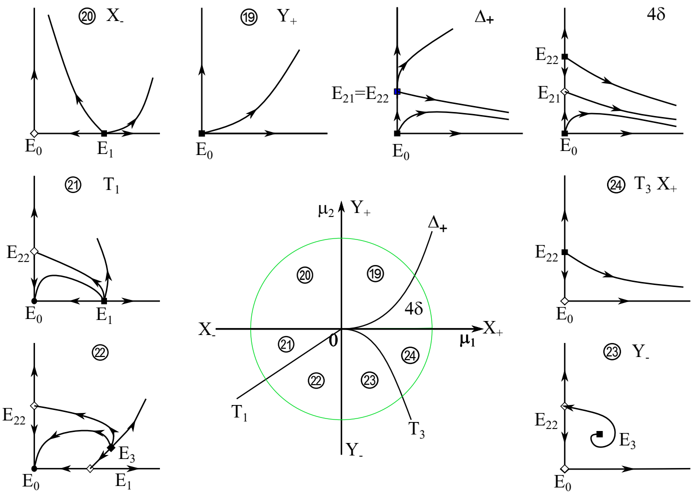

FIGURE 4. Parametric portrait and generic phase portraits in the case $\gamma < 0$, region B_6.

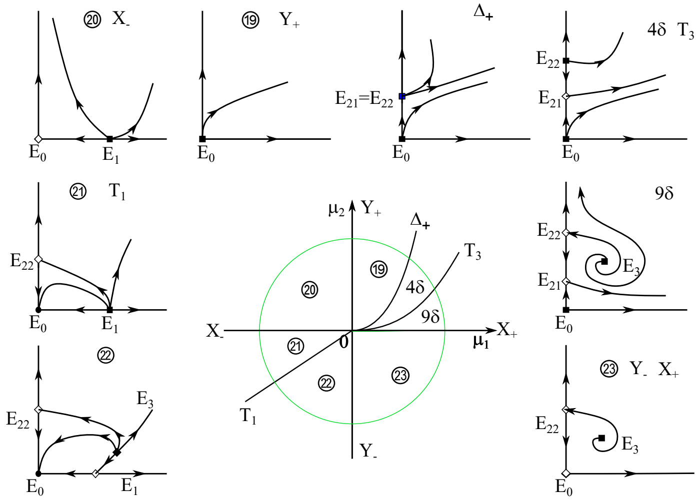

FIGURE 5. Parametric portrait and generic phase portraits in the case $\gamma < 0$, region B_7.

4. Analysis of the system when $\theta(0) = 0$ and $\delta(0) \neq 0$

The local dynamics and bifurcation of system (1.2) in the non-generic case $\theta(0) = 0$, $\delta(0) \neq 0$ can be obtained from the ones in the case $\theta(0) \neq 0$, $\delta(0) = 0$, studied in Sections 2, 3. Indeed, by performing the changes of variables $y_2 = \xi_1, y_1 = \xi_2$, and by reversing the time $\tau = -t$, system (1.2) transforms into

$$\begin{cases} \frac{dy_1}{d\tau} = y_1\left(-\mu_2 - \delta(\mu)y_1 + \frac{1}{\gamma(\mu)}y_2 - P(\mu)y_1^2 + S(\mu)y_1y_2 - Q(\mu)y_2^2\right) \\ \frac{dy_2}{d\tau} = y_2\left(-\mu_1 - \gamma(\mu)y_1 + \theta(\mu)y_2 - L(\mu)y_1^2 + M(\mu)y_1y_2 - N(\mu)y_2^2\right) \end{cases} . \quad (4.1)$$

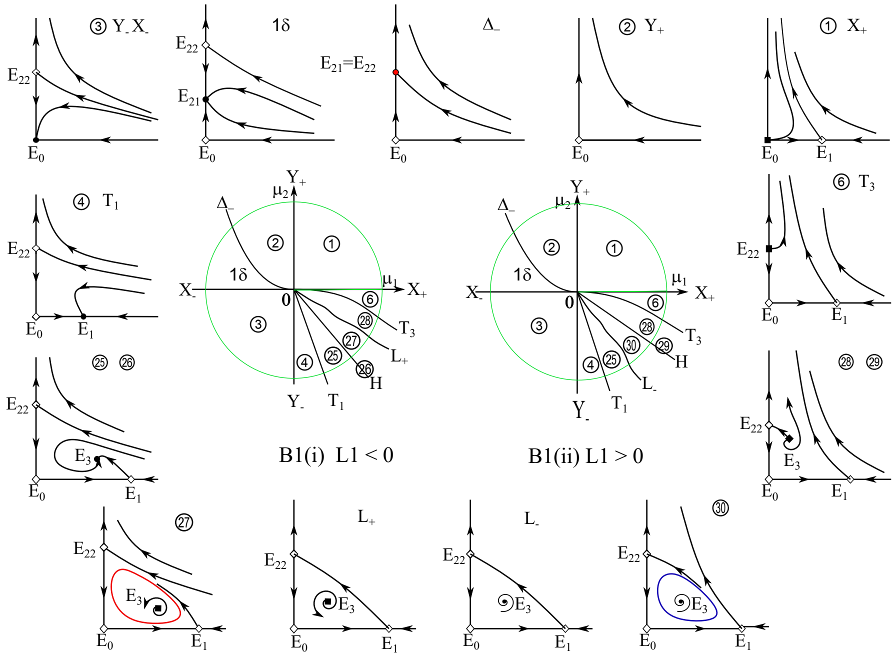

FIGURE 6. Parametric portrait and generic phase portraits in the case $\gamma < 0$, region B_1: (i) $L_1 < 0$, (ii) $L_1 > 0$.

Next, by changing the parameters $\bar{\mu}_1 = -\mu_2$, $\bar{\mu}_2 = -\mu_1$, and denoting $\bar{\theta} = \delta$, $\bar{\delta} = \theta$, $\bar{\gamma} = \frac{1}{\gamma}$, $\bar{N} = -P$, $\bar{M} = -S$, $\bar{L} = -Q$, $\bar{Q} = -L$, $\bar{S} = -M$, $\bar{P} = -N$, system (4.1) reads

$$\begin{cases} \frac{dy_1}{d\tau} = y_1 \left(\bar{\mu}_1 - \bar{\theta}\left(\bar{\mu}\right) y_1 + \bar{\gamma}\left(\bar{\mu}\right) y_2 + \bar{N}\left(\bar{\mu}\right) y_1^2 - \bar{M}\left(\bar{\mu}\right) y_1 y_2 + \bar{L}\left(\bar{\mu}\right) y_2^2\right), \\ \frac{dy_2}{d\tau} = y_2 \left(\bar{\mu}_2 - \frac{1}{\bar{\gamma}(\bar{\mu})} y_1 + \bar{\delta}\left(\bar{\mu}\right) y_2 + \bar{Q}\left(\bar{\mu}\right) y_1^2 - \bar{S}\left(\bar{\mu}\right) y_1 y_2 + \bar{P}\left(\bar{\mu}\right) y_2^2\right), \end{cases} \tag{4.2}$$

and it is obviously equivalent with (1.2).

The hypotheses $\theta(0) = 0$, $\delta(0) \neq 0$ for system (1.2) lead to the hypotheses $\bar{\delta}(0) = 0$, $\bar{\theta}(0) \neq 0$ for system (4.2). Note that in this case the coefficient N plays an important role in describing the local dynamics (1.2).

5. Conclusions

In this paper we studied local dynamics and bifurcation for the cubic Kolmogorov system (1.1), with coefficients depending on two parameters, in the hypothesis $p_{12}(0) \cdot p_{21}(0) < 0$. This study completes the one done in [9], where the case $p_{12}(0) \cdot p_{21}(0) > 0$ was investigated. Compared to the situation treated in [9] (called "the simple case"), we have obtained similar dynamics for certain parameter strata, but also bifurcations that are not present in the simple case. Such bifurcations arose mainly due to the presence of Hopf singularities. Two non-generic cases were analyzed for the equivalent system (1.2), corresponding to the situation when the hypothesis (HH.4): $p_{22}(0) \neq 0$ is not satisfied. In both cases the coefficient P plays a significant role, compared to the non-generic case treated in the first part of the study in [4]. The non-generic

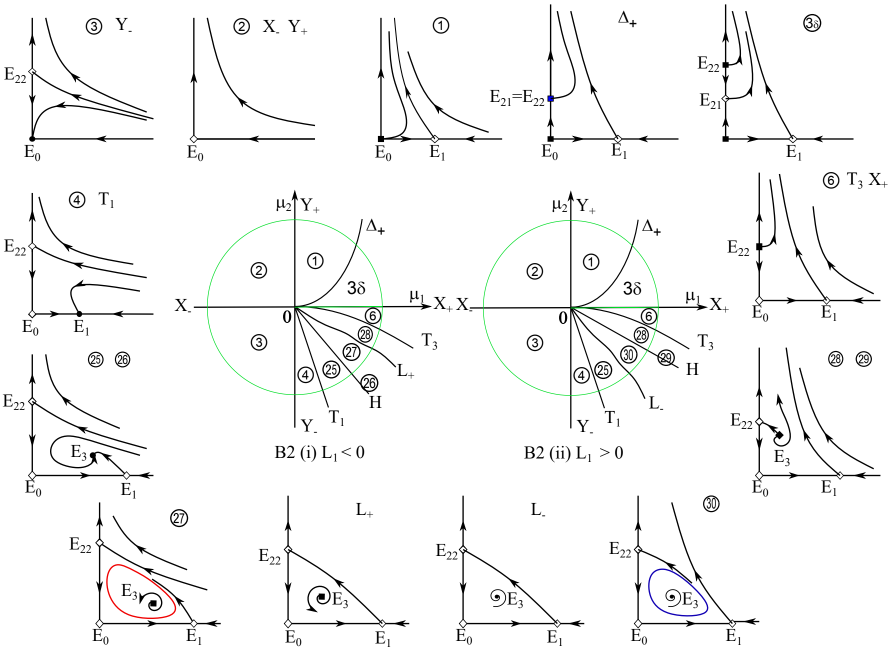

FIGURE 7. Parametric portrait and generic phase portraits in the case $\gamma < 0$, region B_2: (i) $L_1 < 0$, (ii) $L_1 > 0$.

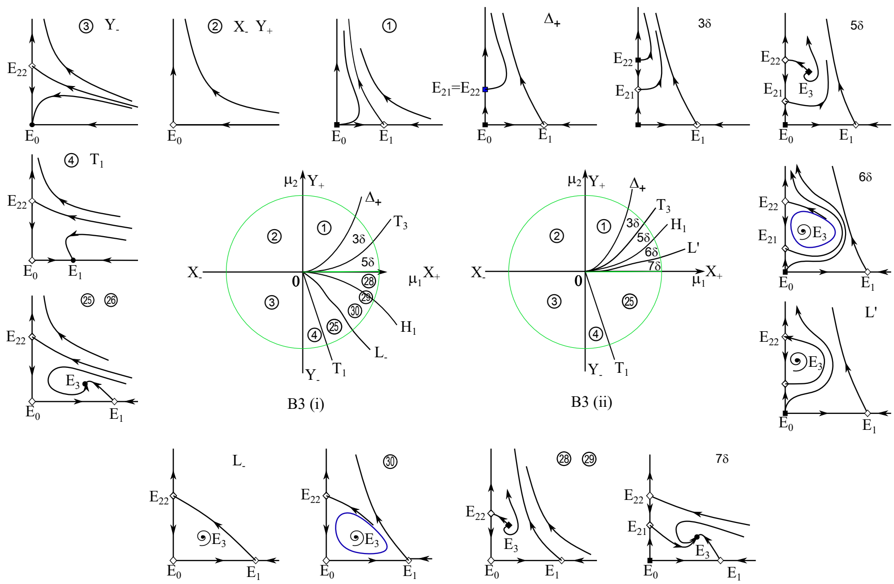

FIGURE 8. Parametric portrait and generic phase portraits in the case $\gamma < 0$, region B_3.

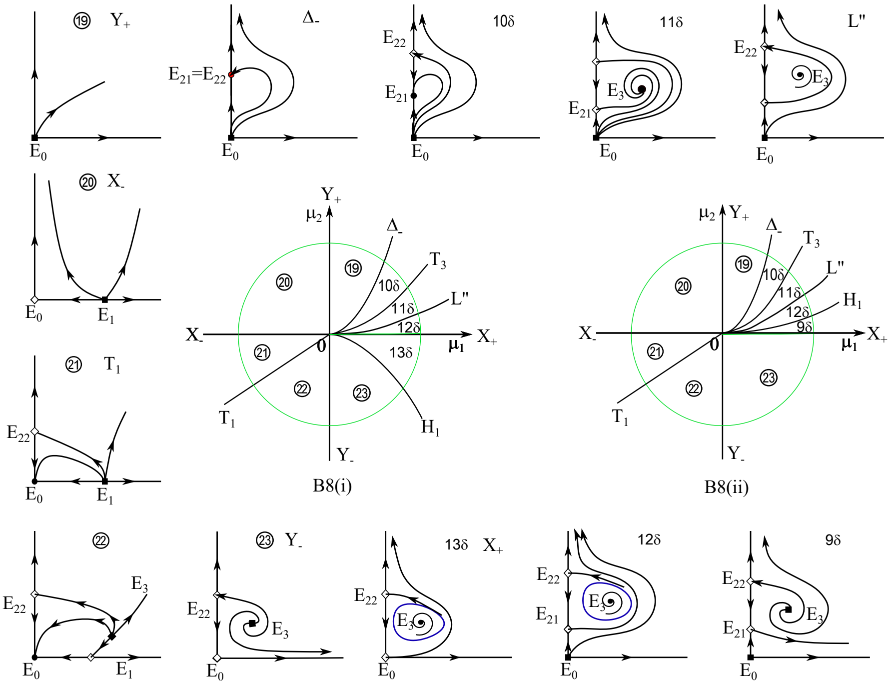

FIGURE 9. Parametric portrait and generic phase portraits in the case $\gamma < 0$, region B_8.

case for system (1.2), corresponding to the situation when the hypothesis (HH.1): $p_{11}(0) \neq 0$ is not satisfied, is reduced to the case when (HH.4) is not satisfied by using appropriate changes of variables, parameters and by reversing the time. In this case the coefficient N plays a significant role.

For the case $\delta \equiv 0, \theta(0) \neq 0$, we found four different situations, determined by the signatures of θ and P, each of them equivalent to one found in the generic case in [4]. For the case $\delta(0) = 0$, $\theta(0) \neq 0$, there where found eight nonequivalent situations, also determined by θ and P, that were not found in the generic case [4], or in [9] for the case $\gamma > 0$.

System (1.1) also appears as the truncated 2D amplitude system in the double Hopf bifurcation [3], [6]. This paper completes the study of the double Hopf bifurcation with the non-generic case when one of the conditions (HH.1) or (HH.4) is not fulfilled.

6. Acknowledgments

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